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Observer-based adaptive fuzzy finite time control for non-strict feedback nonlinear systems with unmodeled dynamics and input delay

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Abstract In this paper, we consider observer-based adaptive fuzzy finite time control scheme for non-strict feedback uncertain nonlinear systems with unmodeled dynamics and input delay. A fuzzy state observer is employed to estimate the unmeasurable states and the unknown nonlinearities are identified by the fuzzy logic systems in each step. The design difficulty caused by the unmodeled dynamics and input delay is tackled by a dynamic signal and a compensation signal, respectively. Based on the proposed compensation signal, the considered input delay can be unknown and time varying. To decrease the computational burden, the dynamic surface control (DSC) scheme is adopted in the design process. In the framework of finite time Lyapunov theory, an effective adaptive fuzzy finite time controller has been obtained by combining the idea of backstepping technology with DSC scheme. The proposed method not only solves the algebraic loop problem, but also realizes the finite time stability performance constraint in the presence of input delay, unmodeled dynamics and unmeasurable states. Finally, the stability analysis shows that all signals of the closed-loop systems are bounded in finite time. Simulation results show the superiority of the devised scheme.

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1 Introduction

Over the past years, the idea of approximated-based control schemes for nonlinear systems by using fuzzy logic systems (FLS) or neural networks (NNS) has been considerable attention by the researchers [1–5]. The main reason is that the unknown nonlinearity inevitably exists in actual system, and it is difficult to get the accurate information of the unknown nonlinearity, which makes the nonlinear system control very difficult [6]. Particularly, some common phenomena, such as external disturbance [1,7], unmodeled dynamics [8,9], time delay [10,11], non-smooth nonlinearity [12,13] and state constraint [14,15], make the control design of nonlinear systems more complicated. Recently, to overcome these design difficulties, dramatic adaptive neural/fuzzy backstepping control schemes have been proposed in [16–18]. Nevertheless, the above-mentioned approaches are limited to the strict feedback nonlinear systems which cannot be extended to nonlinear systems with non-strict feedback form. Regarding non-strict feedback nonlinear systems, the functions of the system are related to all the states, which can cause the algebraic loop problem for the traditional adaptive backstepping control technique [19]. Thus, Chen et al. proposed a variable separation approach for non-strict

feedback systems in [20]. Furthermore, based on FLS or NNS, many effective adaptive control schemes have been proposed to overcome this problem (see [20–23]). However, the aforementioned results are only provided that the system states are all measurable.

In practice, the state variables are usually difficult to be measured, due to the technique difficulty and/or costly expense of measuring [24]. Thus, it is hard to obtain all the information of the system states, which limits the application of the aforementioned approaches. In this case, the idea of output feedback control approach is an alternative effective approach, which is more in line with the actual demand [25,26]. Considering the non-strict feedback nonlinear systems with unmeasurable states and time delay, Chen et al. proposed a linear state observer-based output feedback control scheme in [27]. Furthermore, by using a linear state observer to estimate the unmeasurable states in non-strict feedback stochastic nonlinear systems, the authors in [28,29] considered adaptive fuzzy/neural output-feedback control schemes, respectively. Recently, in [30] Wang et al. proposed an event-triggered fuzzy output-feedback control approach via linear observer. Nevertheless, the linear observer is independent on the controlled systems, it cannot get the effective estimations of the unmeasurable state variables for practical systems. Thus, Tong et al. proposed a fuzzy state observer-based control approach for stochastic nonlinear strict feedback systems in [31], which is constructed according to the system and the unknown nonlinear functions are identified by the FLS. In [32], the authors considered adaptive fuzzy backstepping control for nonlinear systems with sampled and delayed measurements. In [33], regarding switched strict feedback nonlinear systems with dead-zone and unmeasurable states which is investigated by using fuzzy state observer. In [24], based on fuzzy observer, Wang considered repetitive tracking control for strict feedback nonlinear systems. However, as mentioned above, the algebraic loop problem for non-strict feedback nonlinear systems is inevitable for adaptive backstepping technique. To overcome this limitation, by using the property of the fuzzy basis functions, Tong et al. proposed a fuzzy state observer-based control approach for SISO non-strict feedback systems in [34]. Following this study, Wang et al. considered the actuator failures and unmeasurable states in non-strict feedback systems and proposed a composite adaptive fuzzy control via fuzzy state observer in [35]. How-

ever, the aforementioned research results are based on the problem of asymptotic stability in infinite time. Unlike the infinite time stability, finite time control shows higher tracking performance, better disturbance-rejection ability and faster transient response for nonlinear systems. So, it is very significant for us to consider the finite time control scheme for non-strict feedback systems with unmeasurable states.

In recent years, the idea of finite time stability has been widely concerned in strict or pure feedback nonlinear systems, for example, see [36–43]. Furthermore, based on the assumption that all the state variables are measurable in non-strict feedback nonlinear systems, the authors proposed the approximated-based finite time control schemes in [44–49], respectively. However, as mentioned above, the system states are usually unmeasured and it is difficult to get the full information about the system states. Therefore, the aforementioned finite time control schemes are not suitable to non-strict feedback systems with unmeasured states. To overcome this drawback, observer-based finite time control schemes for non-strict feedback systems with unmeasured states were discussed in [50–55]. Based on FLS, Li et al. investigated the finite-time adaptive fuzzy control for MIMO non-strict feedback systems via a fuzzy observer in [50]. Furthermore, considering the actuator faults and saturations in MIMO non-strict feedback systems, Ji et al. considered finite time fuzzy tracking control problem in [51]. Regarding non-strict feedback nonlinear systems with unmeasurable states, the authors in [52–55] studied the phenomena of output constraint, state constraint, prescribed performance constraint and input saturation for finite time control schemes, respectively. Nevertheless, the unmodeled dynamics problem was unconsidered in the aforementioned results. The unmodeled dynamics is inevitable existing in practical systems, due to modeling errors, model simplification and measurement noises. As a result, it is difficult to obtain their accurate information, and the difficulty of system control increases sharply. However, the above-mentioned finite time control schemes are useless for the unmodeled dynamics. By introducing a dynamic signal or small-gain theorem, the problem of unmodeled dynamics has been widely considered in infinite time control of nonlinear systems, for example, in [17,56–62]. However, from the perspective of finite time control strategy, this problem is rarely considered in output feedback mechanism. Particularly, for non-strict feedback systems,

the aforementioned results cannot be extended to this issue. Recently, Sui et al. considered finite time control of stochastic systems with unmodeled dynamics in [63]. Nevertheless, due to the repeated differentiations of virtual controller, which lead to high computational complexity. On the other hand, the input delay problem has not been considered in above research results. It is known that the control forces provided by the actuators are limited in practice. Particularly, the output provided by the observer cannot be used when input delay appears. Thus, the aforementioned results are invalid if the input delay exists in practical systems.

As a kind of time delay, input delay is a common and inevitable phenomenon in practical control systems, such as chemical processes [64], vehicle active suspension system [65] and uncertain mechanical systems [66]. When input delay occurs, the performance of the closed-loop system will be damaged or even be disastrous if the control signal cannot feedback the information provided by the observer in time. To tackle linear systems with input delay, the predictor-based techniques such as smith predictor and truncated predictor feedback are often used. However, it is difficult to deal with nonlinear systems with input delay, due to the unknown nonlinearities exist in actual systems. Thus, the authors in [67–69] extended the predictor-based control approach to tackle nonlinear systems with input delay. Unfortunately, most of these results depend on the exact model information of the nonlinear control plant. It should be noted that the unmodeled dynamics and other uncertainties often exist in actual systems which make the aforementioned results difficult to be applied in practice. In addition, for the predictor-based control methods, the state variables of system are difficult to predict. Based on Laplace transform technique, the Pade approximation is employed to deal with nonlinear systems with input delay. For example, Li [70] investigated strict-feedback systems with input delay and output constraint by combing FLS and the idea of Pade approximation. Furthermore, by combing NNS with Pade approximation, the authors in [71] considered adaptive tracking control approach for strict feedback systems with full state constrained and input delay. Nevertheless, the proposed methods are only applicable to strict feedback nonlinear systems, and the Pade approximation approach is invalid for long input delay. Recently, by using a compensation mechanism to deal with input delay, Wang et al. considered adaptive neural control for non-strict feed-

back systems in [72]. However, the proposed method is only suitable to known constant delay and the state variables are measurable, which might lead to some design conservatism. For the unknown input delay, [73] developed a prediction-based control method based on the exact model assumption of the nonlinear dynamics. Furthermore, [74] proposed proportional integral differential controller for uncertain nonlinear systems. Unfortunately, the considered system model depends on the system model, and the approach is useless when the system model changes. By using finite integral to construct auxiliary signal, Wang et al. considered strict-feedback systems with time-varying input delay and proposed adaptive fuzzy tracking control approach in [75]. Recently, the authors in [76] considered event-triggered dynamic surface control for pure-feedback systems with unknown input time delay and quantized input. However, the aforementioned methods are invalid if the state variables are not measurable or the unmodeled dynamics exist in practice systems. From above narrations, one can observe that no literature is involved in discussion finite time control for non-strict feedback nonlinear systems with unknown time varying input delay. What is more, if this issue involves unmeasured state variables and unmodeled dynamics, the existing results are invalid, which prompt us to carry out this study.

Motivated by above observations, observer-based adaptive fuzzy finite time control for non-strict feedback uncertain nonlinear systems with unknown time varying input delay, unmodeled dynamics and unmeasurable states is investigated in this paper. Fuzzy logical systems are introduced to approximate the unknown nonlinearities and a fuzzy state observer is employed to estimate the unmeasurable state variables. A compensation signal is introduced to tackle the design difficulty caused by the unknown time varying input delay. The unmodeled dynamics is tackled by introduced a dynamical signal and the DSC scheme is employed to tackle the computational burden. Based on the backstepping technique, an effective finite time adaptive fuzzy controller has been obtained in the framework of finite time Lyapunov theory. The main work of this paper is listed as follows:

(1) In the framework of finite time stability, adaptive fuzzy finite time control for non-strict feedback nonlinear systems with unknown time varying input delay, unmodeled dynamics and unmea-

surable states is proposed for the first time. Compared with the nonlinear systems with strict feedback or pure feedback form in [36–43], the proposed method can be applied for a more general structure with unmeasurable states. Compared with the finite time control schemes in [50–55,63], the proposed controller takes into account the unmodeled dynamics and input delay so as to achieve better control performance in practical engineering.

(2) For the unknown and time varying input delay, a compensation signal is introduced to tackle the design difficulty. Compared with the results in [70–72], the restrictions of input delay to be known or constant are removed, compared with [73,74] the assumption of exact model limitation is eliminated, so that the design conservatism is reduced.

(3) The structure of the non-strict feedback system is simplified by using the FLS, and the DSC scheme is adopted in the design process of backstepping technique. Compared with [44–49,63], the computational burden, complexity and difficulty of finite time controller design are reduced. The proposed scheme not only solves the algebraic loop problem, but also realizes the finite time stability performance constraint in the presence of unknown input delay, unmodeled dynamics and unmeasurable states.

The paper is organized as follows: Sect. 2 presents the problem and the preliminary results. Section 3 discusses the design process of the controller and the stability analysis. The simulation examples are considered in Sect. 4. Finally, Sect. 5 gives a brief conclusion.

Notations The following notations will be used throughout this paper $\| \cdot \|$ denotes the 2-norm for a matrix or a vector, $| \cdot |$ denotes the absolute value for a scalar, and the symbol $\circ$ denotes the composition operator between two functions.

2 Problem formulation

Consider the following dynamic system

$$
\begin{cases}
\dot{z} = q(x, z) \\
\dot{x}_i = x_{i+1} + f_i(x) + \Delta_i(x, z), \ i = 1, 2, \cdots, n-1 \\
\dot{x}_n = u(t - \tau(t)) + f_n(x) + \Delta_n(x, z) \\
y = x_1
\end{cases}
$$
(1)

where the state variable $x = [x_1, x_2, \cdots, x_n]^T$ and all variables except x_1 are not measured, z denotes the unmodeled dynamics and $\Delta_i(x, z)$ represents the unknown dynamic disturbance, $f_i(\cdot)$ is unknown smooth nonlinear function, $u(t - \tau(t)) \in R$ is the control input with $\tau(t)$ being the unknown time varying delay, and $y(t) \in R$ is the system output.

Remark 1 In practice, many systems can be described or transformed as the system (1), such as one-link manipulator with motor [53,58], electromechanical system [34] and so on. Compared with the existing results [70–76] which only work on input delay for nonlinear systems with all state variables are measurable, the finite time control problem is unconsidered. In addition, the considered input delay is known and constant or the proposed method depends on the systems model. Compared with the finite time control approaches in [50–55], the design difficulties caused by the phenomena of input delay and unmodeled dynamics are ignored by the authors.

Assumption 1 [62,63] The unknown dynamic disturbance $\Delta_i(x, z)$ satisfy

$$
|\Delta_i(x, z)| \le q_i^* \psi_{i1}(y) + q_i^* \psi_{i2}(|z|) \quad 1 \le i \le n
$$
(2)

where q_i^* is the unknown positive constant, and $\psi_{i1}(\cdot)$ and $\psi_{i2}(\cdot)$ denote the known non-negative smooth functions.

Assumption 2 [62,63] The unmodeled dynamic $\dot{z} = q(x, z)$ in (1) exists an input-to-state practically stable (ISpS) Lyapunov $V(z)$ which yields

$$
\pi_1(|z|) \le V(z) \le \pi_2(|z|)
$$
(3)

$$
\frac{\partial V(z)}{\partial z} \le -aV(z) + \kappa(|x|) + b_0
$$
(4)

where π_1, π_2 and κ are K_∞−functions, and the constants $a > 0$ and $b_0 > 0$.

Lemma 1 [63] *Assume that for a controlled system, if (3) and (4) hold, then for any constant $\bar{a}$ in $(0, a_0)$, function $\bar{\kappa}(x) \le \kappa(|x|)$, and initial value $x_0 = x_0(0)$, there exists a finite $T_0 = T_0(\bar{a}, r_0, z_0)$, $D(t) \ge 0$ for all $t \ge 0$ and a dynamical signal r which described by*

$$\dot{r} = -\bar{a}r + \bar{\kappa}(x_1(t)) + b_0, r(0) = r_0 \qquad (5)$$

satisfying $D(t) = 0, \forall t \geq T_0$

$$V(z(t)) \leq r(t) + D(t, t_0), \quad \forall t \geq 0 \qquad (6)$$

The smooth function $\bar{\kappa}(\cdot)$ is selected as $\bar{\kappa}(s) = s^2\kappa_0(s^2)$. Then, (6) becomes

$$\dot{r} = -\bar{a}r + x_1^2\kappa_0(x_1^2) + b_0, \quad r(0) = r_0 \qquad (7)$$

where $\kappa_0(x_1^2) > 0$ is a smooth function.

Assumption 3 [34] Let $\hat{x} = [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n]^T$ be the estimation of $x = [x_1, x_2, \cdots, x_n]^T$, and l_i $(1 \leq i \leq n)$ be a known constant which satisfying $|f_i(x) - f_i(\hat{x})| \leq l_i\|x - \hat{x}\|$, where $\|\cdot\|$ denotes the 2-norm of a vector.

Remark 2 Assumption 3 is commonly adopted in the literatures (Similar assumptions can be found in [27, 29,34,35]), which is a common basic assumption in the design of nonlinear systems using output feedback. In absence of Assumption 3, it would be extremely difficult to verify the stability of the overall system. As far as the function $f_i(x)$ in (1) which satisfies local Lipschitz condition, Assumption 3 can be relaxed to satisfy the local Lipschitz condition according to the relevant theories, See [28,31,32,59]. In order to make the description of the problem convenient, we employ this Lemma defined in [34].

Assumption 4 [74] For the slowly varying delay $\tau(t)$, there exists a known constant $\hat{\tau} > 0$ satisfying $\tau(t) < \hat{\tau}. \forall t \in R, |\dot{\tau}(t)| < \eta < 1$, and $\eta > 0$ is the unknown constant.

Definition 1 [45] Suppose that the equilibrium point of nonlinear system $\dot{x}(t) = f(x(t))$ is $x = 0$. If for any initial condition $x(t_0) = x_0$, there exists a settling time $T(\varepsilon, x_0) < +\infty$ and $\varepsilon > 0$ which yields $\forall t > t_0 + T$, then $\dot{x}(t) = f(x(t))$ is called semiglobal practical finite-time stable(SGPFS).

Lemma 2 [45] *For system $\dot{x}(t) = f(x(t))$ is SGPFS, if there exist some scalars $c > 0, 0 < \beta < 1, d > 0$ and a smooth function $V(x(t)) > 0$ which yields $\dot{V}(x(t)) \leq -cV^\beta(x(t)) + d$, when $t \geq 0$.*

Lemma 3 [45] *Let ω be a real number and $0 \leq \omega \leq 1$, $x_{i(1 \leq i \leq m)} \in R$, such that*

$$\left(\sum_{i=1}^m |x_i|\right)^\omega \leq \sum_{i=1}^m |x_i|^\omega \leq m^{(1-p)}\left(\sum_{i=1}^m |x_i|\right)^\omega. \qquad (8)$$

Lemma 4 [45] *Let x and y be real variables, for any given positive constants ϑ_1, ϑ_2, and ϑ_3, then*

$$|x|^{\vartheta_1}|y|^{\vartheta_2} \leq \frac{\vartheta_1}{\vartheta_1 + \vartheta_2}\vartheta_3|x|^{\vartheta_1 + \vartheta_2} + \frac{\vartheta_1}{\vartheta_1 + \vartheta_2}\vartheta_3^{-\frac{\vartheta_1}{\vartheta_2}}|y|^{\vartheta_1 + \vartheta_2}. \qquad (9)$$

The purpose of this paper is to design a unified framework of observer-based adaptive fuzzy finite time control scheme for non-strict feedback nonlinear systems with unknown input delay, unmodeled dynamics and unmeasurable state variables such that all signals of the closed-loop systems are SGPFS.

In the following design process, FLS can be employed to identify the unknown continuous nonlinear functions.

Lemma 5 [34] *$f(x)$ is continuous and defined on a compact set Ω and can be approximated by a fuzzy logic system $\theta^T\phi(x)$. Then for any constant ε, which yields*

$$\sup_{x \in \Omega}|f(x) - \theta^T\varphi(x)| \leq \varepsilon \qquad (10)$$

where $\theta^T = [\theta_1, \theta_2, \cdots, \theta_N]^T$ denotes the optimal weight vector with $N > 1$ being the number of the fuzzy rules. $\varphi(x) = [\varphi_1(x), \varphi_2(x), \cdots, \varphi_N(x)]^T$ is the membership function vector with

$$\varphi_l(x) = \frac{\prod_{i=1}^n \mu_{F_i^l(x_i)}}{\sum_{l=1}^N \left[\prod_{i=1}^n \mu_{F_i^l(x_i)}\right]}$$

and $\mu_{F_i^l(x_i)}$ is the membership function of inference antecedent variable of x_i in IF-then rules.

3 Control design

Based on the idea of backstepping technique, the design process of adaptive controller will be discussed in this section.

3.1 Fuzzy observer design

Firstly, let $\hat{x}$ be the estimation of state variable x, then system (1) can be transformed as

$$
\begin{cases}
\dot{x}_1 = x_2 + f_1(\hat{x}) + \Delta f_1 + \Delta_1(x, z) \\
\dot{x}_i = x_{i+1} + f_i(\hat{x}) + \Delta f_i + \Delta_i(x, z) \\
\dot{x}_n = u(t - \tau(t)) + f_n(\hat{x}) + \Delta f_n + \Delta_n(x, z) \\
y = x_1
\end{cases}
\tag{11}
$$

where $\Delta f_i = f_i(x) - f_i(\hat{x})$, $i = 1, 2, \cdots, n$.

For $1 \leq i \leq n$, according to the idea of approximation in Lemma 5, the unknown function $f_i(\hat{x})$ in system (1) can be modeled by a FLS $\hat{f}_i(\hat{x}) = \hat{\theta}_i^T \varphi_i(\hat{x})$, i.e.,

$$
f_i(\hat{x}) = \varepsilon_i(\hat{x}) + \hat{f}_i(\hat{x}) = \varepsilon_i(\hat{x}) + \theta_i^{*T} \varphi_i(\hat{x}), \quad |\varepsilon_i| \leq \varepsilon_i^*
\tag{12}
$$

where $\varepsilon_i(\hat{x})$ is the minimum approximation error, ε_i^* is an unknown constant, and θ_i^* denotes the optimal parameter vector, which defined as

$$
\theta_i^* = \arg \min_{\hat{\theta}_i \in \Omega_i} [\sup_{\hat{x} \in U} |\hat{f}_i(\hat{x}) - f_i(\hat{x})|]
\tag{13}
$$

with $\hat{\theta}_i$ being the estimation of θ_i^*, Ω_i and U being the compact regions for $\hat{\theta}_i$ and $\hat{x}$, respectively.

Substituting (12) into (11), and rewriting system (11) as

$$
\dot{x} = Ax + Ly + \sum_{i=1}^{n} B_i \theta_i^{*T} \varphi_i(\hat{x}) + \varepsilon
$$
$$
+ \Delta F + B_n U(t - \tau(t)) + \Delta_{xz}
$$
$$
y = Cx
\tag{14}
$$

where

$$
A = \begin{bmatrix} -l_1 & & \\ \vdots & & I_{n-1} \\ -l_n & 0 & \cdots & 0 \end{bmatrix}, L = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix},
$$

$$
\Delta F = \begin{bmatrix} \Delta f_1 \\ \vdots \\ \Delta f_n \end{bmatrix}, C = [1, \cdots, 0],
$$

$$
\varepsilon = [\varepsilon_1, \cdots, \varepsilon_n]^T, B_i = [0 \cdots 1 \cdots 0]^T,
$$
$$
\Delta_{xz} = [\Delta_1(x, z), \Delta_2(x, z), \cdots, \Delta_n(x, z)]^T.
$$

Next, we define the fuzzy state observer as

$$
\begin{cases}
\dot{\hat{x}}_1 = \hat{x}_2 + \hat{\theta}_1^T \varphi_1(\hat{x}) + l_1(x_1 - \hat{x}_1) \\
\dot{\hat{x}}_i = \hat{x}_{i+1} + \hat{\theta}_i^T \varphi_i(\hat{x}) + l_i(x_1 - \hat{x}_1) \\
\dot{\hat{x}}_n = u(t - \tau(t)) + \hat{\theta}_n^T \varphi_n(\hat{x}) + l_n(x_1 - \hat{x}_1) \\
y = \hat{x}_1
\end{cases}
\tag{15}
$$

Remark 3 The linear observer proposed in [27–30], which is independent on the controlled systems. This results in the limited information obtained by the linear observer, especially the unmodeled dynamics phenomena exist in real systems. In contrast, the fuzzy state observer (15) uses the information of the controlled system in the design process, which can obtain more system information and achieve better estimations of the unmeasurable states.

Furthermore, system (15) can be described as

$$
\dot{\hat{x}} = A\hat{x} + Ly + \sum_{i=1}^{n} B_i \hat{\theta}_i^T \varphi_i(\hat{x})
$$
$$
+ B_n U(t - \tau(t)) y = Cx
\tag{16}
$$

Let the estimation error e be

$$
e = \frac{1}{q^*}[e_1, e_2, \cdots, e_n]^T = \frac{1}{q^*}(x - \hat{x})
\tag{17}
$$

where $q^* = max\{1, q_1^*, q_2^{*2}|1 \leq i \leq n\}$.

Based on (14) and (16), one can describe the dynamics of the observer error as

$$
\dot{e} = Ae + \frac{1}{q^*} \left(\sum_{i=1}^{n} B_i \tilde{\theta}_i^T \varphi_i(\hat{x}) + \varepsilon + \Delta F + \Delta_{xz} \right)
\tag{18}
$$

where the estimation error $\tilde{\theta}_i = \theta_i^* - \hat{\theta}_i$.

According Assumption 1 and Lemma 1, we define the transformations as

$$
\bar{e} = \Gamma(r)e
\tag{19}
$$

with

$$
\Gamma(r) = \left(1 + 4 \sum_{i=1}^{n} [\psi_{i2} \circ \pi_1^{-1}(2r)]^2 \right)^{-\frac{1}{2}}.
\tag{20}
$$

Obviously, $\Gamma^2(r) \leq 1$, and $\dot{\Gamma}(r)$ can be described as

$$
\begin{aligned}
\dot{\Gamma}(r) = &-\left(1 + 4\sum_{i=1}^{n}[\psi_{i2} \circ \pi_1^{-1}(2r)]^2\right)^{-\frac{3}{2}} \\
&4\sum_{i=1}^{n}[\psi_{i2} \circ \pi_1^{-1}(2r)]^2 \frac{\partial(\psi_{i2} \circ \pi_1^{-1}(2r))}{\partial(2r)}2\dot{r} \\
= &-\Gamma^3(r)4\sum_{i=1}^{n}[\psi_{i2} \circ \pi_1^{-1}(2r)]^2 \\
&\frac{\partial(\psi_{i2} \circ \pi_1^{-1}(2r))}{\partial(2r)}2\dot{r}.
\end{aligned}
\tag{21}
$$

From (18) and (21), one has

$$
\begin{aligned}
\dot{\bar{e}} &= \Gamma(r)\dot{e} + e\dot{\Gamma}(r) \\
&= \Gamma(r)\dot{e} + \Gamma^{-1}(r)\bar{e}\dot{\Gamma}(r) \\
&= \Gamma(r)\left(Ae + \frac{1}{q^*}\left(\sum_{i=1}^{n}B_i\tilde{\theta}_i^T\varphi_i(\hat{x}) + \varepsilon + \Delta F + \Delta_{xz}\right)\right) \\
&\quad + \Gamma^{-1}(r)\bar{e}\dot{\Gamma}(r) \\
&= A\bar{e} + \frac{1}{q^*}\Gamma(r)\left(\sum_{i=1}^{n}B_i\tilde{\theta}_i^T\varphi_i(\hat{x}) + \varepsilon + \Delta F + \Delta_{xz}\right) \\
&\quad - \Gamma^2(r)4\sum_{i=1}^{n}[\psi_{i2} \circ \pi_1^{-1}(2r)] \\
&\quad \frac{\partial([\psi_{i2} \circ \pi_1^{-1}(2r)])}{\partial(2r)}2\dot{r}\bar{e}.
\end{aligned}
\tag{22}
$$

By selecting vector, L which yields matrix A is a strict Hurwitz matrix. Then, for any given matrix $Q = Q^T > 0$, there exists $P = P^T > 0$ which yields $A^T P + PA = -Q$.

Consider the Lyapunov function $V_0 = \bar{e}^T P\bar{e}$, one has

$$
\begin{aligned}
\dot{V}_0 \leq &-\lambda_{\min}(Q)||\bar{e}||^2 + \frac{2\bar{e}^T P}{q^*}\Gamma(r) \\
&\left(\sum_{i=1}^{n}B_i\tilde{\theta}_i^T\varphi_i(\hat{x}) + \varepsilon + \Delta F + \Delta_{xz}\right) \\
&-\Gamma(r)^2 4\sum_{i=1}^{n}[\psi_{i2} \circ \pi_1^{-1}(2r)] \\
&\frac{\partial([\psi_{i2} \circ \pi_1^{-1}(2r)])}{\partial(2r)}4\dot{r}\bar{e}^T P\bar{e}.
\end{aligned}
\tag{23}
$$

By using the complete squares formula and the property of $0 < \varphi_i^T(\hat{x})\varphi_i(\hat{x}) \leq 1$, one can get the following results

$$
\begin{aligned}
&\frac{2}{q^*}\Gamma(r)\bar{e}^T P\sum_{i=1}^{n}B_i\tilde{\theta}_i^T\varphi_i(\hat{x}) \\
&\leq ||P||^2\sum_{i=1}^{n}\tilde{\theta}_i^T\tilde{\theta}_i + n||\bar{e}||^2
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&\frac{2}{q^*}\Gamma(r)\bar{e}^T P(\varepsilon + \Delta F) \\
&\leq 2||\bar{e}||^2 + ||P||^2||\varepsilon^*||^2 + ||P||^2||\Delta F||^2 \\
&\leq 2||\bar{e}||^2 + ||P||^2||\varepsilon^*||^2 + ||P||^2\left(\sum_{j=1}^{n}l_j^2||\bar{e}||^2\right).
\end{aligned}
\tag{25}
$$

Based on Assumption 2 and Lemma 1, one can get $|z| \leq \pi_1^{-1}(r + D_0)$. Furthermore, by using Assumption 1 and the complete squares formula, one can have

$$
\begin{aligned}
&\frac{2}{q^*}\Gamma(r)\bar{e}^T P\Delta_{xz} \\
&\leq 2||\bar{e}||||P||\left(\sum_{i=1}^{n}\psi_{i1}(y) + \sum_{i=1}^{n}\psi_{i2}(|z|)\right) \\
&\leq n^2\sum_{i=1}^{n}\bar{\psi}_{i1}^2(y)y^2 + 2||P||^2||\bar{e}||^2 \\
&\quad + n\Gamma^2(r)\sum_{i=1}^{n}(\psi_{i2} \circ \pi_1^{-1}(2r))^2 \\
&\quad + n\Gamma^2(r)\sum_{i=1}^{n}(\psi_{i2} \circ \pi_1^{-1}(2D))^2 \\
&\leq n^2\sum_{i=1}^{n}\bar{\psi}_{i1}^2(y)y^2 + 2||P||^2||\bar{e}||^2 + n + d_0
\end{aligned}
\tag{26}
$$

where $\Gamma^2(r)\sum_{i=1}^{n}(\psi_{i2}\circ\pi_1^{-1}(2r))^2 \leq 1, d_0 = n\sum_{i=1}^{n}(\psi_{i2}\circ\pi_1^{-1}(2D))^2, \bar{\psi}_{i1}(y)$ is a smooth function and $y\bar{\psi}_{i1}(y) = \psi_{i1}(y)$.

Substituting (24)–(26) into (23) and based on the fact $\frac{\partial([\psi_{i2}\circ\pi_1^{-1}(2r)])}{\partial(2r)} > 0$ and $\bar{e}^T P\bar{e} > 0$, one can get

$$
\dot{V}_0 \leq -\lambda_0||\bar{e}||^2 + ||P||^2\sum_{j=1}^{n}\tilde{\theta}_j^T\tilde{\theta}_j
$$

$$+ n^2 \sum_{i=1}^{n} \bar{\psi}_{i1}^2(y) y^2 + \Upsilon_0 \qquad (27)$$

where $\lambda_0 = \lambda_{\min}(Q) - ||P||^2 (\sum_{j=1}^{n} l_j^2) - 2||P||^2 - 2 - n$ and $\Upsilon_0 = ||P||^2 ||\varepsilon^*||^2 + n + d_0$.

3.2 Controller design

Firstly, the following change of coordinate is employed in the process of backstepping designing

$$\begin{aligned} z_1 &= x_1 \\ z_i &= \hat{x}_i - \omega_i, \quad i = 2, 3, \cdots, n \end{aligned} \qquad (28)$$

where ω_i is the first-order filter output signal, which is defined as

$$\xi_i \dot{\omega}_i + \omega_i = \alpha_{i-1}, \omega_i(0) = \alpha_{i-1}(0), \quad i = 2, 3, \cdots, n \qquad (29)$$

with ξ_i being design parameter and α_i being the first-order filter input signals. The filter errors are given by

$$\chi_i = \omega_i - \alpha_{i-1}, \quad i = 2, 3, \cdots, n \qquad (30)$$

According to the previous change of coordinate, the specific design steps as follows.

Step 1: Based on (28) and (29), the derivative of z_1 is obtained that

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 \\ &= x_2 + f_1(\hat{x}) + \Delta f_1 + \Delta_1(x, z) \\ &= \hat{x}_2 + q^* e_2 + \theta_1^{*T} \varphi_1(\hat{x}) \\ &\quad + \varepsilon_1 + \Delta f_1 + \Delta_1(x, z). \end{aligned} \qquad (31)$$

For (31), a desired virtual input α_1 can be derived to take place of $\hat{x}_2$ based on $z_2 = \hat{x}_2 - \omega_2$ and $\chi_2 = \omega_2 - \alpha_1$, we can get

$$\begin{aligned} \dot{z}_1 &= z_2 + \chi_2 + q^* e_2 + \alpha_1 + \theta_1^{*T} \varphi_1(\hat{x}) \\ &\quad + \varepsilon_1 + \Delta f_1 + \Delta_1(x, z) \\ &= z_2 + \chi_2 + q^* e_2 + \alpha_1 + \theta_1^{*T} \varphi_1(\hat{x}) \\ &\quad + \hat{\theta}_1^T \varphi_1(\hat{x}_1) - \theta_1^{*T} \varphi_1(\hat{x}_1) + \tilde{\theta}_1^T \varphi_1(\hat{x}_1) \\ &\quad + \varepsilon_1 + \Delta f_1 + \Delta_1(x, z). \end{aligned} \qquad (32)$$

Let the Lyapunov function candidate be

$$\begin{aligned} V_{\omega 1} &= V_0 + \frac{1}{2} z_1^2 + \frac{1}{2\gamma_1} \tilde{\theta}_1^T \tilde{\theta}_1 \\ &\quad + \frac{1}{2\bar{\gamma}_1} \tilde{\Theta}_1^2 + \frac{1}{2} \tilde{q}^2 + \frac{r}{\gamma_0} \end{aligned} \qquad (33)$$

where $\tilde{\Theta}_1 = \Theta_1^* - \hat{\Theta}_1$ and $\tilde{q} = q^* - \hat{q}$ represent the estimation errors, $\hat{\Theta}_1$ and $\hat{q}$ are the estimations of $\Theta_1^* = ||\theta_1^*||^2$ and q^*, respectively. The design parameters $\gamma_1 > 0$, $\bar{\gamma}_1 > 0$ and $\gamma_0 > 0$.

Differentiating $V_{\omega 1}$, then

$$\begin{aligned} \dot{V}_{\omega 1} &\leq \dot{V}_0 + z_1 \dot{z}_1 - \frac{1}{\gamma_1} \tilde{\theta}_1^T \dot{\hat{\theta}}_1 \\ &\quad - \frac{1}{\bar{\gamma}_1} \tilde{\Theta}_1 \dot{\hat{\Theta}}_1 + \tilde{q} \dot{\tilde{q}} + \frac{1}{\gamma_0} \dot{r} \\ &\leq -\lambda_0 ||\bar{e}||^2 + ||P||^2 \sum_{j=1}^{n} \tilde{\theta}_j^T \tilde{\theta}_j - \frac{1}{\gamma_1} \tilde{\theta}_1^T \dot{\hat{\theta}}_1 \\ &\quad + n^2 \sum_{i=1}^{n} \bar{\psi}_{i1}^2(y) y^2 + \Upsilon_0 - \frac{1}{\bar{\gamma}_1} \tilde{\Theta}_1 \dot{\hat{\Theta}}_1 \\ &\quad + z_1(z_2 + \chi_2 + q^* e_2 + \alpha_1 + \theta_1^{*T} \varphi_1(\hat{x}) \\ &\quad + \hat{\theta}_1^T \varphi_1(\hat{x}_1) + \tilde{\theta}_1^T \varphi_1(\hat{x}_1) \\ &\quad - \theta_1^{*T} \varphi_1(\hat{x}_1) + \varepsilon_1 + \Delta f_1 + \Delta_1(x, z)) \\ &\quad - \tilde{q} \dot{\hat{q}} + \frac{1}{\gamma_0}(-\bar{a} r + x_1^2 \kappa_0(x_1^2) + b_0). \end{aligned} \qquad (34)$$

Based on the completion of squares and Assumption 1, one can get the following results

$$\begin{aligned} &z_1(q^* e_2 + \varepsilon_1 + \Delta f_1 + \chi_2) \\ &\leq \frac{3}{2} z_1^2 + \frac{z_1^2 q^*}{2\Gamma^2(r)} + \left(\frac{1}{2} + \frac{l_1^2}{2\Gamma^2(r)} \right) ||\bar{e}||^2 \\ &\quad + \frac{1}{2} \varepsilon_1^{*2} + \frac{1}{2} \chi_2^2 \end{aligned} \qquad (35)$$

$$z_1 \left(\theta_1^{*T} \varphi_1(\hat{x}) - \theta_1^{*T} \varphi_1(\hat{x}_1) \right) \leq \frac{\varsigma_1}{2} z_1^2 \Theta_1^* + \frac{2}{\varsigma_1} \qquad (36)$$

where $\varsigma_1 > 0$ is a design parameter.

From (2) and (3), one can get

$$\begin{aligned} z_1 \Delta_1(x, z) &\leq \frac{1}{4} [\psi_{i2} \circ \pi_1^{-1}(2r)]^2 z_1^2 q^* + 1 + \frac{1}{4} z_1^2 q^* \\ &\quad + 4[\psi_{i2} \circ \pi_1^{-1}(2D)]^2 + \bar{\psi}_{i1}(y) y z_1 q^*. \end{aligned} \qquad (37)$$

Furthermore, substituting (35)–(37) into (34), we can have

$$
\begin{aligned}
\dot{V}_{\omega 1} \leq{} & -\lambda_1 ||\bar{e}||^2 + \tilde{\theta}_1^T \left(\varphi_1(\hat{x}_1) z_1 - \frac{1}{\gamma_1} \dot{\hat{\theta}}_1 \right) \\
& + \tilde{\Theta}_1 \left(\frac{\varsigma_1}{2} z_1^2 - \frac{1}{\bar{\gamma}_1} \dot{\hat{\Theta}}_1 \right) \\
& + z_1 \left(\frac{3}{2} z_1 + \alpha_1 + z_2 + \frac{z_1 \kappa_0(x_1^2)}{\gamma_0} + \frac{\varsigma_1 \hat{\Theta}_1 z_1}{2} \right) \\
& + n^2 \sum_{i=1}^{n} \bar{\psi}_{i1}^2(y) y + \hat{\theta}_1^T \varphi_1(\hat{x}_1) \\
& + \delta_1 \hat{q}) + \tilde{q}(\delta_1 y - \dot{\hat{q}}) + \frac{1}{\gamma_0}(-\bar{a}r + b_0) \\
& + ||P||^2 \sum_{j=1}^{n} \tilde{\theta}_j^T \tilde{\theta}_j + \frac{1}{2} \chi_2^2 + \bar{\Upsilon}_1
\end{aligned}
\tag{38}
$$

where $\lambda_1 = \lambda_0 - (\frac{1}{2} + \frac{l_1^2}{2\Gamma^2(r)})$, $\delta_1 = \frac{z_1}{2\Gamma^2(r)} + \frac{1}{4}[\psi_{i2} \circ \pi_1^{-1}(2r)]^2 z_1 + \frac{1}{4} z_1 + \bar{\psi}_{i1}(y) y$ and $\bar{\Upsilon}_1 = \Upsilon_0 + \frac{2}{\varsigma_1} + \frac{1}{2}\varepsilon_1^{*2} + 1 + 4[\psi_{i2} \circ \pi_1^{-1}(2D)]^2$.

Now, we design the virtual control signal as

$$
\begin{aligned}
\alpha_1 ={} & -\frac{3}{2} z_1 - k_1 z_1^{2\beta-1} - \frac{z_1 \kappa_0(x_1^2)}{\gamma_0} - \frac{\varsigma_1 \hat{\Theta}_1 z_1}{2} \\
& - n^2 \sum_{i=1}^{n} \bar{\psi}_{i1}^2(y) y - \hat{\theta}_1^T \varphi_1(\hat{x}_1) - \delta_1 \hat{q}
\end{aligned}
\tag{39}
$$

with the adaptive laws $\dot{\hat{\theta}}_1$ and $\dot{\hat{\Theta}}_1$ as

$$
\dot{\hat{\theta}}_1 = \gamma_1 \phi_1^T(\hat{x}_1) z_1 - \varsigma_1 \hat{\theta}_1
\tag{40}
$$

$$
\dot{\hat{\Theta}}_1 = \frac{1}{2} \bar{\gamma}_1 \varsigma_1 z_1^2 - \bar{\varsigma}_1 \hat{\Theta}_1
\tag{41}
$$

and the turning function ζ_1 as

$$
\zeta_1 = \delta_1 z_1 - \varpi \hat{q}
\tag{42}
$$

where the design parameters $k_1 > 0$, $\varsigma_1 > 0$, $\bar{\varsigma}_1 > 0$ and $\varpi > 0$.

From (38)–(40), the following result can be obtained

$$
\begin{aligned}
\dot{V}_{\omega 1} \leq{} & -\lambda_1 ||\bar{e}||^2 - k_1 z_1^{2\beta} + z_1 z_2 + \frac{\varsigma_1 \tilde{\theta}_1^T \hat{\theta}_1}{\gamma_1} \\
& + \frac{\bar{\varsigma}_1 \tilde{\Theta}_1 \hat{\Theta}_1}{\bar{\gamma}_1} + \varpi_1 \tilde{q} \hat{q} \\
& - \tilde{q}(\dot{\hat{q}} - \zeta_1) + \frac{1}{\gamma_0}(-\bar{a}r + b_0) \\
& + ||P||^2 \sum_{j=1}^{n} \tilde{\theta}_j^T \tilde{\theta}_j + \frac{1}{2} \chi_2^2 + \bar{\Upsilon}_1.
\end{aligned}
\tag{43}
$$

To deal with the problem of "explosion of complexity" caused by repeatedly differentiating α_1, let α_1 pass through a given low-pass filter ω_2, which defined in (29) with the filter time design parameter ξ_2 and the filter error χ_2 defined in (30). Then, one can get

$$
\dot{\omega}_2 = \frac{-\chi_2}{\xi_2}
\tag{44}
$$

and

$$
\begin{aligned}
\dot{\chi}_2 = \dot{\omega}_2 - \dot{\alpha}_1 &= \frac{-\chi_2}{\xi_2} - \dot{\alpha}_1 \\
&= \frac{-\chi_2}{\xi_2} + M_2(\cdot)
\end{aligned}
\tag{45}
$$

where $M_2(\cdot)$ is a continuous function, and $M_2(z_1, z_2, \chi_2, \hat{\theta}_1, \hat{\Theta}_1, y_d, \dot{y}_d, \ddot{y}_d, \mu_1) = -(\frac{\partial \alpha_1}{\partial \hat{x}_1}\dot{\hat{x}}_1 + \frac{\partial \alpha_1}{\partial z_1}\dot{z}_1 + \frac{\partial \alpha_1}{\partial \hat{\theta}_1}\dot{\hat{\theta}}_1 + \frac{\partial \alpha_1}{\partial \hat{\Theta}_1}\dot{\hat{\Theta}}_1 + \frac{\partial \alpha_1}{\partial y_d}\dot{y}_d + \frac{\partial \alpha_1}{\partial \mu_1}\dot{\mu}_1)$. Furthermore, for any given B_0 and σ, the sets $H_0 := \{(y_d, \dot{y}_d, \ddot{y}_d) : (y_d)^2 + (\dot{y}_d)^2 + (\ddot{y}_d)^2 \leq B_0\}$ is compact in R^3, and $H_2 := \{\sum_{j=1}^{2} z_j^2 + \gamma_1^{-1}\tilde{\theta}_1^T \tilde{\theta}_1 + \bar{\gamma}_1^{-1}\tilde{\Theta}_1^T \tilde{\Theta}_1 + \chi_2^2 \leq 2\sigma\}$ is compact in R^{N_1+3} with N_1 being the dimension of $\tilde{\theta}_1^T$. According [77], M_2 has a maximum value B_2. Thus,

$$
\begin{aligned}
\chi_2 \dot{\chi}_2 &= \chi_2(\dot{\omega}_2 - \dot{\alpha}_1) \\
&= \chi_2 \left(\frac{-\chi_2}{\xi_2} \right) \\
&\quad + \chi_2 M_2(z_1, z_2, \chi_2, \hat{\theta}_1, \hat{\Theta}_1, y_d, \dot{y}_d, \ddot{y}_d, \mu_1) \\
&\leq -\frac{1}{\xi_2} \chi_2^2 + \frac{1}{2} \chi_2^2 + \frac{1}{2} B_2^2.
\end{aligned}
\tag{46}
$$

Let the first Lyapunov function of the system be

$$V_1 = V_0 + V_{\omega 1} + \frac{1}{2}\chi_2^2. \tag{47}$$

From (43)–(47), one can get

$$
\begin{aligned}
\dot{V}_1 &= \dot{V}_0 + \dot{V}_{\omega 1} + \chi_2 \dot{\chi}_2 \\
&\leq -\lambda_1 ||\bar{e}||^2 - k_1 z_1^{2\beta} + z_1 z_2 \\
&\quad + \frac{\varsigma_1 \tilde{\theta}_1^T \hat{\theta}_1}{\gamma_1} + \frac{\bar{\varsigma}_1 \tilde{\Theta}_1 \hat{\Theta}_1}{\bar{\gamma}_1} + \varpi \tilde{q} \hat{q} \\
&\quad - \tilde{q}(\dot{\hat{q}} - \zeta_1) + \frac{1}{\gamma_0}(-\bar{a}r + b_0) \\
&\quad + ||P||^2 \sum_{j=1}^{n} \tilde{\theta}_j^T \tilde{\theta}_j + \left(1 - \frac{1}{\xi_2}\right)\chi_2^2 + \Upsilon_1
\end{aligned}
\tag{48}
$$

where $\Upsilon_1 = \bar{\Upsilon}_0 + \frac{1}{2}B_2^2$.

Step i $(2 \leq i \leq n-1)$: Based on $z_{i+1} = \hat{x}_{i+1} - \omega_{i+1}$ and $\chi_{i+1} = \omega_{i+1} - \alpha_i$, from (28)–(30), differentiating z_i then

$$
\begin{aligned}
\dot{z}_i &= \dot{\hat{x}}_i - \dot{\omega}_i \\
&= \hat{x}_{i+1} + \hat{\theta}_i^T \varphi_i(\hat{x}) + l_i q^*(x_1 - \hat{x}_1) - \dot{\omega}_i \\
&= z_{i+1} + \chi_{i+1} + \alpha_i + l_i q^* e_1 - \dot{\omega}_i \\
&\quad - \tilde{\theta}_i^T \varphi_i(\hat{x}) + \tilde{\theta}_i^T \varphi_i(\underline{\hat{x}}_i) \\
&\quad + \hat{\theta}_i^T \varphi_i(\underline{\hat{x}}_i) - \theta_i^{*T} \varphi_i(\underline{\hat{x}}_i) + \theta_i^{*T} \varphi_i(\hat{x})
\end{aligned}
\tag{49}
$$

where $\underline{\hat{x}}_i = [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_i]$.

Let the following Lyapunov function candidate be

$$V_{\omega i} = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2\gamma_i}\tilde{\theta}_i^T \tilde{\theta}_i + \frac{1}{2\bar{\gamma}_i}\tilde{\Theta}_i^2 \tag{50}$$

where the estimation error is $\tilde{\Theta}_i = \Theta_i^* - \hat{\Theta}_i$ and $\hat{\Theta}_i$ represents the estimation of $||\theta_i^*||^2$. The design parameters $\gamma_i > 0$ and $\bar{\gamma}_i > 0$.

Thus, differentiating $V_{\omega i}$ one has

$$
\begin{aligned}
\dot{V}_{\omega i} &\leq -\lambda_{i-1}||\bar{e}||^2 - \sum_{j=1}^{i-1} k_j z_j^{2\beta} + z_{i-1} z_i \\
&\quad + \sum_{j=1}^{i-1} \frac{\varsigma_j \tilde{\theta}_j^T \hat{\theta}_j}{\gamma_j} + \sum_{j=1}^{i-1} \frac{\bar{\varsigma}_j \tilde{\Theta}_j \hat{\Theta}_j}{\bar{\gamma}_j} + \varpi \tilde{q} \hat{q} \\
&\quad + z_i(z_{i+1} + \chi_{i+1} + \alpha_i + l_i q^* e_1 - \dot{\omega}_i)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \tilde{\theta}_i^T \varphi_i(\hat{x}) + \tilde{\theta}_i^T \varphi_i(\underline{\hat{x}}_i) - \theta_i^{*T} \varphi_i(\underline{\hat{x}}_i) \\
&\quad + \hat{\theta}_i^T \varphi_i(\underline{\hat{x}}_i) + \theta_i^{*T} \varphi_i(\hat{x})) - \frac{1}{\gamma_i}\tilde{\theta}_i^T \dot{\hat{\theta}}_i \\
&\quad - \frac{1}{\bar{\gamma}_i}\tilde{\Theta}_i \dot{\hat{\Theta}}_i - \tilde{q}(\dot{\hat{q}} - \zeta_{i-1}) + \sum_{j=2}^{i-1} \frac{\tilde{\theta}_j^T \tilde{\theta}_j}{2} \\
&\quad + \frac{1}{\gamma_0}(-\bar{a}r + b_0) + ||P||^2 \sum_{j=1}^{n} \tilde{\theta}_j^T \tilde{\theta}_j \\
&\quad + \sum_{j=1}^{i-1}\left(1 - \frac{1}{\xi_{j+1}}\right)\chi_{j+1}^2 + \Upsilon_{i-1}.
\end{aligned}
\tag{51}
$$

Based on the completion of squares, one has

$$z_i \chi_{i+1} \leq \frac{1}{2}z_i^2 + \frac{1}{2}\chi_{i+1}^2 \tag{52}$$

$$z_i q^* l_i e_1 \leq \frac{z_i^2 l_i^2 q^*}{2\Gamma^2(r)} + \frac{1}{2}||\bar{e}||^2 \tag{53}$$

$$-z_i \tilde{\theta}_i^T \varphi_i(\hat{x}) \leq \frac{1}{2}z_i^2 + \frac{1}{2}\tilde{\theta}_i^T \tilde{\theta}_i \tag{54}$$

$$z_i(\theta_i^{*T} \varphi_i(\hat{x}) - \theta_i^{*T} \varphi_i(\underline{\hat{x}}_i)) \leq \frac{\varsigma_i}{2}z_i^2 \Theta_i^* + \frac{2}{\varsigma_i} \tag{55}$$

where $\varsigma_i > 0$ is a design parameter.

Based on (51)–(55), we can obtain that

$$
\begin{aligned}
\dot{V}_{\omega i} &\leq -\lambda_i ||\bar{e}||^2 - \sum_{j=1}^{i-1} k_j z_j^{2\beta} + z_{i-1} z_i \\
&\quad + \sum_{j=1}^{i-1} \frac{\varsigma_j \tilde{\theta}_j^T \hat{\theta}_j}{\gamma_j} + \sum_{j=1}^{i-1} \frac{\bar{\varsigma}_j \tilde{\Theta}_j \hat{\Theta}_j}{\bar{\gamma}_j} + \varpi \tilde{q} \hat{q} \\
&\quad + z_i\left(z_i + z_{i+1} + \alpha_i + \frac{\varsigma_i}{2}z_i \hat{\Theta}_i + \hat{\theta}_i^T \varphi_i(\underline{\hat{x}}_i)\right. \\
&\quad \left. + \delta_i \hat{q} - \dot{\omega}_i\right) - \tilde{q}(\dot{\hat{q}} - \zeta_i) \\
&\quad + \tilde{\theta}_i^T\left(\varphi_i(\underline{\hat{x}}_i)z_i - \frac{1}{\gamma_i}\dot{\hat{\theta}}_i\right) + \sum_{j=1}^{i-1}\left(1 - \frac{1}{\xi_{j+1}}\right)\chi_{j+1}^2 \\
&\quad + \frac{1}{2}\chi_{i+1}^2 + ||P||^2 \sum_{j=1}^{n} \tilde{\theta}_j^T \tilde{\theta}_j \\
&\quad + \frac{1}{\gamma_0}(-\bar{a}r + b_0) + \tilde{\Theta}_i\left(\frac{\varsigma_i}{2}z_{i+1}^2 - \frac{1}{\bar{\gamma}_i}\dot{\hat{\Theta}}_i\right) \\
&\quad + \sum_{j=2}^{i} \frac{\tilde{\theta}_j^T \tilde{\theta}_j}{2} + \frac{2}{\varsigma_i} + \Upsilon_{i-1}
\end{aligned}
\tag{56}
$$

where $\lambda_i = \lambda_{i-1} + \frac{1}{2}$ and $\delta_i = \frac{z_i l_i^2}{2\Gamma^2(r)}$.

Then, design the virtual control input signal as

$$\alpha_i = -z_{i-1} - z_i - k_i z_i^{2\beta-1} - \frac{\varsigma_i}{2} z_i \hat{\Theta}_i$$
$$- \hat{\theta}_i^T \varphi_i(\hat{\underline{x}}_i) - \delta_i \hat{q} + \dot{\omega}_i \qquad (57)$$

with the adaptive laws $\dot{\hat{\theta}}_i$ and $\dot{\hat{\Theta}}_i$ as

$$\dot{\hat{\theta}}_i = \gamma_i \phi_i^T(\hat{\underline{x}}_i) z_i - \varsigma_i \hat{\theta}_i \qquad (58)$$
$$\dot{\hat{\Theta}}_i = \frac{1}{2} \bar{\gamma}_i \varsigma_i z_i^2 - \bar{\varsigma}_i \hat{\Theta}_i \qquad (59)$$

and the turning function ζ_i as

$$\zeta_i = \zeta_{i-1} + \delta_i z_i \qquad (60)$$

where the design parameters $k_i > 0$, $\varsigma_i > 0$ and $\bar{\varsigma}_i > 0$.

Furthermore, substituting (57)–(60) into (56), we can have

$$\dot{V}_{\omega i} \leq -\lambda_i ||\bar{e}||^2 - \sum_{j=1}^{i} k_j z_j^{2\beta} + z_i z_{i+1}$$
$$+ \sum_{j=1}^{i} \frac{\varsigma_j \tilde{\theta}_j^T \hat{\theta}_j}{\gamma_j} + \sum_{j=1}^{i} \frac{\bar{\varsigma}_j \tilde{\Theta}_j \hat{\Theta}_j}{\bar{\gamma}_j} + \varpi \tilde{q} \hat{q}$$
$$+ ||P||^2 \sum_{j=1}^{n} \tilde{\theta}_j^T \tilde{\theta}_j - \tilde{q}(\dot{\hat{q}} - \zeta_i)$$
$$+ \frac{1}{\gamma_0}(-\bar{a}r + b_0) \qquad (61)$$
$$+ \sum_{j=1}^{i-1} \left(1 - \frac{1}{\xi_{j+1}}\right) \chi_{j+1}^2 + \frac{1}{2} \chi_{i+1}^2$$
$$+ \Upsilon_{i-1} + \frac{2}{\varsigma_i} + \sum_{j=2}^{i} \frac{\tilde{\theta}_j^T \tilde{\theta}_j}{2}.$$

As in step 1, let α_i pass through a given low-pass filter ω_{i+1}, which defined in (29) with the filter time design parameter ξ_{i+1} and the filter error χ_{i+1} defined in (30). Then, one can get

$$\dot{\omega}_{i+1} = \frac{-\chi_{i+1}}{\xi_{i+1}} \qquad (62)$$

and

$$\dot{\chi}_{i+1} = \frac{-\chi_{i+1}}{\xi_{i+1}} + M_{i+1}(\cdot) \qquad (63)$$

where $M_{i+1}(\cdot)$ is a continuous function, and $M_{i+1}(z_1, \cdots, z_{i+1}, \chi_1, \cdots, \chi_{i+1}, \hat{\theta}_1, \cdots, \hat{\theta}_i, \qquad \hat{\Theta}_1, \cdots, \hat{\Theta}_i, y_d, \dot{y}_d, \ddot{y}_d, \mu_1, \cdots, \mu_i) = -(\frac{\partial \alpha_i}{\partial \hat{x}_i}\dot{\hat{x}}_i + \frac{\partial \alpha_i}{\partial z_i}\dot{z}_i + \frac{\partial \alpha_i}{\partial \hat{\theta}_i}\dot{\hat{\theta}}_i + \frac{\partial \alpha_i}{\partial \hat{\Theta}_i}\dot{\hat{\Theta}}_i + \frac{\partial \alpha_i}{\partial y_d}\dot{y}_d + \frac{\partial \alpha_i}{\partial \mu_i}\dot{\mu}_i)$. Furthermore, for any given B_0 and σ, the sets $H_0 := \{(y_d, \dot{y}_d, \ddot{y}_d) : (y_d)^2 + (\dot{y}_d)^2 + (\ddot{y}_d)^2 \leq B_0\}$ is compact in R^3, and $H_i := \{\sum_{j=1}^{i} z_j^2 + \gamma_i^{-1}\tilde{\theta}_i^T \tilde{\theta}_i + \bar{\gamma}_i^{-1}\tilde{\Theta}_i^T \tilde{\Theta}_i + \chi_i^2 \leq 2\sigma\}$ is compact in R^{N_1+3} with N_1 being the dimension of $\tilde{\theta}_i^T$. According [77], M_{i+1} has a maximum value B_{i+1}. Thus,

$$\chi_{i+1}\dot{\chi}_{i+1} = \chi_{i+1}(\dot{\omega}_{i+1} - \dot{\alpha}_i)$$
$$= \chi_{i+1}\left(\frac{-\chi_{i+1}}{\xi_{i+1}}\right) + \chi_{i+1} M_{i+1}(\cdot) \qquad (64)$$
$$\leq -\frac{1}{\xi_{i+1}}\chi_{i+1}^2 + \frac{1}{2}\chi_{i+1}^2 + \frac{1}{2}B_{i+1}^2.$$

Let the ith Lyapunov function be

$$V_i = V_{\omega i} + \frac{1}{2}\chi_{i+1}^2. \qquad (65)$$

Then, from (61), (64) and (65) we have

$$\dot{V}_i = \dot{V}_{\omega i} + \chi_{i+1}\dot{\chi}_{i+1}$$
$$\leq -\lambda_i ||\bar{e}||^2 - \sum_{j=1}^{i} k_j z_j^{2\beta} + z_i z_{i+1}$$
$$+ \sum_{j=1}^{i} \frac{\varsigma_j \tilde{\theta}_j^T \hat{\theta}_j}{\gamma_j} + \varpi \tilde{q}\hat{q} + \sum_{j=1}^{i} \frac{\bar{\varsigma}_j \tilde{\Theta}_j \hat{\Theta}_j}{\bar{\gamma}_j}$$
$$+ \sum_{j=1}^{i}(1 - \frac{1}{\xi_{j+1}})\chi_{j+1}^2 - \tilde{q}(\dot{\hat{q}} - \zeta_i) \qquad (66)$$
$$+ \frac{1}{\gamma_0}(-\bar{a}r + b_0) + \Upsilon_i$$
$$+ ||P||^2 \sum_{j=1}^{n}\tilde{\theta}_j^T\tilde{\theta}_j + \sum_{j=2}^{i}\frac{\tilde{\theta}_j^T\tilde{\theta}_j}{2}$$

where $\Upsilon_i = \Upsilon_{i-1} + \frac{2}{\varsigma_i} + \frac{1}{2}B_{i+1}^2$.

Step n: In this step, a compensation signal μ is devised to tackle the effect of input delay, and the com-

pensate signal μ is designed as

$$\dot{\mu} = -h\mu + u(t) - u(t - \hat{\tau}(t)). \tag{67}$$

Furthermore, we redefine the signal z_n as follows

$$z_n = \hat{x}_n - \omega_n + \mu. \tag{68}$$

Based on (67) and (68), differentiating z_n

$$\begin{aligned}
\dot{z}_n &= \dot{\hat{x}}_n - \dot{\omega}_n + \dot{\mu} \\
&= u(t - \tau(t)) + \hat{\theta}_n^T \varphi_n(\hat{x}) + l_n q^* e_1 \\
&\quad - \dot{\omega}_n - h\mu - u(t - \hat{\tau}) + u(t).
\end{aligned} \tag{69}$$

Then, let the Lyapunov function be

$$\begin{aligned}
V_n &= V_{n-1} + \frac{1}{2}z_n^2 + \frac{1}{2}\mu^2 + \frac{1}{2\gamma_n}\tilde{\theta}_n^T \tilde{\theta}_n \\
&\quad + D_1 + D_2 + D_3 + D_4
\end{aligned} \tag{70}$$

where $D_1 = \frac{1}{2(1-\eta)}\int_{t-\tau(t)}^t ||u(s)||^2 \mathrm{d}s$, $D_2 = \frac{1}{1-\eta}\int_{t-\tau(t)}^t \int_\theta^t ||u(s)||^2 \mathrm{d}s\mathrm{d}\theta$, $D_3 = \int_{t-\hat{\tau}}^t ||u(s)||^2 \mathrm{d}s$, and $D_4 = 2\int_{t-\hat{\tau}}^t \int_\theta^t ||u(s)||^2 \mathrm{d}s\mathrm{d}\theta$. The derivatives of D_1, D_2, D_3, and D_4 can be described as follows

$$\begin{aligned}
\dot{D}_1 &= \frac{1}{2(1-\eta)} \\
&\quad \left(||u(t)||^2 - ||u(t - \tau(t))||^2(1 - \dot{\tau}(t)) \right)
\end{aligned} \tag{71}$$

$$\begin{aligned}
\dot{D}_2 &= \frac{1}{1-\eta} \\
&\quad \left(||u(t)||^2 \tau(t) - (1 - \dot{\tau}(t))\int_{t-\tau(t)}^t ||u(s)||^2 \mathrm{d}s \right)
\end{aligned} \tag{72}$$

$$\dot{D}_3 = ||u(t)||^2 - ||u(t - \hat{\tau})||^2 \tag{73}$$

$$\dot{D}_4 = 2\left(||u(t)||^2\hat{\tau} - \int_{t-\hat{\tau}}^t ||u(s)||^2 \mathrm{d}s \right). \tag{74}$$

Differentiating V_n as in step i, it is clearly that

$$\begin{aligned}
\dot{V}_n &= \dot{V}_{n-1} + z_n\dot{z}_n + \mu\dot{\mu} + \frac{1}{\gamma_n}\tilde{\theta}_n^T \dot{\tilde{\theta}}_n \\
&\quad + \dot{D}_1 + \dot{D}_2 + \dot{D}_3 + \dot{D}_4 \\
&\leq \dot{V}_{n-1} + z_n(u(t - \tau(t)) + \hat{\theta}_n^T \varphi_n(\hat{x}) + l_n q^* e_1 \\
&\quad - \dot{\omega}_n - h\mu + u(t) - u(t - \hat{\tau})) \\
&\quad + \mu(-h\mu + u(t) - u(t - \hat{\tau})) - z_n\tilde{\theta}_n^T \varphi_n(\hat{x})
\end{aligned}$$

$$\begin{aligned}
&\quad + \tilde{\theta}_n^T \left(\varphi_n(\hat{x}_n)z_n - \frac{1}{\gamma_n}\dot{\hat{\theta}}_n \right) \\
&\quad + \left(\frac{1 + 2\tau(t)}{2(1-\eta)} + 2\hat{\tau} + 1 \right) ||u(t)||^2 \\
&\quad - \int_{t-\tau(t)}^t ||u(s)||^2 \mathrm{d}s \\
&\quad - 2\int_{t-\hat{\tau}}^t ||u(s)||^2 \mathrm{d}s - ||u(t - \hat{\tau})||^2 \\
&\quad - \frac{1}{2}||u(t - \tau(t))||^2.
\end{aligned} \tag{75}$$

It is easy to get the following results

$$z_n q^* l_n e_1 \leq \frac{z_n^2 l_n^2 q^*}{2\Gamma^2(r)} + \frac{1}{2}||\bar{e}||^2 \tag{76}$$

$$\begin{aligned}
&z_n(u(t - \tau(t)) - u(t - \hat{\tau})) \\
&\quad \leq z_n^2 + \frac{1}{2}||u(t - \tau(t))||^2 \\
&\quad + \frac{1}{2}||u(t - \hat{\tau})||^2
\end{aligned} \tag{77}$$

$$\begin{aligned}
&\mu(u(t) - u(t - \hat{\tau})) \\
&\quad \leq \mu^2 + \frac{1}{2}||u(t)||^2 + \frac{1}{2}||u(t - \hat{\tau})||^2
\end{aligned} \tag{78}$$

$$-z_n\tilde{\theta}_n^T \varphi_n(\hat{x}) \leq \frac{1}{2}z_n^2 + \frac{1}{2}\tilde{\theta}_n^T \tilde{\theta}_n. \tag{79}$$

Furthermore, one has

$$\int_{t-\tau(t)}^t \int_\theta^t ||u(s)||^2 \mathrm{d}s\mathrm{d}\theta \leq \tau(t)\int_{t-\tau(t)}^t ||u(s)||^2 \mathrm{d}s. \tag{80}$$

Substituting (76)–(80) into (75), one can get

$$\begin{aligned}
\dot{V}_n &\leq -\lambda_n||\bar{e}||^2 - \sum_{j=1}^{n-1} k_j z_j^{2\beta} + \sum_{j=1}^{n-1} \frac{\varsigma_j \tilde{\theta}_j^T \hat{\theta}_j}{\gamma_j} \\
&\quad + \sum_{j=1}^{n-1} \frac{\bar{\varsigma}_j \tilde{\Theta}_j \hat{\Theta}_j}{\bar{\gamma}_j} + \varpi\tilde{q}\hat{q} \\
&\quad + z_n(z_{n-1} + \frac{3}{2}z_n + \hat{\theta}_n^T \varphi_n(\hat{x}) \\
&\quad + \delta_n\hat{q} - \dot{\omega}_n - h\mu + u(t)) \\
&\quad + \tilde{\theta}_n^T \left(\varphi_n(\hat{x}_n)z_n - \frac{1}{\gamma_n}\dot{\hat{\theta}}_n \right) \\
&\quad - \tilde{q}(\dot{\hat{q}} - \zeta_n) + ||P||^2 \sum_{j=1}^n \tilde{\theta}_j^T \tilde{\theta}_j
\end{aligned}$$

$$+ \frac{1}{\gamma_0}(-\bar{a}r + b_0) + \sum_{j=1}^{n-1}\left(1 - \frac{1}{\xi_{j+1}}\right)\chi_{j+1}^2$$

$$- (h-1)\mu^2$$

$$- (1-\eta)D_1 - \frac{1}{2\hat{\tau}}D_2 - D_3$$

$$- \frac{1}{2\hat{\tau}}D_4 + \sum_{j=2}^{n}\frac{\tilde{\theta}_j^T\tilde{\theta}_j}{2}$$

$$+ \left(\frac{1+2\tau(t)}{2(1-\eta)} + \frac{3}{2} + 2\hat{\tau}\right)||u(t)||^2 + \Upsilon_n \quad (81)$$

where $\lambda_n = \lambda_{n-1} + \frac{1}{2}$, $\delta_n = \frac{z_n l_n^2}{2\Gamma^2(r)}$ and $\Upsilon_n = \Upsilon_{n-1} + \frac{2}{\varsigma_n}$.
Therefore, the real controller and the adaptive laws are designed as

$$u(t) = -z_{n-1} - \frac{3}{2}z_n - k_n z_n^{2\beta-1}$$

$$- \hat{\theta}_n^T\varphi_n(\hat{x}) - \delta_n\hat{q}$$

$$+ \dot{\omega}_n + h\mu \quad (82)$$

$$\dot{\hat{\theta}}_n = \gamma_n\phi_n^T(\hat{x})z_n - \varsigma_n\hat{\theta}_n \quad (83)$$

$$\dot{\hat{q}} = \zeta_{n-1} + \delta_n z_n \quad (84)$$

where the design parameters $k_n > 0$ and $\varsigma_n > 0$.

Consequently, from (81), (82), (83) and (84), we have the following result

$$\dot{V}_n \leq -\lambda_n||\bar{e}||^2 - \sum_{j=1}^{n}k_j z_j^{2\beta} + \sum_{j=1}^{n}\frac{\varsigma_j\tilde{\theta}_j^T\hat{\theta}_j}{\gamma_j}$$

$$+ \sum_{j=1}^{n-1}\frac{\bar{\varsigma}_j\tilde{\Theta}_j\hat{\Theta}_j}{\bar{\gamma}_j} + \varpi\tilde{q}\hat{q}$$

$$+ ||P||^2\sum_{j=1}^{n}\tilde{\theta}_j^T\tilde{\theta}_j + \frac{1}{\gamma_0}(-\bar{a}r + b_0)$$

$$+ \sum_{j=1}^{n-1}\left(1 - \frac{1}{\xi_{j+1}}\right)\chi_{j+1}^2 - (h-1)\mu^2$$

$$- (1-\eta)D_1 - \frac{1}{2\hat{\tau}}D_2 - D_3$$

$$- \frac{1}{2\hat{\tau}}D_4 + \sum_{j=2}^{n}\frac{\tilde{\theta}_j^T\tilde{\theta}_j}{2}$$

$$+ \left(\frac{1+2\tau(t)}{2(1-\eta)} + \frac{3}{2} + 2\hat{\tau}\right)||u(t)||^2 + \Upsilon_n \quad (85)$$

where $\lambda_n = \lambda_{n-1} + \frac{1}{2}$, $\delta_n = \frac{z_n l_n^2}{2\Gamma^2(r)}$ and $\Upsilon_n = \Upsilon_{n-1} + \frac{2}{\varsigma_n}$.

3.3 Stability analysis

Based on the above design process, the stability and boundedness of all signals will be discussed in this section.

Theorem 1 *Considered system (1) in the presence of unmodeled dynamic, unknown time varying input delay and unmeasurable states, based on Assumptions 1-3, for the initial conditions $V(0) \leq H$ and the real controller defined in (82), associated with the virtual signals defined in (39), (57) for $1 \leq i \leq n-1$, the adaptive laws defined in (40), (41), (58), (59), (83), (84) and the turning functions (42), (60), which can ensure that all signals of the closed-loop system are bounded in finite time.*

Proof Firstly, we design the following Lyapunov function.

$$V = V_n. \quad (86)$$

Based on the definitions of $\tilde{\theta}_j$, $\tilde{\Theta}_j$ and $\tilde{q}$, the inequalities $\tilde{\theta}_j^T\hat{\theta}_j \leq \frac{1}{2}\theta_j^{*T}\theta_j^* - \frac{1}{2}\tilde{\theta}_j^T\tilde{\theta}_j$, $\tilde{\Theta}_j\hat{\Theta}_j \leq \frac{1}{2}\Theta_j^{*2} - \frac{1}{2}\tilde{\Theta}_j^2$, and $\tilde{q}\hat{q} \leq \frac{1}{2}q^{*2} - \frac{1}{2}\tilde{q}^2$ are used to deal with the terms $\tilde{\theta}_j^T\hat{\theta}_j$, $\tilde{\Theta}_j\hat{\Theta}_j$ and $\tilde{q}\hat{q}$.

Differentiating V, one can get

$$\dot{V} \leq -\lambda_n||\bar{e}||^2 - \sum_{j=1}^{n}k_j z_j^{2\beta}$$

$$- \left(\frac{\varsigma_1}{2\gamma_1} - ||P||^2\right)\tilde{\theta}_1^T\tilde{\theta}_1 - \sum_{j=1}^{n-1}\left(\frac{1}{\xi_{j+1}} - 1\right)\chi_{j+1}^2$$

$$- \sum_{j=2}^{n}\left(\frac{\varsigma_j}{2\gamma_j} - \frac{1}{2} - ||P||^2\right)\tilde{\theta}_j^T\tilde{\theta}_j - \sum_{j=1}^{n-1}\frac{\bar{\varsigma}_j\tilde{\Theta}_j^2}{2\bar{\gamma}_j}$$

$$- \frac{\varpi\tilde{q}^2}{2} - \frac{\bar{a}r}{\gamma_0} - (h-1)\mu^2$$

$$- (1-\eta)D_1 - \frac{1}{2\hat{\tau}}D_2 - D_3 - \frac{1}{2\hat{\tau}}D_4$$

$$+ \sum_{j=1}^{n}\frac{\varsigma_j\theta_j^{*T}\theta_j^*}{2\gamma_j} + \sum_{j=1}^{n-1}\frac{\bar{\varsigma}_j\Theta_j^{*2}}{2\bar{\gamma}_j}$$

$$+ \frac{\varpi q^{*2}}{2} + \left(\frac{1 + 2\tau(t)}{2(1-\eta)} + \frac{3}{2} + 2\hat{\tau} \right) ||u(t)||^2$$

$$- \frac{b_0}{\gamma_0} + \Upsilon_n. \tag{87}$$

Let

$$\lambda = \frac{\lambda_n}{\lambda_{max}(P)} > 0 \tag{88}$$

$$\hat{\gamma}_1 = \varsigma_1 - 2\gamma_1 ||P||^2 > 0 \tag{89}$$

$$\hat{\gamma}_j = \varsigma_j - \gamma_j - 2\gamma_j ||P||^2 > 0, \quad j = 2, 3, \cdots, n \tag{90}$$

$$\hat{\xi}_{j+1} = \frac{1}{\xi_{j+1}} - 1 > 0, \quad j = 1, 2, \cdots, n-1 \tag{91}$$

Substituting (88)–(91) into (87)

$$\dot{V} \leq -\lambda \bar{e}^T P \bar{e} - \sum_{j=1}^{n} k_j z_j^{2\beta} - \sum_{j=1}^{n} \frac{\hat{\gamma}_j}{2\gamma_j} \tilde{\theta}_j^T \tilde{\theta}_j$$

$$- \sum_{j=1}^{n-1} \frac{\bar{\varsigma}_j \tilde{\Theta}_j^2}{2\bar{\gamma}_j} - \sum_{j=1}^{n-1} \hat{\xi}_{j+1} \chi_{j+1}^2 - \frac{\varpi \tilde{q}^2}{2}$$

$$- \frac{\bar{a}r}{\gamma_0} - (h-1)\mu^2 - (1-\eta)D_1 - \frac{D_2}{2\hat{\tau}}$$

$$- D_3 - \frac{D_4}{2\hat{\tau}} + \sum_{j=1}^{n} \frac{\varsigma_j \theta_j^{*T} \theta_j^*}{2\gamma_j}$$

$$+ \sum_{j=1}^{n-1} \frac{\bar{\varsigma}_j \Theta_j^{*2}}{2\bar{\gamma}_j} + \frac{\varpi q^{*2}}{2}$$

$$+ \left(\frac{1 + 2\tau(t)}{2(1-\eta)} + \frac{3}{2} + 2\hat{\tau} \right) ||u(t)||^2 - \frac{b_0}{\gamma_0} + \Upsilon_n$$

$$\leq -c\bar{e}^T P \bar{e} - c \sum_{j=1}^{n} z_j^{2\beta} - c \sum_{j=1}^{n} \frac{1}{2\gamma_j} \tilde{\theta}_j^T \tilde{\theta}_j$$

$$- c \sum_{j=1}^{n-1} \frac{\tilde{\Theta}_j^2}{2\bar{\gamma}_j} - c \sum_{j=1}^{n-1} \frac{1}{2} \chi_{j+1}^2 - c \frac{\tilde{q}^2}{2}$$

$$- c \frac{r}{\gamma_0} - c \frac{\mu^2}{2} - cD_1 - cD_2 - cD_3 - cD_4$$

$$+ \sum_{j=1}^{n} \frac{\varsigma_j \theta_j^{*T} \theta_j^*}{2\gamma_j} + \frac{\varpi q^{*2}}{2} - \frac{b_0}{\gamma_0} + \sum_{j=1}^{n-1} \frac{\bar{\varsigma}_j \Theta_j^{*2}}{2\bar{\gamma}_j}$$

$$+ \left(\frac{1 + 2\tau(t)}{2(1-\eta)} + \frac{3}{2} + 2\hat{\tau} \right) ||u(t)||^2 + \Upsilon_n \tag{92}$$

where $c = \min\{\lambda, 2k_j, \hat{\gamma}_j : 1 \leq j \leq n, \bar{\varsigma}_j, 2\hat{\xi}_{j+1} : 1 \leq j \leq n-1, \varpi, \bar{a}, 2(h-1), 1-\eta, \frac{1}{2\hat{\tau}}\}$.

Let $x = \bar{e}^T P \bar{e}$, $y = 1$, $\vartheta_1 = \beta$, $\vartheta_2 = 1 - \beta$, $\vartheta_3 = \beta^{-1}$, from Lemma 3, one can get

$$(\bar{e}^T P \bar{e})^\beta \leq \bar{e}^T P \bar{e} + (1-\beta)\beta^{\frac{\beta}{1-\beta}} \tag{93}$$

Furthermore, similar to (93), by using Lemma 3, one can get the following results.

$$\left(\sum_{j=1}^{n} \frac{1}{2} \tilde{\theta}_j^T \tilde{\theta}_j \right)^\beta \leq \sum_{j=1}^{n} \frac{1}{2} \tilde{\theta}_j^T \tilde{\theta}_j + (1-\beta)\beta^{\frac{\beta}{1-\beta}} \tag{94}$$

$$\left(\sum_{j=1}^{n-1} \frac{\tilde{\Theta}_j^2}{2\bar{\gamma}_j} \right)^\beta \leq \sum_{j=1}^{n-1} \frac{\tilde{\Theta}_j^2}{2\bar{\gamma}_j} + (1-\beta)\beta^{\frac{\beta}{1-\beta}} \tag{95}$$

$$\left(\sum_{j=1}^{n-1} \frac{1}{2} \chi_{j+1}^2 \right)^\beta \leq \sum_{j=1}^{n-1} \frac{1}{2} \chi_{j+1}^2 + (1-\beta)\beta^{\frac{\beta}{1-\beta}} \tag{96}$$

$$\left(\frac{\tilde{q}^2}{2} \right)^\beta \leq \sum_{j=1}^{n-1} \frac{1}{2} \tilde{q}^2 + (1-\beta)\beta^{\frac{\beta}{1-\beta}} \tag{97}$$

$$\left(\frac{r}{\gamma_0} \right)^\beta \leq \frac{r}{\gamma_0} + (1-\beta)\beta^{\frac{\beta}{1-\beta}} \tag{98}$$

$$\left(\frac{\mu^2}{2} \right)^\beta \leq \frac{\mu^2}{2} + (1-\beta)\beta^{\frac{\beta}{1-\beta}} \tag{99}$$

$$D_i^\beta \leq D_i + (1-\beta)\beta^{\frac{\beta}{1-\beta}}, i = 1, 2, 3, 4. \tag{100}$$

Substituting (93)–(100) into (92) and by using Lemma 3, one can get

$$\dot{V} \leq -c(\bar{e}^T P \bar{e})^\beta - c \left(\sum_{j=1}^{n} z_j^2 \right)^\beta$$

$$- c \left(\sum_{j=1}^{n} \frac{1}{2} \tilde{\theta}_j^T \tilde{\theta}_j \right)^\beta - c \left(\sum_{j=1}^{n-1} \frac{\tilde{\Theta}_j^2}{2\bar{\gamma}_j} \right)^\beta$$

$$- c \left(\sum_{j=1}^{n-1} \frac{1}{2} \chi_{j+1}^2 \right)^\beta - c \left(\frac{\tilde{q}^2}{2} \right)^\beta$$

$$- c \left(\frac{r}{\gamma_0} \right)^\beta - c \left(\frac{\mu^2}{2} \right)^\beta - cD_1^\beta$$

$$- cD_2^\beta - cD_3^\beta - cD_4^\beta + \sum_{j=1}^{n} \frac{\varsigma_j \theta_j^{*T} \theta_j^*}{2\gamma_j}$$

$$+ \sum_{j=1}^{n-1} \frac{\bar{\varsigma}_j \Theta_j^{*2}}{2\bar{\gamma}_j} + \frac{\varpi q^{*2}}{2} - \frac{b_0}{\gamma_0}$$

$$+ \Upsilon_n + 11(1-\beta)\beta^{\frac{\beta}{1-\beta}}$$

$$+ \left(\frac{1+2\tau(t)}{2(1-\eta)} + \frac{3}{2} + 2\hat{\tau} \right) ||u(t)||^2$$

$$\le -cV^\beta + \sum_{j=1}^{n} \frac{\varsigma_j \theta_j^{*T} \theta_j^*}{2\gamma_j} + \sum_{j=1}^{n-1} \frac{\bar{\varsigma}_j \Theta_j^{*2}}{2\bar{\gamma}_j}$$

$$+ \frac{\varpi q^{*2}}{2} - \frac{b_0}{\gamma_0} + \Upsilon_n + 11(1-\beta)\beta^{\frac{\beta}{1-\beta}}$$

$$+ \left(\frac{1+2\tau(t)}{2(1-\eta)} + \frac{3}{2} + 2\hat{\tau} \right) ||u(t)||^2. \tag{101}$$

If $V(t) < H$, then the error signals $z_j(t)$, $\tilde{\theta}_j(t)$, $\bar{e}_j(t)$, $\tilde{\Theta}_j(t)$, $\chi_{j+1}(t)$, and $\tilde{q}(t)$, the dynamic signal $r(t)$ and the compensation signal $\mu(t)$ are all bounded. It is easy to prove the boundedness of $\hat{\theta}_i(t)$, $\hat{\Theta}_j(t)$, $\hat{q}(t)$, and $e_i(t)$. Furthermore, from (39), (57) and (82) one can get that α_i and $u(t)$ are all bounded. Therefore, we can get

$$\dot{V}(t) \le -cV(t)^\beta + d \tag{102}$$

where $d = \sum_{j=1}^{n} \frac{\varsigma_j \theta_j^{*T} \theta_j^*}{2\gamma_j} + \sum_{j=1}^{n-1} \frac{\bar{\varsigma}_j \Theta_j^{*2}}{2\bar{\gamma}_j} + \frac{\varpi q^{*2}}{2} - \frac{b_0}{\gamma_0} + \Upsilon_n + \left(\frac{1+2\tau(t)}{2(1-\eta)} + \frac{3}{2} + 2\hat{\tau} \right) u_{max} + 11(1-\beta)\beta^{\frac{\beta}{1-\beta}}$ and u_{max} is the upper bound of $||u(t)||^2$.

Based on Lemma 2, the trajectory of the closed-loop system will be derived into $V(t)^\beta \le \frac{d}{(1-\rho)c}$ in finite time T_R, and T_R can be computed by

$$T_R = \left(\frac{1}{(1-\beta)c\rho} \right)$$

$$\left[V(0)^{1-\beta} - \left(\frac{d}{(1-\rho)c} \right)^{\frac{1-\beta}{\beta}} \right] \tag{103}$$

where, $V(0)$ is the initial value of $V(t)$.

If $V(t) = H$, based on the boundedness of $z_j(t)$, $\tilde{\theta}_j(t)$, $\bar{e}_j(t)$, $\tilde{\Theta}_j(t)$, $\chi_{j+1}(t)$, and $\tilde{q}(t)$, we also can get $\dot{V}(t) \le -cV(t)^\beta + d$ with $d = \sum_{j=1}^{n} \frac{\varsigma_j \theta_j^{*T} \theta_j^*}{2\gamma_j} +$

$\sum_{j=1}^{n-1} \frac{\bar{\varsigma}_j \Theta_j^{*2}}{2\bar{\gamma}_j} + \frac{\varpi q^{*2}}{2} - \frac{b_0}{\gamma_0} + \Upsilon_n + \left(\frac{1+2\tau(t)}{2(1-\eta)} + \frac{3}{2} + 2\hat{\tau} \right) u_{max} +$ $11(1-\beta)\beta^{\frac{\beta}{1-\beta}}$. If we choose $c > \frac{d}{H^\beta}$, then the derivative of satisfies $\dot{V}(t) < 0$. This means that the trajectory of will not escape the boundedness of H. That is to say $V(t) \le H, \forall t \ge 0$ for $V(0) \le H$. Then, based on (102) and Lemma 2, all signals will be derived into $V(t)^\beta \le \frac{d}{(1-\rho)c}$ in finite time.

From above discussion, all signals are SGPFS for $V(0) \le H$. The proof is completed. □

Remark 4 Compared with the existing results [50–55, 63,70–72], which only work on the nonlinear system with small input delay or ignoring input delay for finite time control, the proposed scheme in this paper can tackle the problem of finite time control for non-strict feedback uncertain nonlinear systems with unknown time-varying input delay.

Remark 5 The proposed compensation signal can be constructed, which is independent on the system model. The compensation signal is bounded in finite time and the design procedures have nothing to do with the input delay. The fuzzy observer and the dynamical signal function can be easily designed to deal with unmeasurable states and unmodeled dynamics, respectively. In addition, the finite time stability of the closed-loop system can also be guaranteed by choosing the appropriate Lyapunov function and the setting parameters. Thus, the proposed method is tractable for fuzzy finite time control.

4 Examples

In this section, two examples are utilized to show the effectiveness and characteristics of the proposed approach.

Example 1 Consider a second-order system with unmodeled dynamics and input delay described as the following form:

$$\begin{cases} \dot{z} = -2z + 0.25x_1^2 \\ \dot{x}_1 = x_2 + \frac{x_2 \cos(x_1^2) + \sin(x_1 x_2)}{\sin^2(x_1)+1} + \Delta_1 \\ \dot{x}_2 = u(t - \tau(t)) + \frac{x_2}{1+x_1^2 x_2^2} + \Delta_2 \\ y = x_1 \end{cases} \tag{104}$$

where $\Delta_1 = 0.5z^2$, $\Delta_2 = z^2$ and $\tau(t) = 0.4 + 0.1sin(t)s$.

Let $\psi_{i1} = y$, $\psi_{i2} = z^2$ for $i = 1, 2$, it can be easily to prove Assumption 1. Choosing $V(z) = z^2$, one has $\dot{V}(z) = -2z^2 + 0.25z_1x_1^2 \leq -1.6z^2 + 2.5x_1^4 + 0.625$. Let $a = 1.6$, $\kappa(|x|) = 2.5x_1^4$, $b_0 = 0.625$, $\pi_1 = 0.5z^2$ and $\pi_2 = 1.6z^2$ then Assumption 2 is true.

By selecting $\bar{a} = 1.2 \in (0, a)$ and according to Lemma 1 the dynamical signal function r can be designed as

$$\dot{r} = -1.2r + 2.5x_1^4 + 0.625.$$

The virtual control input α_1 and u are designed as (39) and (82), $\dot{\hat{\theta}}_1$, $\dot{\hat{\theta}}_2$, $\dot{\hat{\Theta}}_1$, and $\dot{\hat{q}}$ are designed as (40), (41), (83), (84), and the turning function ζ_1 is designed as (42).

In the simulation, $\mu_{F_i^j(\hat{\underline{x}}_i)} = \exp[-0.5(\hat{\underline{x}}_i + 3 - j)]$, for $i = 1, 2$, $j = 1, 2, \cdots, 5$. The observer parameters are selected as $l_1 = 30$ and $l_2 = 150$. Selecting $Q = 6I$, one can get

$$P = \begin{bmatrix} 0.1007 & 0.0200 \\ 0.0200 & 15.7000 \end{bmatrix}.$$

Let the initial conditions $x_1(0) = x_2(0) = 0.2$, $\hat{x}_1(0) = \hat{x}_2(0) = 0$, $r(0) = 0$, $\mu(0) = 0$, $\omega_2(0) = 0$, $\hat{\theta}_1(0) = \hat{\theta}_2(0) = [0, 0, 0, 0, 0]$, $\hat{\Theta}_1(0) = 0$, $\hat{q}(0) = 0$ and $z(0) = 0.2$. The other parameters are taken as $k_1 = 8$, $k_2 = 15$, $\gamma_1 = \gamma_2 = \bar{\gamma}_1 = \bar{\gamma}_2 = 2$, $\gamma_0 = 1$, $\varsigma_1 = 1$, $\varsigma_2 = 5$, $\bar{\varsigma}_1 = 5$, $\bar{\varsigma}_2 = 6$, $\varpi = 100$, $\xi_1 = 0.01$, $h = 2$, $\beta = 0.92$ and $\hat{\tau} = 0.5$. The simulation time is $10s$, and the simulation results are shown in Figs. 1, 2, 3, 4 and 5.

Figures 1 and 2 draw the trajectories of the state variables x_1 and x_2 and their estimated values $\hat{x}_1$ and $\hat{x}_2$, respectively.

From Figs. 1 and 2, one can observe that the state trajectories of $\hat{x}_1$ and $\hat{x}_2$ can track the state variables x_1 and x_2 quickly, which means that the sate observer can effectively estimate the unmeasurable states by using the proposed fuzzy observer. In addition, it can be seen from Figs. 1 and 2 that the proposed adaptive fuzzy finite-time controller can effectively stabilize the closed-loop systems under the time varying input delay, unmodeled dynamics and unmeasurable states.

Figure 3 depicts the trajectories of auxiliary variable μ.

From Fig. 3, one can observe that under the time varying input delay $\tau(t) = 0.4 + 0.1sin(t)s$, the compensation signal μ is bounded in finite time. This means
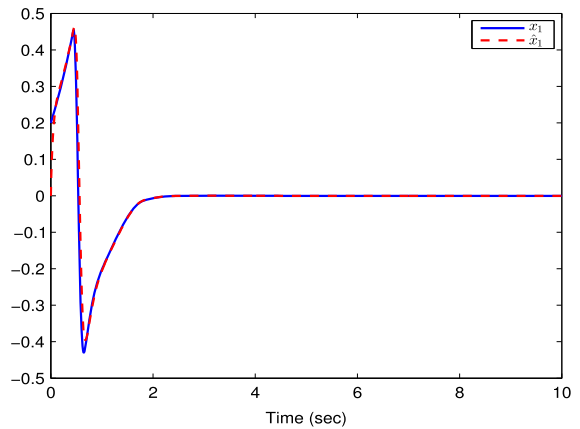
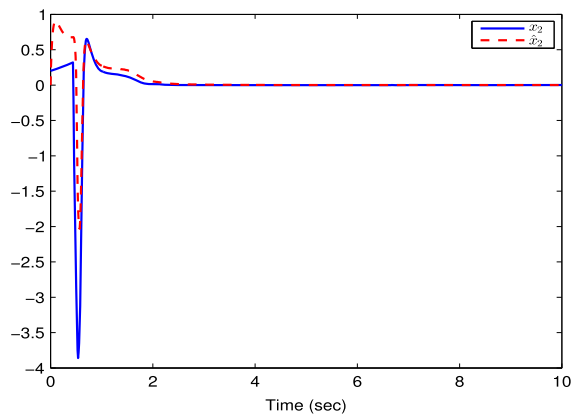**Fig. 1** The trajectories of x_1 and $\hat{x}_1$ in example 1
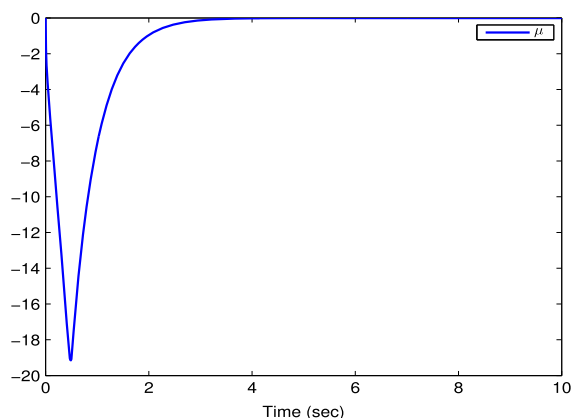

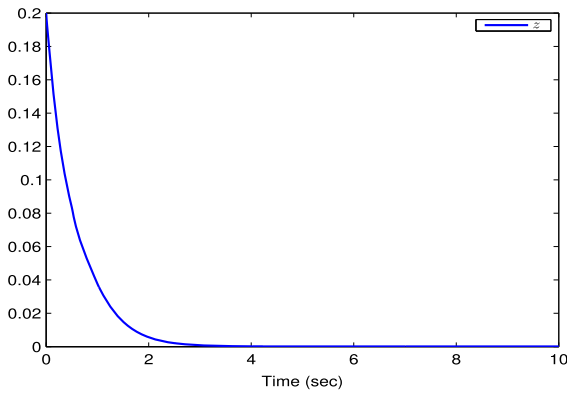**Fig. 2** The trajectories of x_2 and $\hat{x}_2$ in example 1
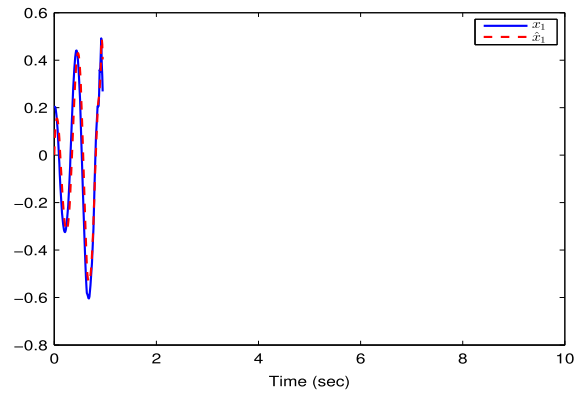

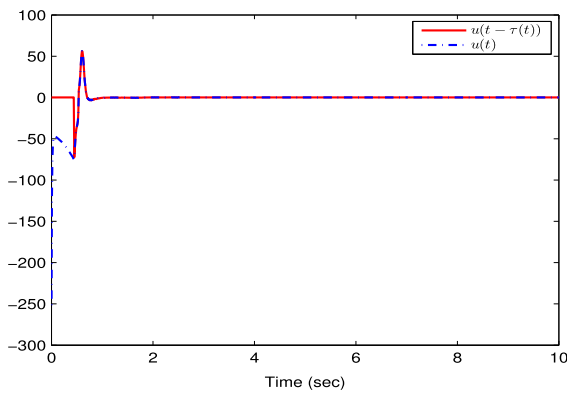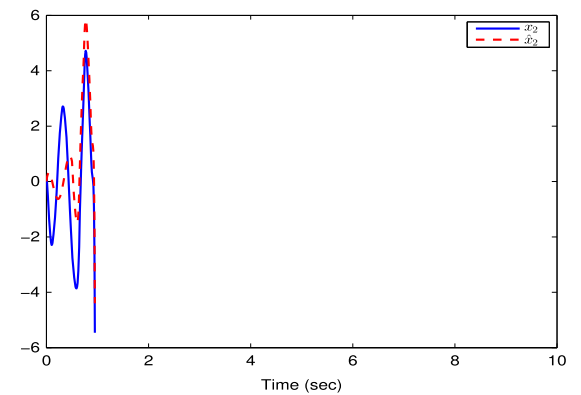
Fig. 3 The trajectory of μ in example 1

Fig. 4 The trajectory of unmodeled dynamics z in example 1

Fig. 6 The trajectories of x_1 and $\hat{x}_1$ in example 1(Case 1)

Fig. 5 The trajectories of $u(t - \tau(t))$ and $u(t)$ in example 1

Fig. 7 The trajectories of x_2 and $\hat{x}_2$ in example 1(Case 1)

that the proposed compensation signal is bounded in finite time, although there are unmodeled dynamics in the real system. On the other hand, Fig. 3 shows that the proposed compensation mechanism cannot destroy the stability of the closed-loop systems.

Figure 4 draws the trajectory of unmodeled dynamics z.

From Fig. 4, one can observe that the trajectory of unmodeled dynamics z is bounded by using the proposed adaptive fuzzy finite time controller.

The control signals $u(t - \tau(t))$ and $u(t)$ are shown in Fig. 5.

From the simulation results in Figs. 1, 2, 3, 4 and 5, one can obtain that for the time varying input delay, unmodeled dynamics and unmeasurable states, all the signals of the closed-loop systems are bounded by using the proposed adaptive fuzzy finite time controller.

In what follows, two cases are considered to verify the effectiveness of the proposed scheme for the

unmodeled dynamics and unknown time varying input delay.

Case 1. The unmodeled dynamics exists in the considered system (104) and without unknown time varying input delay, i.e., $\tau(t) = 0$. If the dynamical signal r does not employed for controller design, then finite time control scheme cannot guarantee the stability of the system. To test this, the compensation signal for unknown time varying input delay is omitted, i.e., $\mu = 0$ and the other parameters are designed as above for system (104), the trajectories of x_1, $\hat{x}_1$, x_2, $\hat{x}_2$ and z are shown in Figs. 6, 7 and 8.

From Figs. 6, 7 and 8, it can be observed that for the finite time control scheme, all signals of the closed-loop system become unstable if the dynamic signal is not used to deal with the unmodeled dynamics.
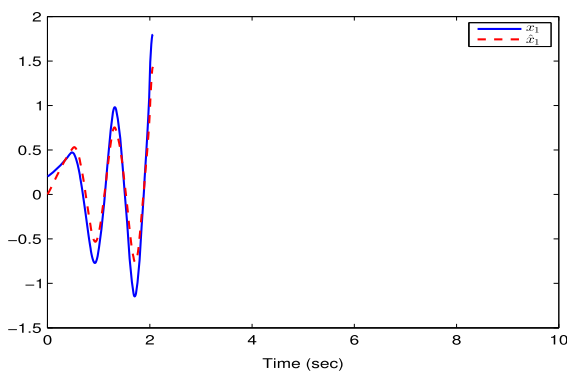
Case 2. To test the validity of the compensation mechanism for unknown time varying input delay, both the
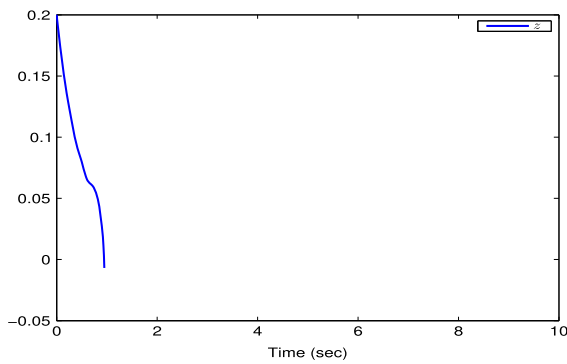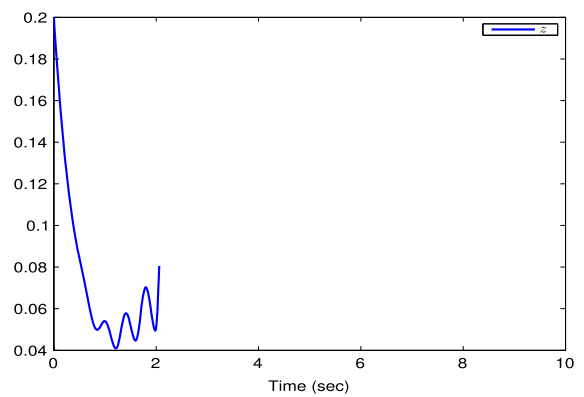
Fig. 8 The trajectory of unmodeled dynamics z in example 1(Case 1)
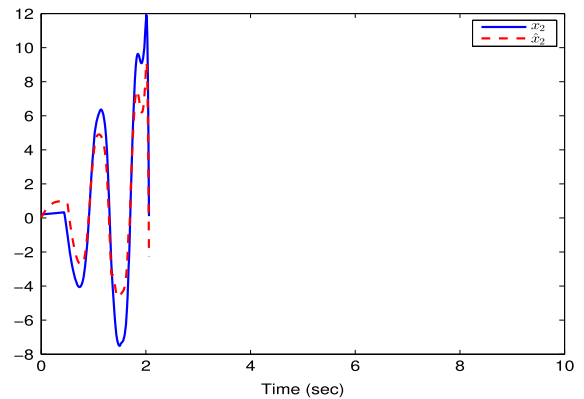
Fig. 9 The trajectories of x_1 and $\hat{x}_1$ in example 1(Case 2)

Fig. 10 The trajectories of x_2 and $\hat{x}_2$ in example 1(Case 2)

Fig. 11 The trajectory of unmodeled dynamics z in example 1(Case 2)

unmodeled dynamics and the unknown time varying input delay are included in the considered system (104). Under such a condition, only the dynamical signal is employed to tackle the unmodeled dynamics, and the compensation signal for unknown time varying input delay is ignored, i.e., $\mu = 0$.

The dynamical signal r and the other parameters are designed as above for system (104), the trajectories of x_1, $\hat{x}_1$, x_2, $\hat{x}_2$ and z are shown in Figs. 9, 10 and 11.

From Figs. 9, 10 and 11, one can observe that the finite time control scheme also cannot ensure the stability of the closed-loop system when the compensation mechanism for unknown time varying input delay is ignored for the controller design. In addition, from the trajectory of unmodeled dynamics z in Figs. 8 and 11, one can observer that the unmodeled dynamics z does not change sharply when the dynamical signal is used in Fig. 11. However, due to the existence of unknown time varying input delay, the finite time controller cannot stabilize the closed-loop system.

From the above simulation results, one can observe that the proposed method in this paper can tackle finite time control for non-strict feedback systems with unmeasurable states, unknown time varying input delay and unmodeled dynamics.

Example 2 Consider an application example the one-link manipulator with motor dynamics and disturbances used in [53,58]. The dynamics model of the system is described as follows:

$$\begin{cases} D\ddot{q} + B\dot{q} + N\sin(q) = I + I_d \\ M\dot{I} + HI = -K_m\dot{q} + V \end{cases} \quad (105)$$

where q is the link position, $\dot{q}$ is the velocity, and $\ddot{q}$ is the acceleration. I denotes the torque produced, $I_d = sin(\dot{q})cos(I)$ denotes the torque disturbance, and V denotes the input control electromechanical torque. The parameters are selected as $D = 1$ kg m^2,

$B = 1\,\text{Nm s/rad}, N = 10\,\text{Nm}, M = 0.3\,\text{H}, H = 1.0\,\Omega$ and $Km = 2\,\text{Nm/A}$.

It is assumed that the time varying input delay, the disturbance and the unmodeled dynamics exist in the system (104). Let $x_1 = q$, $x_2 = \dot{q}$, $x_3 = I/D$ and $u(t - \tau(t)) = V(t - \tau(t))/DM$ then system (105) can be written as

$$\begin{cases} \dot{z} = -2z + 0.25x_1^2 \\ \dot{x}_1 = x_2 + \Delta_1 \\ \dot{x}_2 = x_3 - \dfrac{N}{D}\sin(x_1) - \dfrac{B}{D}x_2 \\ \qquad + \dfrac{1}{D}sin(x_2)cos(Dx_3) + \Delta_2 \\ \dot{x}_3 = -\dfrac{K_m}{MD}x_2 - \dfrac{H}{MD}x_3 + u(t - \tau(t)) + \Delta_3 \\ y = x_1 \end{cases}$$

where $\Delta_1 = z^2 x_1 sin(x_1)$, $\Delta_2 = z^2 x_1 x_2$, $\Delta_3 = zx_1x_2x_3$, and $\tau(t) = 0.4 + 0.1sin(t)s$.

Let $\psi_{i1} = y$, $\psi_{i2} = z^2$ for $i = 1, 2, 3$, then Assumption 1 holds. Choosing $V(z) = z^2$, one has $\dot{V}(z) = -2z^2 + 0.5z_1x_1^2 \le -1.5z^2 + 2.5x_1^4 + 0.625$. Let $a = 1.5$, $\kappa(|x|) = 2.5x_1^4$, $b_0 = 0.625$, $\pi_1 = 0.5z^2$ and $\pi_2 = 1.5z^2$ then Assumption 2 is true.

By selecting $\bar{a} = 1.2 \in (0, a)$, and from Lemma 1 the dynamical signal function r can be designed as

$$\dot{r} = -1.2r + 2.5x_1^4 + 0.625.$$

The virtual control input α_1, α_2 and u are designed as (39), (57) and (82), $\dot{\hat{\theta}}_1$, $\dot{\hat{\theta}}_2$, $\dot{\hat{\theta}}_3$, $\dot{\hat{\Theta}}_1$, $\dot{\hat{\Theta}}_2$, and $\dot{\hat{q}}$ are designed as (40), (41), (58), (59),(83) and (84), the turning function ζ_1 and ζ_2 are designed as (42) and (60).

In the simulation, $\mu_{F_i^j}(\hat{x}_i) = \exp[-0.5(\hat{x}_i + 3 - j)]$ for $i = 1, 2, 3$, $j = 1, 2, \cdots, 5$. The observer parameters are selected as $l_1 = 80$, $l_2 = 2$, $l_3 = 2$. Selecting $Q = I$, one can get

$$P = \begin{bmatrix} 0.1361 & 10.3861 & 0.2500 \\ 10.3861 & 830.908 & 20.2722 \\ 0.2500 & 20.2722 & 21.2722 \end{bmatrix}.$$

Let the initial conditions $x_1(0) = x_2(0) = 0.5$, $x_3(0) = -0.5$, $\hat{x}_1(0) = \hat{x}_2(0) = \hat{x}_3(0) = 0$, $\hat{\theta}_1(0) = \hat{\theta}_2(0) = \hat{\theta}_3(0) = [0, 0, 0, 0, 0]$, $\hat{\Theta}_1(0) = \hat{\Theta}_2(0) = 0$, $r(0) = 0$, $\mu(0) = 0$, $\omega_2(0) = \omega_3(0) = 0$, $\hat{q}(0) = 0$, $z(0) = 0.5$. The other parameters are taken as $k_1 = 15$, $k_2 = 15$, $k_3 = 12$, $\gamma_1 = 3$, $\gamma_2 = 1$, $\gamma_3 = 3$, $\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}_3 = 3$, $\gamma_0 = 2$, $\varsigma_1 = 2$, $\varsigma_2 = 4$, $\varsigma_3 = 3$,
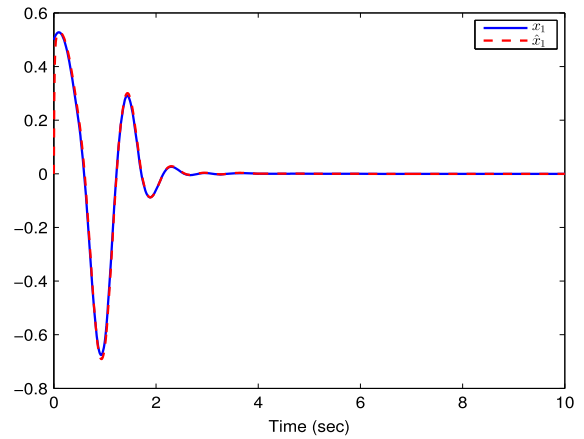

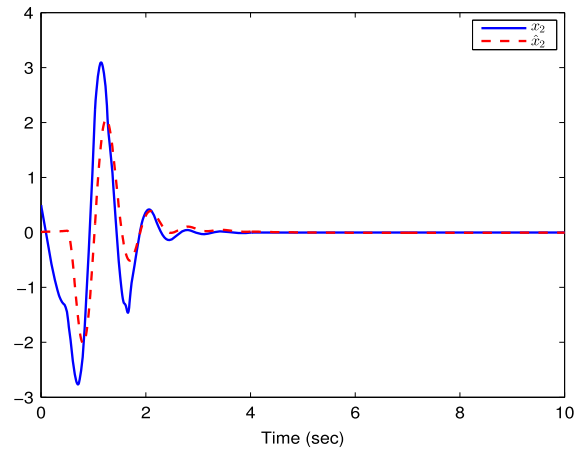Fig. 12 The trajectories of x_1 and $\hat{x}_1$ in example 2

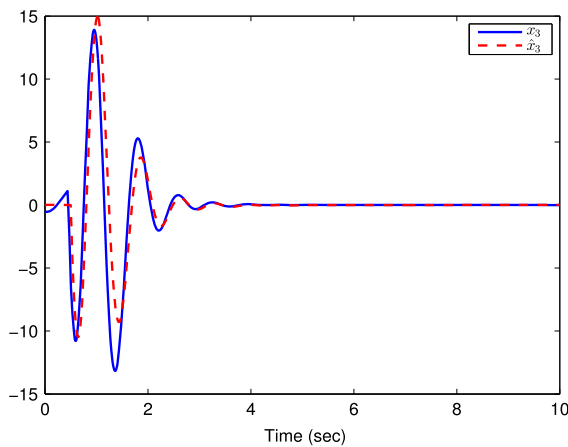Fig. 13 The trajectories of x_2 and $\hat{x}_2$ in example 2

$\bar{\varsigma}_1 = 2$, $\bar{\varsigma}_2 = 4$, $\bar{\varsigma}_3 = 3$, $\varpi = 100$, $\xi_1 = 2$, $\xi_2 = 0.2$, $h = 4$, $\beta = 0.95$, $\hat{\tau} = 0.5$. The simulation time is $10s$, and the simulation results are shown in Figs. 12, 13, 14, 16, 15 and 17.

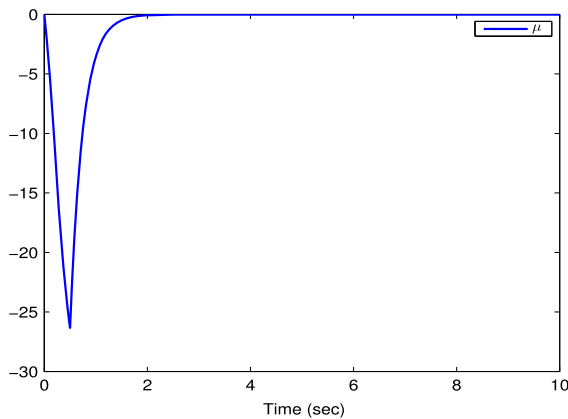Figures 12, 13 and 14 draw the trajectories of x_1, $\hat{x}_1$, x_2, $\hat{x}_2$, x_3 and $\hat{x}_3$.

From Figs. 12, 13 and 14, one can observe that under the time varying input delay $\tau(t) = 0.4 + 0.1sin(t)s$ and the unmodeled dynamics and disturbance, the proposed fuzzy state observer can effectively estimate the unmeasurable states and all state variables are bounded.

The trajectories of auxiliary variable μ are depicted in Fig. 15, which shows that the compensation signal μ is bounded in finite time.

From the simulation result in Fig. 15, one can conclude that the proposed compensation signal is bounded

Fig. 14 The trajectories of x_3 and $\hat{x}_3$ in example 2
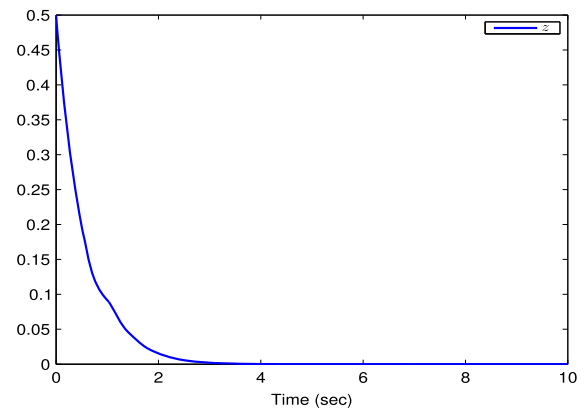

Fig. 15 The trajectory of μ in example 2

Fig. 16 The trajectory of unmodeled dynamics z in example 2

Fig. 17 The trajectories of $u(t - \tau(t))$ and $u(t)$ in example 2

in finite time, although there are unmodeled dynamics in the real system.

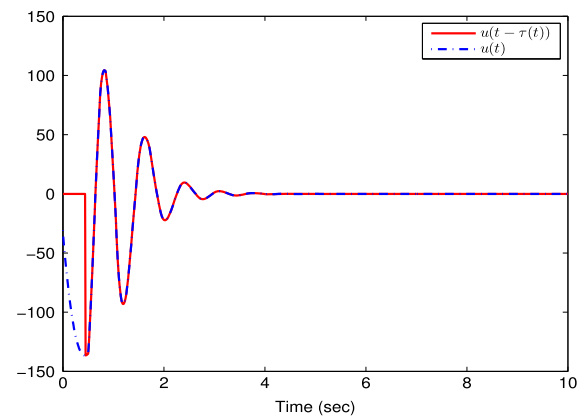Figure 16 draws the trajectory of the unmodeled dynamics z.

From Fig. 16, one can observer that the trajectory of unmodeled dynamics z is bounded. This means that the unmodeled dynamics can be effectively suppressed by introducing dynamic signal mechanism in the design process of adaptive fuzzy finite time controller.

Figure 17 shows that the control signals $u(t - \tau)$ and $u(t)$ are bounded.

From the simulation results in Figs. 12, 13, 14, 16, 15 and 17, one can conclude that for time varying input delay, unmodeled dynamic and unmeasurable states, all signals are bounded by using the proposed adaptive fuzzy finite time controller.

Remark 6 We considered the one-link manipulator with motor dynamics, disturbances, unmodeled dynamics and unknown time varying input delay. The system model is a non-strict feedback form and the finite time control performance includes the time varying input delay, which will cause great difficulty to design finite time controller. The proposed compensation mechanism effectively solves the input delay problem, and the design process does not make the controller design complicated. The effective control performance is verified by the simulation results.

Furthermore, in order to demonstrate the effectiveness of the proposed compensation mechanism to deal with time input delay, a comparison with the Pade approximation method employed in [70,71] is carried out to test the control performance. The simulation results are shown in Figs. 18, 19 and 20.
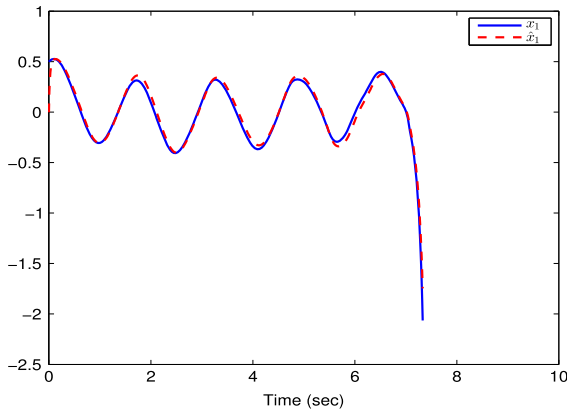
Fig. 18 The trajectories of x_1 and $\hat{x}_1$ in example 2 (Pade approximation method)
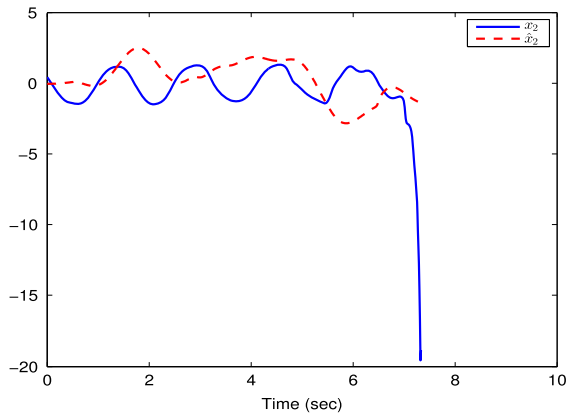
Fig. 19 The trajectories of x_2 and $\hat{x}_2$ in example 2 (Pade approximation method)
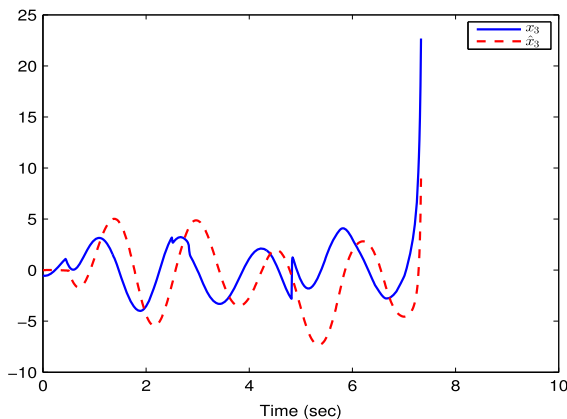
Fig. 20 The trajectories of x_3 and $\hat{x}_3$ in example 2 (Pade approximation method)

From Figs. 18, 19 and 20, one can observe that the fuzzy observer cannot effectively estimate the unmeasurable states, and all state variables of the closed-loop systems are unstable when time varying input delay occurs for Pade approximation method. The simulation results show that the control signal is completely invalid when the input delay occurs. This means that the Pade approximation method is invalid for long time varying input delay.

Comparative explanations The proposed method in this paper gives an effective way for adaptive fuzzy finite time control of non-strict feedback nonlinear systems with unmodeled dynamics, unknown time varying input delay and unmeasurable states. Compared with the existing results [17,36–43,56–63,70,71], the main advantages of the proposed scheme can be summarized in three aspects.

(1) The proposed compensation mechanism can effectively overcome the design difficulty caused by the unknown time varying input delay. Unlike the Pade approximation method in [70,71], which is invalid once the input delay is long, the proposed method can tackle long unknown time varying input delay. In addition, the proposed method does not depend on the system model.

(2) The algebraic loop problem has been overcome and the complexity of the non-strict feedback system is reduced for finite time controller design. Specially, the proposed method is also suitable for nonlinear systems with strict-feedback form. Unlike [36–43], the considered finite time control schemes are limited to the strict feedback nonlinear systems or all state variables are measurable, the proposed method is more general.

(3) Compared with the results in [17,56–63], the unmodeled dynamics and unknown input delay problem are both considered in finite time control schemes, which can improve the robustness of the closed-loop system in practical applications. Most of the existing general finite time control schemes are invalid for this issue. In addition, the computational burden for finite time controller design is reduced by using the idea of DSC scheme.

5 Conclusions

In this paper, observer-based adaptive fuzzy finite time control for non-strict feedback uncertain nonlinear sys-

tems has been addressed. A novel compensation mechanism and dynamical signal function are introduced to solve the challenges caused by unknown time varying input delay and unmodeled dynamics, respectively. A fuzzy state observer is designed to estimate the unknown state variables based on the approximation ability of FLS. The non-strict feedback structure is simplified by using the property of FLS. Moreover, the relationship between input delay $\tau(t)$ and the convergence time T_R will be further studied in future work.

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this paper

References

1. Tong, S., Min, X., Li, Y.: Observer-based adaptive fuzzy tracking control for strict-feedback nonlinear systems with unknown control gain functions. IEEE Trans. Cybern. **50**(9), 3903–3913 (2020)
2. Qi, W., Yang, X., Park, J.H., et al.: Fuzzy SMC for quantized nonlinear stochastic switching systems with semi-Markovian process and application. IEEE Trans. Cybern. (2021). https://doi.org/10.1109/TCYB.2021.3069423
3. Li, S., Ahn, C.K., Xiang, Z.: Sampled-data adaptive output feedback fuzzy stabilization for switched nonlinear systems with asynchronous switching. IEEE Trans. Fuzzy Syst. **27**(1), 200–205 (2019)
4. Wang, X., Niu, B., Song, X., et al.: Neural networks-based adaptive practical preassigned finite-time fault tolerant control for nonlinear time-varying delay systems with full state constraints. Int. J. Robust Nonlinear Control **31**(5), 1497–1513 (2021)
5. Qi, W., Gao, X., Ahn, C.K., et al.: Fuzzy integral sliding-mode control for nonlinear semi-Markovian switching systems with application. IEEE Trans. Syst. Man Cybern.: Syst. **52**(3), 1674–1683 (2022)
6. Zou, W., Shi, P., Xiang, Z., et al.: Consensus tracking control of switched stochastic nonlinear multiagent systems via event-triggered strategy. IEEE Trans. Neural Netw. Learn. Syst. **31**(3), 1036–1045 (2020)
7. Jing, Y.H., Yang, G.H.: Fuzzy adaptive quantized fault-tolerant control of strict-feedback nonlinear systems with mismatched external disturbances. IEEE Trans. Syst. Man Cybern.: Syst. **50**(9), 3424–3434 (2018)
8. Li, S., Ahn, C.K., Xiang, Z.: Adaptive fuzzy control of switched nonlinear time-varying delay systems with prescribed performance and unmodeled dynamics. Fuzzy Sets Syst. **371**, 40–60 (2019)
9. Zou, W., Xiang, Z., Ahn, C.K.: Mean square leader-following consensus of second-order nonlinear multiagent systems with noises and unmodeled dynamics. IEEE Trans. Syst. Man Cybern.: Syst. **49**(12), 2478–2486 (2019)
10. Huang, J., Wang, W., Wen, C., et al.: Adaptive control of a class of strict-feedback time-varying nonlinear systems with unknown control coefficients. Automatica **93**, 98–105 (2018)
11. Lai, G., Liu, Z., Zhang, Y., et al.: Adaptive fuzzy quantized control of time-delayed nonlinear systems with communication constraint. Fuzzy Sets Syst. **314**, 61–78 (2017)
12. Zhu, G., Du, J., Kao, Y.: Command filtered robust adaptive NN control for a class of uncertain strict-feedback nonlinear systems under input saturation. J. Frankl. Inst. **355**(15), 7548–7569 (2018)
13. Yan, X., Chen, M., Feng, G., et al.: Fuzzy robust constrained control for nonlinear systems with input saturation and external disturbances. IEEE Trans. Fuzzy Syst. **29**(2), 345–356 (2019)
14. Ke, C., Li, C., You, L.: Consensus of nonlinear multiagent systems with grouping via state-constraint impulsive protocols. IEEE Trans. Cybern. **51**(8), 4162–4172 (2019)
15. Zhao, K., Song, Y.: Removing the feasibility conditions imposed on tracking control designs for state-constrained strict-feedback systems. IEEE Trans. Autom. Control **64**(3), 1265–1272 (2018)
16. Liu, Y., Zhu, Q., Wen, G.: Adaptive Tracking Control for Perturbed Strict-Feedback Nonlinear Systems Based on Optimized Backstepping Technique. IEEE Trans. Neural Netw. Learn. Syst. **33**(2), 853–865 (2022)
17. Wang, H., Liu, P.X., Xie, X., et al.: Adaptive fuzzy asymptotical tracking control of nonlinear systems with unmodeled dynamics and quantized actuator. Inf. Sci. **575**, 779–792 (2021)
18. Li, Y., Liu, Y., Tong, S.: Observer-based neuro-adaptive optimized control of strict-feedback nonlinear systems with state constraints. IEEE Trans. Neural Netw. Learn. Syst. (2021)
19. Liang, Y., Li, Y.X., Che, W.W., et al.: Adaptive fuzzy asymptotic tracking for nonlinear systems with nonstrict-feedback structure. IEEE Trans. Cybern. **51**(2), 853–861 (2020)
20. Chen, B., Liu, X.P., Ge, S.S., et al.: Adaptive fuzzy control of a class of nonlinear systems by fuzzy approximation approach. IEEE Trans. Fuzzy Syst. **20**(6), 1012–1021 (2012)
21. Chen, B., Lin, C., Liu, X., et al.: Adaptive fuzzy tracking control for a class of MIMO nonlinear systems in nonstrict-feedback form. IEEE Trans. Cybern. **45**(12), 2744–2755 (2014)
22. Wang, H., Liu, X., Liu, K., et al.: Approximation-based adaptive fuzzy tracking control for a class of nonstrict-feedback stochastic nonlinear time-delay systems. IEEE Trans. Fuzzy Syst. **23**(5), 1746–1760 (2014)
23. Yang, Y., Niu, Y.: Event-triggered adaptive neural backstepping control for nonstrict-feedback nonlinear time-delay systems. J. Frankl. Inst. **357**(8), 4624–4644 (2020)
24. Wang, Y., Zheng, L., Zhang, H., et al.: Fuzzy observer-based repetitive tracking control for nonlinear systems. IEEE Trans. Fuzzy Syst. **28**(10), 2401–2415 (2020)

25. Li, S., Xiang, Z.: Sampled-data decentralized output feedback control for a class of switched large-scale stochastic nonlinear systems. IEEE Syst. J. **14**(2), 1602–1610 (2019)

26. Li, S., Ahn, C.K., Guo, J., et al.: Global output feedback sampled-data stabilization of a class of switched nonlinear systems in the p-normal form. IEEE Trans. Syst. Man Cybern.: Syst. **51**(2), 1075–1084 (2021)

27. Chen, B., Lin, C., Liu, X., et al.: Observer-based adaptive fuzzy control for a class of nonlinear delayed systems. IEEE Trans. Syst. Man Cybern.: Syst. **46**(1), 27–36 (2016)

28. Wang, H., Liu, P.X., Shi, P.: Observer-based fuzzy adaptive output-feedback control of stochastic nonlinear multiple time-delay systems. IEEE Trans. Cybern. **47**(9), 2568–2578 (2017)

29. Li, S., Yu, Z., Yu, Z., et al.: Adaptive neural output feedback control for nonstrict-feedback stochastic nonlinear systems with unknown backlash-like hysteresis and unknown control directions. IEEE Trans. Neural Netw. Learn. Syst. **29**(4), 1147–1160 (2017)

30. Wang, A., Liu, L., Qiu, J., et al.: Event-triggered adaptive fuzzy output-feedback control for nonstrict-feedback nonlinear systems with asymmetric output constraint. IEEE Trans. Cybern. **52**(1), 712–722 (2022)

31. Tong, S., Li, Y., Li, Y., et al.: Observer-based adaptive fuzzy backstepping control for a class of stochastic nonlinear strict-feedback systems. IEEE Trans. Syst. Man Cybern. B (Cybernetics) **41**(6), 1693–1704 (2011)

32. Wang, T., Zhang, Y., Qiu, J., et al.: Adaptive fuzzy backstepping control for a class of nonlinear systems with sampled and delayed measurements. IEEE Trans. Fuzzy Syst. **23**(2), 302–312 (2015)

33. Tong, S., Sui, S., Li, Y.: Observed-based adaptive fuzzy tracking control for switched nonlinear systems with deadzone. IEEE Trans. Cybern. **45**(12), 2816–2826 (2015)

34. Tong, S., Li, Y., Sui, S.: Adaptive fuzzy tracking control design for SISO uncertain nonstrict feedback nonlinear systems. IEEE Trans. Fuzzy Syst. **24**(6), 1441–1454 (2016)

35. Wang, L., Basin, M.V., Li, H., et al.: Observer-based composite adaptive fuzzy control for nonstrict-feedback systems with actuator failures. IEEE Trans. Fuzzy Syst. **26**(4), 2336–2347 (2017)

36. Cui, B., Xia, Y., Liu, K., et al.: Finite-time tracking control for a class of uncertain strict-feedback nonlinear systems with state constraints: A smooth control approach. IEEE Trans. Neural Netw. Learn. Syst. **31**(11), 4920–4932 (2020)

37. Zou, W., Shi, P., Xiang, Z., et al.: Finite-time consensus of second-order switched nonlinear multi-agent systems. IEEE Trans. Neural Netw. Learn. Syst. **31**(5), 1757–1762 (2020)

38. Sun, K., Liu, L., Qiu, J., et al.: Fuzzy adaptive finite-time fault-tolerant control for strict-feedback nonlinear systems. IEEE Trans. Fuzzy Syst. **29**(4), 786–796 (2020)

39. Qi, W., Hou, Y., Zong, G., et al.: Finite-time event-triggered control for semi-Markovian switching cyber-physical systems with FDI attacks and applications. IEEE Trans. Circuits Syst. I Regul. Pap. **68**(6), 2665–2674 (2021)

40. Wang, N., Fu, Z., Song, S., et al.: Barrier lyapunov-based adaptive fuzzy finite-time tracking of pure-feedback nonlinear systems with constraints. IEEE Trans. Fuzzy Syst. (2021)

41. Wang, F., Chen, B., Liu, X., et al.: Finite-time adaptive fuzzy tracking control design for nonlinear systems. IEEE Trans. Fuzzy Syst. **26**(3), 1207–1216 (2017)

42. Wu, J., Li, J., Zong, G., et al.: Global finite-time adaptive stabilization of nonlinearly parametrized systems with multiple unknown control directions. IEEE Trans. Syst. Man Cybern.: Syst. **47**(7), 1405–1414 (2016)

43. Wang, H., Bai, W., Zhao, X., et al.: Finite-time-prescribed performance-based adaptive fuzzy control for strict-feedback nonlinear systems with dynamic uncertainty and actuator faults. IEEE Trans. Cybern. https://doi.org/10.1109/TCYB.2020.3046316

44. Sun, K., Qiu, J., Karimi, H.R., et al.: A novel finite-time control for nonstrict feedback saturated nonlinear systems with tracking error constraint. IEEE Trans. Syst. Man Cybern.: Syst. **51**(6), 3968–3979 (2021)

45. Sun, Y., Chen, B., Lin, C., et al.: Finite-time adaptive control for a class of nonlinear systems with nonstrict feedback structure. IEEE Trans. Cybern. **48**(10), 2774–2782 (2017)

46. Xu, Q., Wang, Z., Zhen, Z.: Adaptive neural network finite time control for quadrotor UAV with unknown input saturation. Nonlinear Dyn. **98**(3), 1973–1998 (2019)

47. Liu, Y., Zhu, Q., Zhao, N., et al.: Fuzzy approximation-based adaptive finite-time control for nonstrict feedback nonlinear systems with state constraints. Inf. Sci. **548**, 101–117 (2021)

48. Liu, Y., Liu, X., Jing, Y., et al.: A novel finite-time adaptive fuzzy tracking control scheme for nonstrict feedback systems. IEEE Trans. Fuzzy Syst. **27**(4), 646–658 (2019)

49. Liu, Y., Liu, X., Jing, Y., et al.: Direct adaptive preassigned finite-time control with time-delay and quantized input using neural network. IEEE Trans. Neural Netw. Learn. Syst. **31**(4), 1222–1231 (2020)

50. Li, Y., Li, K., Tong, S.: Finite-time adaptive fuzzy output feedback dynamic surface control for MIMO nonstrict feedback systems. IEEE Trans. Fuzzy Syst. **27**(1), 96–110 (2018)

51. Ji, R., Ma, J., Li, D., et al.: Finite-time adaptive output feedback control for MIMO nonlinear systems with actuator faults and saturations. IEEE Trans. Fuzzy Syst. **29**(8), 2256–2270 (2021)

52. Li, K., Tong, S., Li, Y.: Finite-time adaptive fuzzy decentralized control for nonstrict-feedback nonlinear systems with output-constraint. IEEE Trans. Syst. Man Cybern.: Syst. **50**(12), 5271–5284 (2018)

53. Zhang, H., Liu, Y., Wang, Y.: Observer-based finite-time adaptive fuzzy control for nontriangular nonlinear systems with full-state constraints. IEEE Trans. Cybern. **51**(3), 1110–1120 (2020)

54. Cui, G., Yu, J., Shi, P.: Observer-based finite-time adaptive fuzzy control with prescribed performance for nonstrict-feedback nonlinear systems. IEEE Trans. Fuzzy Syst. **30**(3), 767–778 (2022)

55. Ma, L., Zong, G., Zhao, X., et al.: Observed-based adaptive finite-time tracking control for a class of nonstrict-feedback nonlinear systems with input saturation. J. Frankl. Inst. **357**(16), 11518–11544 (2020)

56. Wu, W., Tong, S.: Robust adaptive fuzzy control for nonstrict feedback switched nonlinear systems with unmodeled dynamics. Int. J. Syst. Sci. **52**(2), 307–320 (2021)

57. Ye, D., Cai, Y., Yang, H., et al.: Adaptive neural-based control for non-strict feedback systems with full-state con-

straints and unmodeled dynamics. Nonlinear Dyn. **97**(1), 715–732 (2019)

58. Wang, H., Shi, P., Li, H., et al.: Adaptive neural tracking control for a class of nonlinear systems with dynamic uncertainties. IEEE Trans. Cybern. **47**(10), 3075–3087 (2016)

59. Tong, S., Wang, T., Li, Y., et al.: A combined backstepping and stochastic small-gain approach to robust adaptive fuzzy output feedback control. IEEE Trans. Fuzzy Syst. **21**(2), 314–327 (2013)

60. Yin, S., Shi, P., Yang, H.: Adaptive fuzzy control of strict-feedback nonlinear time-delay systems with unmodeled dynamics. IEEE Trans. Cybern. **46**(8), 1926–1938 (2015)

61. Li, Y., Tong, S., Liu, Y., et al.: Adaptive fuzzy robust output feedback control of nonlinear systems with unknown dead zones based on a small-gain approach. IEEE Trans. Fuzzy Syst. **22**(1), 164–176 (2013)

62. Wang, L., Li, H., Zhou, Q., et al.: Adaptive fuzzy control for nonstrict feedback systems with unmodeled dynamics and fuzzy dead zone via output feedback. IEEE Trans. Cybern. **47**(9), 2400–2412 (2017)

63. Sui, S., Chen, C.L.P., Tong, S.: Event-trigger-based finite-time fuzzy adaptive control for stochastic nonlinear system with unmodeled dynamics. IEEE Trans. Fuzzy Syst. **29**(7), 1914–1926 (2020)

64. Li, D.J., Lu, S.M., Liu, Y.J., et al.: Adaptive fuzzy tracking control based barrier functions of uncertain nonlinear MIMO systems with full-state constraints and applications to chemical process. IEEE Trans. Fuzzy Syst. **26**(4), 2145–2159 (2017)

65. Zhang, M., Jing, X., Wang, G.: Bioinspired nonlinear dynamics-based adaptive neural network control for vehicle suspension systems with uncertain/unknown dynamics and input delay. IEEE Trans. Industr. Electron. **68**(12), 12646–12656 (2020)

66. Sharma, N., Bhasin, S., Wang, Q., et al.: Predictor-based control for an uncertain Euler-Lagrange system with input delay. Automatica **47**(11), 2332–2342 (2011)

67. Bekiaris-Liberis, N., Krstic, M.: Predictor-feedback stabilization of multi-input nonlinear systems. IEEE Trans. Autom. Control **62**, 516–531 (2017)

68. Bekiaris-Liberis, N., Krstic, M.: Stability of predictor-based feedback for nonlinear systems with distributed input delay. Automatica **70**, 195–203 (2016)

69. Krstic, M.: Input delay compensation for forward complete and strict-feedforward nonlinear systems. IEEE Trans. Autom. Control **55**, 287–303 (2010)

70. Li, H., Wang, L., Du, H., et al.: Adaptive Fuzzy Backstepping Tracking Control for Strict-Feedback Systems With Input Delay. IEEE Trans. Fuzzy Syst. **25**(3), 642–652 (2017)

71. Li, D., Liu, Y., Tong, S., et al.: Neural networks-based adaptive control for nonlinear state constrained systems with input delay. IEEE Trans. Cybern. **49**(4), 1249–1258 (2019)

72. Wang, H., Liu, S., Yang, X.: Adaptive neural control for non-strict-feedback nonlinear systems with input delay. Inf. Sci. **514**, 605–616 (2020)

73. Bresch-Pietri, D., Krstic, M.: Delay-adaptive control for nonlinear systems. IEEE Trans. Autom. Control **59**(5), 1203–1218 (2014)

74. Obuz, S., Klotz, J.R., Kamalapurkar, R., et al.: Unknown time-varying input delay compensation for uncertain nonlinear systems. Automatica **76**, 222–229 (2017)

75. Wang, T., Wu, J., Wang, Y., et al.: Adaptive fuzzy tracking control for a class of strict-feedback nonlinear systems with time-varying input delay and full state constraints. IEEE Trans. Fuzzy Syst. **28**(12), 3432–3441 (2019)

76. Xia, X., Zhang, T., Kang, G., et al.: Adaptive event-triggered control of pure-feedback systems with quantized input and unknown input delay. Int. J. Robust Nonlinear Control **31**(18), 9074–9093 (2021)

77. Wang, D., Huang, J.: Neural network-based adaptive dynamic surface control for a class of uncertain nonlinear systems in strict-feedback form. IEEE Trans. Neural Netw. **16**(1), 195–202 (2005)