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ORIGINAL PAPER

Painlevé integrability and lump solutions for two extended (3 + 1)- and (2 + 1)-dimensional Kadomtsev–Petviashvili equations

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Abstract The current work introduces two extended (3 + 1)- and (2 + 1)-dimensional Painlevé integrable Kadomtsev–Petviashvili (KP) equations. The integrability feature of both extended equations is carried out by using the Painlevé test. We use the Hirota's bilinear strategy to explore multiple-soliton solutions for both extended models. Moreover, we formally furnish a class of lump solutions, for each extended KP equation, by using distinct values of the parameters. Proper graphs are furnished to highlight the characteristics of the lump, contour, and density solutions.

Keywords Kadomtsev–Petviashvili equation · Painlevé integrability · Multiple soliton solutions · Lump solutions

1 Introduction

Higher dimensional integrable systems have been dealt with recently from physical and mathematical points of view. Because of its significant effects in scientific areas, immense research has been invested in constructing and studying integrable models. The higher dimensional integrable equations have attracted lot of studies in various fields, such as solitary waves theory, ion-acoustic waves in plasmas, and many other scientific fields. The Painlevé integrable systems exhibit multiple solitons solutions and infinite conservation laws [1–16].

In [1], a (3 + 1)-dimensional Kadomtsev–Petviashvili-type equation was introduced as

$$au_{xt} - \frac{a^4+b^4-6a^2b^2}{16}u_{xxxx} - \frac{3(b^2-a^2)}{4}(u^2)_{xx} - \frac{3}{4}u_{yy} + \frac{3}{4}u_{zz} = 0, \quad (1)$$

which admits a weak dispersion term u_{xxxx}, was introduced in [1]. The Bäcklund transformation was used to investigate its integrability [1]. However, equation (1) is not Painlevé integrable in its given form.

Researchers invested a great deal of works to extend and generalize the integrable systems [11–20] to higher dimensional models. These works have led to the formation of many higher dimensional integrable systems. The higher-dimensional integrable models allow us to explore the solution dynamics through using a variety of powerful techniques [20–34]. Two comprehensive review papers appeared recently in [33,34] in the area of nonlinear wave structures in many physical settings, including nonlinear optics and photonics and matter waves in Bose–Einstein condensates. Researchers in [33] examined the two- and three-dimensional solitons and related states, such as quantum droplets, that can appear in optical systems, atomic Bose–Einstein condensates, and liquid crystals, among other physical settings. However, in the overview presented in [34], new findings concerning light bullets, the creation and diverse applications of few-cycle (ultra-narrow) optical

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pulses, and the emergence of rogue waves in various media were formally furnished.

We now propose an extended (3 + 1)-dimensional Painlevé integrable equation that reads

$$au_{xt}-\frac{a^4-6a^2b^2+b^4}{16}u_{xxxx}-\frac{3(b^2-a^2)}{4}(u^2)_{xx}\\+\alpha u_{xx}+\beta u_{xy}+\gamma u_{xz}+\lambda u_{zz}+\mu u_{yz}\\+\frac{\mu^2}{4\lambda}u_{yy}=0, \tag{2}$$

that extends (1) by adding more linear terms, where $a, b, \alpha, \beta, \gamma, \lambda$, and μ are real numbers, but $a, \lambda \neq 0$, and $u = u(x, y, z, t)$ is a sufficiently differentiable function with respect to the spatial and the temporal variables.

It is to be noted that for $\gamma = 0, \lambda = 0$, and $\mu = 0$, equation (2) will be transformed to a (2 + 1)-dimensional equation as

$$au_{xt}-\frac{a^4-6a^2b^2+b^4}{16}u_{xxxx}-\frac{3(b^2-a^2)}{4}(u^2)_{xx}\\+\alpha u_{xx}+\beta u_{xy}+\eta u_{yy}=0, \tag{3}$$

where $u = u(x, y, t)$. Equation (3) will be studied later in details. However, each equation (2) or (3) involves one nonlinear term $(u^2)_{xx}$ in addition to the remaining linear terms.

Studies on multiple soliton solutions and lump solutions have been flourishing recently for their important roles in nonlinear scientific fields. Lumps are characterized by locality with high amplitude. Lumps and solitons can be obtained via using the Hirota bilinear form [1–20]. Physically a lump may be detached (or emitted) from a line soliton, survives for a brief transient period in time, and then merges with the next adjacent soliton. Several powerful schemes, such as the Hirota method, and Darboux transformation method, have been formally employed for studying nonlinear integrable models [30–34].

In this article, we will first confirm THE complete integrability of the extended (3 + 1)- and (2 + 1)-dimensional KP equation (2) and (3) via using the Painlevé test to show each retrieves Painlevé integrability, whereas Eq. (1) is not Painlevé integrable. Consequently, multiple soliton solutions can be derived for both extended equations. In addition, a class of lump solutions will be furnished for both extended KP equations (2) and (3) by using distinct values of the employed parameters.

2 Extended (3 + 1)-dimensional nonlinear KP equation

We will first emphasize the Painlevé integrability for the extended (3 + 1)-dimensional nonlinear KP equation (2). Next, the Hirota's method will be used to formally derive the multiple soliton solutions for this extended model. A class of lump solutions will be furnished, and we will select two cases of the employed parameters to derive lump solutions. Other lump solutions will be furnished.

2.1 Painlevé analysis to Eq. (2)

Many powerful tools, such as Lax pair, and Painlevé Analysis, can be used to test integrability of any nonlinear evolution equation [1–10]. To show that Eq. (2) retrieves its integrability, although some linear terms were added, we employ the Painlevé analysis method [1–20].

We assume that Eq. (2) has a solution, given as a Laurent expansion about a singular manifold $\psi = \psi(x, y, z, t)$ as

$$u(x,y,z,t)=\sum_{k=0}^{\infty}u_k(x,y,z,t)\,\psi^{k-\gamma}. \tag{4}$$

This in turn gives a characteristic equation with one branch for resonances at $k = -1, 4, 5$, and 6. The resonance at $k = -1$ corresponds to $\psi(x, y, z, t) = 0$. Moreover, explicit expressions for u_1, u_2, and u_3 were furnished. However, we found that u_4, u_5, u_6 turn out to be arbitrary functions for all real values of the parameters $\alpha, \beta, \gamma, \lambda, \mu, a$ and b. Moreover, it is necessary to note that $a \neq 0, a \neq \pm b$, and $\lambda \neq 0$. This confirms the Painlevé integrability of the (3 + 1)-dimensional Eq. (2).

However, for $\gamma = 0, \lambda = 0$, and $\mu = 0$, the reduced (2+1)-dimensional equation is also Painlevé integrable via using Painlevé analysis. Based on this, we will study briefly this case related to the multiple soliton solutions in the next section.

2.2 Multiple soliton solutions

To derive the multiple soliton solutions of Eq. (2), we substitute

$$u(x, y, z, t) = e^{k_i x + r_i y + s_i z - c_i t}, \tag{5}$$

into the linear terms of (2) to obtain the dispersion relations as

$$c_i = \frac{-(a^4 - 6a^2b^2 + b^4)\,\lambda k_i^4 + 16\lambda k_i(\beta r_i + \gamma s_i) + 4\mu r_i(4\lambda s_i + \mu r_i) + 16\lambda(\alpha k_i^2 + \lambda s_i^2)}{16a\lambda k_i}, \tag{6}$$

and hence, the phase variables

$$\theta_i = k_i x + r_i y + s_i z - \frac{-(a^4 - 6a^2b^2 + b^4)\,\lambda k_i^4 + 16\lambda k_i(\beta r_i + \gamma s_i) + 4\mu r_i(4\lambda s_i + \mu r_i) + 16\lambda(\alpha k_i^2 + \lambda s_i^2)}{16a\lambda k_i}\, t, \tag{7}$$

follow immediately for $i = 1, 2, \cdots, N$. We next use the transformation

$$u(x, y, z, t) = R(\ln\, f(x, y, z, t))_{xx}, \tag{8}$$

into Eq. (2), where the auxiliary function $f(x, y, z, t)$ reads

$$f(x, y, z, t) = 1 + e^{\theta_1} = 1 + e^{k_1 x + r_1 y + s_1 z - \frac{-(a^4 - 6a^2b^2 + b^4)\lambda k_1^4 + 16\lambda k_1(\beta r_1 + \gamma s_1) + 4\mu r_1(4\lambda s_1 + \mu r_1) + 16\lambda(\alpha k_1^2 + \lambda s_1^2)}{16a\lambda k_1}\, t}, \tag{9}$$

used for deriving the single soliton solution. This gives

$$R = -\frac{(a^4 - 6a^2b^2 + b^4)}{2(a^2 - b^2)}. \tag{10}$$

Based on this, the single soliton solution follows upon substituting (10) and (9) into (8) as

$$u(x, y, z, t) = -\frac{(a^4 - 6a^2b^2 + b^4)e^{k_1 x + r_1 y + s_1 z - \frac{-\lambda(a^4 - 6a^2b^2 + b^4)k_1^4 + 16\lambda k_1(\beta r_1 + \gamma s_1) + 4\mu r_1(4\lambda s_1 + \mu r_1) + 16\lambda(\alpha k_1^2 + \lambda s_1^2)}{16a\lambda k_1}}}{2(a^2 - b^2)(1 + e^{k_1 x + r_1 y + s_1 z - \frac{-\lambda(a^4 - 6a^2b^2 + b^4)k_1^4 + 16\lambda k_1(\beta r_1 + \gamma s_1) + 4\mu r_1(4\lambda s_1 + \mu r_1) + 16\lambda(\alpha k_1^2 + \lambda s_1^2)}{16a\lambda k_1}}}. \tag{11}$$

For the two soliton solutions, we use the auxiliary function as

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2}, \tag{12}$$

where a_{12} is the phase shift of the interaction of solitons. To determine the phase shift a_{12}, we substitute (12) into (2), and solving we obtain

$$a_{12}=\frac{3\lambda(a^4-6a^2b^2+b^4)k_1^2k_2^2(k_1-k_2)^2+16\lambda^2(k_1s_2-k_2s_1)^2+4\mu^2(k_1r_2-k_2r_1)^2+16\lambda\mu(k_1r_2-k_2r_1)(k_1s_2-k_2s_1)}{3\lambda(a^4-6a^2b^2+b^4)k_1^2k_2^2(k_1+k_2)^2+16\lambda^2(k_1s_2-k_2s_1)^2+4\mu^2(k_1r_2-k_2r_1)^2+16\lambda\mu(k_1r_2-k_2r_1)(k_1s_2-k_2s_1)}, \tag{13}$$

which can be generalized as

$$a_{ij}=\frac{3\lambda(a^4-6a^2b^2+b^4)k_i^2k_j^2(k_i-k_j)^2+16\lambda^2(k_is_j-k_js_i)^2+4\mu^2(k_ir_j-k_jr_i)^2+16\lambda\mu(k_ir_j-k_jr_i)(k_is_j-k_js_i)}{3\lambda(a^4-6a^2b^2+b^4)k_i^2k_j^2(k_i+k_j)^2+16\lambda^2(k_is_j-k_js_i)^2+4\mu^2(k_ir_j-k_jr_i)^2+16\lambda\mu(k_ir_j-k_jr_i)(k_is_j-k_js_i)}, \tag{14}$$

for $1\le i<j\le 3$. This result shows that the phase shifts (14) depend only on a, b, λ and μ as well as on the coefficients of the spatial parameters k_n, r_n and s_n, $n=1,2,3$. Substituting (13) and (12) into (8) provides the two soliton solutions.

For the three soliton solutions, we apply the auxiliary function $f(x,y,z,t)$ as

$$\begin{aligned}f(x,y,z,t)&=1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}\\&+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+a_{12}a_{23}a_{13}e^{\theta_1+\theta_2+\theta_3}\end{aligned} \tag{15}$$

Substituting (15) into (8) provides three soliton solutions.

2.3 The (3 + 1)-dimensional model: lump solutions

Lump solution is a kind of rational function solution localized in all spatial directions [20–32]. This is unlike a soliton solution which is exponentially localized in all directions in spatial and temporal variables [1–12]. Usually, we use the generalized positive quadratic function to study the lump solutions. To derive lump solutions, we transform the (3 + 1)-dimensional KP equation (2) into a bilinear equation in operators as

$$\left(aD_xD_t-\frac{a^4-6a^2b^2+b^4}{16}D_x^4+\alpha D_x^2+\beta D_xD_y+\gamma D_xD_z+\lambda D_z^2+\mu D_yD_z+\frac{\mu^2}{4\lambda}D_y^2\right)f\cdot f=0, \tag{16}$$

where D_t, D_x, D_y, and D_z are the Hirota's bilinear derivative operators. To ease computational works, we substitute $\alpha=\beta=\gamma$, and $\mu=2\lambda$ in (3 + 1)-dimensional KP equation (2) to obtain

$$\begin{aligned}&au_{xt}-\frac{a^4-6a^2b^2+b^4}{16}u_{xxxx}\\&-\frac{3(b^2-a^2)}{4}(u^2)_{xx}+\alpha(u_{xx}+u_{xy}+u_{xz})\\&+\lambda(u_{zz}+2u_{yz}+u_{yy})=0.\end{aligned} \tag{17}$$

Consequently, equation (16) is transformed to

$$\begin{aligned}&a(ff_{xt}-f_xf_t)-\tfrac{a^4-6a^2b^2+b^4}{16}\\&\times(ff_{xxxx}-4f_{xxx}f_x+3(f_{xx})^2)\\&+\alpha\left((ff_{xx}-f_xf_x)+(ff_{xy}-f_xf_y)\right.\\&\left.+(ff_{xz}-f_xf_z)\right)\\&+\lambda\left((ff_{zz}-f_zf_z)+2(ff_{yz}-f_yf_z)\right.\\&\left.+(ff_{yy}-f_yf_y)\right)=0,\end{aligned} \tag{18}$$

obtained upon using

$$u(x,y,z,t)=-\frac{a^4-6a^2b^2+b^4}{2(a^2-b^2)}(\ln f(x,y,z,t))_{xx}. \tag{19}$$

To obtain the quadratic soliton solutions for (17), we set

$$\begin{aligned}g&=a_1x+a_2y+a_3z+a_4t+a_5,\\h&=a_6x+a_7y+a_8z+a_9t+a_{10},\\f&=g^2+h^2+a_{11},\end{aligned} \tag{20}$$

where a_j, $1\le j\le 11$ are real parameters that we will be derived. Substituting (20) in (18), we get a polynomial of the variables x, y, z, and t. To determine the parameters a_j, $1\le j\le 11$, we build up a system of equations of the coefficients of the variables and the constant terms. In what follows, we highlight some cases of a variety of parameters.

Case 1.

We first select

$$
\begin{aligned}
a_1 &= a_1, a_2 = a_2, a_3 = a_3, a_4 = a_4, a_{10} = a_{10},\\
a_5 &= \frac{a_1 a_{10}\left(\alpha(a_1+a_2+a_3)+a a_4\right)}{\sqrt{\lambda a_1\left(\alpha(a_1+a_2+a_3)+a a_4\right)}\,(a_2+a_3)}, a_2 \neq -a_3,\\
a_6 &= \frac{\sqrt{\lambda a_1\left(\alpha(a_1+a_2+a_3)+a a_4\right)}\,(a_2+a_3)}{\alpha(a_1+a_2+a_3)+a a_4},\\
a_7 &= -\frac{a_1 a_2\left(\alpha(a_1+a_2+a_3)+a a_4\right)}{\sqrt{\lambda a_1\left(\alpha(a_1+a_2+a_3)+a a_4\right)}\,(a_2+a_3)}, a_2 \neq -a_3,\\
a_8 &= -\frac{a_1 a_3\left(\alpha(a_1+a_2+a_3)+a a_4\right)}{\sqrt{\lambda a_1\left(\alpha(a_1+a_2+a_3)+a a_4\right)}\,(a_2+a_3)}, a_2 \neq -a_3,\\
a_9 &= \frac{\alpha^2(a_1^2+a_1a_2+a_1a_3)+\alpha\lambda(a_2+a_3)^2+\lambda a a_4(a_2+a_3)+\alpha a a_1 a_4}{a\sqrt{\lambda a_1\left(\alpha(a_1+a_2+a_3)+a a_4\right)}},\\
a_{11} &= \frac{3\left(a^4-6a^2b^2+b^4\right)\left(\alpha(a_1^2+a_1a_2+a_1a_3)+\lambda(a_2+a_3)^2+a a_1 a_4\right)a_1^2}{16\left(\alpha(a_1+a_2a_3)+a a_4\right)^2},
\end{aligned}
\tag{21}
$$

which needs to satisfy

$$
\begin{aligned}
&\alpha(a_1+a_2a_3)+a a_4 \neq 0, \text{ and}\\
&\lambda a_1(\alpha(a_1+a_2a_3)+a a_4) > 0,
\end{aligned}
\tag{22}
$$

to obtain a well-defined function $f(x, y, z, t)$ and its positiveness. We can furnish a class of lump solutions to Eq. (17) by using $u(x, y, z, t))$ as follows

$$
\begin{aligned}
u(x, y, z, t) &= -\frac{(b^4-6a^2b^2+b^4)}{2(a^2-b^2)}\times(\ln(f(x, y, z, t))_{xx},\\
&= -\frac{(b^4-6a^2b^2+b^4)}{2(a^2-b^2)}\times\frac{2(a_1^2+a_6^2)f-4(a_1g+a_6h)^2}{f^2},
\end{aligned}
\tag{23}
$$

where f, g, and h are given earlier in (20). Note that the obtained lump solutions $u(x, y, z, t) \to 0$ if and only if $g^2+h^2 \to \infty$.

For example, selecting

$$
\begin{aligned}
&a_1 = 1, a_2 = 2, a_3 = 1, a_4 = 2, a_{10} = 2, a = 6,\\
&b = 2, \lambda = 4, \alpha = 1,
\end{aligned}
\tag{24}
$$

gives

$$
\begin{aligned}
&a_5 = \frac{4}{3}, a_6 = \frac{3}{2}, a_7 = -\frac{4}{3}, a_8 = -\frac{2}{3},\\
&a_9 = \frac{49}{12}, a_{11} = \frac{273}{16}.
\end{aligned}
\tag{25}
$$

The lump solution follows as

$$
u = \frac{111384t^2+(90720x+16128y+8064z+120960)\,t+18144x^2-32256y^2-32256yz-8064z^2+48384x-63000}{\left(229t^2+180tx-32ty-16tz+36x^2+64y^2+64yz+16z^2+240t+96x+253\right)^2}. \tag{26}
$$

Using the parameters as given earlier, and substituting $z = 1$, $t = 2$ leads to (Figs. 1, 2, 3)

$$u(x, y, z, t) = \frac{504(36x^2 - 64y^2 + 456x + 1255)}{(36x^2 + 64y^2 + 456x + 163)^2}. \quad (27)$$

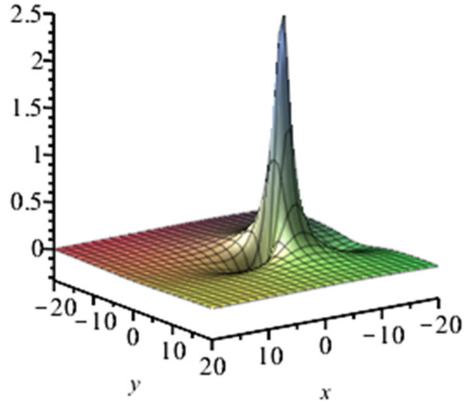

Fig. 1 Profile of the lump solution for (27), $-20 \le x, y \le 20$

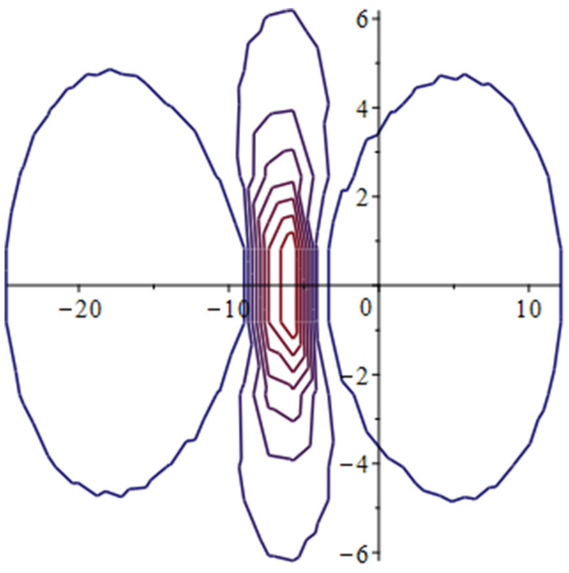

Fig. 2 Profile of the contour plot for (27), $-40 \le x, y \le 40$

Case 2.

We next select

$$\begin{aligned}
&a_1 = a_1, a_2 = a_2, a_3 = a_3, a_5 = a_5, a_6 = a_6, a_7 = a_7, a_8 = a_8, a_{10} = a_{10},\\
&a_4 = -\frac{\alpha(a_1+a_2+a_3)(a_1^2+a_6^2)+\lambda a_1(a_2+a_3)^2-\lambda a_1(a_7+a_8)^2+2\lambda a_6(a_2+a_3)(a_7+a_8)}{a({a_1}^2+{a_6}^2)},\\
&a_9 = -\frac{\alpha(a_6+a_7+a_8)(a_1^2+a_6^2)+2\lambda a_1(a_2+a_3)(a_7+a_8)-\lambda a_6(a_2+a_3)^2+\lambda a_6(a_7+a_8)^2}{a({a_1}^2+{a_6}^2)},\\
&a_{11} = \frac{3(a^4-6a^2b^2+b^4)(a_1^2+a_6^2)^3}{16\lambda(a_1a_7+a_1a_8-a_2a_6-a_3a_6)^2},
\end{aligned} \quad (28)$$

where $a \neq 0$, $\lambda \neq 0$, to derive a well-defined function $f(x, y, z, t)$ as furnished earlier. A class of lump solutions to the Eq. (17) is furnished by using $u(x, y, z, t))$ as follows

$$\begin{aligned}
u(x, y, z, t) &= -\frac{a^4-6a^2b^2+b^4}{2(a^2-b^2)}(\ln(f(x, y, z, t))_{xx},\\
&= -\frac{a^4-6a^2b^2+b^4}{2(a^2-b^2)}\left(\frac{2(a_1^2+a_6^2)f-4(a_1g+a_6h)^2}{f^2}\right),
\end{aligned} \quad (29)$$

where f, g, and h are given earlier in (20). Note that the obtained lump solutions $u(x, y, z, t) \to 0$ if and only if $g^2 + h^2 \to \infty$.

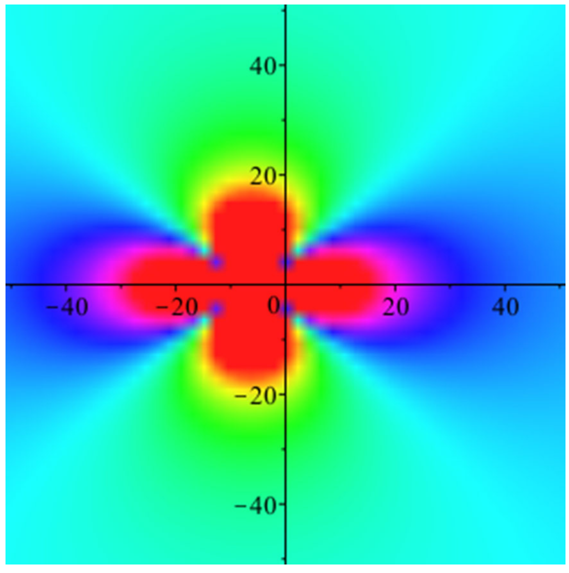

Fig. 3 Profile of the density plot for (27), $-50 \le x, y \le 50$

3 Extended (2 + 1)-dimensional nonlinear KP equation

Setting $\gamma = 0, \lambda = 0$ and $\mu = 0$ in (2), the (3 + 1)-dimensional Painlevé integrable (2) reduces to the (2 + 1)-dimensional equation

$$au_{xt} - \frac{a^4 - 6a^2b^2 + b^4}{16} u_{xxxx} - \frac{3(b^2 - a^2)}{4}(u^2)_{xx} + \alpha u_{xx} + \beta u_{xy} + \eta u_{yy} = 0, \quad (30)$$

where a, b, α, β and η, are real parameters, $a \neq 0$, and $u = u(x, y, t)$ is a sufficiently differentiable function with respect to the spatial and the temporal variables x, y, and t.

3.1 Painlevé analysis

Following the Painlevé analysis that we employed earlier for the (3 + 1)-dimensional structure (2) shows that the (2 + 1)-dimensional equation (30) is completely Painlevé integrable. Moreover, the resonance points were derived as $-1, 4, 5, 6$, the same as obtained for the (3 + 1)-dimensional model (2).

3.2 Multiple soliton solutions

Substituting

$$u(x, y, t) = e^{k_i x + r_i y - c_i t}, \quad (31)$$

into the linear terms of (30) gives the dispersion relation as

$$c_i = \frac{-(a^4 - 6a^2b^2 + b^4)k_i^4 + 16(\alpha k_i^2 + \beta k_i r_i + \eta r_i^2)}{16ak_i}, \quad i = 1, 2, \ldots, N, \quad (32)$$

that gives the phase variable as

$$\theta_i = k_i x + r_i y - \frac{-(a^4 - 6a^2b^2 + b^4)k_i^4 + 16(\alpha k_i^2 + \beta k_i r_i + \eta r_i^2)}{16ak_i} t, \quad i = 1, 2, \ldots, N, \quad (33)$$

follows immediately. We next use the transformation

$$u(x, y, t) = -\frac{a^4 - 6a^2b^2 + b^4}{2(a^2 - b^2)}(\ln f(x, y, t))_{xx}, \quad (34)$$

into Eq. (30), where the auxiliary function $f(x, y, t)$ is

$$f(x, y, t) = 1 + e^{\theta_1} = 1 + e^{k_1 x + r_1 y - \frac{-(a^4 - 6a^2b^2 + b^4)k_1^4 + 16(\alpha k_1^2 + \beta k_1 r_1 + \eta r_1^2)}{16ak_1} t}, \quad (35)$$

and hence, the single soliton solution reads

$$u(x, y, t) = -\frac{(a^4 - 6a^2b^2 + b^4)k_1^2 e^{k_1 x + r_1 y - \frac{-(a^4 - 6a^2b^2 + b^4)k_1^4 + 16(\alpha k_1^2 + \beta k_1 r_1 + \eta r_1^2)}{16ak_1} t}}{2(a^2 - b^2)\left(1 + e^{k_1 x + r_1 y - \frac{-(a^4 - 6a^2b^2 + b^4)k_1^4 + 16(\alpha k_1^2 + \beta k_1 r_1 + \eta r_1^2)}{16ak_1} t}\right)^2}. \quad (36)$$

For the two soliton solutions, the auxiliary function reads

$$f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2}, \quad (37)$$

that will lead to the following phase shifts

$$a_{12} = \frac{3(a^4 - 6a^2b^2 - b^4)k_1^2k_2^2(k_1 - k_2)^2 + 16\eta(k_1r_2 - k_2r_1)^2}{3(a^4 - 6a^2b^2 - b^4)k_1^2k_2^2(k_1 + k_2)^2 + 16\eta(k_1r_2 - k_2r_1)^2}, \quad (38)$$

which can be generalized as

$$a_{ij} = \frac{3(a^4 - 6a^2b^2 - b^4)k_i^2k_2^2(k_i - k_j)^2 + 16\eta(k_ir_j - k_jr_i)^2}{3(a^4 - 6a^2b^2 - b^4)k_i^2k_j^2(k_i + k_j)^2 + 16\eta(k_ir_j - k_jr_i)^2}, \quad 1 \leq i < j \leq 3, \quad (39)$$

where the phase shifts (39) depend only on the parameters a, b, and η, and on the spatial coefficients k_n, and $r_n, n = 1, 2, 3$. Substituting (38) and (37) into (34) gives the two soliton solutions. In addition, the three soliton solutions can be obtained as employed earlier.

3.3 The (2 + 1)-dimensional model: lump solutions

The bilinear equation of the KP equation (30) can be set as

$$\left(aD_xD_t - \frac{a^4-6a^2b^2+b^4}{16}D_x^4 + \alpha D_x^2 + \beta D_xD_y + \eta D_y^2\right) f\cdot f = 0, \tag{40}$$

where D_t, D_x, and D_y are the Hirota's bilinear derivative operators. To ease computational works, we substitute $\alpha = \beta$, in (2 + 1)-dimensional KP equation (30) to obtain

$$au_{xt} - \frac{a^4-6a^2b^2+b^4}{16}u_{xxxx} - \frac{3(b^2-a^2)}{4}(u^2)_{xx} + \alpha(u_{xx}+u_{xy}) + \eta u_{yy} = 0. \tag{41}$$

Consequently, Eq. (40) is transformed to

$$\begin{aligned} &a(ff_{xt} - f_xf_t) \\ &-\tfrac{a^4-6a^2b^2+b^4}{16}(ff_{xxxx} - 4f_{xxx}f_x + 3(f_{xx})^2) \\ &+\alpha\left((ff_{xx} - f_xf_x) + (ff_{xy} - f_xf_y)\right) \\ &+\eta(ff_{yy} - f_yf_y) = 0, \end{aligned} \tag{42}$$

where we applied

$$u(x, y, t) = -\frac{b^4-6a^2b^2+b^4}{2(a^2-b^2)}(\ln f(x, y, t))_{xx}. \tag{43}$$

The quadratic soliton solutions for Eq. (41) can be obtained by using the assumptions

$$\begin{aligned} g &= a_1x + a_2y + a_3t + a_4, \\ h &= a_5x + a_6y + a_7t + a_8, \\ f &= g^2 + h^2 + a_9, \end{aligned} \tag{44}$$

where $a_j, 1 \le j \le 9$ are real parameters that we will be derived. Substituting (44) in (42), we get a polynomial of the variables x, y, and t, where the parameters $a_j, 1 \le j \le 9$, can be obtained as furnished earlier:

Case 1.

In this case, we select

$$\begin{aligned} a_1 &= \frac{a_6\sqrt{\alpha\eta}}{\alpha}, \\ a_2 &= -\frac{aa_3}{\alpha}, \\ a_4 &= \frac{a_6a_8\alpha}{aa_3}, \\ a_5 &= \frac{aa_3\sqrt{\alpha\eta}}{\alpha^2}, \\ a_7 &= -\frac{a_6\alpha}{a}, \\ a_9 &= \frac{2\eta\left(\alpha^2a_6^2(a^4-6a^2b^2+b^4)(\alpha^2a_6^2+a^2a_3^2)\right)}{16\alpha^4}, \end{aligned} \tag{45}$$

where

$$a, a_3, \alpha \ne 0, \tag{46}$$

to obtain a well-defined function $f(x, y, t)$, and to strengthen the localization of $u(x, y, t)$ in all directions in the space. Hence, lump solutions to the KP equation (41) read

$$\begin{aligned} u(x, y, t) &= -\tfrac{b^4-6a^2b^2+b^4}{2(a^2-b^2)}(\ln(f(x, y, z, t))_{xx}, \\ &= -\tfrac{b^4-6a^2b^2+b^4}{2(a^2-b^2)}\tfrac{2(a_1^2+a_5^2)f-4(a_1g+a_5h)^2}{f^2}, \end{aligned} \tag{47}$$

where f, g, and h are given earlier in (44).

For example, selecting

$$\begin{aligned} &a_3 = 1, a_6 = 2, a_8 = 2, a = 6, b = 2, \eta = 2, \\ &\alpha = 1, t = 2, \end{aligned} \tag{48}$$

gives the lump solution as

$$\begin{aligned} &u(x, y, t) \\ &\quad = 1008 \times \frac{24\sqrt{2}x - t^2 + 12\,ty + 72\,x^2 - 36\,y^2 - 6044}{\left(24\sqrt{2}x + t^2 - 12\,ty + 72\,x^2 + 36\,y^2 + 6052\right)^2}. \end{aligned} \tag{49}$$

Case 2.

We next choose

$$a_1 = a_1, a_2 = a_2, a_4 = a_4, a_5 = a_5, a_6 = a_6, a_8 = a_8,$$

$$a_3 = \frac{\alpha(a_1+a_2)(a_1^2+a_5^2)+\eta(a_1a_2^2-a_1a_6^2+2a_2a_5a_6)}{a(a_1^2+a_5^2)},$$
$$a \neq 0,$$
$$a_7 = -\frac{\alpha(a_5+a_6)(a_1^2+a_5^2)+\eta(2a_1a_2a_6-a_5a_2^2+a_5a_6^2)}{a(a_1^2+a_5^2)},$$
$$a \neq 0,$$
$$a_9 = \frac{3(a^4-6a^2b^2+b^4)(a_1^2+a_5^2)^3)}{16\eta(a_1a_6-a_2a_5)^2}, \quad (50)$$

where $a \neq 0$, $\eta \neq 0$, and the determinant condition

$$\Delta = (a_1a_6 - a_2a_5) = \begin{vmatrix} a_1 & a_2 \\ a_5 & a_6 \end{vmatrix} \neq 0, \quad (51)$$

to secure a well-defined function $f(x, y, t)$, and localization of $u(x, y, t)$ in all spatial sides, respectively. This gives lump solutions to the (3 + 1)-dimensional KP equation (41) as

$$u(x,y,t) = -\frac{b^4-6a^2b^2+b^4}{2(a^2-b^2)}(\ln(f(x,y,t))_{xx},$$
$$= -\frac{b^4-6a^2b^2+b^4}{\times}2(a^2-b^2)\frac{2(a_1^2+a_5^2)f-4(a_1g+a_5h)^2}{f^2}, \quad (52)$$

where f, g, and h are given earlier in (44).

For example, selecting the same values of the parameters as in the previous case the lump solution as

$$u(x,y,t) = \frac{35000\,x^2+(56000\,y-8400)x+16800\,y^2-2240\,y-9187892}{(50\,x^2+80\,xy+40\,y^2-12\,x-16\,y+13127)^2}. \quad (53)$$

4 Conclusions

We gave two extended (3 + 1)- and (2 + 1)-dimensional Kadomtsev–Petviashvili (KP) equations in shallow water waves. Shallow water waves play an important role in the study of fluid dynamics, which involves the development of ground water resources, sea water intrusion, marine engineering, and many other fields. The two extended KP equations were proposed to explore new multiple solitons solutions and more lump solutions as well. We used the Painlevé analysis method to ensure the integrability of each extended equation and to confirm that the newly added linear terms did not end the integrability feature. The Hirota's method was employed to exhibit multiple soliton solutions for each examined equation. Two sets of lump solutions were derived for proposed model. The results are helpful to understand the dynamic properties of extended KP equations in fluid mechanics.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

References

1. Tian, H., Niu, Y., Behzad Ghanbari, B., Zhang, Z., Cao, Y.: Integrability and high-order localized waves of the (4 + 1)-dimensional nonlinear evolution equation. Chaos Solitons Fractals **162**, 112406 (2022)
2. Ma, Y.-L., Wazwaz, A.M., Li, B.-Q.: New extended Kadomtsev–Petviashvili equation: multiple soliton solutions, breather, lump and interaction solutions. Nonlinear Dyn. **104**, 1581–1594 (2021)
3. Ma, Y.-L., Wazwaz, A.M., Li, B.-Q.: Novel bifurcation solitons for an extended Kadomtsev–Petviashvili equation in fluids. Phys. Lett. A **413**, 127585 (2021)
4. Saha Ray, S.: Painlevé, analysis, group invariant analysis, similarity reduction, exact solutions, and conservation laws of Mikhailov-Novikov-Wang equation. Int. J. Geom. Methods Mod. Phys. **18**(6), 2150094 (2021)
5. Saha Ray, S.: A numerical solution of the coupled sine-Gordon equation using the modified decomposition method. Appl. Math. Comput. **175**(2), 1046–1054 (2006)
6. Sahoo, S., Saha Ray, S.: Solitary wave solutions for time fractional third order modified KdV equation using two reliable techniques (G'/G)-expansion method and improved (G'/G)-expansion method. Phys. A Stat. Mech. Appl. **448**, 265–282 (2016)
7. Khalique, C.M., Adem, K.R.: Exact solutions of the (2 + 1)-dimensional Zakharov–Kuznetsov modified equal width equation using Lie group analysis. Math. Comput. Modell. **54**(1–2), 184–189 (2011)
8. Khalique, C.M.: Solutions and conservation laws of Benjamin–Bona–Mahony–Peregrine equation with power-law and dual power-law nonlinearities. Pramana J. Phys. **80**(6), 413–427 (2013)
9. Khalique, C.M., Mhalanga, I.: Travelling waves and conservation laws of a (2 + 1)-dimensional coupling system with Korteweg–de Vries equation. Appl. Math. Nonlinear Sci. **3**(1), 241–254 (2018)
10. Zhou, Q., Zhu, Q.: Optical solitons in medium with parabolic law nonlinearity and higher order dispersion. Waves Random Complex Media **25**(1), 52–59 (2014)
11. Zhou, Q.: Optical solitons in the parabolic law media with high-order dispersion. Optik **125**(18), 5432–5435 (2014)

12. Wang, G.: A new (3 + 1)-dimensional Schrödinger equation: derivation, soliton solutions and conservation laws. Nonlinear Dyn. **104**, 1595–1602 (2021)
13. Wang, G., Yanga, K., Guc, H., Guana, F., Kara, A.H.: A (2 + 1)-dimensional sine-Gordon and sinh-Gordon equations with symmetries and kink wave solutions. Nuclear Phys. B **953**, 114956 (2020)
14. Ebaid, A., Khaled, S.: New types of exact solutions for nonlinear Schrödinger equation with cubic nonlinearity. J. Comput. Appl. Math. **235**(8), 1984–1992 (2011)
15. Ebaid, A.: Exact solitary wave solutions for some nonlinear evolution equations via Exp-function method. Phys. Lett. A **365**, 213–219 (2007)
16. Wazwaz, A.M., Tantawy, S.A.E.: Solving the (3 + 1)-dimensional KP Boussinesq and BKP-Boussinesq equations by the simplified Hirota method. Nonlinear Dyn. **88**, 3017–3021 (2017)
17. Wazwaz, A.M.: Painlevé analysis for a new integrable equation combining the modified Calogero–Bogoyavlenskii–Schiff (MCBS) equation with its negative-order form. Nonlinear Dyn. **91**, 877–883 (2018)
18. Kaur, L., Wazwaz, A.M.: Painlevé analysis and invariant solutions of generalized fifth-order nonlinear integrable equation. Nonlinear Dyn. **94**, 2469–2477 (2018)
19. Xu, G.Q., Wazwaz, A.M.: Bidirectional solitons and interaction solutions for a new integrable fifth-order nonlinear equation with temporal and spatial dispersion. Nonlinear Dyn. **101**, 581–595 (2020)
20. Xu, G.Q., Wazwaz, A.M.: Characteristics of integrability, bidirectional solitons and localized solutions for a (3 + 1)-dimensional generalized breaking soliton equation. Nonlinear Dyn. **96**, 1989–2000 (2019)
21. Schelte, C., Camelin, P., Marconi, M., Garnache, A., Huyet, G., Beaudoin, G., Sagnes, I., Giudici, M., Javaloyes, J., Gurevich, S.V.: Third order dispersion in time-delayed systems. Phys. Rev. Lett. **123**, 043902 (2019)
22. Hirota, R.: The Direct Method in Soliton Theory. Cambridge University Press, Cambridge (2004)
23. Biswas, A., Khalique, C.M.: Stationary solution of nonlinear Schrö dinger equation with log law nonlinearity by Lie symmetry analysis. Waves Random Complex Media **21**(4), 554–558 (2011)
24. Mihalache, D.: Multidimensional localized structures in optics and Bose–Einstein condensates: a selection of recent studies. Rom. J. Phys. **59**(3/4), 295–312 (2014)
25. Mihalache, D.: Multidimensional localized structures in optical and matter-wave media: a topical survey of recent literature. Rom. Rep. Phys. **69**, 403 (2017)
26. Leblond, H., Mihalache, D.: Models of few optical cycle solitons beyond the slowly varying envelope approximation. Phys. Rep. **523**, 61–126 (2013)
27. Leblond, H., Mihalache, D.: Few-optical-cycle solitons: modified Korteweg–de Vries sine-Gordon equation versus other non-slowly-varying-envelope-approximation models. Phys. Rev. A **79**(1–7), 063835 (2009)
28. Khuri, S.A.: Soliton and periodic solutions for higher order wave equations of KdV type (I). Chaos Solitons Fractals **26**(8), 25–32 (2005)
29. Khuri, S.: Exact solutions for a class of nonlinear evolution equations: a unified Ansätze approach. Chaos Solitons Fractals **36**(5), 1181–1188 (2008)
30. Wazwaz, A.M.: Multiple soliton solutions for the (2 + 1)-dimensional asymmetric Nizhanik–Novikov–Veselov equation. Nonlinear Anal. Ser. A Theory Methods Appl. **72**, 1314–1318 (2010)
31. Wazwaz, A.M.: Multiple soliton solutions and multiple complex soliton solutions for two distinct Boussinesq equations. Nonlinear Dyn. **85**, 731–737 (2016)
32. Wazwaz, A.M.: Two wave mode higher-order modified KdV equations: essential conditions for multiple soliton solutions to exist. Int. J. Heat Fluid Flow **27**, 2223–2230 (2017)
33. Kartashov, Y.V., Astrakharchik, G.E., Malomed, B.A., Torner, L.: Frontiers in multidimensional self-trapping of nonlinear fields and matter. Nat. Rev. Phys. **1**, 185–197 (2019)
34. Mihalache, D.: Localized structures in optical and matter-wave media: a selection of recent studies. Rom. Rep. Phys. **73**, 403 (2021)