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Some new kink type solutions for the new (3+1)-dimensional Boiti–Leon–Manna–Pempinelli equation

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Abstract Exact solutions of higher-dimensional nonlinear equations takes a major place in the study of nonlinear phenomena observed in nature. In this article, some new kink type solutions are investigated for the new (3+1)-dimensional Boiti-Leon-Manna-Pempinelli(BLMP) equation. Firstly, a variety of solutions are obtained by Hirota's bilinear form, which include kink type wave solution, periodic solitary wave solutions and singular solitary wave solutions using extended homoclinic test approach. Secondly, solutions with three wave form are obtained by generalized three wave method. The extended homoclinic test approach is also used to construct solutions with a tail which explain some physical phenomenon. Moreover, some figures of the solutions are shown behind.

Keywords Extended homoclinic test approach · New (3+1)-dimensional BLMP equation · Hirota bilinear form · Three wave method

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1 Introduction

Nonlinear science is a vital discovery in the area of natural science since the 20th century, and its rapid development has made it one of the popular topics in mathematical physics. In recent decades, researches on soliton solutions [1],rogue waves solutions [2], periodic wave solutions [3], interaction solutions [4] and lump solutions [5] of nonlinear partial differential equations (NLPDEs) have received increasing attention from experts and scholars.

Recently, nonlinear evolution equations (NLEEs) has wide application in various fields. Various methods have been proposed to explore nonlinear phenomena. For example, three-wave method [6,7], tanh method [8], Bäcklund transformation [9], Darboux transformation [10], Exp-function method [11], Hirota bilinear method [12], homogeneous balance method [13], the generally projective F-expansions [14], (G'/G)-expansion method [15], auxiliary equation method [16], Riccati equation method [17], simplest equation method [18,19], etc.

Wazwaz proposed a new (3+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation which has time dependent coefficients. It's written as the form [20]

$$F(t)(u_x+u_y+u_z)_t+G(t)(u_x+u_y+u_z)_{xxx}$$
$$+H(t)(u_x(u_x+u_y+u_z))_x=0. \tag{1.1}$$

It comes from one of the most typical NLEEs, which is the (3+1)-dimensional BLMP equation

$$(u_y+u_z)_t+(u_y+u_z)_{xxx}+(u_x(u_y+u_z))_x=0, \tag{1.2}$$

which includes three expressions which consist the derivatives u_y+u_z. The Eq. (1.1) has the additional derivative u_x to every expressions.

Let $F(t)=1, G(t)=1, H(t)=1$, Eq.(1.1) reduces to the a equation mainly discussed in this paper with constant coefficients

$$(u_x+u_y+u_z)_t+(u_x+u_y+u_z)_{xxx} +(u_x(u_x+u_y+u_z))_x=0. \tag{1.3}$$

These equations describe a kind of physical phenomenon in the natural world, that is, the propagation of waves in incompressible fluid. To confirm its integrability, the Painlév analysis is used by Wazwaz to obtain the compatibility condition. By the simplified Hirota's method and the complex Hirota's criteria, the multiple soliton solutions and multiple complex soliton solutions are determined. [20]

Based on the new (3+1)-dimensional BLMP equation given by Wazwaz, scholars have obtained many new research achievements. Liu and Wazwaz further get breather wave solutions, lump type solutions of equation (1.3) in Ref. [21]. Yuan presented more kinds of the interaction solutions, including lump and N-soliton solutions. The breather-wave solution is studied in Ref. [22]. Han and Bao investigated the equation (1.1) with time-dependent coefficients on the basis of the Hirota bilinear method, and obtained the mixed high-order lump N-soliton solutions and the hybrid solutions in Ref. [23]. To construct test functions, Qiao, Zhang and Yue used a specific bilinear neural network framework. Three kinds of periodic-type solutions of Eq. (1.3) are given in Ref. [24].

The paper is structured as follows, Sect. 2 obtains kink-shaped solitary wave solutions, singular solitary wave solutions, periodic wave solutions and periodic kink wave solutions by using the extended homoclinic test approach. In Sect. 3 a three-wave method is used to find three-wave solutions. We get periodic kink wave solutions, periodic cross kink wave solutions and periodic wave solutions. Section 4 obtains kink-shaped solitary wave solutions with tails, by using the extended homoclinic test approach. Section 5 is dedicated to giving the conclusion.

2 Extended homoclinic test approach

The new (3+1)-dimensional BLMP equation with constant coefficients has been written as

$$(u_x+u_y+u_z)_t+(u_x+u_y+u_z)_{xxx} +(u_x(u_x+u_y+u_z))_x=0. \tag{2.1}$$

According to the function transformation

$$u(x,y,z,t)=6[\ln f(x,y,z,t)]_x\,, \tag{2.2}$$

the Hirota bilinear form of Eq. (2.1) is

$$[D_x^4+D_yD_x^3+D_zD_x^3+D_tD_x+D_tD_y +D_tD_z]f\cdot f=0. \tag{2.3}$$

Then Eq. (2.1) becomes

$$3{f_{xx}}^2+3f_{xz}f_{xx}+3f_{xy}f_{xx}-f_tf_z-f_tf_y-f_tf_x -3f_xf_{xxz}-3f_xf_{xxy}-f_zf_{xxx} -f_yf_{xxx}-4f_xf_{xxx}+f(f_{zt}+f_{yt}+f_{xt} +f_{xxxz}+f_{xxxy}+f_{xxxx})=0. \tag{2.4}$$

Let f take the form

$$f=k_1\cos(\xi_1)+k_2\exp(\xi_2)+\exp(-\xi_2), \tag{2.5}$$

where $\xi_i=a_ix+b_iy+c_iz+d_it$, $k_i\in\mathbb{R}$, $a_i,b_i,c_i,d_i\in\mathbb{C}(i=1,2)$are undetermined constants.

Substituting Eq. (2.5) into (2.3) and setting coefficients of

$$\cos(\xi_1)\exp(-\xi_2),\ \cos(\xi_1)\exp(\xi_2),$$
$$\sin(\xi_1)\exp(-\xi_2),\ \sin(\xi_1)\exp(\xi_2)$$

and the constant term to zero, we obtain a set of algebraic equations (2.6) with respect to k_i and a_i, b_i, c_i, d_i, $(i=1,2)$.

$$\begin{cases} k_1(-a_1^4-(b_1+c_1)a_1^3+3a_2(c_2+2a_2+b_2)a_1^2 \\ \quad +[(3b_1+3c_1)a_2^2+d_1]a_1 \\ -a_2^4+(-c_2-b_2)a_2^3-a_2d_2+(b_1+c_1)d_1 \\ \quad -d_2(c_2+b_2))=0, \\ k_1((-c_2-4a_2-b_2)a_1^3-3a_2(b_1+c_1)a_1^2 \\ \quad +[4a_2^3+(3c_2+3b_2)a_2^2+d_2]a_1 \\ +(b_1+c_1)a_2^3+d_1a_2+(c_2+b_2)d_1+d_2(b_1+c_1))=0, \\ k_1k_2(a_1^4+(b_1+c_1)a_1^3-3a_2(c_2+2a_2+b_2)a_1^2 \\ \quad -[(3b_1+3c_1)a_2^2+d_1]a_1 \\ a_2^4+(c_2+b_2)a_2^3+a_2d_2+(-b_1-c_1)d_1+d_2(c_2+b_2))=0, \\ k_1k_2((-c_2-4a_2-b_2)a_1^3-3a_2(b_1+c_1)a_1^2 \\ \quad +[4a_2^3+(3c_2+3b_2)a_2^2+d_2]a_1 \\ +(b_1+c_1)a_2^3+d_1a_2+(c_2+b_2)d_1+d_2(b_1+c_1))=0, \\ -(a_1+b_1+c_1)(-4a_1^3+d_1)k_1^2+(a_2+b_2+c_2)(4a_2^3+d_2)4k_2=0. \end{cases} \tag{2.6}$$

Solving Eq. (2.6), we have following conclusion.

2.1 Solution in the situation $k_1 = 0$

If $k_1 = 0$, we have some cases.

Case 1
I.

$$k_1 = 0,\ a_2 = -c_2 - b_2, \tag{2.7}$$

where $a_1, b_1, b_2, c_1, c_2, d_1, d_2$, and k_2 are free parameters.
II.

$$k_1 = 0,\ d_2 = -4a_2^3, \tag{2.8}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1$ and k_2 are free parameters.

Substituting Eq. (2.7–2.8) into Eq. (2.5) and Eq. (2.2), we obtain the solutions in the case of $i = 1$ and $i = 2$. The corresponding solutions are given by

$$u = C_i \frac{k_2 e^{\zeta_i} - e^{-\zeta_i}}{k_2 e^{\zeta_i} + e^{-\zeta_i}}, \quad i = 1, 2, \tag{2.9}$$

where

$$\zeta_1 = (-c_2 - b_2)\, x + b_2 y + c_2 z + d_2 t,$$
$$\zeta_2 = -a_2 x - b_2 y - c_2 z + 4a_2^3 t,\ \ C_1 = -6\,(c_2 + b_2)\,,\ \ C_2 = 6a_2.$$

In particular, solutions Eq. (2.9) can be expressed as

$$u_1 = C_i \tanh\left(\zeta_i + \frac{1}{2}\ln k_2\right), \quad k_2 > 0, i = 1, 2, \tag{2.10}$$

$$u_2 = C_i \coth\left(\zeta_i + \frac{1}{2}\ln(-k_2)\right), \quad k_2 < 0, i = 1, 2, \tag{2.11}$$

where

$$\zeta_1 = (-c_2 - b_2)\, x + b_2 y + c_2 z + d_2 t,$$
$$\zeta_2 = -a_2 x - b_2 y - c_2 z + 4a_2^3 t,$$
$$C_1 = -6\,(c_2 + b_2)\,,\ C_2 = 6a_2.$$

It is easy to know that u_1, u_2 are kink-shaped solitary wave solutions.

To simplify the results, the sign of k_2 will not be discussed in each case later.

2.2 Solution in the situation $k_2 = 0$

If $k_2 = 0$, we have

Case 2

$$b_1 = -a_1 - c_1,\ d_1 = a_1^3 - 3a_1 a_2^2,\ d_2 = 3a_1^2 a_2 - a_2^3,\ k_2 = 0, \tag{2.12}$$

where a_1, a_2, b_2, c_1, c_2, and k_1 are free parameters.

Substituting Eq. (2.12) into Eq. (2.2) and Eq. (2.5), the solutions are yielded that

$$u = 6\frac{-k_1 a_1 \sin(\zeta_1) - a_2 e^{\zeta_2}}{k_1 \cos(\zeta_1) + e^{\zeta_2}}, \tag{2.13}$$

where $\zeta_1 = \left(a_1 x + (-a_1 - c_1)\, y + c_1 z + \left({a_1}^3 - 3a_1 {a_2}^2\right) t\right)$, $\zeta_2 = -a_2 x - b_2 y - c_2 z - \left(3{a_1}^2 a_2 - {a_2}^3\right) t$.

In particular, solution Eq. (2.13) can be expressed as

$$u_3 = 6\frac{-k_1 a_1 \sin(\zeta_1) - a_2 (\cosh \zeta_2 + \sinh \zeta_2)}{k_1 \cos(\zeta_1) + \cosh \zeta_2 + \sinh \zeta_2}, \tag{2.14}$$

where $\zeta_1 = \left(a_1 x + (-a_1 - c_1)\, y + c_1 z + \left({a_1}^3 - 3a_1 {a_2}^2\right) t\right)$, $\zeta_2 = -a_2 x - b_2 y - c_2 z - \left(3{a_1}^2 a_2 - {a_2}^3\right) t$.

Case 3
The case of complex solutions.
I.

$$b_2 = -\frac{1}{3a_1({a_1}^2 + {a_2}^2)}(b_{2a} + b_{2b} + b_{2c} + b_{2d}),$$
$$d_1 = 4a_1^3, d_2 = \pm\left(3\,\mathrm{i}{a_1}^3 + 3\,\mathrm{i}a_1{a_2}^2\right) + 3a_2{a_1}^2 - {a_2}^3, k_2 = 0, \tag{2.15}$$

where a_1, a_2, b_1, c_1, c_2, and k_1 are free parameters. and

$$b_{2a} = 3{a_1}^3 c_2 - 3a_2\,{a_1}^2 b_1 - 3a_2\,{a_1}^2 c_1 + 4a_1\,{a_2}^3 + 3a_1\,{a_2}^2 c_2 + {a_2}^3 b_1 + {a_2}^3 c_1,$$
$$b_{2b} = (a_1(\pm(3\,\mathrm{i}{a_1}^3 + 3\,\mathrm{i}a_1\,{a_2}^2) + 3a_2\,{a_1}^2 - {a_2}^3),$$
$$b_{2c} = (\pm(3\,\mathrm{i}{a_1}^3 + 3\,\mathrm{i}a_1\,{a_2}^2) + 3a_2\,{a_1}^2 - {a_2}^3)b_1,$$
$$b_{2d} = \left(\pm\left(3\,\mathrm{i}{a_1}^3 + 3\,\mathrm{i}a_1\,{a_2}^2\right) + 3a_2\,{a_1}^2 - {a_2}^3\right) c_1),$$

II.

$$a_1 = \pm\mathrm{i}a_2,\ b_1 = \mp\mathrm{i}a_2 - c_1,$$
$$d_1 = \mp 4\mathrm{i}a_2^3,\ b_2 = -a_2 - c_2,\ k_2 = 0, \tag{2.16}$$

where a_2, c_1, c_2, d_2 and k_1 are free parameters.

Substituting Eq. (2.15–2.16) into Eq. (2.5) and Eq. (2.2), we obtain $a^* = a_1, i = 1$ and $a^* = \pm\mathrm{i}a_2, i = 2$, respectively. The corresponding solutions are given by

$$u = -6\frac{k_1 a^* \sin(\zeta_i) + a_2 e^{\eta_i}}{k_1 \cos(\zeta_i) + e^{\eta_i}}, \ i = 1, 2, \tag{2.17}$$

where $\zeta_1 = a_1 x + b_1 y + c_1 z + 4{a_1}^3 t$, $\eta_1 = -a_2 x - b_2 y - c_2 z - (\pm\left(3\,\mathrm{i}{a_1}^3 + 3\,\mathrm{i}a_1{a_2}^2\right) + 3a_2{a_1}^2 - {a_2}^3)t$, $\zeta_2 = (\pm\mathrm{i})\, a_2 x + ((\mp\mathrm{i})\, a_2 - c_1)\, y + c_1 z + 4\,(\mp\mathrm{i})\, a_2^3 t$, $\eta_2 = -a_2 x - (-a_2 - c_2)\, y - c_2 z - d_2 t$.

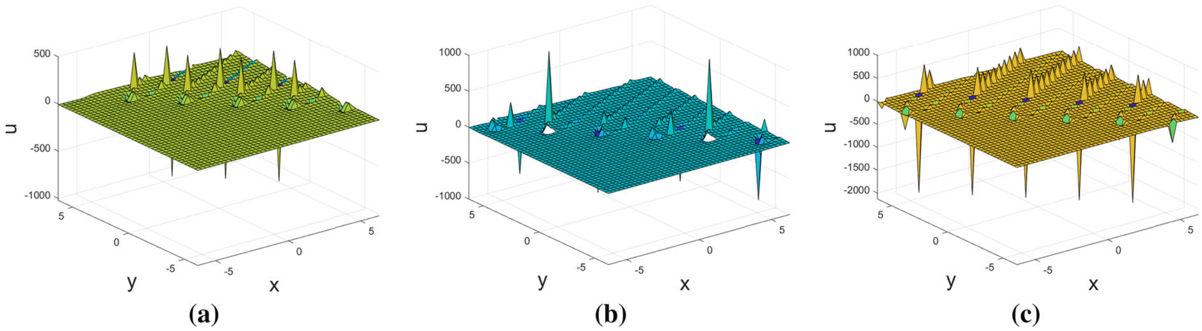

Fig. 1 The singular solitary wave solution u_3 at $a_1 = 1, c_1 = 1, a_2 = 1, b_2 = 1, k_1 = 1, z = 0$, **a**$t$ =-1, **b**$t = 0$, **c**$t = 1$

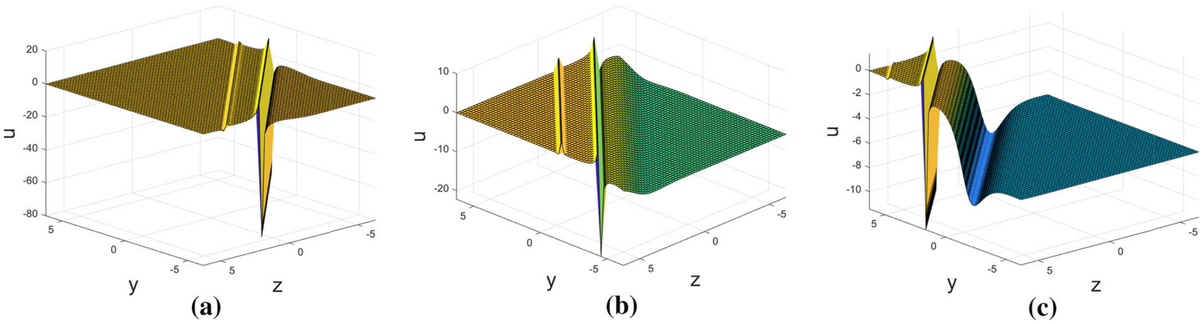

Fig. 2 The singular solitary wave solution u_5 at $a_2 = 1, b_1 = 1, b_2 = 1, k_1 = 1, x = 0$, **a**$t = -5$, **b**$t = 0$, **c**$t = 5$

Here, the form of b_2 is shown in Eq. (2.15). In particular, solution Eq. (2.17) can be expressed as

$$u_4 = -6\frac{k_1 a^* \sin(\zeta_i) + a_2 (\cosh\eta_i + \sinh\eta_i)}{k_1\cos(\zeta_i) + \cosh\eta_i + \sinh\eta_i},\ i = 1, 2, \tag{2.18}$$

where $\zeta_1 = a_1x + b_1y + c_1z + 4{a_1}^3t$, $\eta_1 = -a_2x - b_2y - c_2z - (\pm(3\,\mathrm{i}{a_1}^3 + 3\,\mathrm{i}a_1{a_2}^2) + 3a_2{a_1}^2 - {a_2}^3)t$, $\zeta_2 = (\pm\mathrm{i})\,a_2x + ((\mp\mathrm{i})\,a_2 - c_1)\,y + c_1z + 4\,(\mp\mathrm{i})\,a_2^3t$, $\eta_2 = -a_2x - (-a_2 - c_2)\,y - c_2z - d_2t$.

Case 4

$$a_1 = 0,\ d_1 = 0,\ d_2 = -a_2^3,\ k_2 = 0, \tag{2.19}$$

where a_2, b_1, b_2, c_1, c_2, and k_1 are free parameters.

Substituting Eq. (2.19) into Eq. (2.2) and Eq. (2.5), the solutions are yielded that

$$u = -\frac{6a_2\,\mathrm{e}^{\zeta_1}}{k_1\cos(b_1y + c_1z) + \mathrm{e}^{\zeta_1}}, \tag{2.20}$$

where $\zeta_1 = {a_2}^3t - a_2x - b_2y - c_2z$. In particular, solution Eq. (2.20) can be expressed as

$$u_5 = -6a_2\frac{\cosh\zeta_1 + \sinh\zeta_1}{k_1\cos(b_1y + c_1z) + \cosh\zeta_1 + \sinh\zeta_1}, \tag{2.21}$$

where $\zeta_1 = {a_2}^3t - a_2x - b_2y - c_2z$.

2.3 Solution in the situation $k_1 \neq 0$ and $k_2 \neq 0$

If $k_1 \neq 0$ and $k_2 \neq 0$, we have following conclusion.

Case 5

$$a_1 = 0,\ d_1 = 0,\ b_2 = -a_2 - c_2,\ d_2 = -a_2^3, \tag{2.22}$$

where a_2, b_1, c_1, c_2, k_1 and k_2 are free parameters.

Substituting Eq. (2.22) into Eq. (2.2) and Eq. (2.5), the solutions are yielded that

$$u = 6a_2\frac{k_2\mathrm{e}^{\zeta_1} - \mathrm{e}^{-\zeta_1}}{k_1\cos(\zeta_2) + k_2\,\mathrm{e}^{\zeta_1} + \mathrm{e}^{-\zeta_1}}, \tag{2.23}$$

where $\zeta_1 = a_2x + (-a_2 - c_2)\,y + c_2z - {a_2}^3t$, $\zeta_2 = b_1y + c_1z$.

In particular, the solution Eq. (2.23) can be expressed as

$$u_6 = 12\sqrt{k_2}a_2\frac{\sinh\left(\zeta_1 + \frac{1}{2}\ln k_2\right)}{k_1\cos(\zeta_2) + 2\sqrt{k_2}\cosh\left(\zeta_1 + \frac{1}{2}\ln k_2\right)} \tag{2.24}$$

where $\zeta_1 = a_2x + (-a_2 - c_2)\,y + c_2z - {a_2}^3t$, $\zeta_2 = b_1y + c_1z$.

Case 6

$$b_1 = -a_1 - c_1,\ d_1 = a_1^3,\ a_2 = 0,\ d_2 = 0, \tag{2.25}$$

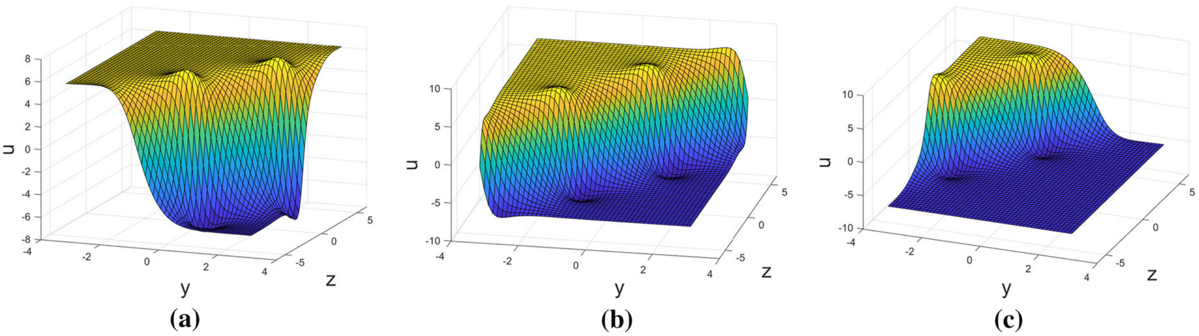

Fig. 3 The periodic kink wave solution u_6 at $a_2 = 1, b_1 = 1, c_1 = 1, c_2 = 1, k_1 = 1, k_2 = 1, x = 0$, **a**$t = -5$, **b**$t = 0$, **c**$t = 5$

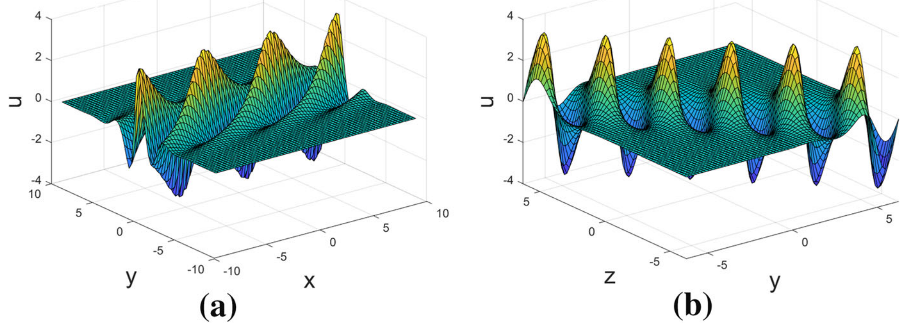

Fig. 4 The periodic solitary wave solution u_7 at $a_1 = 1, b_2 = 1, c_1 = 1, c_2 = 1, k_1 = 1, k_2 = 1$, **a**$z = 0, t = 0$, **b**$x = 0, t = 0$

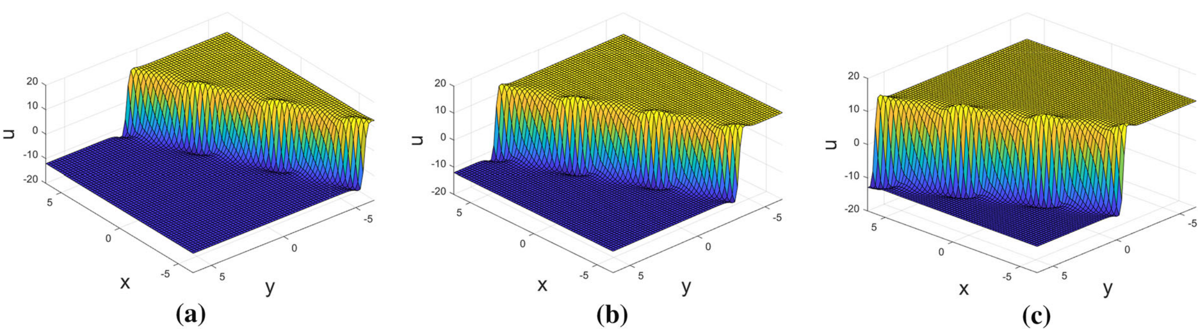

Fig. 5 The periodic kink wave solution u_8 at $i = 1, a_2 = 2, b_1 = 1, c_1 = 1, c_2 = 2, d_2 = 2, k_1 = 1, k_2 = 2, z = 0$, at $= -5$, bt $= 0$, ct $= 5$

where a_1, b_2, c_1, c_2, k_1 and k_2 are free parameters.

Substituting Eq. (2.25) into Eq. (2.2) and Eq. (2.5), we obtain the solution

$$u = -6k_1a_1\frac{\sin(\zeta_1)}{k_1\cos(\zeta_1) + k_2\,\mathrm{e}^{\zeta_2} + \mathrm{e}^{-\zeta_2}}, \tag{2.26}$$

where $\zeta_1 = a_1x + (-a_1 - c_1)\,y + c_1z + a_1^3t$, $\zeta_2 = b_2y + c_2z$.

In particular, the solution Eq. (2.26)can be expressed as

$$u_7 = -6k_1a_1\frac{\sin(\zeta_1)}{k_1\cos(\zeta_1) + 2\sqrt{k_2}\cosh\left(\zeta_2 + \frac{1}{2}\ln k_2\right)} \tag{2.27}$$

where $\zeta_1 = a_1x + (-a_1 - c_1)\,y + c_1z + a_1^3t$, $\zeta_2 = b_2y + c_2z$.

Case 7

$$b_1 = -a_1 - c_1,\ a_2 = -c_2 - b_2, \tag{2.28}$$

where $a_1, b_2, c_1, c_2, d_1, d_2, k_1$ and k_2 are free parameters.

Substituting Eq. (2.28) into Eq. (2.5) and Eq. (2.2), we obtain

$$u = 6\frac{-k_1a_1\sin(\zeta_1) + (-c_2 - b_2)\,(k_2\,\mathrm{e}^{\zeta_2} - \mathrm{e}^{-\zeta_2})}{k_1\cos(\zeta_1) + k_2\,\mathrm{e}^{\zeta_2} + \mathrm{e}^{-\zeta_2}}, \tag{2.29}$$

where $\zeta_1 = a_1x + (-a_1 - c_1)\,y + c_1z + d_1t$, $\zeta_2 = (-c_2 - b_2)\,x + b_2y + c_2z + d_2t$.

In particular, the solution Eq. (2.29) can be expressed as

$$u_8 = 6\frac{-k_1a_1\sin(\zeta_1) + (-c_2 - b_2)\left(2\sqrt{k_2}\sinh\left(\zeta_2 + \frac{1}{2}\ln k_2\right)\right)}{k_1\cos(\zeta_1) + 2\sqrt{k_2}\cosh\left(\zeta_2 + \frac{1}{2}\ln k_2\right)}, \tag{2.30}$$

where $\zeta_1 = a_1x + (-a_1 - c_1)\,y + c_1z + d_1t$, $\zeta_2 = (-c_2 - b_2)\,x + b_2y + c_2z + d_2t$.

Case 8

Compared with the case7, the solution obtained is complex solutions.

$$a_1 = \pm \mathrm{i}a_2,\ d_1 = \mp 4\mathrm{i}a_2^3,\ d_2 = -4a_2^3, \tag{2.31}$$

where $a_2, b_1, b_2, c_1, c_2, k_1$ and k_2 are free parameters.

Substituting Eq. (2.31) into Eq. (2.2) and Eq. (2.5), we obtain

$$u = 6\frac{\mp \mathrm{i}a_2k_1\sin(\zeta_1) + a_2(k_2\,\mathrm{e}^{\zeta_2} - \mathrm{e}^{-\zeta_2})}{k_1\cos(\zeta_1) + k_2\,\mathrm{e}^{\zeta_2} + \mathrm{e}^{-\zeta_2}}, \tag{2.32}$$

where $\zeta_1 = \pm \mathrm{i}a_2x + b_1y + c_1z \mp 4\mathrm{i}{a_2}^3t$, $\zeta_2 = -4{a_2}^3t + a_2x + b_2y + c_2z$.

In particular, the solution Eq. (2.32) can be expressed as

$$u_9 = 6\frac{\mp \mathrm{i}a_2k_1\sin(\zeta_1) + a_2\left(2\sqrt{k_2}\sinh\left(\zeta_2 + \frac{1}{2}\ln k_2\right)\right)}{k_1\cos(\zeta_1) + 2\sqrt{k_2}\cosh\left(\zeta_2 + \frac{1}{2}\ln k_2\right)}, \tag{2.33}$$

where $\zeta_1 = \pm \mathrm{i}a_2x + b_1y + c_1z \mp 4\mathrm{i}{a_2}^3t$, $\zeta_2 = -4{a_2}^3t + a_2x + b_2y + c_2z$.

3 Three wave method

Now, the equation (1.3) is considered by three wave method. We assume it has three wave solutions, which takes the form

$$f(x, y, z, t) = \exp(\xi_1) + \delta_1\cos(\xi_2) + \delta_2\cosh(\xi_3) + \delta_3\exp(-\xi_1), \tag{3.1}$$

where

$$\xi_i = P_ix + Q_iy + R_iz + w_it;\ \delta_i \in \mathbb{R};\ R_i, Q_i, R_i, w_i \in \mathbb{R}(i = 1, 2, 3)$$

are undetermined constants.

Substituting Eq. (3.1) into Eq. (2.5) and setting coefficients of

$\cosh(\xi_3)\exp(\pm\xi_1)$, $\cos(\xi_2)\exp(\pm\xi_1)$, $\sinh(\xi_3)\exp(\pm\xi_1)$, $\sin(\xi_2)\exp(\pm\xi_1)$, $\cos(\xi_2)\cosh(\xi_3)$, $\sin(\xi_2)\sinh(\xi_3)$,

and the constant term to zero, a set of nonlinear algebraic equations Eq. (3.2).

$$\begin{cases}
\delta_2\delta_3[P_1^4 + (R_1+Q_1)\,P_1^3 + 3P_3\,(R_3+Q_3+2P_3)\,P_1^2 \\
\quad + \left((3R_1+3Q_1)\,P_3^2 + w_1\right)P_1 \\
\quad + P_3^4 + (Q_3+R_3)\,P_3^3 + w_3P_3 + (R_1+Q_1)\,w_1 \\
\quad + w_3\,(Q_3+R_3)] = 0, \\
\delta_1\delta_3[P_1^4 + (R_1+Q_1)\,P_1^3 - 3P_2\,(R_2+Q_2+2P_2)\,P_1^2 \\
\quad + \left((-3R_1-3Q_1)\,P_2^2 + w_1\right)P_1 \\
\quad + P_2^4 + (Q_2+R_2)\,P_2^3 - w_2P_2 + (R_1+Q_1)\,w_1 \\
\quad + w_2\,(R_2+Q_2)] = 0, \\
\delta_2\delta_3[(R_3+Q_3+4P_3)\,P_1^3 + 3P_3\,(R_1+Q_1)\,P_1^2 \\
\quad + \left(4P_3^3 + (3R_3+3Q_3)\,P_3^2 + w_3\right)P_1 \\
\quad + (R_1+Q_1)\,P_3^3 + w_1P_3 + (R_3+Q_3)\,w_1 \\
\quad + w_3\,(R_1+Q_1)] = 0, \\
\delta_1\delta_3[(R_2+Q_2+4P_2)\,P_1^3 + 3P_2\,(R_1+Q_1)\,P_1^2 \\
\quad + \left(-4P_2^3 + (-3R_2-3Q_2)\,P_2^2 + w_2\right)P_1 \\
\quad + (-R_1-Q_1)\,P_2^3 + w_1P_2 + (R_2+Q_2)\,w_1 \\
\quad + w_2\,(R_1+Q_1)] = 0, \\
\delta_2[P_1^4 + (R_1+Q_1)\,P_1^3 + 3P_3\,(R_3+Q_3+2P_3)\,P_1^2 \\
\quad + \left((3R_1+3Q_1)\,P_3^2 + w_1\right)P_1 \\
\quad + P_3^4 + (Q_3+R_3)\,P_3^3 + w_3P_3 + (R_1+Q_1)\,w_1 \\
\quad + w_3\,(Q_3+R_3)] = 0, \\
\delta_1[P_1^4 + (R_1+Q_1)\,P_1^3 - 3P_2\,(R_2+Q_2+2P_2)\,P_1^2 \\
\quad + \left((-3R_1-3Q_1)\,P_2^2 + w_1\right)P_1 \\
\quad + P_2^4 + (Q_2+R_2)\,P_2^3 - w_2P_2 + (R_1+Q_1)\,w_1 \\
\quad + w_2\,(R_2+Q_2)] = 0, \\
\delta_2((R_3+Q_3+4P_3)\,P_1^3 + 3P_3\,(R_1+Q_1)\,P_1^2 \\
\quad + \left(4P_3^3 + (3R_3+3Q_3)\,P_3^2 + w_3\right)P_1 \\
\quad (R_1+Q_1)\,P_3^3 + w_1P_3 + (R_3+Q_3)\,w_1 \\
\quad + w_3\,(R_1+Q_1)) = 0, \\
\delta_1((R_2+Q_2+4P_2)\,P_1^3 + 3P_2\,(R_1+Q_1)\,P_1^2 \\
\quad + \left(4P_2^3 + (-3R_2-3Q_2)\,P_2^2 + w_2\right)P_1 \\
\quad (-R_1-Q_1)\,P_2^3 + w_1P_2 + (R_2+Q_2)\,w_1 \\
\quad + w_2\,(R_1+Q_1)) = 0, \\
\delta_1\delta_2(P_2^4 + (R_2+Q_2)\,P_2^3 - 3P_3\,(R_3+Q_3+2P_3)\,P_3^2 \\
\quad + \left((-3R_2-3Q_2)\,P_3^2 - w_2\right)P_2 \\
\quad + P_3^4 + (Q_3+R_3)\,P_3^3 - w_3P_3 + (-R_2-Q_2)\,w_2 \\
\quad + w_3\,(R_3+Q_3)) = 0, \\
\delta_1\delta_2((-R_3-Q_3-4P_3)\,P_2^3 - 3P_3\,(R_2+Q_2)\,P_2^2 \\
\quad + \left(4P_3^3 + (3R_3+3Q_3)\,P_3^2 + w_3\right)P_2 \\
\quad + (R_2+Q_2)\,P_3^3 + w_2P_3 + (R_3+Q_3)\,w_2 \\
\quad + w_3\,(R_2+Q_2)) = 0, \\
(-4P_2^3 + w_2)\cdot(R_2+P_2+Q_2) = 0, \\
(4P_3^3 + w_3)\cdot(R_3+P_3+Q_3) = 0, \\
(4P_1^3 + w_1)\cdot(R_1+P_1+Q_1) = 0.
\end{cases} \tag{3.2}$$

Solve Eq. (3.2), we have following conclusion.

In general, we only consider the case where the free parameters are real numbers, and let $\delta_1 \neq 0, \delta_2 \neq 0$ and $\delta_3 \neq 0$.

In addition, if $\delta_2 = 0$, Eq. (3.1) has the same form as Eq. (2.5).

Case 1

$$R_1 = -P_1 - Q_1,\ R_2 = -Q_2 - P_2,\ R_3 = -Q_3 - P_3, \tag{3.3}$$

where $Q_1, Q_2, Q_3, P_1, P_2, P_3, w_1, w_2$ and w_3 are free parameters.

Substituting Eq. (3.3) into Eq. (2.2) and Eq. (3.1), we obtain the solution

$$u = 6\frac{P_1 e^{\eta_1} - \delta_1 P_2 \sin(\eta_2) + \delta_2 P_3 \sinh(\eta_3) - \delta_3 P_1 e^{-\eta_1}}{e^{\eta_1} + \delta_1 \cos(\eta_2) + \delta_2 \cosh(\eta_3) + \delta_3 e^{-\eta_1}} \tag{3.4}$$

where $\eta_1 = P_1 x + Q_1 y + (-Q_1 - P_1) z + w_1 t$. $\eta_2 = P_2 x + Q_2 y + (-Q_2 - P_2) z + w_2 t$. $\eta_3 = P_3 x + Q_3 y + (-Q_3 - P_3) z + w_3 t$.

In particular, solution Eq. (3.4) can be expressed as

$$u_1 = 6\frac{-\delta_1 P_2 \sin(\eta_2) + \delta_2 P_3 \sinh(\eta_3) + 2P_1\sqrt{\delta_3}\sinh\left(\eta_1 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}{\delta_1 \cos(\eta_2) + \delta_2 \cosh(\eta_3) + 2\sqrt{\delta_3}\cosh\left(\eta_1 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}, \tag{3.5}$$

where $\eta_1 = P_1 x + Q_1 y + (-Q_1 - P_1) z + w_1 t$. $\eta_2 = P_2 x + Q_2 y + (-Q_2 - P_2) z + w_2 t$. $\eta_3 = P_3 x + Q_3 y + (-Q_3 - P_3) z + w_3 t$.

Case 2

$$P_1 = -\sqrt[3]{w_1},\ R_1 = -Q_1 + \sqrt[3]{w_1},\ P_2 = \sqrt[3]{w_2},\ R_2 = -Q_2 - \sqrt[3]{w_2},\ P_3 = 0,\ w_3 = 0, \tag{3.6}$$

where Q_1, Q_2, Q_3, R_3, w_1 and w_2 are free parameters.

Substituting Eq. (3.5) into Eq. (2.2) and Eq. (3.1), we obtain the solution

$$u = 6\frac{-\sqrt[3]{w_1}e^{\eta_1} - \delta_1\sqrt[3]{w_2}\sin(\eta_2) + \delta_3\sqrt[3]{w_1}e^{-\eta_1}}{e^{\eta_1} + \delta_1 \cos(\eta_2) + \delta_2 \cosh(\eta_3) + \delta_3 e^{-\eta_1}}, \tag{3.7}$$

where $\eta_1 = -\sqrt[3]{w_1}x + Q_1 y + \left(-Q_1 + \sqrt[3]{w_1}\right) z + w_1 t$, $\eta_2 = \sqrt[3]{w_2}x + Q_2 y + \left(-Q_2 - \sqrt[3]{w_2}\right) z + w_2 t$, $\eta_3 = Q_3 y + R_3 z$.

In particular, solution Eq. (3.7) can be expressed as

$$u_2 = 6\frac{-\delta_1\sqrt[3]{w_2}\sin(\eta_2) - 2\sqrt[3]{w_1}\sqrt{\delta_3}\sinh\left(\eta_1 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}{\delta_1 \cos(\eta_2) + \delta_2 \cosh(\eta_3) + 2\sqrt{\delta_3}\cosh\left(\eta_1 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}, \tag{3.8}$$

where $\eta_1 = -\sqrt[3]{w_1}x + Q_1 y + \left(-Q_1 + \sqrt[3]{w_1}\right) z + w_1 t$, $\eta_2 = \sqrt[3]{w_2}x + Q_2 y + \left(-Q_2 - \sqrt[3]{w_2}\right) z + w_2 t$, $\eta_3 = Q_3 y + R_3 z$.

Case 3

$$P_1 = -\sqrt[3]{w_1},\ R_1 = -Q_1 + \sqrt[3]{w_1},\ P_2 = 0,\ w_2 = 0,\ P_3 = -\sqrt[3]{w_3},\ R_3 = -Q_3 + \sqrt[3]{w_3}, \tag{3.9}$$

where Q_1, Q_2, Q_3, R_2, w_1 and w_3 are free parameters.

Substituting Eq. (3.9) into Eq. (2.2) and Eq. (3.1), we obtain the solution

$$u = 6\frac{-\sqrt[3]{w}_1 e^{\eta_1} - \delta_2\sqrt[3]{w_3}\sinh(\eta_2) + \delta_3\sqrt[3]{w_1}e^{-\eta_1}}{e^{\eta_1} + \delta_1 \cos(\eta_3) + \delta_2 \cosh(\eta_2) + \delta_3 e^{-\eta_1}}, \tag{3.10}$$

where $\eta_1 = -\sqrt[3]{w_1}x + Q_1 y + R_1 z + w_1 t$, $\eta_2 = -\sqrt[3]{w_3}x + Q_3 y + \left(-Q_3 + \sqrt[3]{w_3}\right) z + w_3 t$, $\eta_3 = Q_2 y + R_2 z$.

In particular, solution Eq. (3.10) can be expressed as

$$u_3 = 6\frac{-\delta_2\sqrt[3]{w_3}\sinh(\eta_2) - 2\sqrt[3]{w_1}\sqrt{\delta_3}\sinh\left(\eta_1 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}{\delta_1 \cos(\eta_3) + \delta_2 \cosh(\eta_2) + 2\sqrt{\delta_3}\cosh\left(\eta_1 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}, \tag{3.11}$$

where $\eta_1 = -\sqrt[3]{w_1}x + Q_1 y + R_1 z + w_1 t$, $\eta_2 = -\sqrt[3]{w_3}x + Q_3 y + \left(-Q_3 + \sqrt[3]{w_3}\right) z + w_3 t$, $\eta_3 = Q_2 y + R_2 z$.

Case 4

$$P_1 = -\sqrt[3]{w_1},\ R_1 = -Q_1 + \sqrt[3]{w_1},\ P_2 = 0,\ w_2 = 0,\ P_3 = 0,\ w_3 = 0, \tag{3.12}$$

where Q_1, Q_2, Q_3, R_2, R_3, and w_1 are free parameters.

Substituting Eq. (3.12) into Eq. (2.2) and Eq. (3.1), we obtain the solution

$$u = -6\frac{\sqrt[3]{w_1}(e^{\eta_1} - \delta_3 e^{-\eta_1})}{e^{\eta_1} + \delta_1 \cos(\eta_2) + \delta_2 \cosh(\eta_3) + \delta_3 e^{-\eta_1}} \tag{3.13}$$

where $\eta_1 = -\sqrt[3]{w_1}x + Q_1 y + \left(-Q_1 + \sqrt[3]{w_1}\right) z + w_1 t$, $\eta_2 = Q_2 y + R_2 z$, $\eta_3 = Q_3 y + R_3 z$.

In particular, solution Eq. (3.13) can be expressed as

$$u_4 = -6\frac{2\sqrt{\delta_3}\sqrt[3]{w_1}\sinh\left(\eta_1 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}{\delta_1 \cos(\eta_2) + \delta_2 \cosh(\eta_3) + 2\sqrt{\delta_3}\cosh\left(\eta_1 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}, \tag{3.14}$$

where $\eta_1 = -\sqrt[3]{w_1}x + Q_1 y + \left(-Q_1 + \sqrt[3]{w_1}\right) z + w_1 t$, $\eta_2 = Q_2 y + R_2 z$, $\eta_3 = Q_3 y + R_3 z$.

Case 5

$$P_1 = 0,\ w_1 = 0,\ P_2 = \sqrt[3]{w_2},\ R_2 = -Q_2 - \sqrt[3]{w_2},\ P_3 = -\sqrt[3]{w_3},\ R_3 = -Q_3 + \sqrt[3]{w_3}, \tag{3.15}$$

where Q_1, Q_2, Q_3, R_1, w_2 and w_3 are free parameters.

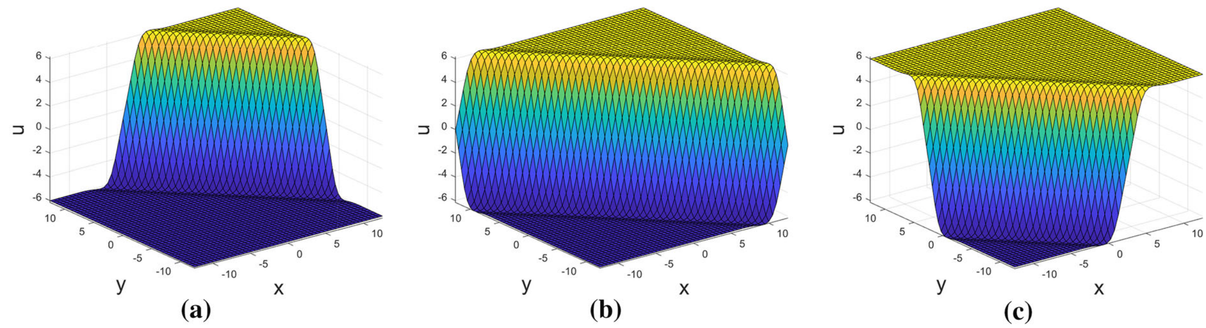

Fig. 6 The periodic kink wave solution u_1 at $\delta_1 = 1, \delta_2 = 1, \delta_3 = 1, Q_1 = 1, Q_2 = 1, Q_3 = 1, P_1 = 1, P_2 = 1, P_3 = 1, w_1 = 2, w_2 = 2, w_3 = 2, z = 0$, **a**$t = -5$, **b**$t = 0$, **c**$t = 5$

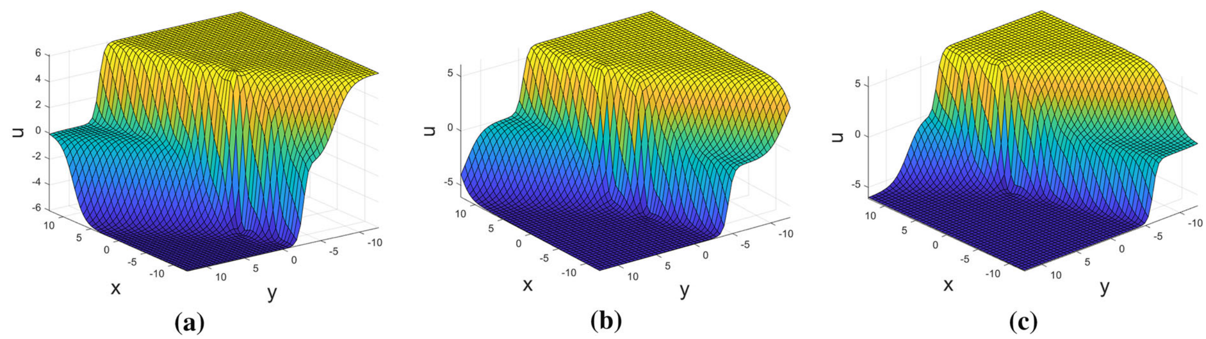

Fig. 7 The periodic cross kink wave solution u_2 at $\delta_1 = 1, \delta_2 = 1, \delta_3 = 1, Q_1 = 2, Q_2 = 1, Q_3 = 1, R_3 = 1, w_1 = 1, w_2 = 1, z = 0$, **a**$t = -5$, **b**$t = 0$, **c**$t = 5$

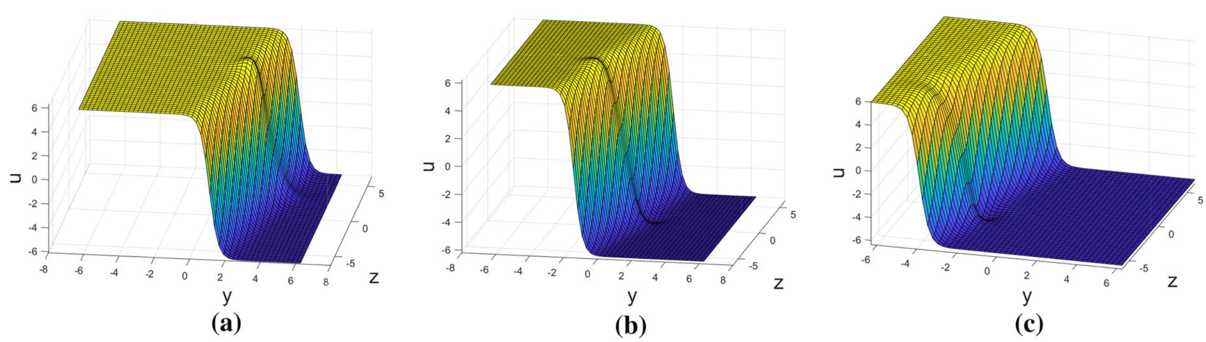

Fig. 8 The periodic kink wave solution u_3 at $\delta_1 = 1, \delta_2 = 1, \delta_3 = 1, Q_1 = 1, Q_2 = 1, Q_3 = 3, R_1 = 1, R_2 = 1, w_1 = 1, w_2 = 1, w_3 = 1, x = 0$, **a**$t = -5$, **b**$t = 0$, **c**$t = 5$

Substituting Eq. (3.15) into (2.2) and Eq. (3.1), we obtain the solution

$$u = -6\frac{\delta_1 \sqrt[3]{w_2}\sin(\eta_1) + \delta_2 \sqrt[3]{w_3}\sinh(\eta_2)}{\mathrm{e}^{\eta_3} + \delta_1\cos(\eta_1) + \delta_2\cosh(\eta_2) + \delta_3\,\mathrm{e}^{-\eta_3}}, \tag{3.16}$$

where $\eta_1 = \sqrt[3]{w_2}x + Q_2 y + \left(-Q_2 - \sqrt[3]{w_2}\right) z + w_2 t$, $\eta_2 = -\sqrt[3]{w_3}x + Q_3 y + \left(-Q_3 + \sqrt[3]{w_3}\right) z + w_3 t$, $\eta_3 = Q_1 y + R_1 z$.

In particular, solution Eq. (3.16) can be expressed as

$$u_5 = -6\frac{\delta_1 \sqrt[3]{w_2}\sin(\eta_1) + \delta_2 \sqrt[3]{w_3}\sinh(\eta_2)}{\delta_1\cos(\eta_1) + \delta_2\cosh(\eta_2) + 2\sqrt{\delta_3}\cosh\left(\eta_3 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}, \tag{3.17}$$

where $\eta_1 = \sqrt[3]{w_2}x + Q_2 y + \left(-Q_2 - \sqrt[3]{w_2}\right) z + w_2 t$, $\eta_2 = -\sqrt[3]{w_3}x + Q_3 y + \left(-Q_3 + \sqrt[3]{w_3}\right) z + w_3 t$, $\eta_3 = Q_1 y + R_1 z$.

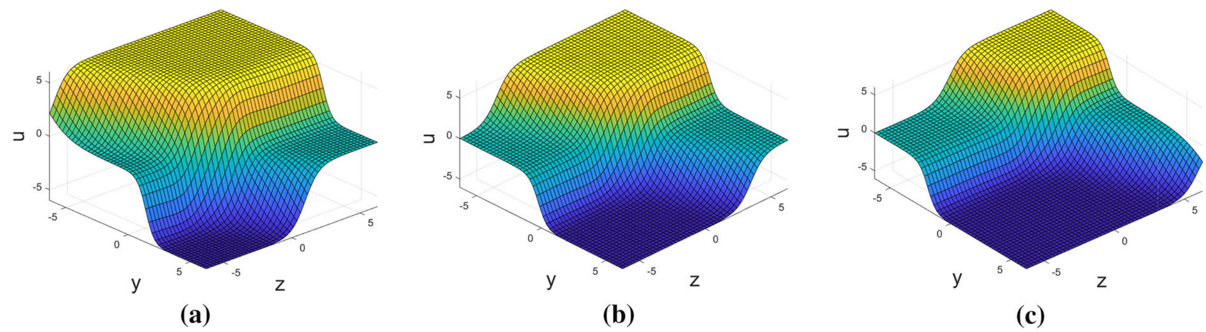

Fig. 9 The periodic cross kink wave solution u_4 at $\delta_1 = 1, \delta_2 = 1, \delta_3 = 1, Q_1 = 2, Q_2 = 1, Q_3 = 1, R_2 = 1, R_3 = 1, w_1 = 1, x = 0$, **a** $t = -5$, **b** $t = 0$, **c** $t = 5$

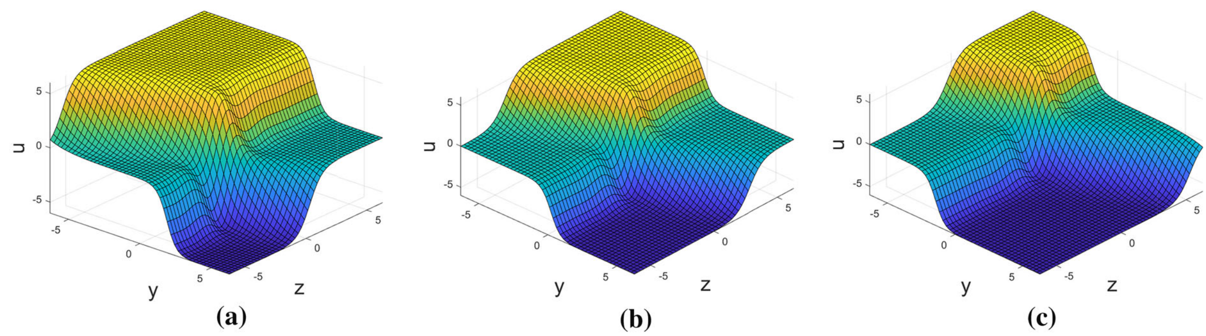

Fig. 10 The periodic cross kink wave solution u_5 at $\delta_1 = 1, \delta_2 = 1, \delta_3 = 1, Q_1 = 1, Q_2 = 1, Q_3 = 2, R_1 = 1, w_2 = 1, w_3 = 1, x = 0$, **a** $t = -5$, **b** $t = 0$, **c** $t = 5$

Case 6

$$P_1 = 0,\ w_1 = 0,\ P_2 = \sqrt[3]{w_2},\ R_2 = -Q_2 - \sqrt[3]{w_2},\\ P_3 = 0,\ w_3 = 0, \tag{3.18}$$

where Q_1, Q_2, Q_3, R_1, R_3 and w_2 are free parameters.

Substituting Eq. (3.18) into Eq. (2.2) and Eq. (3.1), we obtain the solution

$$u = -6\frac{\delta_1 \sqrt[3]{w_2}\sin(\eta_1)}{e^{\eta_2} + \delta_1\cos(\eta_1) + \delta_2\cosh(\eta_3) + \delta_3 e^{-\eta_2}}, \tag{3.19}$$

where $\eta_1 = \sqrt[3]{w_2}x + Q_2 y + \left(-Q_2 - \sqrt[3]{w_2}\right)z + w_2 t$, $\eta_2 = Q_1 y + R_1 z$, $\eta_3 = Q_3 y + R_3 z$.

In particular, solution Eq. (3.19) can be expressed as

$$u_6 = -6\frac{\delta_1 \sqrt[3]{w_2}\sin(\eta_1)}{\delta_1\cos(\eta_1) + \delta_2\cosh(\eta_3) + 2\sqrt{\delta_3}\cosh\left(\eta_2 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}, \tag{3.20}$$

where $\eta_1 = \sqrt[3]{w_2}x + Q_2 y + \left(-Q_2 - \sqrt[3]{w_2}\right)z + w_2 t$, $\eta_2 = Q_1 y + R_1 z$, $\eta_3 = Q_3 y + R_3 z$.

Case 7

$$P_1 = 0,\ w_1 = 0,\ P_2 = 0,\ w_2 = 0,\ P_3 = -\sqrt[3]{w_3},\\ R_3 = -Q_3 + \sqrt[3]{w_3}, \tag{3.21}$$

where Q_1, Q_2, Q_3, R_1, R_2 and w_3 are free parameters.

Substituting Eq. (3.21) into Eq. (3.1) and Eq. (2.2), we obtain the solution

$$u = -6\frac{\delta_2 \sqrt[3]{w_3}\sinh(\eta_1)}{e^{\eta_2} + \delta_1\cos(\eta_3) + \delta_2\cosh(\eta_1) + \delta_3 e^{-\eta_2}}, \tag{3.22}$$

where $\eta_1 = -\sqrt[3]{w_3}x + Q_3 y + \left(-Q_3 + \sqrt[3]{w_3}\right)z + w_3 t$, $\eta_2 = Q_1 y + R_1 z$, $\eta_3 = Q_2 y + R_2 z$.

In particular, solution Eq. (3.22) can be expressed as

$$u_7 = -6\frac{\delta_2 \sqrt[3]{w_3}\sinh(\eta_1)}{\delta_1\cos(\eta_3) + \delta_2\cosh(\eta_1) + 2\sqrt{\delta_3}\cosh\left(\eta_2 + \frac{1}{2}\ln\frac{1}{\delta_3}\right)}, \tag{3.23}$$

where $\eta_1 = -\sqrt[3]{w_3}x + Q_3 y + \left(-Q_3 + \sqrt[3]{w_3}\right)z + w_3 t$, $\eta_2 = Q_1 y + R_1 z$, $\eta_3 = Q_2 y + R_2 z$.

The figure of u_7 is similar to the figure of u_4, and it is a periodic cross kink wave solution.

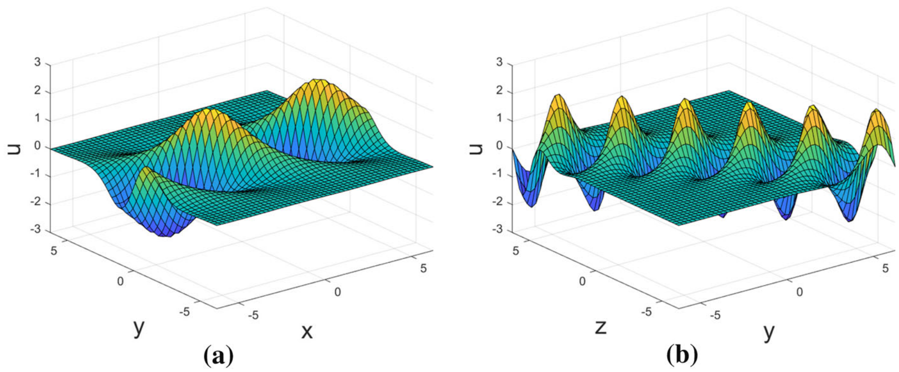

Fig. 11 The periodic wave solution u_6 at $\delta_1 = 1, \delta_2 = 1, \delta_3 = 1, Q_1 = 1, Q_2 = 1, Q_3 = 1, R_1 = 1, R_3 = 1, w_2 = 1$, **a** $z = 0, t = 0$, **b** $x = 0, t = 0$

4 Non-traveling wave solutions

In this section, we use the extended homoclinic test approach in Ref. [25] to get non-traveling wave solutions, which in form

$$u(x, y, z, t) = \varphi(\xi, t) + q(z), \tag{4.1}$$

where $\xi = x+my+nt+\theta(z)$, m, n are two nonzero constants, $\varphi(\xi, t)$, $q(z)$ and $\theta(z)$ are three functions undetermined. Substituting Eq. (4.1) into (2.1), we obtain

$$\begin{aligned}&\left(1 + m + \theta'(z)\right) \varphi_{\xi\xi\xi\xi} + \big(n + mn + n\theta'(z)\\ &\quad +q'(z)\big) \varphi_{\xi\xi}\\ &\quad + \left(2 + 2m + 2\theta'(z)\right) \varphi_\xi \varphi_{\xi\xi} + \left(1 + m + \theta'(z)\right)\\ &\quad \varphi_{\xi t} = 0.\end{aligned} \tag{4.2}$$

To simplify Eq. (4.2), we let

$$n + mn + n\theta'(z) + q'(z) = 0. \tag{4.3}$$

From Eq. (4.3), we get

$$\begin{aligned}q(z) &= -\int [(n + mn + n\theta'(z))\mathrm{d}z] + c\\ &= -n\theta(z) - (nm + n)z + c.\end{aligned} \tag{4.4}$$

where c is the integral constant. Therefore, in the condition of $(1 + m + \theta'(z)) \neq 0$, Eq. (4.2) reduces to

$$\varphi_{\xi\xi\xi\xi} + 2\varphi_\xi \varphi_{\xi\xi} + \varphi_{\xi t} = 0. \tag{4.5}$$

Integrating Eq. (4.5) once with respect to ξ. Let constant $c = 0$, we get

$$\varphi_{\xi\xi\xi} + (\varphi_\xi)^2 + \varphi_t = 0. \tag{4.6}$$

Let

$$\psi(\xi, t) = -\frac{1}{3}\varphi(\xi, t). \tag{4.7}$$

Substituting Eq. (4.7) into (4.6), one gets

$$\psi_{\xi\xi\xi} - 3\psi_\xi^2 + \psi_t = 0. \tag{4.8}$$

In order to solving Eq. (4.8), a nonlinear function transformation of dependent variable are used

$$\psi = -2(\ln \phi)_\xi, \tag{4.9}$$

where $\phi(\xi, t)$ will be determined later. Substituting Eq. (4.9) into Eq. (4.8), one can get a bilinear equation

$$\left(D_\xi D_t + D_\xi^4\right) \phi \cdot \phi = 0. \tag{4.10}$$

Let the solution in the form

$$\phi = k_1 \cos(\zeta_1) + k_2 \exp(\zeta_2) + \exp(-\zeta_2), \tag{4.11}$$

where $\zeta_i = a_i\xi + b_i t$, $k_i \in \mathbb{R}$; $a_i, b_i \in \mathbb{C} (i = 1, 2)$ are undetermined constants.

Substituting Eq. (4.11) into (4.10) and setting coefficients of $\cos^2(\zeta_1)$, $\cos(\zeta_1)\exp(\zeta_2)$, $\cos(\zeta_1)\exp(-\zeta_2)$, $\sin^2(\zeta_1)$, $\sin(\zeta_1)\exp(\zeta_2)$, $\sin(\zeta_1)\exp(-\zeta_2)$ and the constant term to zero, a set of nonlinear algebraic equations with respect to a_i, b_i and k_i, $(i = 1, 2)$ are given

$$\begin{cases} k_1^2(4a_1^4 - a_1b_1) = 0,\\ k_1k_2(a_1^4 + a_2^4 - 6a_1^2a_2^2 + a_2b_2 - a_1b_1) = 0,\\ k_1(a_1^4 + a_2^4 - 6a_1^2a_2^2 + a_2b_2 - a_1b_1) = 0,\\ k_2(16a_2^4 + 4a_2b_2) = 0,\\ k_1k_2(4a_1a_2^3 - 4a_1^3a_2 + a_1b_2 + a_2b_1) = 0,\\ k_1(-4a_1a_2^3 + 4a_1^3a_2 - a_1b_2 - a_2b_1) = 0.\end{cases} \tag{4.12}$$

Solving Eq. (4.12), we have the following results.

Case1

$$k_1 = 0, \ b_2 = -4a_2^3, \tag{4.13}$$

where a_1, a_2, b_1 and k_2 are free parameters.

Collecting Eq. (4.1), (4.4), (4.7), (4.9), (4.11), (4.13), we obtain the solution

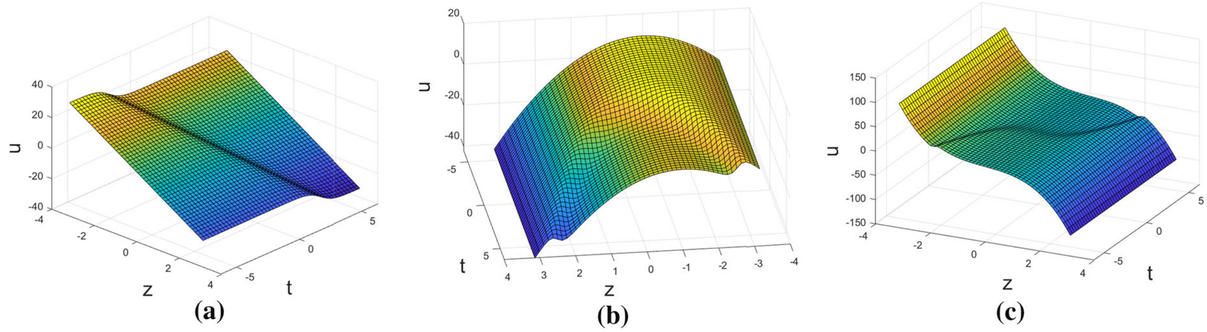

Fig. 12 The exact kink-like solution with tail u_1 at $x = y = 0, a_2 = 1, k_2 = 1, n = 3, m = 1, c = 0$, **a**$\theta(z) = z$, **b**$\theta(z) = z^2$, **c**$\theta(z) = z^3$

$$u = 6a_2\frac{k_2 e^{\lambda_1} - e^{-\lambda_1}}{k_2 e^{\lambda_1} + e^{-\lambda_1}} - n\theta(z) - (nm+n)z + c. \tag{4.14}$$

where$\lambda_1 = a_2\left(x + my + \left(n - 4a_2^2\right)t + \theta(z)\right)$.

In particular, solution Eq. (4.14) can be expressed as

$$u_1 = 6a_2 \tanh\left(\lambda_1 + \frac{1}{2}\ln k_2\right) - n\theta(z) - (nm+n)z + c, \quad k_2 > 0 \tag{4.15}$$

$$u_2 = 6a_2 \coth\left(\lambda_1 + \frac{1}{2}\ln(-k_2)\right) - n\theta(z) - (nm+n)z + c, \quad k_2 < 0 \tag{4.16}$$

where$\lambda_1 = a_2\left(x + my + \left(n - 4a_2^2\right)t + \theta(z)\right)$.

Case 2

$$a_1 = \frac{3^{3/4}\sqrt{2}}{6}((\pm i) \pm 1)a_2,\ b_1 = \frac{2\cdot 3^{3/4}\sqrt{2}}{3}((\pm i) \pm 1)a_2^3,\quad b_2 = -4a_2^3,\ k_2 = 0, \tag{4.17}$$

where a_2 and k_1 are free parameters.

Collecting Eq. (4.1), (4.4), (4.7), (4.9), (4.11), (4.17), we obtain the solution

$$u = -a_2\frac{3^{\frac{3}{4}}\sqrt{2}\ ((\pm i) \pm 1)\, k_1 \sin\lambda_2 + 6\, e^{-a_2\lambda_3}}{k_1\cos\lambda_2 + e^{-a_2\lambda_3}}, \tag{4.18}$$

where $\lambda_2 = \frac{3^{\frac{3}{4}}\sqrt{2}}{6}\ ((\pm i) \pm 1)\, a_2\left(4{a_2}^2 t + my + nt + \theta(z) + x\right)$and $\lambda_3 = -4{a_2}^2 t + my + nt + \theta(z) + x$.

In particular, solution Eq. (4.18) can be expressed as

$$u_3 = -a_2\frac{3^{\frac{3}{4}}\sqrt{2}\ ((\pm i) \pm 1)\, k_1 \sin\lambda_2 + 6\ (\cosh(-a_2\lambda_3) + \sinh(-a_2\lambda_3))}{k_1\cos\lambda_2 + \cosh(-a_2\lambda_3) + \sinh(-a_2\lambda_3)}, \tag{4.19}$$

where $\lambda_2 = \frac{3^{\frac{3}{4}}\sqrt{2}}{6}\ ((\pm i) \pm 1)\, a_2\left(4{a_2}^2 t + my + nt + \theta(z) + x\right)$ and $\lambda_3 = -4{a_2}^2 t + my + nt + \theta(z) + x$.

Case 3

$$a_1 = 0,\ b_1 = 0,\ b_2 = -a_2^3,\ k_2 = 0, \tag{4.20}$$

where a_2 and k_1 are free parameters.

Collecting Eqs. (4.20, 4.11, 4.9, 4.7, 4.4) with Eq. (4.1), we obtain the solution

$$u = -6a_2\frac{e^{-\lambda_4}}{k_1 + e^{-\lambda_4}} - n\theta(z) - (nm+n)z + c, \tag{4.21}$$

where $\lambda_4 = a_2\left(x + my + \left(n - a_2^2\right)t + \theta(z)\right)$.

In particular, solution Eq. (4.21) can be expressed as

$$u_4 = -6a_2\frac{(\cosh(-\lambda_4) + \sinh(-\lambda_4))}{k_1 + (\cosh(-\lambda_4) + \sinh(-\lambda_4))} - n\theta(z) - (nm+n)z + c, \tag{4.22}$$

The figure of u_4 is similar to the figure of u_1, and it is a kink-like solution with tail.

5 Discussion and conclusions

In this work, we mainly investigate the new (3+1)-dimensional BLMP equation, which is firstly proposed by Wazwaz. In Sect. 2, it is devoted to use the extended homoclinic test approach to construct solutions. If $k_1 = 0$, a kink-shaped solitary wave solution is obtained, if $k_2 = 0$, different kinds of singualr solitary wave solutions are obtained; if $k_1 \neq 0$ and $k_2 \neq 0$, we get 2 kinds of periodic kink wave solutions and periodic solitary wave solution. In Sect. 3, we use the three wave method to construct three wave solutions. It is obviously that, if $\delta_2 = 0$, the form of the solution constructed is the same as extended homoclinic test approach. In this section, we let the free parameters are real numbers and let $\delta_1 \neq 0$, $\delta_2 \neq 0$, and $\delta_3 \neq 0$. And the periodic kink wave solutions, periodic cross kink wave solutions and periodic wave solutions are obtained. In Sect. 4, we also use the extended homoclinic test approach to construct kink-shaped solitary wave solutions, what is different from the second part is that these solutions have a tail. These results reflect that the methods used in this paper are effective for seeking solutions of higher dimensional NLEEs.

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Data Availability The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation, to any qualified researcher.

Declarations

Competing interests The authors have no relevant financial or non-financial interests to disclose.

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