Nonlinear Dyn (2023) 111:319–339
https://doi.org/10.1007/s11071-022-07839-z

ORIGINAL PAPER

Fixed-time stabilization of high-order nonlinear systems with an asymmetric output constraint

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Received: 6 June 2022 / Accepted: 22 August 2022 / Published online: 2 September 2022

Abstract This article studies the problem of fixed-time stabilization for a class of *uncertain* high-order nonlinear systems subjected to an *asymmetric* output constraint. A *tangent*-type barrier function is first developed as an intermediate design ingredient by subtly extracting and utilizing the inherent features of system nonlinearities. Next, the proposed barrier function along with the intrinsic attributes of signum functions is exploited to elegantly renovate the celebrated technique of adding a power integrator, thereby establishing a *unified* approach by which a *tangent*-type asymmetric barrier *Lyapunov* function together with a continuous state feedback fixed-time stabilizer can be constructed systematically while guaranteeing the achievement of pre-specified output constraints successfully. A technical novelty of the presented scheme is ascribed to the unified nature enabling us to design a fixed-time stabilizer simultaneously workable for the system subjected to or free from output constraints without needing to revamp the controller structure. A numerical example is provided to show the effectiveness and superiority of the developed method.

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Keywords High-order nonlinear systems · Fixed-time stabilization · Asymmetric output constraint · Adding a power integrator

1 Introduction

Without any doubt, the stabilization task of high-order nonlinear systems [1] (also known as *p*-normal form systems [2]) has been fairly recognized as a significantly formidable and challenging problem in the field of nonlinear control. The primary difficulty of this issue is the inherent nonlinearities of the system together with the uncontrollability and potential nonexistence of Jacobian linearization at the origin, which prevents the applicability of various existing nonlinear feedback design methods, including backstepping strategy [3] (also called adding an integrator [4]). Such a critical obstruction was conquered with a technological breakthrough achieved by Qian and Lin in seminal papers [5,6], where a renowned scheme named adding a power integrator was proposed. The cardinal philosophy underlying the technique of adding a power integrator is the maneuvering of feedback domination, which not only provides distinctive insights into overcoming the ingrained obstacles originating in system inherent nonlinearities but also contributes to a revolutionary perspective on constructing a feedback stabilizer defeating the uncontrollability and nonexistence of Jacobian linearization and thereby invigorating a series of elegant works dedicated to the stabilization problem of high-order nonlinear systems in the past two decades; see, for instance, [7–19] and the references therein.

In addition to the pure stabilization mission, a more aspiring goal is to stabilize nonlinear systems in consideration of a pre-specified output constraint since system operation safety and/or performance specifications are crucial and critical to be pursued in practice. For example, the constraints on the joint angles of a robot manipulator during stabilization/tracking operation are imperative for preventing potential structural damage or injury to the persons nearby [20]; some practical examples can be also found in [21–27]. For high-order nonlinear systems (i.e., p-normal form systems), compared with the great advances in the pure stabilization issue (e.g., [7,9–18,28–30]), much less progress has been achieved toward investigating the problem of stabilization subjected to a pre-specified output constraint [31–38]. Specifically, the standard barrier Lyapunov function (BLF) (for the definition, see [39]), the *tangent*-type symmetric BLF[1], and the nonlinear transformation methods were adopted in the works [31–34], [35], and [36], respectively, to deal with output constraints in the stabilization task; however, the schemes proposed in [31–35] only consider symmetric output constraints and the strategies in [31–36] are merely applicable to a rather limited class of high-order nonlinear systems because they essentially suffer from restrictive structural assumptions that either system powers must obey a monotone inequality relation or system nonlinear terms are forced necessarily to comply with several complicated growth conditions. By fully exploring the characteristics of system structures and inherent nonlinearities, a *fraction*-type BLF along with an explicit stabilizer design was presented in our recent results [37] and [38], where the structural restrictions in [31–34] were relatively lifted; thus, the methods in [37] and [38] are applicable to a broader class of high-order nonlinear systems, and in particular, the strategy in [38] further secures finite-time state convergence in the stabilization mission without violating output constraints.

Although stabilization can be successfully realized in a finite time horizon by the designs in [33,34] and our previous study [38], an apparent defect included in [33,34,38] is the dependence between initial states and the estimation of the settling (convergence) time, which restrains to some extent the application scope of the manners in [33,34,38] due to the unavailability of precise initial states. Additionally, the main treatment/idea taken in [33,34,38] for rendering finite-time state convergence is asymptotic state convergence plus local finite-time stability [13,40], which potentially prohibits analytically estimating the settling time, even when exact information on the initial states is available, and therefore leads to technical shortcomings. Interestingly, a notion named fixed-time state convergence (stability) was recently studied in [41,42], depicting the property of finite-time state convergence with a guaranteed upper bound of the settling time independent of initial states and stimulating numerous studies devoted to the fixed-time stabilization[2] issue of various nonlinear systems (see, for example, [43,44]). However, in the literature of which we are aware, the stabilization problem of high-order nonlinear systems (i.e., p-normal form systems) has never been exhaustively advanced with ensuring the property of fixed-time state convergence as well as the fulfillment of output constraints, that is, a fundamental question of how to construct a stabilizer for high-order nonlinear systems, achieving simultaneously the requirement of output constraints and the performance of fixed-time state convergence, remains largely open until now and deserves an in-depth investigation.

In this article, we concentrate on the problem of fixed-time stabilization for a class of *uncertain* high-order nonlinear systems subjected to a pre-specified *asymmetric* output constraint[3] described by the equations of the form

[1] It should be emphasized that for the stabilization problem of high-order nonlinear systems (e.g., system (1)) with output constraints, the work [35] is the very first result in the literature proposing a solution by using *tangent*-type BLFs.

[2] The task of fixed-time stabilization is to perform the finite-time stabilization with securing an upper bound of the settling time irrelative to initial states [41–43].

[3] Here, the asymmetric output constraint is the constraint with nonidentical (asymmetric) upper and lower bounds (constraints); e.g., $-\varepsilon_L < y(t) = x_1(t) < \varepsilon_U$ for all $t \geq t_0$ with $\varepsilon_L > 0$, $\varepsilon_U > 0$ and $\varepsilon_L \neq \varepsilon_U$.

$$
\begin{aligned}
\dot{x}_1 &= d_1(t,x)x_2^{p_1} + f_1(t,x,u)\\
\dot{x}_2 &= d_2(t,x)x_3^{p_2} + f_2(t,x,u)\\
&\vdots\\
\dot{x}_{n-1} &= d_{n-1}(t,x)x_n^{p_{n-1}} + f_{n-1}(t,x,u)\\
\dot{x}_n &= d_n(t,x)u^{p_n} + f_n(t,x,u)y = x_1
\end{aligned} \tag{1}
$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$ are the system state, control input, and system output, respectively; with $t_0 \in \mathbb{R}_+$ being the initial time instant, the initial state is represented by $x(t_0) \in \mathbb{R}^n$. For each $i = 1, \ldots, n$, the system power $p_i \in \mathbb{R}_+^{\text{odd}} = \{s \in \mathbb{R}_+ \mid s = s_1/s_2$ with s_1 and s_2 being two positive odd integers$\}$ with $p_n = 1$, and the nonlinear term $f_i : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and parameter $d_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ are *uncertain* (unknown) continuous functions. The primary objective is to develop a methodology effectively addressing and guiding the design of a state feedback controller u under which each trajectory $x(t)$ of the closed-loop system (1) converges to the origin in fixed time, that is, $x(t) \to 0$ in finite time and $x(t) = 0$ for all $t \geq T_r$ for some $T_r \in (t_0, \infty)$ being an upper bound of the settling (convergence) time, which is independent of the initial state $x(t_0)$ but related to certain design parameters. Meanwhile, the system output $y(t) = x_1(t)$ fulfills the pre-specified asymmetric constraint $-\varepsilon_L < y(t) = x_1(t) < \varepsilon_U$ for all $t \geq t_0$ with ε_L and ε_U being positive real constants. Notably, pursuing this problem is nontrivial and quite challenging. A key obstruction inhibiting us from seeking a solution is essentially the privation of constructive designs of BLFs and controllers (stabilizers) in efficiently performing the fixed-time stabilization as well as achieving the demand of asymmetric output constraints. Another difficult impediment is the lack of explicit methods for analyzing fixed-time state convergence involving pre-specified output constraints because, in the presence of output constraints imposed on the closed-loop system, the Lyapunov-like criterion presented in [41] is no longer applicable. Being aware of the aforementioned difficulties, in this article, we first design a new *tangent*-type barrier function[4] as an intermediate/unformed design ingredient by artfully extracting and exploiting the inherent characteristics of system nonlinearities. Based on a skillful implantation of the presented barrier function and a delicate utilization of the intrinsic traits of signum functions, the technique of adding a power integrator is elegantly renovated to subtly establish a novel approach that, in a systematic fashion, guides us in constructing a *tangent*-type asymmetric BLF together with a continuous state feedback stabilizer (controller) and thereby fulfilling simultaneously the fixed-time stabilization task and the requirement of the pre-specified constraint on the system output.

To the best of our knowledge, this article is the first work in the literature coping with and offering an affirmative solution to the problem of fixed-time stabilization for *uncertain* high-order nonlinear systems (e.g., system (1)) subjected to *asymmetric* output constraints. Technically, the appealing innovations and contributions of this article can be summarized in the following three aspects.

(i) The constructed *tangent*-type asymmetric BLF acting as a subtle leverage equipped in the proposed method in dealing with asymmetric output constraints differs significantly from the commonly utilized *tangent*-type [20,21,24,25,35] and *logarithm*-type [23,33,34,39] BLFs; specifically, the inherent features of system nonlinearities $f_i(t, x, u)$'s are comprehensively taken into account and skillfully assimilated in the construction of a *tangent*-type barrier function as an intermediate/unformed design ingredient so that the resultant *tangent*-type asymmetric BLF under the presented approach inherits the dynamic characteristics of system (1), thereby providing the feasibility of achieving fixed-time stabilization for system (1).

(ii) A new tool (i.e., Lemma 6) with the inspiration of modifying the so-called Bihari-type inequality [45] is introduced to facilitate the analysis of fixed-time state convergence; to be more specific, on the basis of the developed tool, the property of fixed-time state convergence can be evaluated/scrutinized directly and analytically without relying on the idea of asymptotic convergence plus local finite-time stability employed in [33,34,38], and an upper bound of the settling (convergence) time independent of initial states $x(t_0)$ can be acquired explicitly.

(iii) The proposed approach offers and enjoys a unified nature that enables one to synthesize a fixed-time stabilizer simultaneously workable for system (1)

[4] Such an intermediate/unformed design ingredient in the design processes is named a barrier function rather than a BLF since extra requirements are necessary for a function to be a BLF.

subjected to or free from asymmetric output constraints, without needing to revamp the controller structure; in other words, when the output constraints are intentionally set to be infinity (or equivalently a very large value) so as to correspond to the scenario where the output constraints are no longer obligatory/imperative for system (1), the presented strategy will directly evolve into the design procedure applicable to tackling the pure mission of fixed-time stabilization for system (1) without output constraints.

Notation: All notations throughout this article are listed below. $\mathbb{R}$ and $\mathbb{R}_+$ represent the set of real numbers and the set of nonnegative real numbers, respectively. $\mathbb{R}^n$ is the standard n-dimensional Euclidean space and $\mathbb{R}_+^{\text{odd}} := \{s \in \mathbb{R}_+ \mid s = s_1/s_2$ with s_1 and s_2 being two positive odd integers$\}$. Suppose that $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and $\mathbb{B} \subset \mathbb{R}^n$ is an open connected set (i.e., a domain); for the sake of simplicity, we let $\overline{x}_i := (x_1, x_2, \ldots, x_i)^T \in \mathbb{R}^i$ for all $i = 1, \ldots, n$ with $\overline{x}_1 = x_1$ and $\overline{x}_n = x$, and $\partial\mathbb{B}$ be the boundary of $\mathbb{B} \subset \mathbb{R}^n$. Given $\beta_i \in (0, \infty)$ with $i = 1, 2, 3$, we also define $\lceil s \rfloor^{\beta_1} := |s|^{\beta_1} sign(s)$ for all $s \in \mathbb{R}$, where $sign(\cdot)$ is the signum function, and $\mathbb{M}_i(\beta_2, \beta_3) := \{\overline{x}_i \in \mathbb{R}^i \mid -\beta_2 < x_1 < \beta_3\} \subset \mathbb{R}^i$ for all $i = 1, \ldots, n$.

2 Technical preliminaries and assumptions

First, we introduce an important lemma that contributes to handling an asymmetric output constraint imposed on a continuous non-autonomous (time-varying) nonlinear system, which can be non-Lipschitz continuous particularly; the detailed proof is referred to our previous work [38].

Lemma 1 ([38]) *Consider a non-autonomous nonlinear system*

$$\dot{z} = \phi(t, z), \quad y = x_1 \tag{2}$$

where $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n$ and $\phi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and may not be Lipschitz. Let $-\varepsilon_L < y(t) = x_1(t) < \varepsilon_U$ for all $t \geq t_0$ be the asymmetric constraint imposed on the output $y = x_1$, where $\varepsilon_L, \varepsilon_U \in (0, \infty)$ are two pre-specified constants. Suppose that $V_1 : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ and $V_2 : \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ are two continuously differentiable functions, where $V_1(z_1)$ is positive definite while meeting the property $V_1(z_1) \to \infty$ as $z_1 \to \partial\mathbb{M}_1(\varepsilon_L, \varepsilon_U)$, and $V_2(z)$ is nonnegative. If $V(z) := V_1(z_1) + V_2(z)$ fulfills the following two conditions:

(i) $V(z)$ is radially unbounded with respect to $(z_2, z_3, \ldots, z_n)$; in other words, there holds

$$V(z) \to \infty \quad as \quad \|(z_2, z_3, \ldots, z_n)\| \to \infty$$

for any fixed $z_1 \in \mathbb{M}_1(\varepsilon_L, \varepsilon_U)$

(ii) the derivative of $V(z)$ along system (2) is nonpositive; i.e., there holds[5]

$$\frac{\partial V(z)}{\partial z}\phi(t, z) \leq 0$$

for all $(t, z) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$

then every solution[6] *$z(t)$ of system (2) starting at any initial state $z(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ is defined on $[t_0, \infty)$ (i.e., $z(t)$ is forward complete*[7]*) and satisfies $z(t) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ for all $t \geq t_0$, thereby fulfilling the output constraint $-\varepsilon_L < y(t) = x_1(t) < \varepsilon_U$ for all $t \geq t_0$.*

Remark 1 Remarkably, system (2) involved in Lemma 1 is continuous only (i.e., $\phi(t, z)$ is merely set as a continuous function); thus, Lemma 1 can be utilized to tackle non-autonomous nonlinear systems without needing both the Lipschitz continuity of $\phi(t, z)$ and the uniqueness of solutions corresponding to a given initial state. In fact, since fixed-time (or finite-time) state convergence would take place only in non-Lipschitz continuous systems [17,41], Lemma 1 goes beyond Lemma 1 of the work [39], in which the systems are strictly restricted to be Lipschitz continuous, and further provides the technical possibility of investigating fixed-time state convergence in consideration of asymmetric output constraints.

Remark 2 It can be observed from Lemma 1 that a function $V(z)$ (or equivalently $V_1(z_1)$ and $V_2(z)$) satisfying the differentiability, the positive definiteness and

[5] It can be found from the proof of Lemma 1 in [38] that the nonpositivity of the derivative of $V(z)$ along system (2) is sufficient to ensure the boundedness of the solution $z(t)$; thus, the condition (ii) here is simplified, compared to the one in [38].

[6] Because $\phi(t, z)$ is continuous and probably not Lipschitz on $\mathbb{R}_+ \times \mathbb{R}^n$, the solution of system (2) satisfying a given initial state $z(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ is in general not unique [46].

[7] For the definition of forward completeness, please refer to [46].

the conditions (i) and (ii) of Lemma 1 definitely induces the properties $V(z) \to \infty$ as $z \to \partial\mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ and $V(z(t)) \leq B < \infty$ for all $t \geq t_0$ with $B \in \mathbb{R}_+$ and $z(t)$ being an arbitrary solution of system (2) starting at the initial state $z(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$; these two induced properties are generally adopted in defining the so-called BLF [38,39]. In other words, with the implication of Lemma 1 in mind, the organization of an asymmetric BLF $V(z)$ for a continuous non-autonomous nonlinear system can be performed via designing $V_1(z_1)$ and $V_2(z)$ by fitting the conditions of Lemma 1. Based on this reason, in the design processes later, a positive definite and continuously differentiable function $V_1 : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ satisfying $V_1(z_1) \to \infty$ as $z_1 \to \partial\mathbb{M}_1(\varepsilon_L, \varepsilon_U)$ will be specifically referred to as a barrier function.

We next present four lemmas in aid of deriving our main results; the proofs of the first three can be readily found in the literature (e.g., [5,14,18,37]), and the last one is proven accordingly.

Lemma 2 ([5]) *Let $m_1 > 0$ and $m_2 \geq 1$ be real constants. For any $s_1, s_2 \in \mathbb{R}$, one has*

$$\left| \lceil s_1 \rfloor^{\frac{m_1}{m_2}} - \lceil s_2 \rfloor^{\frac{m_1}{m_2}} \right| \leq 2^{1-\frac{1}{m_2}} \left| \lceil s_1 \rfloor^{m_1} - \lceil s_2 \rfloor^{m_1} \right|^{\frac{1}{m_2}}.$$

Lemma 3 ([14,18]) *Let $m_1, m_2 > 0$ be real constants and $\gamma : \mathbb{R}^2 \to (0, \infty)$ be a function. For any $s_1, s_2 \in \mathbb{R}$, there holds*

$$|s_1|^{m_1}|s_2|^{m_2} \leq \frac{m_1}{m_1+m_2}\gamma(s_1,s_2)|s_1|^{m_1+m_2} + \frac{m_2}{m_1+m_2}\gamma^{-\frac{m_1}{m_2}}(s_1,s_2)|s_1|^{m_1+m_2}.$$

Lemma 4 ([18,37]) *Let $m > 0$ be a real constant. For any $s_i \in \mathbb{R}$ with $i = 1, \ldots, n$, one has*

$$(|s_1|+\cdots+|s_n|)^m \leq \max(1, n^{m-1}) \left(|s_1|^m + \cdots + |s_n|^m\right).$$

Lemma 5 *For any $s \in [0, \pi/2)$, there holds* $\tan(s) \leq s\sec(s)$.

Proof It suffices to prove the case where $s \in (0, \pi/2)$; to this end, one can deduce by the mean value theorem that for all $s \in (0, \pi/2)$ there always exists $s^* \in (0, s)$ such that $\cos(s^*) = \sin(s)/s$, which implies $\sin(s) \in (0, s]$ for all $s \in (0, \pi/2)$. This readily gives $\tan(s) \leq s/\cos(s) = s\sec(s)$ for all $s \in (0, \pi/2)$. □

The lemma below is newly developed; in this article, it will work as a new tool in helping and facilitating the analysis of fixed-time state convergence for nonlinear systems.

Lemma 6 *Let $s_0 \in \mathbb{R}_+$, $c_1, c_2 > 0$, $k > 1$, and $0 < m_i < 1$ for all $i = 1, \ldots, n$ be real constants. Suppose that $\Phi : [s_0, \infty) \to \mathbb{R}_+$ is a decreasing function of the form $\Phi(s) := \sum_{i=1}^n \Phi_i(s)$ with $\Phi_i : [s_0, \infty) \to \mathbb{R}_+$ being continuous for all $i = 1, \ldots, n$, and $m = \max_{i=1,\ldots,n}\{m_i\}$. If $\Phi(s_0) > 0$ and there holds*

$$\Phi(s) \leq \Phi(s_0) - \int_{s_0}^{s} \left[c_1 \left(\sum_{i=1}^{n} \Phi_i^{m_i}(\tau) \right) + c_2 \Phi^k(\tau) \right] \mathrm{d}\tau \quad (3)$$

for all $s \in [s_0, \infty)$, then $\Phi(s) = 0$ for all $s \in [s_r, \infty)$ with

$$s_r = s_0 + \frac{1}{c_1(1-m)} + \frac{1}{c_2(k-1)}.$$

Proof Two cases are considered as follows.
Case 1: When $\Phi(s_0) > 1$, we shall show that $\Phi(s) \leq 1$ for all $s \in [s_1^*, \infty)$ and for some $s_1^* \in (s_0, \infty)$; this is, instead, carried out by proving that $\Phi(s) \leq \Theta(s)$ for all $s \in [s_0, \bar{s}_1^*]$ with $\Theta : [s_0, \bar{s}_1^*] \to \mathbb{R}$ being a continuous function of the form

$$\Theta(s) = \left[\Phi^{1-k}(s_0) + c_2(k-1)(s-s_0) \right]^{\frac{1}{1-k}}$$

and

$$\bar{s}_1^* = s_0 + \frac{\Phi^{(1-k)}(s_0) - 1}{c_2(1-k)}.$$

Toward this end, we assume that there exists $s_1 \in (s_0, \bar{s}_1^*]$ such that $\Phi(s_1) > \Theta(s_1)$. Define

$$\Omega = \{s \mid s \in (s_0, s_1) \text{ such that } \Phi(\tau) > \Theta(\tau) \text{ for all } \tau \in (s, s_1]\} \subseteq (s_0, s_1)$$

and $s_2 = \inf \Omega$. From the continuity of $\Phi(\cdot)$ and $\Theta(\cdot)$, it follows that $\Phi^k(s) \geq \Theta^k(s) > 0$ for all $s \in [s_2, s_1]$; in addition, it is direct to see that for all $s \in [s_0, \bar{s}_1^*]$

$$-c_2 \int_{s_0}^{s} \Theta^k(\tau)\, \mathrm{d}\tau = \Theta(s) - \Theta(s_0).$$

Using these two results, we can directly verify that

$$\Phi(s_0) - c_2 \int_{s_0}^{s} \Phi^k(\tau)\,\mathrm{d}\tau \geq \Phi(s) > \Theta(s) = \Theta(s_0) - c_2 \int_{s_0}^{s} \Theta^k(\tau)\,\mathrm{d}\tau \geq 0$$

for all $s \in [s_2 + \delta, s_1]$ and for a real constant $\delta > 0$ satisfying $s_2 + \delta < s_1$. Since $\Phi(s_0) = \Theta(s_0) > 0$ and $c_2 > 0$, it further gives

$$\int_{s_0}^{s} \Phi^k(\tau)\,\mathrm{d}\tau < \int_{s_0}^{s} \Theta^k(\tau)\,\mathrm{d}\tau$$

for all $s \in [s_2 + \delta, s_1]$, which implies that there exists $s_3 \in [s_2 + \delta, s_1]$ such that $\Phi^k(s_3) < \Theta^k(s_3)$ and thus leads to a contradiction. Hence, $\Phi(s) \leq \Theta(s)$ for all $s \in [s_0, \bar{s}_1^*]$. Observing $\Theta(\bar{s}^*) = 1$ and noticing that $\Phi(\cdot)$ is continuous and decreasing, one has $\Phi(s) \leq 1$ for all $s \in [s_1^*, \infty)$ with s_1^* being larger than $\bar{s}_1^*$ and, in point of fact, having the form

$$s_1^* = s_0 + \frac{1}{c_2(k-1)}.$$

Case 2: In the case of $0 < \Phi(s_0) \leq 1$, one has $\Phi(s) \leq 1$ for all $s \in [s_0, \infty)$ since $\Phi(\cdot)$ is continuous and decreasing. Moreover, it follows from Lemma 4 that

$$\Phi(s) - \Phi(s_0) \leq -c_1 \int_{s_0}^{s} \Phi_i^{m_i}(\tau)\,\mathrm{d}\tau \leq -c_1 \int_{s_0}^{s} \Phi^m(\tau)\,\mathrm{d}\tau$$

for all $s \in [s_0, \infty)$. Using this inequality and an almost identical argument in Case 1, we can easily show that there exists $s_2^* \in (s_0, \infty)$ taking the following form

$$s_2^* = s_0 + \frac{1}{c_1(1-m)}$$

such that $\Phi(s_2^*) = 0$, which together with the fact that $\Phi(\cdot)$ is continuous and decreasing gives $\Phi(s) = 0$ for all $s \in [s_2^*, \infty)$.

Combining the two cases one immediately obtains $\Phi(s) = 0$ for all $s \in [s_r, \infty)$ with

$$s_r = s_0 + \frac{1}{c_1(1-m)} + \frac{1}{c_2(k-1)}$$

and therefore completes the proof. □

Remark 3 It is worth pointing out that establishing Lemma 6 is inspired by modifying the so-called Bihari-type inequality [45]. As a matter of fact, in the case when $c_1 < 0$ and $c_2 = 0$, the condition (3) is exactly a special case of the Bihari-type inequality. Because Lemma 6 is truly developed with involving the positive real constants c_1 and c_2, the condition (3) is in a Bihari-like form with enlarged applicability, and thus, Lemma 6 can be technically thought of as a counterpart of the Bihari-type inequality. A crucial benefit supplied by Lemma 6 is the facile utility of investigating the convergence of a continuous and decreasing function $\Phi(\cdot)$ within a fixed time (i.e., s_r in Lemma 6), without demanding its differentiability; such a benefit also purifies Lemma 6 to a new tool able to verify analytically the property of fixed-time state convergence of nonlinear systems that are continuous and probably not Lipschitz, and acquire explicitly an upper bound of the settling (convergence) time independent of initial states without relying on the idea of asymptotic convergence plus local finite-time stability [33,34,38].

Noting that appropriate conditions restricting the growth rates of both system nonlinearities and uncertainties are essentially imperative even for the task of purely stabilizing system (1) without any constraint on the output [1,5,6], we impose the following two assumptions on the uncertain parameters $d_i(t,x)$'s and the nonlinearities $f_i(t,x,u)$'s, respectively, before starting the development of the primary approach.

Assumption 1 There exist smooth functions $\underline{d}_i : \mathbb{R}^i \to (0,\infty)$ and $\overline{d}_i : \mathbb{R}^i \to (0,\infty)$ such that

$$\underline{d}_i(\overline{x}_i) \leq d_i(t,x) \leq \overline{d}_i(\overline{x}_i)$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and for all $i = 1,\ldots,n$.

Assumption 2 There exist real constants $\omega_n \leq \cdots \leq \omega_2 \leq \omega_1 < 0$ and a smooth function $\overline{f}_i : \mathbb{R}^i \to \mathbb{R}_+$ such that

$$|f_i(t,x,u)| \leq \overline{f}_i(\overline{x}_i)\left(|x_1|^{\frac{\sigma_i+\omega_i}{\sigma_1}} + \cdots + |x_i|^{\frac{\sigma_i+\omega_i}{\sigma_i}}\right) \tag{4}$$

for all $(t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}$ and for all $i = 1,\ldots,n$, where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are real constants defined by, for all $j = 1, \ldots, n$,

$$\sigma_1 = 1 \quad \text{and} \quad \sigma_{j+1} = \frac{\sigma_j + \omega_j}{p_j} > 0. \tag{5}$$

Remark 4 Obviously, the inequality (4) in Assumption 2 means that $f_i(t, x, u)$, for $i = 1, \ldots, n$, has an upper bounded function (i.e., the right-hand side of (4)) depending only on $\overline{x}_i \in \mathbb{R}^i$ though $f_i(t, x, u)$ is related to $(t, x, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}$. Besides, as a clever appliance assisting in dipping up the intrinsic characteristics of system nonlinearities $f_i(t, x, u)$'s, Assumption 2 additionally includes several properties as follows.

(i) Assumption 2 can be in reality treated as a locally homogeneous-like growth condition of system nonlinearities $f_i(t, x, u)$'s. Precisely, in the case when $\omega_i = \omega$ for all $i = 1, \ldots, n$, it is easy to derive that, on any compact set $\mathbb{U} \subset \mathbb{R}^n$, the right-hand side of (4) degenerates into a (weighted) homogeneous function in regard to the dilation weight $(\sigma_1, \ldots, \sigma_i)$, thereby working as a homogeneous upper bound of the nonlinearities $f_i(t, x, u)$; i.e., on any compact set $\mathbb{U} \subset \mathbb{R}^n$, the relation (4) along with with $c = \sup_{(\overline{x}_i, x_{i+1}, \ldots, x_n) \in \mathbb{U}} \overline{f}_i(\overline{x}_i)$ becomes

$$|f_i(t, x, u)| \leq c \left(|x_1|^{\frac{\sigma_i + \omega}{\sigma_1}} + \cdots + |x_i|^{\frac{\sigma_i + \omega}{\sigma_i}} \right) \tag{6}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{U} \times \mathbb{R}$, where the right-hand side of (6) forms a homogeneous function with respect to the dilation weight $(\sigma_1, \ldots, \sigma_i)$; such a degenerated relation (6) has been widely stated/used in the literature [7,9,13,16,30]. Including the homogeneous one as a special case, the relation (4) not only retains genetically the spirit of weighted homogeneity in depicting the upper bounds of nonlinearities $f_i(t, x, u)$'s but also widens the applicability of Assumption 2 to more general circumstances.

(ii) Assumption 2 holds the flexibility arising from the degree of freedom in selecting the monotone parameters $\omega_n \leq \cdots \leq \omega_2 \leq \omega_1$ so that it broadly encompasses diverse assumptions stated in existing studies. For example, if the system power p_i's are all larger than one, that is, $p_i \geq 1$ for all $i = 1, \ldots, n$, Assumption 2 boils down to Assumption 2.1 in [7] when letting $\omega_i = \omega$ (a fixed real constant) for all $i = 1, \ldots, n$; in addition, if an extra monotone relation $p_1 \geq p_2 \geq \cdots \geq p_n \geq 1$ is satisfied, Assumption 2 becomes exactly Assumption 1 in [6] by choosing $\omega_i = p_i - 1$. In the case where the system power p_i's are identical to one and ω_i's are all set to be zero, i.e., $p_i = 1$ and $\omega_i = 0$ for all $i = 1, \ldots, n$, Assumption 2 directly reduces to the one used in Theorem 1 of the study [8].

(iii) Assumption 2 can be fulfilled by $f_i(t, x, u)$'s with certain smoothness properties and structural conditions. More specifically, when the system nonlinearities $f_i(t, x, u)$'s depend only on $\overline{x}_i \in \mathbb{R}^i$ and are continuously differentiable with $f_i(0) = 0$ for all $i = 1, \ldots, n$, such as the nonlinearities included in the under-actuated, weakly coupled, unstable mechanical system [5] and the liquid-level system [47], one can always find, in virtue of the Taylor expansion theorem [48], a smooth function $\hat{f}_i : \mathbb{R}^i \to \mathbb{R}_+$ such that

$$|f_i(t, x, u)| \leq \hat{f}_i(\overline{x}_i)\, (|x_1| + \cdots + |x_i|) \tag{7}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}$; this inequality is involved in the formulation of (4) with $(\sigma_i + \omega_i)/\sigma_j = 1$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, i$. If, in addition, there holds $(\sigma_i + \omega_i)/\sigma_j \leq 1$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, i$, one can also verify that the relation $|x_j| \leq \tilde{f}_{ij}(x_j)|x_j|^{(\sigma_i + \omega_i)/\sigma_j}$ is true for all $x_j \in \mathbb{R}$ and for all $i = 1, \ldots, n$ and $j = 1, \ldots, i$, where $\tilde{f}_{ij} : \mathbb{R} \to \mathbb{R}_+$ is a smooth function; with this fact in mind, the inequality (7) readily becomes (4).

3 Main results

With the aid of Assumptions 1 and 2, this section is dedicated to developing and presenting a new approach by which a continuous state feedback fixed-time stabilizer along with a *tangent*-type asymmetric BLF can be systematically constructed, thereby successfully guaranteeing the achievement of a pre-specified output constraint $-\varepsilon_L < y(t) = x_1(t) < \varepsilon_U$ for all $t \geq t_0$ with ε_L and ε_U being positive real constants. To be more specific, the development procedure is divided into two phases. It begins with organizing a new *tangent*-type barrier function as an intermediate/unformed design ingredient by delicately extracting and using the inherent characteristics of system nonlinearities. In the second phase, with an artful implantation of the presented barrier function along with an exquisite utilization of the intrinsic features of signum functions, the adding a power integrator technique is skillfully renovated to establish a novel approach that guides us, in a systematic fashion, in constructing a *tangent*-type asymmetric BLF together with a continuous state feedback controller capable of fulfilling simultaneously the fixed-time stabilization task and the requirement of the pre-specified asymmetric constraint on the system output.

3.1 Design of a *tangent*-type barrier function

Taking into consideration the pre-specified asymmetric output constraint $-\varepsilon_L < y(t) = x_1(t) < \varepsilon_U$ for all $t \geq t_0$ with two positive real constants ε_L and ε_U, and exploiting the intrinsic feature of system nonlinearities characterized by Assumption 2, we set $\eta \geq \mu \geq \max_{1\leq i\leq n}\{\sigma_i\} \geq 1$ and design $V_T : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ as

$$V_T(x_1) = \gamma(x_1)\frac{2\varepsilon_U^{2\eta-\omega_1}}{(2\eta-\omega_1)\pi}\tan\left(\frac{|x_1|^{2\eta-\omega_1}\pi}{2\varepsilon_U^{2\eta-\omega_1}}\right) + (1-\gamma(x_1))\frac{2\varepsilon_L^{2\eta-\omega_1}}{(2\eta-\omega_1)\pi}\tan\left(\frac{|x_1|^{2\eta-\omega_1}\pi}{2\varepsilon_L^{2\eta-\omega_1}}\right) \tag{8}$$

where $\gamma : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ is of the form

$$\gamma(x_1) = \begin{cases} 1 \text{ if } & x_1 > 0 \\ 0 \text{ if } & x_1 \leq 0 \end{cases}.$$

By the structure of $V_T(x_1)$, it is clear that $V_T(x_1)$ is positive definite on $\mathbb{M}_1(\varepsilon_L, \varepsilon_U)$ and meets $V_T(x_1) \to \infty$ as $x_1 \to \partial\mathbb{M}_1(\varepsilon_L, \varepsilon_U)$. Additionally, a straightforward calculation shows that the derivative of $V_T(x_1)$ with respect to x_1 is

$$\frac{\partial V_T(x_1)}{\partial x_1} = \Upsilon(x_1)\lceil x_1\rfloor^{2\eta-\omega_1-1} \tag{9}$$

for all $x_1 \in \mathbb{M}_1(\varepsilon_L, \varepsilon_U)$, where $\Upsilon : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to [1, \infty)$ is continuous and has the following form

$$\Upsilon(x_1) = \gamma(x_1)\sec^2\left(\frac{|x_1|^{2\eta-\omega_1}\pi}{2\varepsilon_U^{2\eta-\omega_1}}\right) + (1-\gamma(x_1))\sec^2\left(\frac{|x_1|^{2\eta-\omega_1}\pi}{2\varepsilon_L^{2\eta-\omega_1}}\right). \tag{10}$$

Because of the continuity of $\Upsilon(x_1)$ and $\lceil x_1\rfloor^{2\eta-\omega_1-1}$ on $\mathbb{M}_1(\varepsilon_L, \varepsilon_U)$, the function $V_T(x_1)$ described in (8) is continuously differentiable and, according to Remark 2, is indeed a *tangent*-type barrier function with the asymmetry coming from the possible deviation between ε_L and ε_U. As will be explicitly performed later, $V_T(x_1)$ serves as an intermediate design ingredient and also a key role for deriving the *tangent*-type asymmetric BLF and renovating the adding a power integrator technique with a view to achieving both the fixed-time stabilization and the demand of the pre-specified asymmetric output constraints.

Remark 5 Some distinctive traits/merits behind the presented *tangent*-type barrier function $V_T(x_1)$ are revealed as follows.

(i) The design and construction of $V_T(x_1)$ are fundamentally in connection with the innate essence of system nonlinearities $f_i(t, x, u)$'s. More specifically, with the help of Assumption 2, the intrinsic attributes of system nonlinearities $f_i(t, x, u)$'s are skillfully dipped up and deposited/stored potentially in the parameters σ_i's and ω_i's. By completely taking into account those parameters σ_i's and ω_i's, the *tangent*-type barrier function $V_T(x_1)$ is constructed as an unformed design ingredient fully absorbing the inherent features of $f_i(t, x, u)$'s so that the subsequent *tangent*-type asymmetric BLF induced by the proposed approach later, which includes as a built-in core the *tangent*-type barrier function $V_T(x_1)$, attains and inherits the dynamic characteristics of system (1) while offering the feasibility of achieving fixed-time stabilization for system (1).

(ii) Because of the elaborate absorption of the intrinsic essence of system nonlinearities $f_i(t, x, u)$'s, the *tangent*-type barrier function $V_T(x_1)$ acquires a particular structure such that the barrier nature of $V_T(x_1)$ will diminish spontaneously and dwindle away when the constraints on the output tend to infinity. To be more specific, if the pre-specified output constraints are deliberately assigned to be infinity, that is, $\varepsilon_L = \varepsilon_U = \varepsilon$ with $\varepsilon \to \infty$, in order to reflect correspondingly the circumstance that the constraints are no longer necessary/imperative, then one obtains

$$\lim_{\varepsilon\to\infty} V_T(x_1) = \frac{1}{(2\eta-\omega_1)}|x_1|^{2\eta-\omega_1} =: V_c(x_1)$$

which is clearly a continuously differentiable function without any barrier (namely, $V_c(x_1) \to \infty$ only if $|x_1| \to \infty$). As will be shown later, this property of $V_T(x_1)$ further endows, under the presented approach, the resultant *tangent*-type asymmetric BLF with the usability in unconstrained cases (also, see Remark 8).

3.2 Design of a fixed-time stabilizing controller

After completing the design of $V_T(x_1)$ as an intermediate ingredient, we are now in a position to show the development of our main approach which guides one in constructing a fixed-time stabilizer able to ensure the fulfillment of pre-specified output constraints. Details are presented in the theorem below and its proof.

Theorem 1 *Under Assumptions 1 and 2, there exists a continuous state feedback stabilizer for system (1) such that for any $x(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$, every trajectory (solution) $x(t)$ of system (1) starting at $x(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ satisfies the following*

(i) $x(t)$ is forward complete; that is, $x(t)$ is defined on $[t_0, \infty)$
(ii) the output constraint is fulfilled; i.e., $-\varepsilon_L < y(t) = x_1(t) < \varepsilon_U$ for all $t \geq t_0$ where ε_L and ε_U are pre-specified positive real constants
(iii) $x(t)$ converges to the origin in fixed time; namely, $x(t) \to 0$ in finite time and $x(t) = 0$ for all $t \geq T_r$ with $T_r \in (t_0, \infty)$ being an upper bound of the settling (convergence) time and independent of $x(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ and $t_0 \in \mathbb{R}_+$.

Proof The proof inclusive of the methodology of designing a stabilizer is separated into two parts. With a skillful implantation of the presented barrier function $V_T(x_1)$ as well as a delicate utilization of the intrinsic traits of signum functions, a recursive (systematic) approach is explicitly presented in the first part to exhibit the construction of a continuous state feedback controller and validate the requirement of output constraints. Under the presented controller, the second part is concerned with the verification of the fixed-time state convergence of the closed-loop system and therefore confirms the validity of our approach.

Part I—Construction of a continuous state feedback fixed-time stabilizing controller

Step 1: To begin with, we let $\xi_1(x_1) = \lceil x_1 \rceil^{\mu/\sigma_1}$ and define $V_1 : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ as $V_1(x_1) = V_T(x_1)$ with $V_T(x_1)$ being exactly the *tangent*-type barrier function given by (8); thus, $V_1(x_1)$ is positive definite and continuously differentiable on $\mathbb{M}_1(\varepsilon_L, \varepsilon_U)$. By virtue of Assumption 2 and (9), it follows from system (1) that

$$\begin{aligned}\dot{V}_1(x_1) &= \frac{\partial V_1(x_1)}{\partial x_1}\dot{x}_1\\ &= \Upsilon(x_1)\lceil x_1\rceil^{2\eta-\omega_1-1}\left(d_1(t,x)x_2^{p_1} + f_1(t,x,u)\right)\\ &\leq d_1(t,x)\Upsilon(x_1)\lceil \xi_1(x_1)\rceil^{\frac{2\eta-\omega_1-1}{\mu}}x_2^{p_1}\\ &\quad + \Upsilon(x_1)\overline{f}_1(x_1)|\xi_1(x_1)|^{\frac{2\eta}{\mu}} \end{aligned} \tag{11}$$

for all[8] $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$, where $\Upsilon(x_1) \geq 1$ for all $x_1 \in \mathbb{M}_1(\varepsilon_L, \varepsilon_U)$ is defined by (10). Let $\beta > 0$ and $1 < \theta < 2$ be two real constants that are free adjustable design parameters. Then, by choosing $x_2^* : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ as a continuous virtual controller below

$$x_2^*(x_1) = -g_1(x_1)\lceil \xi_1(x_1)\rceil^{\frac{\sigma_2}{\mu}} \tag{12}$$

with

$$g_1^{p_1}(x_1) = \frac{n\beta + \psi_1(x_1)\beta + \overline{f}_1(x_1)}{\underline{d}_1(x_1)}$$

where $g_1 : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to (0, \infty)$ and $\psi_1 : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ are both smooth functions, and $\psi_1(x_1)$ additionally suits the relation[9]

$$|\xi_1(x_1)|^{\frac{2\eta(\theta-1)-\omega_1\theta}{\mu}} \leq \psi_1(x_1)$$

[8] Because the functions on the right-hand side of (11) depend on $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$, the valid region of the inequality (11) is explicitly presented.

[9] For any real constant $1 < \theta < 2$, the existence of the smooth function $\psi_1(x_1)$ is guaranteed by [49] due the continuity of $|\xi_1(x_1)|^{(2\eta(\theta-1)-\omega_1\theta)/\mu}$ with regard to x_1. Notably, it is shown in [49, Theorem 6.21, p. 136] that for any continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ one can always find a smooth function $\overline{\varphi} : \mathbb{R}^n \to \mathbb{R}_+$ such that $|\varphi(s)| \leq \overline{\varphi}(s)$ for all $s \in \mathbb{R}^n$. This truth will be used repeatedly in this article.

for all $x_1 \in \mathbb{M}_1(\varepsilon_L, \varepsilon_U)$, it can be derived from Assumption 1 and (11) that

$$
\begin{aligned}
&\dot{V}_1(x_1) \\
&\le d_1(t,x)\Upsilon(x_1)\lceil\xi_1(x_1)\rceil^{\frac{2\eta-\omega_1-1}{\mu}} x_2^{*p_1}(x_1) \\
&\quad + \Upsilon(x_1)\overline{f}_1(x_1)|\xi_1(x_1)|^{\frac{2\eta}{\mu}} \\
&\quad + d_1(t,x)\Upsilon(x_1)\lceil\xi_1(x_1)\rceil^{\frac{2\eta-\omega_1-1}{\mu}}\left(x_2^{p_1}-x_2^{*p_1}(x_1)\right) \\
&\le -\Upsilon(x_1)n\beta|\xi_1(x_1)|^{\frac{2\eta}{\mu}} - \Upsilon(x_1)\beta|\xi_1(x_1)|^{\frac{2\eta\theta-\omega_1\theta}{\mu}} \\
&\quad + d_1(t,x)\Upsilon(x_1)\lceil\xi_1(x_1)\rceil^{\frac{2\eta-\omega_1-1}{\mu}}\left(x_2^{p_1}-x_2^{*p_1}(x_1)\right)
\end{aligned}
\tag{13}
$$

for all $(t,x,u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L,\varepsilon_U) \times \mathbb{R}$. Interestingly, if $n = 1$ and $x_2 = u$ for system (1), the controller $u = x_2^*(x_1)$ being unrelated to the function $\Upsilon(x_1)$ is such that the inequality (13) turns into $\dot{V}_1(x_1) \le -\Upsilon(x_1)\beta |\xi_1(x_1)|^{2\eta/\mu} - \Upsilon(x_1)\beta|\xi_1(x_1)|^{(2\eta\theta-\omega_1\theta)/\mu} \le 0$ for all $(t,x,u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L,\varepsilon_U) \times \mathbb{R}$, which by Lemma 1 indicates that the output constraint $-\varepsilon_L < y(t) = x_1(t) < \varepsilon_U$ is surely achieved for all $t \ge t_0$. This also discloses the fact that when system (1) is a scalar system (i.e., $n = 1$), the stabilization in view of output constraints is always achievable by a simple controller $u = x_2^*(x_1)$ which is exclusive of the extra control effort produced by the function $\Upsilon(x_1)$. In contrast, in the case where $n \ge 2$, as described later, the term $\Upsilon(x_1)$ is necessarily contained somehow in controller gains for responding to the domination of output constraints.

Step 2: To continue the design for $n \ge 2$, here we pick the case $n = 2$ and let $\xi_2(\overline{x}_2) = \lceil x_2\rceil^{\mu/\sigma_2} - \lceil x_2^*(x_1)\rceil^{\mu/\sigma_2}$ where $x_2^*(x_1)$ is presented in Step 1. We also consider $V_2 : \mathbb{M}_2(\varepsilon_L,\varepsilon_U) \to \mathbb{R}$ defined as $V_2(\overline{x}_2) = V_1(x_1) + W_2(\overline{x}_2)$ with $W_2 : \mathbb{M}_2(\varepsilon_L,\varepsilon_U) \to \mathbb{R}$ being of the following form

$$
W_2(\overline{x}_2) = \int_{x_2^*(x_1)}^{x_2} \left\lceil \lceil s\rceil^{\frac{\mu}{\sigma_2}} - \lceil x_2^*(x_1)\rceil^{\frac{\mu}{\sigma_2}}\right\rceil^{\frac{2\eta-\omega_2-\sigma_2}{\mu}} ds.
$$

By the definitions of $V_1(x_1)$ and $W_2(\overline{x}_2)$, it is clear that $V_2(\overline{x}_2)$ is positive definite. In addition, a simple derivation according to the standard formulas of the partial derivatives $\partial W_2(\overline{x}_2)/\partial x_1$ and $\partial W_2(\overline{x}_2)/\partial x_2$ affirms that $V_2(\overline{x}_2)$ is continuously differentiable on $\mathbb{M}_2(\varepsilon_L,\varepsilon_U)$; specifically, it confirms for all $\overline{x}_2 \in \mathbb{M}_2(\varepsilon_L,\varepsilon_U)$

$$
\begin{aligned}
\frac{\partial W_2(\overline{x}_2)}{\partial x_1} &= \frac{\partial \lceil x_2^*(x_1)\rceil^{\frac{\mu}{\sigma_2}}}{\partial x_1}\left(\frac{\omega_2+\sigma_2-2\eta}{\mu}\right)\times \\
&\qquad \int_{x_2^*(x_1)}^{x_2}\left|\lceil s\rceil^{\frac{\mu}{\sigma_2}} - \lceil x_2^*(x_1)\rceil^{\frac{\mu}{\sigma_2}}\right|^{\frac{2\eta-\omega_2-\sigma_2}{\mu}-1} ds \\
\frac{\partial W_2(\overline{x}_2)}{\partial x_2} &= \left\lceil \lceil x_2\rceil^{\frac{\mu}{\sigma_2}} - \lceil x_2^*(x_1)\rceil^{\frac{\mu}{\sigma_2}}\right\rceil^{\frac{2\eta-\omega_2-\sigma_2}{\mu}} \\
&= \lceil \xi_2(\overline{x}_2)\rceil^{\frac{2\eta-\omega_2-\sigma_2}{\mu}}
\end{aligned}
$$

where $\partial W_2(\overline{x}_2)/\partial x_1$ is bounded by the following

$$
\left|\frac{\partial W_2(\overline{x}_2)}{\partial x_1}\right| \le \rho_1(x_1)|\xi_1(x_1)|^{\frac{\mu-\sigma_1}{\mu}}|\xi_2(\overline{x}_2)|^{\frac{2\eta-\omega_2-\mu}{\mu}}
$$

for all $x_1 \in \mathbb{M}_1(\varepsilon_L,\varepsilon_U)$, with $\rho_1 : \mathbb{M}_1(\varepsilon_L,\varepsilon_U) \to \mathbb{R}_+$ being a smooth function.

Based on these facts and Assumptions 1 and 2, one can derive from system (1) and (13) that

$$
\begin{aligned}
\dot{V}_2(\overline{x}_2) &= \frac{\partial V_2(\overline{x}_2)}{\partial x_1}\dot{x}_1 + \frac{\partial V_2(\overline{x}_2)}{\partial x_2}\dot{x}_2 \\
&= \dot{V}_1(x_1) + \frac{\partial W_2(\overline{x}_2)}{\partial x_1}\dot{x}_1 + \frac{\partial W_2(\overline{x}_2)}{\partial x_2}\dot{x}_2 \\
&\le -\Upsilon(x_1)n\beta|\xi_1(x_1)|^{\frac{2\eta}{\mu}} - \Upsilon(x_1)\beta|\xi_1(x_1)|^{\frac{2\eta\theta-\omega_1\theta}{\mu}} \\
&\quad + d_2(t,x)\lceil\xi_2(\overline{x}_2)\rceil^{\frac{2\eta-\omega_2-\sigma_2}{\mu}} x_3^{p_2} \\
&\quad + d_1(t,x)\Upsilon(x_1)\lceil\xi_1(x_1)\rceil^{\frac{2\eta-\omega_1-1}{\mu}}\left(x_2^{p_1}-x_2^{*p_1}(x_1)\right) \\
&\quad + |\xi_2(\overline{x}_2)|^{\frac{2\eta-\omega_2-\sigma_2}{\mu}}\overline{f}_2(\overline{x}_2)\left(|x_1|^{\frac{\sigma_2+\omega_2}{\sigma_1}} + |x_2|^{\frac{\sigma_2+\omega_2}{\sigma_2}}\right) \\
&\quad + \rho_1(x_1)|\xi_1(x_1)|^{\frac{\mu-\sigma_1}{\mu}}|\xi_2(\overline{x}_2)|^{\frac{2\eta-\omega_2-\mu}{\mu}} \\
&\quad \times\left(\overline{d}_1(x_1)|x_2|^{p_1} + \overline{f}_1(x_1)|x_1|^{\frac{\sigma_1+\omega_1}{\sigma_1}}\right)
\end{aligned}
\tag{14}
$$

for all $(t,x,u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L,\varepsilon_U) \times \mathbb{R}$. To proceed forward, it is necessary to pursue the estimations of the last three terms on the right-hand side of (14); this can be performed as follows.

Noticing $(\sigma_1+\omega_1)/\mu < 1$ and Lemma 2, we first have

$$
\left(x_2^{p_1} - x_2^{*p_1}(x_1)\right) \le 2^{1-\frac{\sigma_1+\omega_1}{\mu}}|\xi_2(\overline{x}_2)|^{\frac{\sigma_1+\omega_1}{\mu}}
$$

for all $\overline{x}_2 \in \mathbb{M}_2(\varepsilon_L,\varepsilon_U)$. Additionally, the continuity of $\Upsilon(x_1)$ with regard to x_1 implies that there exists a smooth function $\hat{\Upsilon} : \mathbb{M}_1(\varepsilon_L, \varepsilon_U) \to [2, \infty)$ such that $\Upsilon(x_1) \le \hat{\Upsilon}(x_1)$ for all $x_1 \in \mathbb{M}_1(\varepsilon_L, \varepsilon_U)$. These two facts together with Assumption 1 and Lemma 3 yield

$$\begin{aligned} &d_1(t,x)\Upsilon(x_1)\lceil \xi_1(x_1)\rfloor^{\frac{2\eta-\omega_1-1}{\mu}} \left(x_2^{p_1} - x_2^{*p_1}(x_1)\right) \\ &\le 2^{1-\frac{\sigma_1+\omega_1}{\mu}} \Upsilon(x_1)\overline{d}_1(x_1)\,|\xi_1(x_1)|^{\frac{2\eta-\omega_1-1}{\mu}}\,|\xi_2(\overline{x}_2)|^{\frac{\sigma_1+\omega_1}{\mu}} \\ &\le \frac{1}{3}\beta\Upsilon(x_1)\,|\xi_1(x_1)|^{\frac{2\eta}{\mu}} + \alpha_2(\overline{x}_2)\hat{\Upsilon}(x_1)\,|\xi_2(\overline{x}_2)|^{\frac{2\eta}{\mu}} \end{aligned} \tag{15}$$

for all $(t, x) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ where $\alpha_2 : \mathbb{M}_2(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ is a smooth function.

Next, by Lemma 2 and the definitions of $\xi_1(x_1)$ and $\xi_2(\overline{x}_2)$, one has the relations $|x_1| = |\xi_1(x_1)|^{\sigma_1/\mu}$ and

$$|x_2| \le g_1(x_1)|\xi_1(x_1)|^{\frac{\sigma_2}{\mu}} + 2^{1-\frac{\sigma_2}{\mu}}|\xi_2(\overline{x}_2)|^{\frac{\sigma_2}{\mu}}$$

for all $\overline{x}_2 \in \mathbb{M}_2(\varepsilon_L, \varepsilon_U)$. It further follows from Lemma 4 that $|x_1|^{(\sigma_1+\omega_1)/\sigma_1} = |\xi_1(x_1)|^{(\sigma_1+\omega_1)/\mu}$ and

$$\begin{aligned} |x_2|^{p_1} \le{}& \left(2^{p_1-1}+1\right) g_1^{p_1}(x_1)\,|\xi_1(x_1)|^{\frac{\sigma_1+\omega_1}{\mu}} \\ &+ \left(2^{p_1-1}+1\right) 2^{p_1-\frac{p_1\sigma_2}{\mu}}\,|\xi_2(\overline{x}_2)|^{\frac{\sigma_1+\omega_1}{\mu}} \end{aligned}$$

for all $\overline{x}_2 \in \mathbb{M}_2(\varepsilon_L, \varepsilon_U)$. Keeping these in mind and utilizing Lemma 3, we further obtain

$$\begin{aligned} &\rho_1(x_1)|\xi_1(x_1)|^{\frac{\mu-\sigma_1}{\mu}}|\xi_2(\overline{x}_2)|^{\frac{2\eta-\omega_2-\mu}{\mu}} \\ &\quad\times\left(\overline{d}_1(x_1)|x_2|^{p_1} + \overline{f}_1(x_1)|x_1|^{\frac{\sigma_1+\omega_1}{\sigma_1}}\right) \\ &\le 2^{p_1-\frac{p_1\sigma_2}{\mu}}\left(2^{p_1-1}+1\right)\overline{d}_1(x_1)\rho_1(x_1) \\ &\quad\times|\xi_1(x_1)|^{\frac{\mu-\sigma_1}{\mu}}|\xi_2(\overline{x}_2)|^{\frac{2\eta-\mu+\sigma_1+\omega_1-\omega_2}{\mu}} \\ &\quad+\left(2^{p_1-1}+1\right)\overline{d}_1(x_1)\rho_1(x_1)g_1^{p_1}(x_1) \\ &\quad\times|\xi_1(x_1)|^{\frac{\mu+\omega_1}{\mu}}|\xi_2(\overline{x}_2)|^{\frac{2\eta-\omega_2-\mu}{\mu}} \\ &\quad+\rho_1(x_1)\overline{f}_1(x_1)|\xi_1(x_1)|^{\frac{\mu+\omega_1}{\mu}}|\xi_2(\overline{x}_2)|^{\frac{2\eta-\omega_2-\mu}{\mu}} \\ &\le \frac{1}{3}\beta\Upsilon(x_1)|\xi_1(x_1)|^{\frac{2\eta}{\mu}} + \tilde{\alpha}_2(\overline{x}_2)\hat{\Upsilon}(x_1)\,|\xi_2(\overline{x}_2)|^{\frac{2\eta}{\mu}} \end{aligned} \tag{16}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$, where $\tilde{\alpha}_2 : \mathbb{M}_2(\varepsilon_l, \varepsilon_u) \to \mathbb{R}_+$ is a smooth function.

Under the same line of argument, it is easy to see that $|x_1|^{(\sigma_2+\omega_2)/\sigma_1} = |\xi_1(x_1)|^{(\sigma_2+\omega_2)/\mu}$ and

$$\begin{aligned} |x_2|^{\frac{\sigma_2+\omega_2}{\sigma_2}} \le{}& g_1^{\frac{\sigma_2+\omega_2}{\sigma_2}}(x_1)\,|\xi_1(x_1)|^{\frac{\sigma_2+\omega_2}{\mu}} \\ &+ 2^{\frac{(\mu-\sigma_2)(\sigma_2+\omega_2)}{\mu\sigma_2}}\,|\xi_2(\overline{x}_2)|^{\frac{\sigma_2+\omega_2}{\mu}} \end{aligned}$$

for all $\overline{x}_2 \in \mathbb{M}_2(\varepsilon_L, \varepsilon_U)$. By employing these relations and Lemma 3, it is not difficult to derive

$$\begin{aligned} &|\xi_2(\overline{x}_2)|^{\frac{2\eta-\omega_2-\sigma_2}{\mu}}\overline{f}_2(\overline{x}_2)\left(|x_1|^{\frac{\sigma_2+\omega_2}{\sigma_1}} + |x_2|^{\frac{\sigma_2+\omega_2}{\sigma_2}}\right) \\ &\le \overline{f}_2(\overline{x}_2)|\xi_1(x_1)|^{\frac{\sigma_2+\omega_2}{\mu}}|\xi_2(\overline{x}_2)|^{\frac{2\eta-\omega_2-\sigma_2}{\mu}} \\ &\quad+2^{\frac{(\mu-\sigma_2)(\sigma_2+\omega_2)}{\mu\sigma_2}}\overline{f}_2(\overline{x}_2)\,|\xi_2(\overline{x}_2)|^{\frac{2\eta}{\mu}} \\ &\quad+\overline{f}_2(\overline{x}_2)g_1^{\frac{\sigma_2+\omega_2}{\sigma_2}}(x_1)\,|\xi_1(x_1)|^{\frac{\sigma_2+\omega_2}{\mu}}\,|\xi_2(\overline{x}_2)|^{\frac{2\eta-\omega_2-\sigma_2}{\mu}} \\ &\le \frac{1}{3}\beta\Upsilon(x_1)\,|\xi_1(x_1)|^{\frac{2\eta}{\mu}} + \hat{\alpha}_2(\overline{x}_2)\hat{\Upsilon}(x_1)\,|\xi_2(\overline{x}_2)|^{\frac{2\eta}{\mu}} \end{aligned} \tag{17}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$, where $\hat{\alpha}_2 : \mathbb{M}_2(\varepsilon_l, \varepsilon_u) \to \mathbb{R}_+$ is a smooth function.

Substituting the estimations provided by (15)–(17) into (14) yields

$$\begin{aligned} &\dot{V}_2(\overline{x}_2) \\ &\le -\Upsilon(x_1)(n-1)\beta|\xi_1(x_1)|^{\frac{2\eta}{\mu}} - \Upsilon(x_1)\beta|\xi_1(x_1)|^{\frac{2\eta\theta-\omega_1\theta}{\mu}} \\ &\quad+\left(\alpha_2(\overline{x}_2)+\tilde{\alpha}_2(\overline{x}_2)+\hat{\alpha}_2(\overline{x}_2)\right)\hat{\Upsilon}(x_1)\,|\xi_2(\overline{x}_2)|^{\frac{2\eta}{\mu}} \\ &\quad+d_2(t,x)\lceil\xi_2(\overline{x}_2)\rfloor^{\frac{2\eta-\omega_2-\sigma_2}{\mu}}x_3^{p_2} \end{aligned} \tag{18}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$. A continuous virtual controller $x_3^* : \mathbb{M}_2(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ being composed of

$$x_3^*(\overline{x}_2) = -g_2(\overline{x}_2)\hat{\Upsilon}^{\frac{1}{p_2}}(x_1)\lceil\xi_2(\overline{x}_2)\rfloor^{\frac{\sigma_3}{\mu}}$$

with a smooth gain function $g_2 : \mathbb{M}_2(\varepsilon_L, \varepsilon_U) \to (0, \infty)$ as below

$$g_2^{p_2}(\overline{x}_2) = \frac{(n-1)\beta+\psi_2(\overline{x}_2)\beta+\alpha_2(\overline{x}_2)+\tilde{\alpha}_2(\overline{x}_2)+\hat{\alpha}_2(\overline{x}_2)}{\underline{d}_2(\overline{x}_2)}$$

in which $\psi_2 : \mathbb{M}_2(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ is a smooth function satisfying

$$|\xi_2(\overline{x}_2)|^{\frac{2\eta(\theta-1)-\omega_2\theta}{\mu}} \le \psi_2(\overline{x}_2)$$

for all $\overline{x}_2 \in \mathbb{M}_2(\varepsilon_L, \varepsilon_U)$, along with Assumption 1 is such that (18) becomes

$$\begin{aligned}
&\dot{V}_2(\overline{x}_2) \\
&\le -\Upsilon(x_1)(n-1)\beta|\xi_1(x_1)|^{\frac{2\eta}{\mu}} - \Upsilon(x_1)\beta|\xi_1(x_1)|^{\frac{2\eta\theta-\omega_1\theta}{\mu}} \\
&\quad + d_2(t,x)\lceil\xi_2(\overline{x}_2)\rfloor^{\frac{2\eta-\omega_2-\sigma_2}{\mu}} x_3^{*p_2}(\overline{x}_2) \\
&\quad + d_2(t,x)\lceil\xi_2(\overline{x}_2)\rfloor^{\frac{2\eta-\omega_2-\sigma_2}{\mu}}(x_3^{p_2} - x_3^{*p_2}(\overline{x}_2)) \\
&\quad + \left(\alpha_2(\overline{x}_2) + \tilde{\alpha}_2(\overline{x}_2) + \hat{\alpha}_2(\overline{x}_2)\right)\hat{\Upsilon}(x_1)|\xi_2(\overline{x}_2)|^{\frac{2\eta}{\mu}} \\
&\le -\Upsilon(x_1)(n-1)\beta\left(|\xi_1(x_1)|^{\frac{2\eta}{\mu}} + |\xi_2(\overline{x}_2)|^{\frac{2\eta}{\mu}}\right) \\
&\quad - \Upsilon(x_1)\beta\left(|\xi_1(x_1)|^{\frac{2\eta\theta-\omega_1\theta}{\mu}} + |\xi_2(\overline{x}_2)|^{\frac{2\eta\theta-\omega_2\theta}{\mu}}\right) \\
&\quad + d_2(t,x)\lceil\xi_2(\overline{x}_2)\rfloor^{\frac{2\eta-\omega_2-\sigma_2}{\mu}}\left(x_3^{p_2} - x_3^{*p_2}(\overline{x}_2)\right) \qquad (19)
\end{aligned}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$. It can be observed that compared to $x_2^*(x_1)$ described by (12) the virtual controller $x_3^*(\overline{x}_2)$ clearly contains an extra term $\hat{\Upsilon}^{1/p_2}(x_1)$ that assures $\Upsilon(x_1) \le \hat{\Upsilon}(x_1)$ for all $x_1 \in \mathbb{M}_1(\varepsilon_L, \varepsilon_U)$ so as to produce/activate adequate control effort in dominating output constraints.

Inductive Step: At step $k-1$ with $k = 3, \ldots, n$, we suppose that there exist a positive definite and continuously differentiable function $V_{k-1} : \mathbb{M}_{k-1}(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ and a set of continuous virtual controllers $x_1^* = 0$ and $x_i^* : \mathbb{M}_{i-1}(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ with $i = 2, \ldots, k$ defined as

$$x_2^*(x_1) = -g_1(x_1)\lceil\xi_1(x_1)\rfloor^{\frac{\sigma_2}{\mu}}$$

and

$$x_i^*(\overline{x}_{i-1}) = -g_{i-1}(\overline{x}_{i-1})\hat{\Upsilon}^{\frac{1}{p_{i-1}}}(x_1)\lceil\xi_{i-1}(\overline{x}_{i-1})\rfloor^{\frac{\sigma_i}{\mu}}$$

for all $i = 3, \ldots, k$, where $g_i : \mathbb{M}_i(\varepsilon_L, \varepsilon_U) \to (0, \infty)$ are smooth gain functions and

$$\xi_i(\overline{x}_i) = \lceil x_i\rfloor^{\frac{\mu}{\sigma_i}} - \lceil x_i^*(\overline{x}_{i-1})\rfloor^{\frac{\mu}{\sigma_i}}$$

for all $i = 1, \ldots, k-1$, such that the derivative of $V_{k-1}(\overline{x}_{k-1})$ along system (1) is

$$\begin{aligned}
\dot{V}_{k-1}(\overline{x}_{k-1}) &= \sum_{i=1}^{k-1}\frac{\partial V_{k-1}(\overline{x}_{k-1})}{\partial x_i}\dot{x}_i \\
&\le -\Upsilon(x_1)(n+2-k)\beta\sum_{i=1}^{k-1}|\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}} \\
&\quad - \Upsilon(x_1)\beta\sum_{i=1}^{k-1}|\xi_i(\overline{x}_i)|^{\frac{2\eta\theta-\omega_i\theta}{\mu}} \\
&\quad + d_{k-1}(t,x)\lceil\xi_{k-1}(\overline{x}_{k-1})\rfloor^{\frac{2\eta-\omega_{k-1}-\sigma_{k-1}}{\mu}} \\
&\quad \times \left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}(\overline{x}_{k-1})\right) \qquad (20)
\end{aligned}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$. Obviously, the inequality (20) is precisely identical to (19) in the situation $k = 3$. Subsequently, we shall verify that at step k with $k = 3, \ldots, n$, there also exists a continuous virtual controller $x_{k+1}^* : \mathbb{M}_k(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ such that (20) is valid as well for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$ and for a function $V_k : \mathbb{M}_k(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ being positive definite and continuously differentiable. For this purpose, we consider $V_k(\overline{x}_k) = V_{k-1}(\overline{x}_{k-1}) + W_k(\overline{x}_k)$ with $W : \mathbb{M}_k(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ defined as

$$W_k(\overline{x}_k) = \int_{x_k^*(\overline{x}_{k-1})}^{x_k}\left\lceil\lceil s\rfloor^{\frac{\mu}{\sigma_k}} - \lceil x_k^*(\overline{x}_{k-1})\rfloor^{\frac{\mu}{\sigma_k}}\right\rfloor^{\frac{2\eta-\omega_k-\sigma_k}{\mu}} ds.$$

By letting $\xi_k(\overline{x}_k) = \lceil x_k\rfloor^{\mu/\sigma_k} - \lceil x_k^*(\overline{x}_{k-1})\rfloor^{\mu/\sigma_k}$ similarly, the partial derivatives of $W_k(\overline{x}_k)$ in respect of x_i for all $i = 1, \ldots, k$ can be obtained through a simple verification as below

$$\begin{aligned}
\frac{\partial W_k(\overline{x}_k)}{\partial x_i} &= \frac{\partial\lceil x_k^*(\overline{x}_{k-1})\rfloor^{\frac{\mu}{\sigma_k}}}{\partial x_i}\left(\frac{\omega_k + \sigma_k - 2\eta}{\mu}\right) \\
&\quad \times \int_{x_k^*(\overline{x}_{k-1})}^{x_k}\left|\lceil s\rfloor^{\frac{\mu}{\sigma_k}} - \lceil x_k^*(\overline{x}_{k-1})\rfloor^{\frac{\mu}{\sigma_k}}\right|^{\frac{2\eta-\omega_k-\sigma_k}{\mu}-1} ds \\
\frac{\partial W_k(\overline{x}_k)}{\partial x_k} &= \left\lceil\lceil x_k\rfloor^{\frac{\mu}{\sigma_k}} - \lceil x_k^*(\overline{x}_{k-1})\rfloor^{\frac{\mu}{\sigma_k}}\right\rfloor^{\frac{2\eta-\omega_k-\sigma_k}{\mu}} \\
&= \lceil\xi_k(\overline{x}_k)\rfloor^{\frac{2\eta-\omega_k-\sigma_k}{\mu}}
\end{aligned}$$

for all $\overline{x}_k \in \mathbb{M}_k(\varepsilon_L, \varepsilon_U)$ and for all $i = 1, \ldots, k-1$; moreover, $\partial W_k(\overline{x}_k)/\partial x_i$ is indeed bounded in the sense that

$$\left|\frac{\partial W_k(\overline{x}_k)}{\partial x_i}\right| \le \left(\sum_{j=1}^{k-1}|\xi_j(\overline{x}_j)|^{\frac{\mu-\sigma_i}{\mu}}\right)$$

$$\times \rho_{k-1}(\overline{x}_{k-1})|\xi_k(\overline{x}_k)|^{\frac{2\eta-\omega_k-\mu}{\mu}}$$

for all $\overline{x}_k \in \mathbb{M}_k(\varepsilon_L, \varepsilon_U)$ and for all $i = 1, \ldots, k-1$, where $\rho_{k-1} : \mathbb{M}_{k-1}(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ is a smooth function. Then, from (20) one knows that the derivative of $V_k(\overline{x}_k)$ along system (1) is

$$\begin{aligned}
\dot{V}_k(\overline{x}_k) &= \sum_{i=1}^{k} \frac{\partial V_k(\overline{x}_k)}{\partial x_i}\dot{x}_i \\
&= \dot{V}_{k-1}(\overline{x}_{k-1}) + \sum_{i=1}^{k-1} \frac{\partial W_k(\overline{x}_k)}{\partial x_i}\dot{x}_i + \frac{\partial W_k(\overline{x}_k)}{\partial x_k}\dot{x}_k \\
&\le -\Upsilon(x_1)(n+2-k)\beta \sum_{i=1}^{k-1} |\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}} \\
&\quad - \Upsilon(x_1)\beta \sum_{i=1}^{k-1} |\xi_i(\overline{x}_i)|^{\frac{2\eta\theta-\omega_i\theta}{\mu}} \\
&\quad + d_k(t,x)\lceil \xi_k(\overline{x}_k)\rfloor^{\frac{2\eta-\omega_k-\sigma_k}{\mu}} x_{k+1}^{p_k} \\
&\quad + |\xi_k(\overline{x}_k)|^{\frac{2\eta-\omega_k-\sigma_k}{\mu}} \overline{f}_k(\overline{x}_k) \sum_{i=1}^{k} |x_i|^{\frac{\sigma_k+\omega_k}{\sigma_i}} \\
&\quad + d_{k-1}(t,x)\lceil \xi_{k-1}(\overline{x}_{k-1})\rfloor^{\frac{2\eta-\omega_{k-1}-\sigma_{k-1}}{\mu}} \\
&\quad \times \left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}(\overline{x}_{k-1})\right) \\
&\quad + \sum_{i=1}^{k-1} \frac{\partial W_k(\overline{x}_k)}{\partial x_i}\left(d_i(t,x)x_{i+1}^{p_i} + f_i(t,x,u)\right)
\end{aligned} \tag{21}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$. Following a similar line of argument in deducing (15)–(17), we can derive, respectively, the estimations of the last three terms on the right-hand side of (21) as follows

$$\begin{aligned}
&d_{k-1}(t,x)\lceil \xi_{k-1}(\overline{x}_{k-1})\rfloor^{\frac{2\eta-\omega_{k-1}-\sigma_{k-1}}{\mu}} \\
&\quad \times \left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}(\overline{x}_{k-1})\right) \\
&\quad \le \frac{1}{3}\beta\Upsilon(x_1)|\xi_{k-1}(\overline{x}_{k-1})|^{\frac{2\eta}{\mu}} + \alpha_k(\overline{x}_k)\hat{\Upsilon}(x_1)|\xi_k(\overline{x}_k)|^{\frac{2\eta}{\mu}}
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\sum_{i=1}^{k-1} \frac{\partial W_k(\overline{x}_k)}{\partial x_i}\left(d_i(t,x)x_{i+1}^{p_i} + f_i(t,x,u)\right) \\
&\quad \le \frac{1}{3}\beta\Upsilon(x_1)|\xi_{k-1}(\overline{x}_{k-1})|^{\frac{2\eta}{\mu}} + \frac{1}{2}\beta\Upsilon(x_1)\sum_{i=1}^{k-2}|\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}} \\
&\quad + \tilde{\alpha}_k(\overline{x}_k)\hat{\Upsilon}(x_1)|\xi_k(\overline{x}_k)|^{\frac{2\eta}{\mu}}
\end{aligned} \tag{23}$$

$$\begin{aligned}
&|\xi_k(\overline{x}_k)|^{\frac{2\eta-\omega_k-\sigma_k}{\mu}} \overline{f}_k(\overline{x}_k) \sum_{i=1}^{k} |x_i|^{\frac{\sigma_k+\omega_k}{\sigma_i}} \\
&\quad \le \frac{1}{3}\beta\Upsilon(x_1)|\xi_{k-1}(\overline{x}_{k-1})|^{\frac{2\eta}{\mu}} + \frac{1}{2}\beta\Upsilon(x_1)\sum_{i=1}^{k-2}|\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}} \\
&\quad + \hat{\alpha}_k(\overline{x}_k)\hat{\Upsilon}(x_1)|\xi_k(\overline{x}_k)|^{\frac{2\eta}{\mu}}
\end{aligned} \tag{24}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$, where $\alpha_k, \tilde{\alpha}_k, \hat{\alpha}_k : \mathbb{M}_k(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ are smooth functions. Substituting the three estimations (22)–(24) into (21) leads to

$$\begin{aligned}
\dot{V}_k(\overline{x}_k) \le& -\Upsilon(x_1)(n+1-k)\beta \sum_{i=1}^{k-1} |\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}} \\
& - \Upsilon(x_1)\beta \sum_{i=1}^{k-1} |\xi_i(\overline{x}_i)|^{\frac{2\eta\theta-\omega_i\theta}{\mu}} \\
& + d_k(t,x)\lceil \xi_k(\overline{x}_k)\rfloor^{\frac{2\eta-\omega_k-\sigma_k}{\mu}} x_{k+1}^{p_k} \\
& + \left(\alpha_k(\overline{x}_k) + \tilde{\alpha}_k(\overline{x}_k) + \hat{\alpha}_k(\overline{x}_k)\right)\hat{\Upsilon}(x_1)|\xi_k(\overline{x}_k)|^{\frac{2\eta}{\mu}}
\end{aligned}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$. Under Assumption 1, selecting a continuous virtual controller $x_{k+1}^* : \mathbb{M}_k(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ taking the form

$$x_{k+1}^*(\overline{x}_k) = -g_k(\overline{x}_k)\hat{\Upsilon}^{\frac{1}{p_k}}(x_1)\lceil \xi_k(\overline{x}_k)\rfloor^{\frac{\sigma_{k+1}}{\mu}} \tag{25}$$

with a smooth gain function $g_k : \mathbb{M}_k(\varepsilon_L, \varepsilon_U) \to (0, \infty)$ described as

$$\begin{aligned}
g_k^{p_k}(\overline{x}_k) =& \frac{1}{\underline{d}_k(\overline{x}_k)}\left((n-k+1)\beta + \psi_k(\overline{x}_k)\beta + \alpha_k(\overline{x}_k)\right. \\
& \left. + \tilde{\alpha}_k(\overline{x}_k) + \hat{\alpha}_k(\overline{x}_k)\right)
\end{aligned}$$

where $\psi_k : \mathbb{M}_k(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ is a smooth function fulfilling

$$|\xi_k(\overline{x}_k)|^{\frac{2\eta(\theta-1)-\omega_k\theta}{\mu}} \le \psi_k(\overline{x}_k)$$

for all $\overline{x}_k \in \mathbb{M}_k(\varepsilon_L, \varepsilon_U)$, implies that

$$\begin{aligned}
&\dot{V}_k(\overline{x}_k) \\
&\quad \le -\Upsilon(x_1)(n+1-k)\beta \sum_{i=1}^{k-1} |\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}}
\end{aligned}$$

$$
\begin{aligned}
&- \Upsilon(x_1)\beta \sum_{i=1}^{k-1} |\xi_i(\overline{x}_i)|^{\frac{2\eta\theta-\omega_i\theta}{\mu}} \\
&+ d_k(t,x)\lceil \xi_k(\overline{x}_k)\rceil^{\frac{2\eta-\omega_k-\sigma_k}{\mu}} x_{k+1}^{*p_k}(\overline{x}_k) \\
&+ d_k(t,x)\lceil \xi_k(\overline{x}_k)\rceil^{\frac{2\eta-\omega_k-\sigma_k}{\mu}} \left(x_{k+1}^{p_k} - x_{k+1}^{*p_k}(\overline{x}_k)\right) \\
&+ \left(\alpha_k(\overline{x}_k) + \tilde{\alpha}_k(\overline{x}_k) + \hat{\alpha}_k(\overline{x}_k)\right)\hat{\Upsilon}(x_1)|\xi_k(\overline{x}_k)|^{\frac{2\eta}{\mu}} \\
\le& -\Upsilon(x_1)(n+1-k)\beta \sum_{i=1}^{k} |\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}} \\
&- \Upsilon(x_1)\beta \sum_{i=1}^{k} |\xi_i(\overline{x}_i)|^{\frac{2\eta\theta-\omega_i\theta}{\mu}} \\
&+ d_k(t,x)\lceil \xi_k(\overline{x}_k)\rceil^{\frac{2\eta-\omega_k-\sigma_k}{\mu}} \left(x_{k+1}^{p_k} - x_{k+1}^{*p_k}(\overline{x}_k)\right)
\end{aligned} \tag{26}
$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$. Notably, the derivation so far solidly completes the inductive proof and validates that, under the continuous virtual controller $x_{k+1}^*(\overline{x}_k)$ shown in (25), the formula (20) still holds at step k with $k = 3, \dots, n$ for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$; this therefore allows us to conclude that the inequality (26) is true for all $k = 3, \dots, n$ and for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$.

Final Step: In accordance with the inductive proof, at step n where system (1) has $p_n = 1$ and $x_{n+1} = u$, we can find, respectively, a continuous virtual controller $x_{n+1}^* : \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ and a positive definite and continuously differentiable function $V_n : \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ as below

$$x_{n+1}^*(x) = -g_n(x)\hat{\Upsilon}(x_1)\lceil \xi_n(x)\rceil^{\frac{\sigma_{n+1}}{\mu}}$$

and

$$V_n(x_n) = V_{n-1}(\overline{x}_{n-1}) + W_n(x_n)$$

where

$$\xi_n(x) = \lceil x_n\rceil^{\mu/\sigma_n} - \lceil x_n^*(\overline{x}_{n-1})\rceil^{\mu/\sigma_n}$$

and $g_n : \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \to (0, \infty)$ is a smooth function designed as

$$g_n(x) = \frac{\beta + \psi_n(x)\beta + \alpha_n(x) + \tilde{\alpha}_n(x) + \hat{\alpha}_n(x)}{\underline{d}_n(x)}$$

with $\psi_n : \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \to \mathbb{R}_+$ being a smooth function satisfying

$$|\xi_n(x)|^{\frac{2\eta(\theta-1)-\omega_n\theta}{\mu}} \le \psi_n(x)$$

for all $x \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$, and $W_n : \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \to \mathbb{R}$ is a function set to be

$$W_n(x) = \int_{x_n^*(\overline{x}_{n-1})}^{x_n} \left\lceil \lceil s\rceil^{\frac{\mu}{\sigma_n}} - \lceil x_n^*(\overline{x}_{n-1})\rceil^{\frac{\mu}{\sigma_n}} \right\rceil^{\frac{2\eta-\omega_n-\sigma_n}{\mu}} ds$$

such that the derivative of $V_n(x)$ along system (1) is

$$
\begin{aligned}
\dot{V}_n(x) &= \sum_{i=1}^{n} \frac{\partial V_n(x)}{\partial x_i}\dot{x}_i \\
&= \dot{V}_{n-1}(\overline{x}_{n-1}) + \sum_{i=1}^{n-1} \frac{\partial W_n(x)}{\partial x_i}\dot{x}_i + \frac{\partial W_n(x)}{\partial x_n}\dot{x}_n \\
&\le -\Upsilon(x_1)\beta \sum_{i=1}^{n} |\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}} - \Upsilon(x_1)\beta \sum_{i=1}^{n} |\xi_i(\overline{x}_i)|^{\frac{2\eta\theta-\omega_i\theta}{\mu}} \\
&\quad + d_n(t,x)\lceil \xi_n(x)\rceil^{\frac{2\eta-\omega_n-\sigma_n}{\mu}} \left(u - x_{n+1}^*(x)\right)
\end{aligned} \tag{27}
$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$. By observing (27) and simply constructing a continuous state feedback stabilizer of the following form

$$
\begin{aligned}
&u(x) \\
&= x_{n+1}^*(x) \\
&= -g_n(x)\hat{\Upsilon}(x_1)\lceil \xi_n(x)\rceil^{\frac{\sigma_{n+1}}{\mu}} \\
&= -g_n(x)\hat{\Upsilon}(x_1)\Bigg\lceil \lceil x_n\rceil^{\frac{\mu}{\sigma_n}} + g_{n-1}^{\frac{\mu}{\sigma_n}}(\overline{x}_{n-1}) \\
&\quad \times \hat{\Upsilon}^{\frac{\mu}{\sigma_n p_{n-1}}}(x_1)\lceil x_{n-1}\rceil^{\frac{\mu}{\sigma_{n-1}}} \\
&\quad + g_{n-1}^{\frac{\mu}{\sigma_n}}(\overline{x}_{n-1}) g_{n-2}^{\frac{\mu}{\sigma_{n-1}}}(\overline{x}_{n-2})\hat{\Upsilon}^{\frac{\mu}{\sigma_n p_{n-1}}+\frac{\mu}{\sigma_{n-1}p_{n-2}}}(x_1) \\
&\quad \times \lceil x_{n-2}\rceil^{\frac{\mu}{\sigma_{n-2}}} + \cdots \\
&\quad + \cdots + g_{n-1}^{\frac{\mu}{\sigma_n}}(\overline{x}_{n-1}) g_{n-2}^{\frac{\mu}{\sigma_{n-1}}}(\overline{x}_{n-2}) \cdots g_1^{\frac{\mu}{\sigma_2}}(x_1) \\
&\quad \times \hat{\Upsilon}^{\frac{\mu}{\sigma_n p_{n-1}}+\frac{\mu}{\sigma_{n-1}p_{n-2}}+\cdots+\frac{\mu}{\sigma_3 p_2}}(x_1)\lceil x_1\rceil^{\frac{\mu}{\sigma_1}} \Bigg\rceil^{\frac{\sigma_n+\omega_n}{\mu}}
\end{aligned} \tag{28}
$$

one immediately has

$$\dot{V}_n(x) \le -\Upsilon(x_1)\beta \sum_{i=1}^{n} |\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}}$$

$$- \Upsilon(x_1)\beta \sum_{i=1}^{n} |\xi_i(\overline{x}_i)|^{\frac{2\eta\theta-\omega_i\theta}{\mu}} \tag{29}$$

for all $(t, x) \in \mathbb{R}_+ \times \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$. In addition, from the definition and structure of $V_n(x)$, it is not hard to verify that

$$\begin{aligned} V_n(x) &= V_1(x_1) + \sum_{i=2}^{n} W_i(\overline{x}_i) \\ &\geq V_1(x_1) + \sum_{i=2}^{n} \hat{c} \left| x_i - x_i^*(\overline{x}_{i-1}) \right|^{\frac{2\eta-\omega_i}{\sigma_i}} \end{aligned} \tag{30}$$

for all $x \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ and for a real constant $\hat{c} > 0$; this as well as the definitions of $x_i^*(\overline{x}_{i-1})$'s in turn implies that $V_n(x) \to \infty$ as $\|(x_2, \ldots, x_n)\| \to \infty$ for any fixed $x_1 \in \mathbb{M}_1(\varepsilon_L, \varepsilon_U)$. Based on this fact and the inequality (29), it follows from Lemma 1 that with the initial state $x(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$, every trajectory $x(t)$ of system (1) under the controller (28) is defined on $[t_0, \infty)$ (that is, $x(t)$ is forward complete) and certainly fulfills the output constraint $-\varepsilon_L < y(t) = x_1(t) < \varepsilon_U$ for all $t \geq t_0$ with ε_L and ε_U being pre-specified positive real constants.

Part II—Verification of the fixed-time state convergence of the closed-loop system

The remaining part of the proof is devoted to corroborating the fixed-time state convergence of system (1) under the controller (28). To this end, we consider the scenario where the initial state $x(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$; then, every trajectory (solution) $x(t)$ of the closed-loop system (1) under the controller (28) is defined on $[t_0, \infty)$ and carries out $x(t) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ for all $t \geq t_0$. In what follows we shall show that $x(t) \to 0$ in finite time and $x(t) = 0$ for all $t \geq T_r$ for some $T_r \in (t_0, \infty)$. Now, applying Lemmas 4 and 5 to $V_1(x_1)$ readily yields

$$\begin{aligned} V_1(x_1) &\leq \frac{|x_1|^{2\eta-\omega_1}}{(2\eta-\omega_1)} \gamma(x_1) \sec\left(\frac{|x_1|^{2\eta-\omega_1}\pi}{2\varepsilon_U^{2\eta-\omega_1}} \right) \\ &\quad + \frac{|x_1|^{2\eta-\omega_1}}{(2\eta-\omega_1)} (1-\gamma(x_1)) \sec\left(\frac{|x_1|^{2\eta-\omega_1}\pi}{2\varepsilon_L^{2\eta-\omega_1}} \right) \\ &\leq |\xi_1(x_1)|^{\frac{2\eta-\omega_1}{\mu}} \Upsilon^{\frac{1}{2}}(x_1) \end{aligned}$$

for all $x_1 \in \mathbb{M}_1(\varepsilon_L, \varepsilon_U)$; besides, by using Lemma 2, it also follows that

$$\begin{aligned} W_i(\overline{x}_i) &\leq \left| \int_{x_i^*(\overline{x}_{i-1})}^{x_i} \left\lceil \lceil s \rceil^{\frac{\mu}{\sigma_i}} - \lceil x_i^*(\overline{x}_{i-1}) \rceil^{\frac{\mu}{\sigma_i}} \right\rceil^{\frac{2\eta-\omega_i-\sigma_i}{\mu}} ds \right| \\ &\leq 2|\xi_i(\overline{x}_i)|^{\frac{2\eta-\omega_i}{\mu}} \end{aligned}$$

for all $x \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ and for all $i = 2, \ldots, n$. Considering these facts and utilizing Lemma 4, one further obtains

$$V_1^{\frac{2\eta}{2\eta-\omega_1}}(x_1) \leq 2|\xi_1(x_1)|^{\frac{2\eta}{\mu}} \Upsilon(x_1)$$

and

$$W_i^{\frac{2\eta}{2\eta-\omega_i}}(\overline{x}_i) \leq 2|\xi_i(\overline{x}_i)|^{\frac{2\eta}{\mu}}$$

for all $x \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ and for all $i = 2, \ldots, n$, which in turn make (29) become

$$\begin{aligned} &\dot{V}_n(x(t)) + \frac{\beta}{2} \left(V_1^{\frac{2\eta}{2\eta-\omega_1}}(x_1(t)) + \sum_{i=2}^{n} W_i^{\frac{2\eta}{2\eta-\omega_i}}(\overline{x}_i(t)) \right) \\ &\quad + \frac{\beta}{4n} V_n^{\theta}(x(t)) \leq 0 \end{aligned} \tag{31}$$

for all $t \in [t_0, \infty)$. Observing the inequality above, one knows that $V_n(x(t))$ is of course continuous and decreasing on $[t_0, \infty)$; this also means that if $V_n(x(t_0)) = 0$, the positive definiteness and continuity of $V_n(x)$ easily lead to $x(t) = 0$ for all $t \in [t_0, \infty)$. In the case when $V_n(x(t_0)) \neq 0$, it can be deduced from (31) that

$$\begin{aligned} &V_n(x(t)) - V_n(x(t_0)) \\ &\leq \int_{t_0}^{t} -\frac{\beta}{2} \left(V_1^{\frac{2\eta}{2\eta-\omega_1}}(x_1(s)) + \sum_{i=2}^{n} W_i^{\frac{2\eta}{2\eta-\omega_i}}(\overline{x}_i(s)) \right) \\ &\quad - \frac{\beta}{4n} V_n^{\theta}(x(s))\, ds \end{aligned}$$

for all $t \in [t_0, \infty)$, which by Lemma 6 renders $V_n(x(t)) = 0$ for all $t \in [T_r, \infty)$ where

$$T_r = t_0 + \frac{2}{\beta} \left(\frac{2n}{\theta - 1} - \frac{2\eta - \omega_1}{\omega_1} \right) \in (t_0, \infty) \tag{32}$$

and thus $x(t) = 0$ for all $t \in [T_r, \infty)$. Therefore, a direct combination of both cases implies that when $x(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$, every trajectory $x(t)$ converges to the origin in fixed time; that is to say, $x(t) \to 0$

in finite time and $x(t) = 0$ for all $t \geq T_r$ where $T_r \in (t_0, \infty)$ given by (32) is an upper bound of the settling (convergence) time and is clearly independent of both $x(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ and $t_0 \in \mathbb{R}_+$. The proof is completed. □

Remark 6 It is worthwhile noting from (32) that the upper bound $T_r \in (t_0, \infty)$ of the settling (convergence) time given by (32) is related to several tunable parameters. Specifically, when η and ω_1 are explicitly determined and fixed in accordance with the selection rule $\eta \geq \mu \geq \max_{1 \leq i \leq n}\{\sigma_i\}$ and Assumption 2 which definitely depicts the innate features of system nonlinearities $f_i(t, x, u)$'s, the upper bound T_r can be adjusted facilely by tuning the parameters $\beta > 0$ and $1 < \theta < 2$, which are involved in the gain functions $g_i(\overline{x}_i)$'s and of course the resultant stabilizer (28). For example, the upper bound T_r can be reduced by enlarging the value(s) of β and/or θ; in this situation, a smaller settling (convergence) time of the closed-loop system may be obtained at the expense of increased control efforts.

Remark 7 It can be observed from (29) and (30) that the positive definite and continuously differentiable function $V_n(x) = V_1(x_1) + \sum_{i=2}^{n} W_i(\overline{x}_i)$ constructed in Final Step and used for validating the fixed-time state convergence and the fulfillment of output constraints clearly satisfies the conditions (i) and (ii) of Lemma 1 and directly induces the properties $V_n(x) \to \infty$ as $x \to \partial\mathbb{M}_n(\varepsilon_L, \varepsilon_U)$ and $V_n(x(t)) \leq V_n(x(t_0)) < \infty$ for all $t \geq t_0$ with $x(t)$ being any solution of the closed-loop system starting at $x(t_0) \in \mathbb{M}_n(\varepsilon_L, \varepsilon_U)$; therefore, by Remark 2, $V_n(x)$ is indeed a BLF. Furthermore, since the BLF $V_n(x)$ includes as a built-in core the *tangent*-type (asymmetric) barrier function $V_1(x_1) = V_T(x_1)$ described by (8), it is particularly called the *tangent*-type asymmetric BLF for system (1), and the intrinsic features of system nonlinearities $f_i(t, x, u)$'s absorbed into the barrier function $V_T(x_1)$ are entirely retained in $V_n(x)$ so that $V_n(x)$ inherits the dynamic characteristics of system (1) and technically provides the feasibility of performing fixed-time stabilization for system (1) in consideration of output constraints concurrently.

Remark 8 From the definition of $\Upsilon(x_1)$ given by (10), it is easy to see that if the pre-specified output constraints are purposefully set to be infinity, namely, $\varepsilon_L = \varepsilon_U = \varepsilon \to \infty$, for reflecting the situation of no constraint required on the output, there holds

$$\lim_{\varepsilon\to\infty} \Upsilon(x_1) = \lim_{\varepsilon\to\infty} \sec^2\left(\frac{|x_1|^{2\eta-\omega_1}\pi}{2\varepsilon^{2\eta-\omega_1}}\right) = 1.$$

Keeping this in mind, the smooth function $\hat{\Upsilon}(x_1)$ required to satisfy the requisite relation $\Upsilon(x_1) \leq \hat{\Upsilon}(x_1)$ for all $x_1 \in \mathbb{R}$ (noting that $\mathbb{M}_1(\varepsilon_L, \varepsilon_U)$ is replaced by $\mathbb{R}$) in designing the virtual controllers $x_i^*(\overline{x}_{i-1})$'s and the continuous state feedback stabilizer $u(x)$ can be simply/directly assigned as $\hat{\Upsilon}(x_1) = 1$ without destroying the effectiveness/validity of the design; thus, under the proposed scheme, the resultant continuous state feedback stabilizer $u(x)$ becomes

$$\begin{aligned} u(x) = -g_n(x)\Bigg\lceil &\lceil x_n\rfloor^{\frac{\mu}{\sigma_n}} + g_{n-1}^{\frac{\mu}{\sigma_n}}(\overline{x}_{n-1})\lceil x_{n-1}\rfloor^{\frac{\mu}{\sigma_{n-1}}} \\ &+ g_{n-1}^{\frac{\mu}{\sigma_n}}(\overline{x}_{n-1})g_{n-2}^{\frac{\mu}{\sigma_{n-1}}}(\overline{x}_{n-2})\lceil x_{n-2}\rfloor^{\frac{\mu}{\sigma_{n-2}}} + \cdots \\ &+ \cdots + g_{n-1}^{\frac{\mu}{\sigma_n}}(\overline{x}_{n-1})g_{n-2}^{\frac{\mu}{\sigma_{n-1}}}(\overline{x}_{n-2})\cdots \\ &\times g_1^{\frac{\mu}{\sigma_2}}(\overline{x}_1)\lceil x_1\rfloor^{\frac{\mu}{\sigma_1}}\Bigg\rfloor^{\frac{\sigma_n+\omega_n}{\mu}} \end{aligned} \tag{33}$$

which surely has the same structure as (28) with only the gain function $\hat{\Upsilon}(x_1)$ changing into $\hat{\Upsilon}(x_1) = 1$. Besides, from Remark 5 and the definition of $V_n(x)$, it is also simple to verify that under the same scenario $\varepsilon_L = \varepsilon_U = \varepsilon \to \infty$, the following is true

$$\begin{aligned} &\lim_{\varepsilon\to\infty} V_n(x) \\ &= \frac{1}{(2\eta-\omega_1)}|x_1|^{2\eta-\omega_1} \\ &\quad + \sum_{i=2}^{n}\int_{x_i^*(\overline{x}_{i-1})}^{x_i}\left\lceil\lceil s\rfloor^{\frac{\mu}{\sigma_i}} - \lceil x_i^*(\overline{x}_{i-1})\rfloor^{\frac{\mu}{\sigma_i}}\right\rfloor^{\frac{2\eta-\omega_i-\sigma_i}{\mu}} ds \\ &=: V_U(x) \end{aligned} \tag{34}$$

where $V_U(x)$ is well defined on $\mathbb{R}^n$, positive definite, continuously differentiable and proper (i.e., the pre-image of any compact set under $V_U(x)$ is also compact). Taking $u(x)$ and $V_U(x)$ given by (33) and (34), respectively, and using the similar argument of the proof of Theorem 1, one can validate straightforward that the continuous state feedback controller (33) remains valid while indeed being a pure global fixed-time stabilizer for system (1). That is, in the circumstance that there has no constraint required on the output or equivalently output constraints are no longer necessary, by letting the pre-specified asymmetric output constraint turn to infinity, i.e., $\varepsilon_L = \varepsilon_U = \varepsilon \to \infty$, the proposed approach will evolve automatically into the design technique under which the resultant continuous state feedback controller possesses the same structure as the one organized in view of output constraints and is capable of performing the pure fixed-time stabilization task of system (1) without output constraints. This in turn discloses that the presented approach enjoys and offers a unified nature that enables one to synthesize a fixed-time stabilizer simultaneously workable for system (1) subjected to or free from asymmetric output constraints without needing to revamp the controller structure.

Remark 9 Compared with our previous work [38] concerning the finite-time stabilization issue of high-order nonlinear systems with asymmetric output constraints, the main difficulties encountered here are basically the compound tasks arising from the composite demands of handling output constraints and achieving the fixed-time state convergence, which hinder possible utilization or direct parallel extensions of the method in [38] due to the instinctive capability and feasibility limitations on both the designed BLF and controllers, and thus deserve further developments of new design and analysis methodologies. In this article, under the proposed approach, the gain functions as well as structures of both the corresponding stabilizer (28) and BLF $V_n(x)$ (including the built-in *tangent*-type barrier function $V_1(x_1) = V_T(x_1)$) are constructed based on new synthesis strategies along with mild Assumptions 1 and 2, which not only substantially weaken the parameters restrictions of Assumptions 1 and 2 in [38] but also help with comprehensively extracting and exploiting the inherent characteristics of system nonlinearities in design; thus, both the BLF $V_n(x)$ and the stabilizer (28) are granted the ability in advancing and enhancing the state convergence without violating pre-specified output constraints. Moreover, in order to fill the gap regarding the lack of explicit methods in the literature for analyzing and assuring the fixed-time state convergence in connection with output constraints, a new tool (i.e., Lemma 6) is also developed by which the resultant stabilizer (28) collaborating with the BLF $V_n(x)$ can be easily verified to be capable of performing simultaneously the fixed-time stabilization task of system (1) and the requirement of the pre-specified asymmetric output constraint.

4 An illustrative example

To illustrate how the proposed approach can be utilized, we now consider a planar system as below

$$\begin{aligned}\dot{x}_1 &= d_1(t,x)x_2^3 + \sin(x_1)\cos(t+x_2+3u)\\ \dot{x}_2 &= d_2(t,x)u + \cos(x_2+u)\ln(1+x_1^2)\\ y &= x_1\end{aligned} \tag{35}$$

where $d_1(t,x) = 1 + 0.2\cos(2t + x_2^3)$ and $d_2(t,x) = 1 + 0.4\sin(3t + x_1)$. Note that, system (35) has the same structure as system (1) with $p_1 = 3$, $p_2 = 1$, $f_1(t,x,u) = \sin(x_1)\cos(t+x_2+3u)$ and $f_2(t,x,u) = \cos(x_2+u)\ln(1+x_1^2)$. Assumption 1 is obviously fulfilled with $\underline{d}_1(x_1) = 0.8$, $\overline{d}_1(x_1) = 1.2$, $\underline{d}_2(x) = 0.6$ and $\overline{d}_2(x) = 1.4$. In light of the selection $\sigma_1 = 1$, $\omega_1 = -1/10$ and $\omega_2 = -1/5$, which lead to $\sigma_2 = 3/10$ and $\eta = \mu = 1$, and the facts $|\sin(x_1)\cos(t+x_2+3u)| \le |x_1|^{9/10}$ and $|\cos(x_2+u)\ln(1+x_1^2)| \le 19|x_1|^{1/10}$ for all $(t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}$, Assumption 2 is satisfied with $\overline{f}_1(x_1) = 1$ and $\overline{f}_2(x) = 19$, respectively. Based on Theorem 1, we first pick $\theta = 1.5$ and $\beta = 8$, which gives $T_r = t_0 + 7.25$, and let

$$\begin{aligned}V_1(x_1) = {}& \gamma(x_1)\frac{20\varepsilon_U^{\frac{21}{10}}}{21\pi}\tan\left(\frac{|x_1|^{\frac{21}{10}}\pi}{2\varepsilon_U^{\frac{21}{10}}}\right)\\ &+ (1-\gamma(x_1))\frac{20\varepsilon_L^{\frac{21}{10}}}{21\pi}\tan\left(\frac{|x_1|^{\frac{21}{10}}\pi}{2\varepsilon_L^{\frac{21}{10}}}\right)\end{aligned}$$

and the continuous virtual controller

$$x_2^*(x_1) = -\left(31.25 + 10x_1^2\right)^{\frac{1}{3}} \lceil x_1 \rfloor^{\frac{3}{10}}.$$

Then, one has

$$\begin{aligned}\dot{V}_1(x_1) \le {}& -16\Upsilon(x_1)|\xi_1(x_1)|^2 - 8\Upsilon(x_1)|\xi_1(x_1)|^{\frac{63}{20}}\\ &+ d_1(t,x)\Upsilon(x_1)\lceil \xi_1(x_1) \rfloor^{\frac{11}{10}}\left(x_2^3 - x_2^{*3}(x_1)\right)\end{aligned}$$

for all $(t,x,u) \in \mathbb{R}_+ \times \mathbb{M}_2(\varepsilon_L,\varepsilon_U) \times \mathbb{R}$, where $\Upsilon(x_1)$ is of the form

$$\Upsilon(x_1) = \gamma(x_1)\sec^2\left(\frac{|x_1|^{\frac{21}{10}}\pi}{2\varepsilon_U^{\frac{21}{10}}}\right) + (1-\gamma(x_1))\sec^2\left(\frac{|x_1|^{\frac{21}{10}}\pi}{2\varepsilon_L^{\frac{21}{10}}}\right).$$

Next, we choose

$$V_2(x) = V_1(x_1) + \int_{x_2^*(x_1)}^{x_2} \left\lceil \lceil s \rfloor^{\frac{10}{3}} - \lceil x_2^*(x_1) \rfloor^{\frac{10}{3}} \right\rfloor^{\frac{19}{10}} ds$$

and calculate the derivative of $V_2(x)$ along system (35) as follows

$$\begin{aligned}\dot{V}_2(x) \le & -16\Upsilon(x_1)|\xi_1(x_1)|^2 - 8\Upsilon(x_1)|\xi_1(x_1)|^{\frac{63}{20}} \\ & + d_2(t,x)\lceil \xi_2(x) \rfloor^{\frac{19}{10}} u \\ & + d_1(t,x)\Upsilon(x_1)\lceil \xi_1(x_1) \rfloor^{\frac{11}{10}}\left(x_2^3 - x_2^{*3}(x_1)\right) \\ & + \rho_1(x_1)|\xi_2(x)|^{\frac{6}{5}}\left(|x_1|^{\frac{9}{10}} + 1.2|x_2|^3\right) \\ & + 19\,|\xi_2(x)|^{\frac{19}{10}}\left(|x_1|^{\frac{1}{10}} + |x_2|^{\frac{1}{3}}\right)\end{aligned}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_2(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$, in which $\rho_1(x_1)$ is a smooth function directly calculated as $\rho_1(x_1) = -(31.25 + 10x_1^2)^{10/9} - 22.22\ (31.25 + 10x_1^2)^{1/9}x_1^2$. By employing Lemmas 1–3, the estimations of the last three terms on the right-hand side of the above inequality can be easily found as below

$$\begin{aligned}& d_1(t,x)\Upsilon(x_1)\lceil \xi_1(x_1) \rfloor^{\frac{11}{10}}\left(x_2^3 - x_2^{*3}(x_1)\right) \\ & \quad \le \frac{8}{3}\Upsilon(x_1)|\xi_1(x_1)|^2 + \alpha_2(x)\hat{\Upsilon}(x_1)\,|\xi_2(x)|^2 \\ & \rho_1(x_1)|\xi_2(x)|^{\frac{6}{5}}\left(|x_1|^{\frac{9}{10}} + 1.2|x_2|^3\right) \\ & \quad \le \frac{8}{3}\Upsilon(x_1)|\xi_1(x_1)|^2 + \tilde{\alpha}_2(x)\hat{\Upsilon}(x_1)\,|\xi_2(x)|^2 \\ & 19\,|\xi_2(x)|^{\frac{19}{10}}\left(|x_1|^{\frac{1}{10}} + |x_2|^{\frac{1}{3}}\right) \\ & \quad \le \frac{8}{3}\Upsilon(x_1)|\xi_1(x_1)|^2 + \hat{\alpha}_2(x)\hat{\Upsilon}(x_1)\,|\xi_2(x)|^2\end{aligned}$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_2(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$, where $\alpha_2(x) = 0.1143$, $\tilde{\alpha}_2(x) = 0.1694(38.5 + 12x_1^2)^{5/3}\rho_1^{5/3}(x_1)$, $\hat{\alpha}_2(x) = 17.0956(1 + (31.25 + 10x_1^2)^{1/9})^{20/19}$ and

$$\hat{\Upsilon}(x_1) = \gamma(x_1)\sec^2\left(\frac{x_1^2\pi}{2\varepsilon_U^2}\right) + (1-\gamma(x_1))\sec^2\left(\frac{x_1^2\pi}{2\varepsilon_L^2}\right).$$

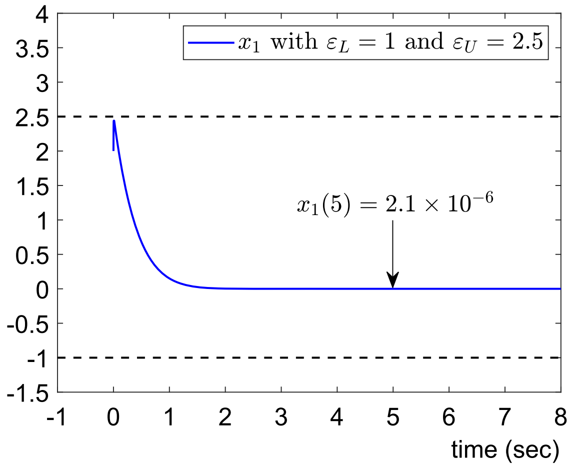

Fig. 1 Responses of x_1 with $\varepsilon_L = 1$ and $\varepsilon_U = 2.5$

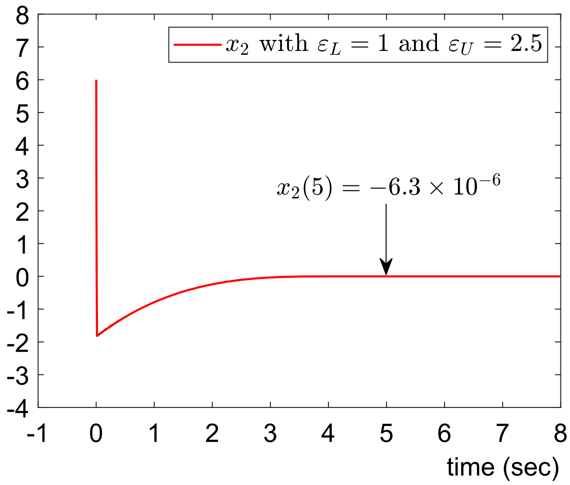

Fig. 2 Responses of x_2 with $\varepsilon_L = 1$ and $\varepsilon_U = 2.5$

Hence, constructing the continuous state feedback controller

$$u(x) = -g_2(x)\hat{\Upsilon}(x_1) \times \left\lceil \lceil x_2 \rfloor^{\frac{10}{3}} + \left(31.25 + 10x_1^2\right)^{\frac{10}{9}} x_1 \right\rfloor^{\frac{1}{10}} \tag{36}$$

with $g_2(x) = 13.52 + 13.33(1 + x_2^2)^{13/6} + 1.67(\alpha_2(x) + \tilde{\alpha}_2(x) + \hat{\alpha}_2(x))$, one simply obtains

$$\begin{aligned}\dot{V}_2(x) \le & -8\Upsilon(x_1)\left(|\xi_1(x_1)|^2 + |\xi_2(x)|^2\right) \\ & - 8\Upsilon(x_1)\left(|\xi_1(x_1)|^{\frac{63}{20}} + |\xi_2(x)|^{\frac{63}{20}}\right)\end{aligned}$$

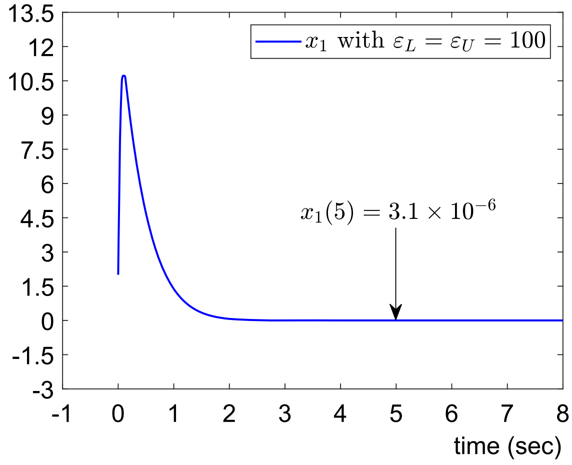

Fig. 3 Responses of x_1 with $\varepsilon_L = \varepsilon_U = 100$

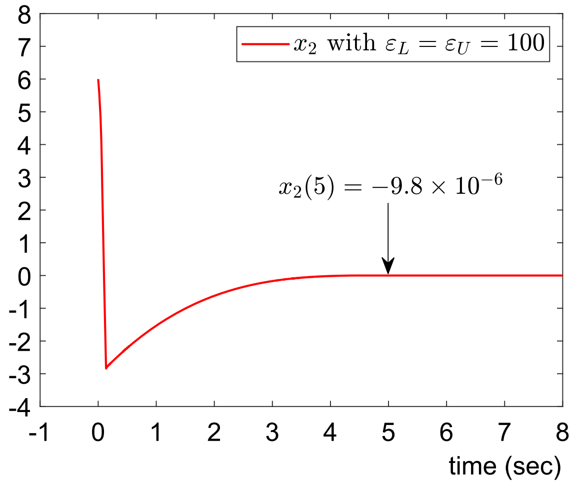

Fig. 4 Responses of x_2 with $\varepsilon_L = \varepsilon_U = 100$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{M}_2(\varepsilon_L, \varepsilon_U) \times \mathbb{R}$. By letting the initial time $t_0 = 0$ and the initial state $(x_1(0), x_2(0))^T = (2, 6)^T$, it can be clearly seen from the simulation results depicted in Fig.1–4 that the continuous state feedback controller (36) successfully stabilizes system (35) in fixed time, i.e., $x(t) \to 0$ in finite time and $x(t) = 0$ for all $t \geq T_r = 7.25$, while fulfilling the requirement of the asymmetric constraint $-1 = -\varepsilon_L < y_1(t) = x_1(t) < \varepsilon_U = 2.5$ for all $t \geq 0$. Furthermore, in the circumstance that the constraint is fairly large, e.g., $\varepsilon_L = \varepsilon_U = 100$, which corresponds to the scenario where the output constraint is no longer imperative, the controller (36) with the same structure remains valid and serves as a pure global fixed-time stabilizer for system (35), almost without limiting/restricting the amplitude of the output $y_1(t) = x_1(t)$. Notably, this example explicitly demonstrates that the proposed approach provides a unified nature enabling us to perform the design of a fixed-time stabilizer for the system simultaneously with and without (asymmetric) output constraints.

5 Conclusion

This article has provided a solution to the problem of fixed-time stabilization for a class of uncertain high-order nonlinear systems subjected to an asymmetric output constraint. A new design methodology was established by subtly renovating the technique of adding a power integrator along with the subtle utilization of the intrinsic attributes of signum functions as well as a novel *tangent*-type barrier function designed with fully absorbing the inherent feature of system nonlinearities. Under the developed approach, a *tangent*-type asymmetric barrier Lyapunov function that helps with the convergence analysis, and a continuous state feedback fixed-time stabilizer that ensures both the fixed-time state convergence and the fulfillment of prespecified output constraints can be constructed, respectively, in a systematic fashion. A distinctive technical novelty of the presented method is the unified nature that enables one to synthesize a fixed-time stabilizer simultaneously workable for the system subjected to or free from asymmetric output constraints, without needing to change the controller structure.

Funding This work was supported in part by the Ministry of Science and Technology (MOST), Taiwan, under Grants MOST 110-2221-E-006-173- and MOST 111-2221-E-006-204-MY2.

Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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