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ORIGINAL PAPER

Robust tracking control of nonlinear unmatched uncertain systems via event-based adaptive dynamic programming

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Abstract In this paper, a novel robust tracking control strategy for nonlinear unmatched uncertain systems is formulated using the event-based adaptive dynamic programming (ADP) approach. First, an augmented system is constructed based on the nonlinear system and the reference trajectory. Then, by forming an auxiliary system and introducing a discounted cost function, the event-based robust tracking control problem is transformed into the event-based optimal control problem of the auxiliary system. The event-based Hamilton–Jacobi–Bellman (HJB) equation associated with the event-based optimal control problem is solved using a single critic neural network (NN) under the ADP framework. A novel weight tuning rule for the critic network is formulated to avoid the necessity of an initial admissible control at the beginning of the weights tuning process. The obtained event-based controller is updated only at the triggering instants decided by the designed triggering condition, which helped in a significant reduction of resources used in computation and communication. Meanwhile, it is demonstrated that the obtained event-based controller can guarantee the tracking error's uniform ultimate boundedness. Furthermore, using the Lyapunov method, it is guaranteed that the established novel event-triggering rule ensures uniform ultimate boundedness of all signals associated with the closed-loop auxiliary system. Finally, the applicability of the proposed control scheme is demonstrated by providing two simulation examples.

Keywords Robust tracking · Unmatched uncertainty · Event-triggered control · Adaptive dynamic programming

1 Introduction

Uncertainties are inevitable in practical nonlinear systems because of the presence of external disturbances and modeling errors. So, considering the requirement of the robustness of the designed feedback controller to uncertainties, many robust control design schemes have been developed over several decades [1–3]. Especially the method developed by Lin [3], in which the optimal control approach is utilized to obtain the robust controller, got remarkable attention [4,5]. In the case of linear systems, the optimal controller can be derived conveniently by solving the algebraic Riccati equation (ARE) associated with it [6]. However, for nonlinear systems, instead of the ARE, one needs to find the solution of the Hamilton–Jacobi–Bellman (HJB) equation [7]. Since the HJB equation is a nonlinear partial differential equation, solving it with an analytical method is challenging. Although dynamic programming is generally used to solve the optimal control problem of nonlinear systems, it suffers heavily from

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the notorious "curse of dimensionality" [8]. The neural network (NN)-based function approximation technique called adaptive dynamic programming (ADP) has been employed to address this difficulty [9,10]. The ADP approach was initially developed by Werbos to determine the solution of the optimal control problem effectively [11]. The ADP algorithm has a close relation with the reinforcement learning (RL) technique [12]. In the literature, ADP is also known as approximate dynamic programming [13], adaptive critic designs (ACDs) [14], neural dynamic programming (NDP) [15], and Q-learning [16].

In most practical applications, the system states need to track the desired trajectory rather than converge to zero merely [17,18]. In the past several years, significant work has been done on tracking control by combining the aforementioned robust control method and the ADP algorithms [19–22]. In [19], the integral reinforcement learning technique is utilized to develop an optimal tracking controller for constrained input systems. For nonlinear matched uncertain systems, in [20], a robust tracking controller is designed via neural network approach, and in [21], a guaranteed cost tracking controller is developed. In [22], tracking controller for nonlinear systems considering unmatched uncertainties is derived via ACDs. However, all the work mentioned above is developed under the classical time-triggering framework, which suffers from inefficient use of computational and communicational resources.

Compared to the classical time-triggered approach, in the event-based or event-triggered strategy, the controller is only updated if a predefined triggering rule is not fulfilled, which helps in the effective use of computational and communicational resources [23–26]. Thus, many works have been done by combining the ADP-based robust control approach and the event-based framework. In [27], an actor-critic NN structure is utilized to derive an event-based optimal adaptive controller for nonlinear systems. In [28], an event-based guaranteed cost controller is derived for nonlinear systems utilizing a single critic NN. In [29], adaptive critic is used to design event-based near-optimal controller for heating, ventilation, and air conditioning (HVAC) systems. In [30], event-triggered optimal controller is designed for continuous stirred tank reactor (CSTR) system using ADP approach. The event-based ADP framework has been further utilized in designing controller for nonlinear systems with constrained input [31], with matched uncertainties [32], and with unmatched uncertainties [33].

Under the event-based ADP framework, the tracking controller for nonlinear systems has been designed in [34–43]. In [34–36,38,44], the ADP approach is used to formulate event-based optimal tracking controller for nonlinear systems without considering any uncertainties. The event-based ADP approach is utilized to derive an optimal tracking controller for modular reconfigurable robots in [39], and in [40], the tracking controller is design with application in wastewater treatment. In [41], event-based ADP is utilized to develop a tracking controller for constrained input systems. In [42], Cui et al. established event-based H_∞ tracking controller via RL method. In our previous work [43], nonlinear matched uncertain system was considered while designing the event-based robust tracking controller. In [44], the event-triggered ADP approach is used to design a tracking controller for partially unknown matched uncertain constrained systems. Nonetheless, papers in the existing literature have not focused on developing a robust tracking controller for continuous-time nonlinear systems with unmatched uncertainty via event-based ADP approach, particularly without using the H_∞ control approach [42]. Unlike the matched uncertainty, the unmatched uncertainty enters the system through a different channel than the control input. The unmatched uncertainties are a more general kind of uncertainties and can be widely seen in most practical systems. So, it is vital to consider unmatched uncertainty while designing a controller for nonlinear systems. This is what drives the research developed in this paper.

The following are the major contributions of this work.

1. Compared with [34–38,41], in this work uncertainty is considered while designing the ADP-based event-triggered robust tracking controller. As unmatched uncertainties are the most common form of uncertainty, they must be taken into account while developing a controller for nonlinear systems.
2. Unlike [42], in this work, the event-based robust tracking controller is derived without using the H_∞ control strategy. In the H_∞ optimal control approach, the existence of the saddle point must be judged, but this is a challenging task.

3. Rather than considering matched uncertainty as in [43,44], unmatched uncertainty is considered in this work. Moreover, unlike [43], the need for the initial stabilizing control at the beginning of the critic weights tuning process is also relaxed by modifying the tuning rule.

The remaining part of this work is organized in the following manner. In Sect. 2, the original tracking control problem is transformed into the optimal control problem of an auxiliary system. The event-based HJB equation is formulated, and the event-triggering rule is derived in Sect. 3. In Sect. 4, the HJB equation is solved via the ADP approach. In Sect. 5, the Lyapunov approach is used to show that all the signals associated with the closed-loop auxiliary system are uniformly ultimately bounded. In Sect. 6, two simulation examples are presented. Finally, a concluding remark is given in Sect. 7. Moreover, limitations and future scope of the proposed work are also mentioned in Sect. 7.

Notation: In this work, the maximum and minimum eigenvalues of a matrix are denoted by $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$, respectively. The transform operation is represented by the superscript $\top$. $\nabla(\cdot)$ denotes the gradient operation. I_n is the identity matrix of dimension $n \times n$ and $0_{n\times m}$ is the zero matrix of dimension $n \times m$. Ω is a compact subset of $\mathbb{R}^2$.

2 Problem transformation

Consider the continuous-time nonlinear uncertain system given in the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) + \Delta f(x(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^b$ are the state vector and control input, respectively. Let $x(0) = x_0$ be the initial state. $f(\cdot)$ and $g(\cdot)$ are smooth functions in their arguments with $f(0) = 0$ and $f + gu$ satisfies the Lipschitz continuity. The unmatched uncertainty $\Delta f(x) = l(x)d(x)$, where $l(x) \in \mathbb{R}^{n\times p}$, $d(x) \in \mathbb{R}^p$ and if $b = p$ then $l(x) \neq g(x)$. Let $d(x)$ be bounded by a known function $\lambda_d(x)$, i.e., $\|d(x)\| \leq \lambda_d(x)$. Furthermore, $\lambda_d(0) = 0$ and $d(0) = 0$. In addition, there exists a nonnegative function $g_M(x)$ satisfying

$$\left\|g^+(x)\Delta f(x)\right\| \leq g_M(x),$$

where $g^+(x)$ is the pseudoinverse of $g(x)$. Let the desired trajectory $x_d(t) \in \mathbb{R}^n$ be generated from

$$\dot{x}_d(t) = \Theta(x_d(t)), \tag{2}$$

where $\Theta(x_d)$ satisfies the Lipschitz continuity and $\Theta(0) = 0$. Let $x_d(0) = x_{do}$ be the initial condition.

The objective of this work is to derive an event-based robust controller for system (1) so that the system state $x(t)$ follows the desired trajectory $x_d(t)$. Define the tracking error as $e_t(t) = x(t) - x_d(t)$. From (1) and (2), the tracking error dynamics can be presented as

$$\begin{aligned}\dot{e}_t(t) &= f(e_t(t) + x_d(t)) + g(e_t(t) + x_d(t))u(t)\\ &\quad + \Delta f(e_t(t) + x_d(t)) - \Theta(x_d(t)).\end{aligned} \tag{3}$$

Now, based on the tracking error and the desired trajectory, an augmented state vector $\xi(t) = [e_t^\top(t), x_d^\top(t)]^\top \in \mathbb{R}^{2n}$ is formed. Then, using (2) and (3), the augmented system dynamics is formulated as

$$\dot{\xi}(t) = F(\xi(t)) + G(\xi(t))u(\xi(t)) + \Delta F(\xi(t)), \tag{4}$$

where $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n\times b}$ are new system matrices while $\Delta F(\xi(t)) \in \mathbb{R}^{2n}$ is the new uncertain term. They can be expressed as

$$F(\xi(t)) = \begin{bmatrix} f(e_t(t) + x_d(t)) - \Theta(x_d(t)) \\ \Theta(x_d(t)) \end{bmatrix},$$

$$G(\xi(t)) = \begin{bmatrix} g(e_t(t) + x_d(t)) \\ 0 \end{bmatrix}$$

and

$$\Delta F(\xi(t)) = \begin{bmatrix} \Delta f(e_t(t) + x_d(t)) \\ 0 \end{bmatrix} = L(\xi(t))d(\xi(t)).$$

The terms $d(\xi)$ and $G^+(\xi)\Delta F(\xi)$ are still upper bounded and the bound can be derived as

$$d(\xi) = d(x) \leq \lambda_d(x) = \lambda_d(e_t + x_d) \triangleq \lambda_d(\xi) \tag{5}$$

and

$$\begin{aligned}\left\|G^+(\xi)\Delta F(\xi)\right\| &= \left\|g^+(x)\Delta f(x)\right\| \\ &\leq g_M(x) = g_M(e_t + x_d) \triangleq g_M(\xi),\end{aligned} \tag{6}$$

respectively.

Next, the uncertain term $L(\xi)d(\xi)$ is projected onto the range of matrix $G(\xi)$ and decomposed into sum of matched and unmatched component, that is

$$L(\xi)d(\xi) = G(\xi)G^{+}(\xi)L(\xi)d(\xi) + (I - G(\xi)G^{+}(\xi))L(\xi)d(\xi).$$

Then following auxiliary system is formed

$$\dot{\xi} = F(\xi) + G(\xi)u(\xi) + (I - G(\xi)G^{+}(\xi))L(\xi)v(\xi), \tag{7}$$

where $v(\xi) \in \mathbb{R}^p$ is an auxiliary control that handles the unmatched component.

3 Event-based robust tracking control strategy

In this section, the event-based HJB equation is developed for the auxiliary system (7). Moreover, the event-triggering rule is also obtained using Lyapunov approach. The cost function associated with the auxiliary system (7) is defined as

$$J(\xi(t)) = \int_t^{\infty} \mathrm{e}^{-\gamma(\tau-t)}\{U(\xi(\tau), u(\xi(\tau)), v(\xi(\tau))) + \beta^2\|m^\top\|^2\lambda_d^2(\xi) + \|r^\top\|^2 g_M^2(\xi)\}\mathrm{d}\tau, \tag{8}$$

where γ and β are positive constant, $U(\xi, u(\xi), v(\xi)) = \xi^\top \bar{Q}\xi + u^\top(\xi)Ru(\xi) + \beta^2 v^\top(\xi)Mv(\xi)$ and $\bar{Q} = diag\{Q, 0_{n\times n}\}$. Q, M and R are positive definite matrices with appropriate dimension. Let r and m be lower triangular matrices with appropriate dimension. Then, using Cholesky decomposition one can write $R = rr^\top$ and $M = mm^\top$.

Remark 1 The discount term $\mathrm{e}^{-\gamma(\tau-t)}$ in cost function (8) is employed to make sure that (8) is bounded. Otherwise, the control policy pair $[u^\top(e_t(t), x_d(t)), v^\top(e_t(t), x_d(t))]^\top$ may cause (8) to become unbounded since it depends on reference trajectory $x_d(t)$. In many practical systems, we need to consider reference trajectory which does not converge to zero. In that situation $x_d(t)$ makes (8) unbounded [45,46].

Let $\Psi(\Omega)$ be the set of admissible controls on Ω. We assume that the optimal control policy pair is admissible. If the cost function $J(\xi)$ is continuously differentiable then one can write

$$\|r^\top\|^2 g_M^2(\xi) + \beta^2\|m^\top\|^2\lambda_d^2 + U(\xi, u(\xi), v(\xi)) - \gamma J(\xi) + \dot{J}(\xi) = 0 \tag{9}$$

with $J(0) = 0$. Here (9) is called the infinitesimal version of (8). The Hamiltonian for the auxiliary system (7) is given as

$$\begin{aligned} &H(\xi, u(\xi), v(\xi), \nabla J(\xi)) \\ &= (\nabla J(\xi))^\top (F(\xi) + G(\xi)u(\xi) \\ &\quad + (I - G(\xi)G^{+}(\xi))L(\xi)v(\xi)) + \|r^\top\|^2 g_M^2(\xi) \\ &\quad + \beta^2\|m^\top\|^2\lambda_d^2 + U(\xi, u(\xi), v(\xi)) - \gamma J(\xi). \end{aligned} \tag{10}$$

The optimal cost function is given by

$$\begin{aligned} &J^*(\xi(t)) \\ &= \min_{u,v\in\Psi(\Omega)} \int_t^{\infty} \mathrm{e}^{-\gamma(\tau-t)}\{U(\xi(\tau), u(\xi(\tau)), v(\xi(\tau))) \\ &\quad + \beta^2\|m^\top\|^2\lambda_d^2(\xi) + \|r^\top\|^2 g_M^2(\xi)\}\mathrm{d}\tau. \end{aligned} \tag{11}$$

By the Bellman's principle, $J^*(\xi(t))$ holds the HJB equation

$$\min_{u,v\in\Psi(\Omega)} H(\xi, u(\xi), v(\xi), \nabla J^*(\xi)) = 0 \tag{12}$$

with $J^*(0) = 0$. Define $(I - G(\xi)G^{+}(\xi))L(\xi) = K(\xi)$. The optimal control policies are obtained as

$$u^*(\xi) = -\frac{1}{2}R^{-1}G^\top(\xi)\nabla J^*(\xi) \tag{13}$$

and

$$v^*(\xi) = -\frac{1}{2\beta^2}M^{-1}K^\top(\xi)\nabla J^*(\xi). \tag{14}$$

Substituting (13) and (14) into (12), we present the HJB equation as

$$\begin{aligned} &(\nabla J^*(\xi))^\top F(\xi) + \xi^\top\bar{Q}\xi + \beta^2\|m^\top\|^2\lambda_d^2(\xi) \\ &\quad + \|r^\top\|^2 g_M^2(\xi) - \gamma J^*(\xi) \\ &\quad - \frac{1}{4}\nabla(J^*(\xi))^\top G(\xi)R^{-1}G^\top(\xi)\nabla J^*(\xi) \\ &\quad - \frac{1}{4\beta^2}\nabla(J^*(\xi))^\top K(\xi)M^{-1}K^\top(\xi)\nabla J^*(\xi) = 0. \end{aligned} \tag{15}$$

3.1 The event-based HJB equation formulation

Here, we present the HJB equation (15) in event-based form. Before proceeding, the event-based strategy is explained.

Let us consider a monotonically increasing sequence $\{t_k\}_{k=0}^{\infty}$, where the kth triggering instant is represented as t_k and $k \in \mathbb{N}$. Let the system state be sampled at every triggering instants and $\{\xi_k\}_{k=0}^{\infty}$ be the sequence of sampled state, where $\xi_k = \xi(t_k)$ is the sampled state at t_k. The triggering error is described as the difference between the current state $\xi(t)$ and the sampled state ξ_k and is represented as

$$e_k(t) = \xi_k - \xi(t), \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \tag{16}$$

Based on (16), the event-based mechanism can be explained. If a predefined triggering rule is not satisfied, then the triggering error becomes zero, i.e., $e_k(t) = 0$, and the control law will be updated. When the triggering rule is fulfilled, the control law is held constant between the two consecutive triggering instants. This principle is similar to the familiar zero-order hold (ZOH) principle, and it can be expressed as

$$u(t) = u(\xi_k) \triangleq \mu(\xi_k), \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}.$$

From (16), the event-based control policy is obtained as

$$u(t) = \mu(\xi(t) + e_k(t)), \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \tag{17}$$

Now, using the control law (17), we obtain the sampled version of auxiliary system (7) as

$$\dot{\xi} = F(\xi) + G(\xi)\mu(\xi(t) + e_k(t)) + K(\xi)v(\xi). \tag{18}$$

The optimal control (13), under event-triggered mechanism, can be expressed as

$$\mu^*(\xi_k) = -\frac{1}{2}R^{-1}G^\top(\xi_k)\nabla J^*(\xi_k). \tag{19}$$

Now, using (19), we formulate the HJB equation under event-based framework as

$$H(\xi, \mu^*(\xi_k), v^*(\xi), \nabla J^*(\xi)) = 0,$$

that is,

$$\begin{aligned}(\nabla J^*(\xi))^\top F(\xi) &+ \xi^\top \bar{Q}\xi + \beta^2\|m^\top\|^2\lambda_d^2(\xi) \\ &+ \|r^\top\|^2 g_M^2(\xi) - \gamma J^*(\xi) \\ &- \frac{1}{2}\nabla(J^*(\xi))^\top G(\xi)R^{-1}G^\top(\xi_k)\nabla J^*(\xi_k) \\ &+ \frac{1}{4}\nabla(J^*(\xi_k))^\top G(\xi_k)R^{-1}G^\top(\xi_k)\nabla J^*(\xi_k) \\ &- \frac{1}{4\beta^2}\nabla(J^*(\xi))^\top K(\xi)M^{-1}K^\top(\xi)\nabla J^*(\xi) = 0,\end{aligned} \tag{20}$$

where $J^*(0) = 0$.

3.2 Event-triggering condition

In this subsection, we obtain the event-triggering condition using the Lyapunov approach. Before continuing, following statement is made which will be required to derive the triggering rule. The following statement is satisfied in many applications when the controller is affine with respect to the event-triggering error signal [27,47].

Assumption 1 Let $\mathcal{L}$ be a positive constant. We consider that the optimal control policy $u^*(\xi)$ fulfills the Lipschitz continuity on Ω such that

$$\begin{aligned}\|u^*(\xi(t)) - u^*(\xi_k)\| &= \|u^*(\xi(t)) - u^*(\xi(t) + e_k(t))\| \\ &\le \mathcal{L}\|e_k(t)\|.\end{aligned}$$

Theorem 1 *Let Assumption 1 be true, $J^*(\xi)$ satisfies the HJB equation (12), the control policies are described by (14) and (19), and the event-triggering law is formulated as*

$$\begin{aligned}\|e_k(t)\|^2 &\le \frac{(1-\eta_1^2)\lambda_m(Q)\|e_t\|^2 - 2\beta^2\|m^\top v^*(\xi)\|^2}{2\|r^\top\|^2\mathcal{L}^2} \\ &\triangleq \|e_T\|^2,\end{aligned} \tag{21}$$

then for $\eta_1 \in (0, 1)$ and $\gamma = 0$, the closed-loop augmented system (4) is asymptotically stable under $\mu^*(\xi_k)$ and for $\gamma \neq 0$ the tracking error e_t is uniformly ultimately bounded.

Proof Consider $J^*(\xi)$ is the Lyapunov function candidate. Differentiating $J^*(\xi)$ along the trajectory of

$\dot{\xi}(t) = F(\xi(t)) + G(\xi(t))\mu^*(\xi_k) + \Delta F(\xi(t))$, one can write

$$\dot{J}^*(\xi) = (\nabla J^*(\xi))^\top (F(\xi) + G(\xi)\mu^*(\xi_k) + \Delta F(\xi(t))) \\ = (\nabla J^*(\xi))^\top F(\xi) + (\nabla J^*(\xi))^\top G(\xi)\mu^*(\xi_k) \\ + (\nabla J^*(\xi))^\top (G(\xi)G^+(\xi)L(\xi) + K(\xi))d(\xi). \tag{22}$$

From (12), we obtain

$$(\nabla J^*(\xi))^\top F(\xi) \\ = -\xi^\top \bar{Q}\xi - \beta^2\|m^\top\|^2\lambda_d^2(\xi) - \|r^\top\|^2 g_M^2(\xi) \\ + \gamma J^*(\xi) + \frac{1}{4}\nabla(J^*(\xi))^\top G(\xi)R^{-1}G^\top(\xi)\nabla J^*(\xi) \\ + \frac{1}{4\beta^2}\nabla(J^*(\xi))^\top K(\xi)M^{-1}K^\top(\xi)\nabla J^*(\xi), \tag{23}$$

from (13), we can write

$$G^\top(\xi)\nabla J^*(\xi) = -2Ru^*(\xi) \tag{24}$$

and from (14), we obtain

$$K^\top(\xi)\nabla J^*(\xi) = -2\beta^2 M v^*(\xi) \tag{25}$$

Using (23), (24) and (25) we derived

$$\dot{J}^*(\xi) = -\xi^\top \bar{Q}\xi - \beta^2\|m^\top\|^2\lambda_d^2(\xi) - \|r^\top\|^2 g_M^2(\xi) \\ + \gamma J^*(\xi) + u^{*\top}(\xi)Ru^*(\xi) \\ + \beta^2 v^{*\top}(\xi)Mv^*(\xi) - 2u^{*\top}(\xi)R\mu^*(\xi_k) \\ - 2u^{*\top}(\xi)RG^+(\xi)L(\xi)d(\xi) \\ - 2\beta^2 v^{*\top}(\xi)Md(\xi). \tag{26}$$

Now,

$$u^{*\top}(\xi)Ru^*(\xi) - 2u^{*\top}(\xi)R\mu^*(\xi_k) \\ - 2u^{*\top}(\xi)RG^+(\xi)L(\xi)d(\xi) \\ = \|r^\top(u^*(\xi) - u^*(\xi_k) - G^+(\xi)L(\xi)d(\xi))\|^2 \\ - \|r^\top(u^*(\xi_k) + G^+(\xi)L(\xi)d(\xi))\|^2 \\ \leq 2\|r^\top\|^2 + 2\|r^\top G^+(\xi)L(\xi)d(\xi)\|^2 \\ - \|r^\top(u^*(\xi_k) + G^+(\xi)L(\xi)d(\xi))\|^2 \\ \leq 2\|r^\top\|^2\mathcal{L}^2\|e_k\|^2 + 2\|r^\top G^+(\xi)L(\xi)d(\xi)\|^2 \\ - \|r^\top(u^*(\xi_k) + G^+(\xi)L(\xi)d(\xi))\|^2 \tag{27}$$

and

$$-2\beta^2 v^{*\top}(\xi)Md(\xi) \leq \beta^2(\|m^\top v^*(\xi)\|^2 \\ + \|m^\top d(\xi)\|^2). \tag{28}$$

Since, $\bar{Q} = diag\{Q, 0_{n\times n}\}$, one can write $\xi^\top \bar{Q}\xi = e_t^\top Q e_t$. Now, using (27), (28), and Assumption 1 we derive

$$\dot{J}^*(\xi) \leq -\lambda_m(Q)\|e_t\|^2 - \beta^2\|m^\top\|^2\lambda_d^2(\xi) \\ - \|r^\top\|^2 g_M^2(\xi) + 2\beta^2 v^{*\top}(\xi)Mv^*(\xi) + \gamma J^*(\xi) \\ + \beta^2\|m^\top d(\xi)\|^2 + 2\|r^\top\|^2\mathcal{L}^2\|e_k\|^2 \\ + 2\|r^\top G^+(\xi)L(\xi)d(\xi)\|^2 \\ - \|r^\top(u^*(\xi_k) + G^+(\xi)L(\xi)d(\xi))\|^2 \\ \leq -\lambda_m(Q)\|e_t\|^2 - \|r^\top\|^2(g_M^2(\xi) \\ - 2\|G^+(\xi)L(\xi)d(\xi)\|^2) + 2\|r^\top\|^2\mathcal{L}^2\|e_k\|^2 \\ - \beta^2\|m^\top\|^2(\lambda_d^2(\xi) - \|d(\xi)\|^2) \\ + 2\beta^2 v^{*\top}(\xi)Mv^*(\xi) + \gamma J^*(\xi) \\ - \|r^\top(u^*(\xi_k) + G^+(\xi)L(\xi)d(\xi))\|^2 \\ \leq -\eta_1^2\lambda_m(Q)\|e_t\|^2 + (\eta_1^2 - 1)\lambda_m(Q)\|e_t\|^2 \\ + 2\|r^\top\|^2\mathcal{L}^2\|e_k\|^2 + 2\beta^2\|m^\top v^*(\xi)\|^2 \\ + \gamma J^*(\xi). \tag{29}$$

Hence, when the triggering rule stated in Theorem 1 is satisfied and $\gamma = 0$, then using (29) we can write

$$\dot{J}^*(\xi) \leq -\eta_1^2\lambda_m(Q)\|e_t(t))\|^2. \tag{30}$$

Thus, the system is asymptotically stable for $\gamma = 0$. When $\gamma \neq 0$, then

$$\dot{J}^*(\xi) \leq \gamma J^*(\xi) - \eta_1^2\lambda_m(Q)\|e_t(t))\|^2. \tag{31}$$

Since $J^*(\xi)$ is positive definite and bounded on Ω, let J^*_{max} be the maximum value of $J^*(\xi)$. So, from (31), $\dot{J}^*(\xi) \leq 0$ only if e_t lies out of the set

$$\Omega_{e_t} = \left\{ e_t : \|e_t\| \leq \frac{1}{\eta_1}\sqrt{\frac{\gamma J^*_{max}}{\lambda_m(Q)}} \right\}. \tag{32}$$

Thus we conclude that for $\gamma \neq 0$, the tracking error $e_t(t)$ is uniformly ultimately bounded and the ultimate bound is $\frac{1}{\eta_1}\sqrt{\frac{\gamma J^*_{max}}{\lambda_m(Q)}}$. □

Remark 2 In this work, the control policy $\mu^*(\xi_k)$ is formulated under the event-triggered framework, but the augmented control policy $v^*(\xi)$ is formulated under the time-triggered framework. There are two reasons behind this. First, the control policy to be used in the uncertain system is $\mu^*(\xi_k)$ not the augmented control $v^*(\xi)$. Second, if we also consider the augmented control in the event-triggering framework, then it becomes very difficult to obtain the event-triggering rule (21).

Remark 3 The lower bond of the minimum event interval $\Delta t_{\min}$ can be expressed as

$$\Delta t_{\min} \geq \frac{1}{\mathcal{P}} \ln(1 + T_{min}),$$

where

$$T_{min} = \min_{k\in\mathbb{N}} \left\{ \frac{||e_k(t_{k+1})||}{||\xi_k|| + \pi} \right\} > 0$$

and $e_k(t_{k+1}) = \xi_k - \xi(t_{k+1})$, $\mathcal{P}$ and π are positive constant satisfying $F(\xi) + G(\xi)u(\xi) + \Delta F(\xi) \leq \mathcal{P}\|\xi\|+\pi$. Note that the positive constants $\mathcal{P}$ and π exist because $F(\xi)+G(\xi)u$ is Lipschitz continuous and the terms $d(\xi)$ and $G^+(\xi)\Delta F(\xi)$ are upper bounded. The theoretical proof is similar to [32]. We have excluded the proof to avoid repetition. In the simulation result, we have presented that the intersample time indeed has a lower limit which is larger than zero. As a result, the infamous Zeno behavior is avoided.

4 ACDs for solving event-based HJB equation

In this section, a single critic network is employed to approximate the optimal value of the cost function under the ADP framework. The optimal cost function can be reconstructed on Ω, utilizing the neural network's universal approximation property and l number of hidden layer neurons, as

$$J^*(\xi) = \omega_c^\top \sigma_c(\xi) + \epsilon_c(\xi), \tag{33}$$

where $\omega_c \in \mathbb{R}^l$ is the actual weight vector of critic network, $\sigma_c(\xi) \in \mathbb{R}^l$ is the activation function, and $\epsilon_c(\xi)$ is the reconstruction error. Next, we obtain the gradient of (33) as

$$\nabla J^*(\xi) = (\nabla\sigma_c(\xi))^\top \omega_c + \nabla\epsilon_c(\xi). \tag{34}$$

Due to the unavailability of the actual weight vector ω_c, the approximate weight vector $\hat{\omega}_c$ is used to form a critic network to estimate the value of the optimal cost function $J^*(\xi)$ as follows

$$\hat{J}(\xi) = \hat{\omega}_c^\top \sigma_c(\xi). \tag{35}$$

Then the gradient of (35) is

$$\nabla\hat{J}(\xi) = (\nabla\sigma_c(\xi))^\top \hat{\omega}_c. \tag{36}$$

Considering (34) we present the augmented control law (14) and the event-based control law (19) as

$$v^*(\xi) = -\frac{1}{2\beta^2} M^{-1} K^\top(\xi)((\nabla\sigma_c(\xi))^\top \omega_c + \nabla\epsilon_c(\xi)) \tag{37}$$

and

$$\mu^*(\xi_k) = -\frac{1}{2} R^{-1} G^\top(\xi_k)((\nabla\sigma_c(\xi_k))^\top \omega_c + \nabla\epsilon_c(\xi_k)), \tag{38}$$

respectively. Then by using (36), the approximate value of $v^*(\xi)$ and $\mu^*(\xi_k)$ can be obtained as

$$\hat{v}(\xi) = -\frac{1}{2\beta^2} M^{-1} K^\top(\xi)(\nabla\sigma_c(\xi))^\top \hat{\omega}_c \tag{39}$$

and

$$\hat{\mu}(\xi_k) = -\frac{1}{2} R^{-1} G^\top(\xi_k)(\nabla\sigma_c(\xi_k))^\top \hat{\omega}_c, \tag{40}$$

respectively. Substituting $J^*(\xi)$ from (33) into (10), we obtain

$$\begin{aligned} &H(\xi, \omega_c, \mu^*(\xi_k), v^*(\xi)) \\ &\quad = \omega_c^\top \nabla\sigma_c(\xi)(F(\xi) + G(\xi)\mu^*(\xi_k) + K(\xi)v^*(\xi)) \\ &\qquad + \|r^\top\|^2 g_M^2(\xi) + \beta^2\|m^\top\|^2\lambda_d^2(\xi) \\ &\qquad + U(\xi, \mu^*(\xi_k), v^*(\xi)) - \gamma\omega_c^\top\sigma_c(\xi) \\ &\quad \triangleq e_{cH}, \end{aligned} \tag{41}$$

where $e_{cH} = -(\nabla\epsilon_c(\xi))^\top(F(\xi) + G(\xi)\mu^*(\xi_k) + K(\xi)v^*(\xi)) + \gamma\epsilon_c(\xi)$ is the residual error because of the reconstruction error associated with the NN approximation. Now the Hamiltonian (10) is approximated as

$$\begin{aligned}&\hat{H}(\xi, \hat{\omega}_c, \hat{\mu}(\xi_k), \hat{v}(\xi))\\&\quad= \hat{\omega}_c^\top \nabla\sigma_c(\xi)(F(\xi) + G(\xi)\hat{\mu}(\xi_k) + K(\xi)\hat{v}(\xi))\\&\qquad+ \|r^\top\|^2 g_M^2(\xi) + \beta^2\|m^\top\|^2\lambda_d^2(\xi)\\&\qquad+ U(\xi, \hat{\mu}(\xi_k), \hat{v}(\xi)) - \gamma\hat{\omega}_c^\top\sigma_c(\xi).\end{aligned} \tag{42}$$

From the HJB equation it is evident that $H(\xi, \omega_c, \mu^*(\xi_k), v^*(\xi)) = 0$. So, the approximation error of Hamiltonian is given by

$$\begin{aligned}e_c &= \hat{H}(\xi, \hat{\omega}_c, \hat{\mu}(\xi_k), \hat{v}(\xi)) - H(\xi, \omega_c, \mu^*(\xi_k),\\ v^*(\xi)) &= \hat{\omega}_c^\top\nabla\sigma_c(\xi)(F(\xi) + G(\xi)\hat{\mu}(\xi_k)\\&\quad+ K(\xi)\hat{v}(\xi)) + \|r^\top\|^2 g_M^2(\xi)\\&\quad+ \beta^2\|m^\top\|^2\lambda_d^2(\xi)\\&\quad+ U(\xi, \hat{\mu}(\xi_k), \hat{v}(\xi)) - \gamma\hat{\omega}_c^\top\sigma_c(\xi)\\&= \|r^\top\|^2 g_M^2(\xi) + \beta^2\|m^\top\|^2\lambda_d^2(\xi)\\&\quad+ U(\xi, \hat{\mu}(\xi_k), \hat{v}(\xi)) + \hat{\omega}_c^\top\phi,\end{aligned} \tag{43}$$

where $\phi = \nabla\sigma_c(\xi)(F(\xi)+G(\xi)\hat{\mu}(\xi_k)+K(\xi)\hat{v}(\xi)) - \gamma\sigma_c(\xi)$.

Now, to ensure e_c given in (43) to be sufficiently small, we need to train the critic network to obtain appropriate weights. For that, the objective function constructed as $E = (1/2)e_c^\top e_c$ is minimized by using the steepest descent technique. Based on this approach the tuning rule is given as

$$\begin{aligned}\dot{\hat{\omega}}_{c1} &= \frac{-l_c}{(1+\phi^\top\phi)^2}\frac{\partial E}{\partial\hat{\omega}_c}\\&= \frac{-l_c\phi}{(1+\phi^\top\phi)^2}(\hat{\omega}_c^\top\phi + \|r^\top\|^2 g_M^2(\xi)\\&\quad+ \beta^2\|m^\top\|^2\lambda_d^2(\xi) + U(\xi, \hat{\mu}(\xi_k), \hat{v}(\xi)),\end{aligned} \tag{44}$$

where $l_c > 0$ is a design parameter which is also known as the critic network's learning rate and $1/(1+\phi^\top\phi)^2$ is introduced to normalize ϕ. However, the tuning rule (44) has following drawbacks

1. An initial stabilizing control is needed at the beginning of the critic weight vector learning process while using the tuning rule provided in (44). However, in some practical applications, determining the initial admissible control can be difficult.
2. The term $\phi/(1+\phi^\top\phi)^2$ in (44) should be held persistently exciting to guarantee the convergence of the critic weights to valid optimal values. To meet the persistency of excitation (PE) condition, usually a probing noise is applied to the control input during the initial period of the critic weights tuning process. However, the probing noise can cause the system to become unstable.

To overcome the above drawbacks, we modify the tuning rule (44) via the Lyapunov approach. Before continuing the following assumption, which is similar to [20], is presented.

Assumption 2 Let us consider $V(\xi)$ be a continuously differentiable Lyapunov function candidate for the system (7) under the action of control policies given by (37) and (38), and satisfy

$$\begin{aligned}&\dot{V}(\xi)\\&\quad= (\nabla V(\xi))^\top(F(\xi) + G(\xi)\mu^*(\xi_k) + K(\xi)v^*(\xi))\\&\quad< 0.\end{aligned} \tag{45}$$

Moreover, there exists a symmetric positive definite matrix $\Lambda \in \mathbb{R}^{2n}$ defined on Ω ensuring

$$\begin{aligned}&(\nabla V(\xi))^\top(F(\xi) + G(\xi)\mu^*(\xi_k) + K(\xi)v^*(\xi))\\&\quad= -(\nabla V(\xi))^\top\Lambda\nabla V(\xi)\\&\quad\le -\lambda_m(\Lambda)\|V(\xi)\|^2.\end{aligned} \tag{46}$$

Remark 4 $F(\xi) + G(\xi)\mu^*(\xi_k) + K(\xi)v^*(\xi)$ is frequently considered to be bounded by a positive constant on a compact set Ω [48]. In other words, there exist a constant z_1 such that $\|F(\xi) + G(\xi)\mu^*(\xi_k) + K(\xi)v^*(\xi)\| \le z_1$. Here we assumed that $F(\xi) + G(\xi)\mu^*(\xi_k)+K(\xi)v^*(\xi)$ is bounded by a function with respect to ξ, which is less stringent than the constant upper bound assumption. Without loss of generality, we consider that $\|(F(\xi) + G(\xi)\mu^*(\xi_k) + K(\xi)v^*(\xi))\| \le z_2\|\nabla V(\xi)\|$, where z_2 is a positive constant. In this regard, we can write $\|(\nabla V(\xi))^\top(F(\xi)+G(\xi)\mu^*(\xi_k)+K(\xi)v^*(\xi))\| \le z_2\|\nabla V(\xi)\|^2$. Observing (45), one can find that (46) is reasonable. In simulation, a polynomial with respect to ξ is chosen as $V(\xi)$.

While we apply the approximated control policies (39) and (40) to the auxiliary system (7), to avoid instability we need to avoid the possibility

$$(\nabla V(\xi))^{\top}(F(\xi) + G(\xi)\hat{\mu}(\xi_k) + K(\xi)\hat{\upsilon}(\xi)) > 0. \quad (47)$$

To avoid (47), the training process is enhanced by introducing an additional term which is obtained using the steepest descent method as given below

$$\begin{aligned}\dot{\hat{\omega}}_{c2} &= l_s \frac{\partial((\nabla V(\xi))^{\top}(F(\xi) + G(\xi)\hat{\mu}(\xi_k) + K(\xi)\hat{\upsilon}(\xi)))}{\partial\hat{\omega}_c} \\ &= \frac{l_s}{2}(\nabla\sigma_c(\xi_k)G(\xi_k)R^{-1}G^{\top}(\xi)\nabla V(\xi) \\ &\quad + \frac{1}{\beta^2}\nabla\sigma_c(\xi)K(\xi)M^{-1}K^{\top}(\xi)\nabla V(\xi)), \quad (48)\end{aligned}$$

where $l_s > 0$ is a design parameter. Now, the modified critic weights tuning rule is obtained by adding the stabilizing term (48) to the traditional tuning rule (44) as

$$\begin{aligned}\dot{\hat{\omega}}_c &= \dot{\hat{\omega}}_{c1} + \dot{\hat{\omega}}_{c2} \\ &= \frac{-l_c\phi}{(1+\phi^{\top}\phi)^2}(\hat{\omega}_c^{\top}\phi + \|r^{\top}\|^2 g_M^2(\xi) \\ &\quad + \beta^2\|m^{\top}\|^2\lambda_d^2(\xi) + U(\xi, \hat{\mu}(\xi_k), \hat{\upsilon}(\xi))) \\ &\quad + \frac{l_s}{2}(\nabla\sigma_c(\xi_k)G(\xi_k)R^{-1}G^{\top}(\xi)\nabla V(\xi) \\ &\quad + \frac{1}{\beta^2}\nabla\sigma_c(\xi)K(\xi)M^{-1}K^{\top}(\xi)\nabla V(\xi)). \quad (49)\end{aligned}$$

Remark 5 The new tuning rule (49) can eliminate the need of initial admissible control. Hence, we can initialize the critic weight vector to zero while learning the appropriate critic weights. Moreover, the risk of instability due to the addition of probing noise to fulfill the PE condition is also eliminated.

The critic weights approximation error $\tilde{\omega}_c$ is defined as the difference between the ideal and the approximate weight vector, i.e., $\tilde{\omega}_c = \omega_c - \hat{\omega}_c$. From (41) and (43) we obtain

$$e_c = -\tilde{\omega}_c^{\top}\phi + e_{cH}. \quad (50)$$

Then, using (49) and (50), the critic weights approximation error dynamics is presented as

$$\begin{aligned}\dot{\tilde{\omega}}_c &= \frac{-l_c\phi}{(1+\phi^{\top}\phi)^2}(\tilde{\omega}_c^{\top}\phi - e_{cH}) \\ &\quad - \frac{l_s}{2}(\nabla\sigma_c(\xi_k)G(\xi_k)R^{-1}G^{\top}(\xi)\nabla V(\xi) \\ &\quad + \frac{1}{\beta^2}\nabla\sigma_c(\xi)K(\xi)M^{-1}K^{\top}(\xi)\nabla V(\xi)). \quad (51)\end{aligned}$$

The closed-loop system functions as an impulsive dynamical system consisting of flow dynamics and jump dynamics under the event-based control law. Let us consider an augmented state vector $\psi = [\xi^{\top}, \xi_k^{\top}, \tilde{\omega}_c^{\top}]^{\top}$. Then, the flow dynamics of the closed-loop system, which occurs for all $t \in [t_k, t_{k+1})$, can be presented as

$$\dot{\psi}(t) = \begin{bmatrix} F(\xi) + G(\xi)\hat{\mu}(\xi_k) + K(\xi)\hat{\upsilon}(\xi) \\ \hdashline 0 \\ \hdashline \frac{-l_c\phi}{(1+\phi^{\top}\phi)^2}(\tilde{\omega}_c^{\top}\phi - e_{cH}) \\ -\frac{l_s}{2}(\nabla\sigma_c(\xi_k)G(\xi_k)R^{-1}G^{\top}(\xi)\nabla V(\xi) \\ +\frac{1}{\beta^2}\nabla\sigma_c(\xi)K(\xi)M^{-1}K^{\top}(\xi)\nabla V(\xi)) \end{bmatrix},$$
$$\forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \quad (52)$$

and the jump dynamics of the closed-loop system, which occurs for all $t \in t_{k+1}$, can be presented as

$$\psi(t^+) = \psi(t) + \begin{bmatrix} 0 \\ \xi_k - \xi(t) \\ 0 \end{bmatrix}, \quad \forall t \in t_{k+1}, k \in \mathbb{N}, \quad (53)$$

where $\psi(t^+) = \lim_{\varsigma\to 0^+}\psi(t+\varsigma)$ and $\varsigma \in (0, t_{k+1} - t_k)$.

5 Stability analysis

In this section, the stability of impulsive dynamical representation of closed-loop system, given by (52) and (53), is studied. Prior to moving forward some assumptions, which are common in the literature, are stated below [32].

Assumption 3 The augmented system dynamics $G(\xi)$ and $K(\xi)$ satisfy the following assumptions, where A, G_M, and K_M are positive constants.

1. The dynamics $G(\xi)$ satisfies the Lipschitz continuity such that $\|G(\xi) - G(\xi_k)\| \le A\|e_k(t)\|$.
2. The dynamics $G(\xi)$ and $K(\xi)$ are upper bounded by G_M and K_M, respectively.

Assumption 4 The following conditions hold on Ω, where B, $\nabla\sigma_{cM}$, $\nabla\epsilon_{cM}$, ω_{cM}, and e_{cHM} are positive constants.

1. The gradient of activation function satisfies the Lipschitz continuity such that $\|\nabla\sigma_c(\xi) - \nabla\sigma_c(\xi_k)\| \le B\|e_k(t)\|$.
2. The gradient of activation function $\nabla\sigma_c(\xi)$ and the gradient of neural approximation error $\nabla\epsilon_c(\xi)$ are upper bounded by $\nabla\sigma_{cM}$ and $\nabla\epsilon_{cM}$, respectively.
3. The ideal weight vector ω_c and the residual error e_{cH} are upper bounded by ω_{cM} and e_{cHM}, respectively.

Theorem 2 *Let Assumptions* 1 *to* 4 *be true. Then, under the control policies* (39) *and* (40), *the closed-loop auxiliary system* (7) *is asymptotically stable and the critic weights approximation error is uniformly ultimately bounded if the inequalities* (54) *and* (55) *hold, where* ($\eta_2 \in (0, 1)$) *is a design parameter, and the values of* Γ_1 *and* Γ_2 *are give by* (63) *and* (69), *respectively.*

$$\|e_k(t)\|^2 \le \frac{(1-\eta_2^2)\lambda_m(Q)\|e_t(t)\|^2 + \|r^\top\hat{\mu}(\xi_k)\|^2 - \beta^2\|m^\top\|^2\|\hat{v}(\xi)\|^2}{\|R^{-1}\|^2\|r^\top\|^2(A^2\nabla\sigma_{cM}^2 + B^2G_M^2)\|\hat{\omega}_c\|^2} \triangleq \|\hat{e}_T\|^2 \tag{54}$$

$$\|\tilde{\omega}_c\| > \sqrt{\frac{2\|R\|^2(1+\phi^\top\phi)(\Gamma_1+\Gamma_2+\gamma J_{max}^*)}{2(1+\phi^\top\phi)(l_c\|R\|^2\lambda_{\varphi m} - G_M^2\nabla\sigma_{cM}^2) - l_c\|R\|^2\lambda_{\varphi M}}} \tag{55}$$

Proof In light of the flow dynamics (52) and the jump dynamics (53) we consider the Lyapunov function candidate as

$$\Upsilon(t) = \Upsilon_1(t) + \Upsilon_2(t) + \Upsilon_3(t) + \Upsilon_4(t), \tag{56}$$

where $\Upsilon_1(t) = J^*(\xi)$, $\Upsilon_2(t) = J^*(\xi_k)$, $\Upsilon_3(t) = \frac{1}{2}\tilde{\omega}_c^\top\tilde{\omega}_c$ and $\Upsilon_4(t) = l_s V(\xi)$. Now, the analysis is separated into two cases.

Case 1. Events are not triggered, i.e., $t \in [t_k, t_{k+1})$. Taking the differentiation of (56) one can write

$$\dot{\Upsilon}(t) = \dot{\Upsilon}_1(t) + \dot{\Upsilon}_2(t) + \dot{\Upsilon}_3(t) + \dot{\Upsilon}_4(t). \tag{57}$$

It is evident that for $t \in [t_k, t_{k+1})$ $\dot{\Upsilon}_2(t) = 0$. Now, differentiating $\Upsilon_1(t)$ along the trajectory of $\dot{\xi}(t) = F(\xi) + G(\xi)\hat{\mu}(\xi_k) + K(\xi)\hat{v}(\xi)$, we obtain

$$\dot{\Upsilon}_1(t) = (\nabla J^*(\xi))^\top(F(\xi) + G(\xi)\hat{\mu}(\xi_k) + K(\xi)\hat{v}(\xi)).$$

Now using Eqs. (23) and (24), we derived

$$\begin{aligned}\dot{\Upsilon}_1(t) = &-\xi^\top\bar{Q}\xi - \beta^2\|m^\top\|^2\lambda_d^2(\xi) - \|r^\top\|^2 g_M^2(\xi) \\ &+ \gamma J^*(\xi) + u^{*\top}(\xi)Ru^*(\xi) \\ &- 2u^{*\top}(\xi)R\hat{\mu}(\xi_k) + \beta^2 v^{*\top}(\xi)Mv^*(\xi) \\ &- 2\beta^2 v^{*\top}(\xi)M\hat{v}(\xi).\end{aligned} \tag{58}$$

We can write

$$\begin{aligned}&u^{*\top}(\xi)Ru^*(\xi) - 2u^{*\top}(\xi)R\hat{\mu}(\xi_k) \\ &\le \|r^\top(u^*(\xi) - \hat{\mu}(\xi_k))\|^2 - \|r^\top\hat{\mu}(\xi_k)\|^2 \\ &\le \|r^\top\|^2\|\frac{1}{2}R^{-1}G^\top(\xi)(\nabla\sigma_c(\xi))^\top\hat{\omega}_c \\ &\quad - \frac{1}{2}R^{-1}G^\top(\xi_k)(\nabla\sigma_c(\xi_k))^\top\hat{\omega}_c \\ &\quad + \frac{1}{2}R^{-1}G^\top(\xi)(\nabla\sigma_c(\xi))^\top(\tilde{\omega}_c + \nabla\epsilon_c(\xi))\|^2 \\ &\quad - \|r^\top\hat{\mu}(\xi_k)\|^2 \\ &\le \frac{\|r^\top\|^2}{2}\|R^{-1}(G^\top(\xi)(\nabla\sigma_c(\xi))^\top \\ &\quad - G^\top(\xi_k)(\nabla\sigma_c(\xi_k))^\top)\hat{\omega}_c\|^2 - \|r^\top\hat{\mu}(\xi_k)\|^2 \\ &\quad + \frac{1}{2}\|R^{-1}G^\top(\xi)((\nabla\sigma_c(\xi))^\top\tilde{\omega}_c + \nabla\epsilon_c(\xi))\|^2 \\ &\le \|r^\top\|^2\|R^{-1}\|^2(A^2\nabla\sigma_{cM}^2 + B^2G_M^2)\|e_k\|^2\|\hat{\omega}_c\|^2 \\ &\quad + \frac{1}{\|R\|^2}G_M^2\nabla\sigma_{cM}^2\|\tilde{\omega}_c\|^2 + \frac{1}{\|R\|^2}G_M^2\nabla\epsilon_{cM} \\ &\quad - \|r^\top\hat{\mu}(\xi_k)\|^2,\end{aligned} \tag{59}$$

$$\begin{aligned}&- 2\beta^2 v^{*\top}(\xi)M\hat{v}(\xi) \\ &\le \beta^2 v^{*\top}(\xi)Mv^*(\xi) + \beta^2\hat{v}^\top(\xi)M\hat{v}(\xi)\end{aligned} \tag{60}$$

and

$$\begin{aligned}&2\beta^2 v^{*\top}(\xi)Mv^*(\xi) \\ &\le \frac{1}{2\beta^2\|M\|}\|K^\top(\xi)(\nabla\sigma_c(\xi))^\top(\omega_c + \nabla\epsilon_c)\|^2 \\ &\le \frac{1}{2\beta^2\|M\|}K_M^2\nabla\sigma_{cM}^2(\omega_{cM}^2 + \nabla\epsilon_{cM}).\end{aligned} \tag{61}$$

Based on the above three inequalities, (58) can be expressed as

$$\begin{aligned}\dot{\Upsilon}_1(t)\\ &\le -\xi^\top \bar{Q}\xi - \beta^2\|m^\top\|^2\lambda_d^2(\xi) - \|r^\top\|^2 g_M^2(\xi)\\ &\quad + \gamma J^*(\xi) + \beta^2\|m^\top\|^2\|\hat{v}(\xi)\|^2 - \|r^\top\hat{\mu}(\xi_k)\|^2\\ &\quad + \|r^\top\|^2\|R^{-1}\|^2(A^2\nabla\sigma_{cM}^2 + B^2G_M^2)\|e_k\|^2\|\hat{\omega}_c\|^2\\ &\quad + \frac{1}{\|R\|^2}G_M^2\nabla\sigma_{cM}^2\|\tilde{\omega}_c\|^2 + \Gamma_1, \qquad (62)\end{aligned}$$

where Γ_1 is a positive constant and it is expressed as

$$\Gamma_1 = \frac{G_M^2}{\|R\|^2}\nabla\epsilon_{cM}^2 + \frac{K_M^2\nabla\sigma_{cM}^2}{2\beta^2\|M\|}(\omega_{cM}^2 + \nabla\epsilon_{cM}^2). \quad (63)$$

We have $\omega_c^\top\phi = \phi^\top\omega_c$. Let $\varphi = \phi/(1+\phi^\top\phi)$. Now, using (51), the time derivative of $\Upsilon_3(t)$ is found as

$$\begin{aligned}\dot{\Upsilon}_3(t) &= -l_c\tilde{\omega}_c^\top\varphi\varphi^\top\tilde{\omega}_c + \frac{l_c}{(1+\phi^\top\phi)}\tilde{\omega}_c^\top\varphi e_{cH}\\ &\quad - \frac{l_s}{2}\tilde{\omega}_c^\top\nabla\sigma_c(\xi_k)G(\xi_k)R^{-1}G^\top(\xi)\nabla V(\xi)\\ &\quad - \frac{l_s}{2\beta^2}\tilde{\omega}_c^\top\nabla\sigma_c(\xi)K(\xi)M^{-1}K^\top(\xi)\nabla V(\xi). \qquad (64)\end{aligned}$$

Let $\lambda_M(\varphi\varphi^\top) = \lambda_{\varphi M}$ and $\lambda_m(\varphi\varphi^\top) = \lambda_{\varphi m}$. Then, Considering Young's inequality $2c^\top d \le c^\top c + d^\top d$ and Assumption 4, (64) can be expressed as

$$\begin{aligned}\dot{\Upsilon}_3(t)\\ &\le -l_c\lambda_{\varphi m}\|\tilde{\omega}_c\|^2\\ &\quad + \frac{l_c}{2(1+\phi^\top\phi)}(\lambda_{\varphi M}\|\tilde{\omega}_c\|^2 + e_{cHM}^2)\\ &\quad - \frac{l_s}{2}\tilde{\omega}_c^\top\nabla\sigma_c(\xi_k)G(\xi_k)R^{-1}G^\top(\xi)\nabla V(\xi)\\ &\quad - \frac{l_s}{2\beta^2}\tilde{\omega}_c^\top\nabla\sigma_c(\xi)K(\xi)M^{-1}K^\top(\xi)\nabla V(\xi). \qquad (65)\end{aligned}$$

Now, substituting $\tilde{\omega}_c = \omega_c - \hat{\omega}_c$ in last two terms of (65) and considering the control policies (39) and (40), one can write

$$\begin{aligned}\dot{\Upsilon}_3(t)\\ &\le -l_c\lambda_{\varphi m}\|\tilde{\omega}_c\|^2\\ &\quad + \frac{l_c}{2(1+\phi^\top\phi)}(\lambda_{\varphi M}\|\tilde{\omega}_c\|^2 + e_{cHM}^2)\\ &\quad - \frac{l_s}{2}(\nabla V(\xi))^\top G(\xi)R^{-1}G^\top(\xi_k)\nabla\sigma_c(\xi_k)\omega_c\\ &\quad - l_s(\nabla V(\xi))^\top G(\xi)\hat{\mu}(\xi_k)\\ &\quad - \frac{l_s}{2\beta^2}(\nabla V(\xi))^\top K(\xi)M^{-1}K^\top(\xi)\nabla\sigma_c(\xi)\omega_c\\ &\quad - l_s(\nabla V(\xi))^\top K(\xi)\hat{v}(\xi). \qquad (66)\end{aligned}$$

The derivative of $\Upsilon_4(t)$ is

$$\dot{\Upsilon}_4(t) = l_s\nabla V(\xi)(F(\xi) + G(\xi)\hat{\mu}(\xi_k) + K(\xi)\hat{v}(\xi)). \quad (67)$$

Now, combining (66) and (67) and using the control policies (37) and (38), one can write

$$\begin{aligned}\dot{\Upsilon}_3(t) + \dot{\Upsilon}_4(t)\\ &\le -l_c\lambda_{\varphi m}\|\tilde{\omega}_c\|^2\\ &\quad + \frac{l_c}{2(1+\phi^\top\phi)}(\lambda_{\varphi M}\|\tilde{\omega}_c\|^2 + e_{cHM}^2)\\ &\quad + l_s(\nabla V(\xi))^\top(F(\xi) + G(\xi)\mu^*(\xi_k) + K(\xi)v^*(\xi))\\ &\quad + \frac{l_s}{2}(\nabla V(\xi))^\top G(\xi)R^{-1}G^\top(\xi_k)\nabla\epsilon_c(\xi_k)\\ &\quad + \frac{l_s}{2\beta^2}(\nabla V(\xi))^\top K(\xi)M^{-1}K^\top(\xi)\nabla\epsilon_c(\xi).\end{aligned}$$

Now, utilizing Assumptions 1 to 4, we can write

$$\begin{aligned}\dot{\Upsilon}_3(t) + \dot{\Upsilon}_4(t)\\ &\le -l_c\lambda_{\varphi m}\|\tilde{\omega}_c\|^2\\ &\quad + \frac{l_c}{2(1+\phi^\top\phi)}(\lambda_{\varphi M}\|\tilde{\omega}_c\|^2 + e_{cHM}^2)\\ &\quad - l_s\lambda_m(\Lambda)\|V(\xi)\|^2 + l_s\kappa\|\nabla V(\xi)\|\\ &\le -l_c\lambda_{\varphi m}\|\tilde{\omega}_c\|^2 + \frac{l_c}{2(1+\phi^\top\phi)}\lambda_{\varphi M}\|\tilde{\omega}_c\|^2\\ &\quad + \frac{l_c}{2(1+\phi^\top\phi)}e_{cHM}^2 + \frac{l_s\kappa^2}{4\lambda_m(\Lambda)}\\ &\quad - l_s\lambda_m(\Lambda)\left(\|V(\xi)\| - \frac{\kappa}{2\lambda_m(\Lambda)}\right)^2\\ &\le -l_c\lambda_{\varphi m}\|\tilde{\omega}_c\|^2 + \frac{l_c\lambda_{\varphi M}}{2(1+\phi^\top\phi)}\|\tilde{\omega}_c\|^2 + \Gamma_2, \qquad (68)\end{aligned}$$

where $\kappa = \frac{1}{2}\nabla\epsilon_{cM}(G_M^2\|R^{-1}\| + \frac{1}{\beta^2}K_M^2\|M^{-1}\|)$ and the positive constant Γ_2 is given by

$$\Gamma_2 = \frac{l_c e_{cHM}^2}{2(1+\phi^\top\phi)} + \frac{l_s\kappa^2}{4\lambda_m(\Lambda)}. \tag{69}$$

Substituting (62) and (68) into (57), we obtain

$$\begin{aligned}\dot{\Upsilon}(t) &\le -\xi^\top\bar{Q}\xi - \beta^2\|m^\top\|^2\lambda_d^2(\xi) - \|r^\top\|^2 g_M^2(\xi)\\ &+ \gamma J^*(\xi) + \beta^2\|m^\top\|^2\|\hat{v}(\xi)\|^2\\ &+ \|r^\top\|^2\|R^{-1}\|^2(A^2\nabla\sigma_{cM}^2 + B^2G_M^2)\|e_k\|^2\|\hat{\omega}_c\|^2\\ &- \|r^\top\hat{\mu}(\xi_k)\|^2 + \frac{1}{\|R\|^2}G_M^2\nabla\sigma_{cM}^2\|\tilde{\omega}_c\|^2\\ &- l_c\lambda_{\varphi m}\|\tilde{\omega}_c\|^2 + \frac{l_c\lambda_{\varphi M}}{2(1+\phi^\top\phi)}\|\tilde{\omega}_c\|^2 + \Gamma_1 + \Gamma_2.\end{aligned} \tag{70}$$

Since $\bar{Q} = diag\{Q, 0_{n\times n}\}$, one can write $\xi^\top(t)\bar{Q}\xi(t) = e_t^\top(t)Qe_t(t)$. Now, introducing the design parameter η_2, (70) can be presented as

$$\begin{aligned}\dot{\Upsilon}(t) &\le -\eta_2^2\lambda_m(Q)\|e_t(t)\|^2 - (1-\eta_2^2)\lambda_m(Q)\|e_t(t)\|^2\\ &- \beta^2\|m^\top\|^2\lambda_d^2(\xi) - \|r^\top\|^2 g_M^2(\xi)\\ &+ \beta^2\|m^\top\|^2\|\hat{v}(\xi)\|^2 - \|r^\top\hat{\mu}(\xi_k)\|^2\\ &+ \|r^\top\|^2\|R^{-1}\|^2(A^2\nabla\sigma_{cM}^2 + B^2G_M^2)\|e_k\|^2\|\hat{\omega}_c\|^2\\ &+ \frac{G_M^2}{\|R\|^2}\nabla\sigma_{cM}^2\|\tilde{\omega}_c\|^2 - l_c\lambda_{\varphi m}\|\tilde{\omega}_c\|^2\\ &+ \frac{l_c\lambda_{\varphi M}}{2(1+\phi^\top\phi)}\|\tilde{\omega}_c\|^2 + \gamma J_{max}^* + \Gamma_1 + \Gamma_2.\end{aligned} \tag{71}$$

If the inequalities (54) and (55) mentioned in Theorem 2 hold then (71) implies

$$\begin{aligned}\dot{\Upsilon}(t) &\le -\eta_2^2\lambda_m(Q)\|e_t(t)\|^2 - \beta^2\|m^\top\|^2\lambda_d^2(\xi)\\ &\quad - \|r^\top\|^2 g_M^2(\xi)\\ &\quad < 0,\end{aligned}$$

i.e., the proposed Lyapunov function candidate has negative time derivative for all $t \in [t_k, t_{k+1})$.
Case 2. Events are triggered, i.e., $t \in t_{k+1}$. We derive the difference of the Lyapunov function candidate as

$$\begin{aligned}\Delta\Upsilon(t_k) &= J^*(\xi(t_k^+)) - J^*(\xi(t_k))\\ &+ \frac{1}{2}\tilde{\omega}_c^\top(t_k^+)\tilde{\omega}_c(t_k^+) - \frac{1}{2}\tilde{\omega}_c^\top(t_k)\tilde{\omega}_c(t_k))\\ &+ J^*(\xi_{k+1}) - J^*(\xi_k) + l_s(V(t_k^+) - V(t_k)),\end{aligned} \tag{72}$$

where $\xi(t_k^+) = \lim_{\varsigma\to 0^+}\xi(t_k+\varsigma)$ and $\varsigma \in (0, t_{k+1} - t_k)$. In Case 1 we derived that $\dot{\Upsilon}(t) < 0$ for all $t \in [t_k, t_{k+1})$, so

$$\begin{aligned}\Upsilon(t_k) &\ge \lim_{\varsigma\to 0^+}\Upsilon(t_k+\varsigma) \quad \forall\varsigma \in (0, t_{k+1}-t_k), k \in \mathbb{N}\\ &\triangleq \Upsilon(t_k^+).\end{aligned} \tag{73}$$

Thus, one can write

$$\begin{aligned}&J^*(\xi(t_k^+)) + \frac{1}{2}\tilde{\omega}_c^\top(t_k^+)\tilde{\omega}_c(t_k^+) + l_sV(t_k^+)\\ &\quad - J^*(\xi_k) - \frac{1}{2}\tilde{\omega}_c^\top(t_k)\tilde{\omega}_c(t_k)) - l_sV(t_k^+) \le 0.\end{aligned} \tag{74}$$

Hence, we can further express that

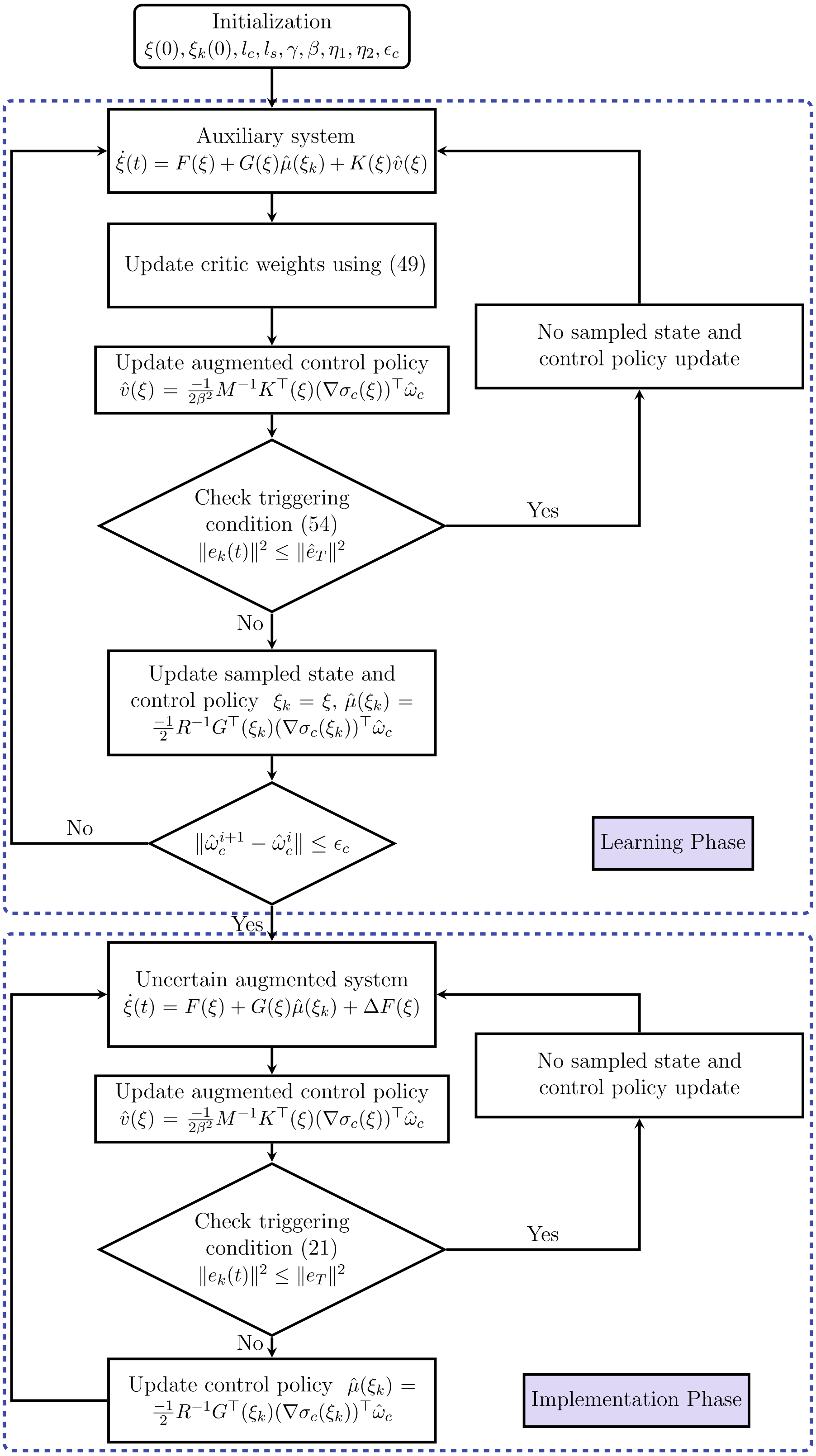
$$(J^*(\xi_{k+1}) - J^*(\xi_k)) \le -\vartheta\|e_{k+1}(t_k)\|, \tag{75}$$

where ϑ is a class k function and $e_{k+1}(t_k) = \xi_{k+1} - \xi_k$. The inequalities (74) and (75) imply the monotonically decreasing property of the proposed Lyapunov function candidate for all $t \in t_{k+1}$.

Thus from the two cases presented above, we conclude that the closed-loop system is asymptotically stable and the critic weights approximation error is uniformly ultimately bounded. □

A flowchart is given in Fig. 1 to explain the fundamental methodology of the proposed work, which comprises the learning and implementation phases. In the learning phase, the converged critic weights are obtained after sufficient iterations while using the event-triggering rule (54). Then the converged weights are passed to the implementation phase to obtain the approximate values of the optimal control policies $\mu^*(\xi_k)$ and $v^*(\xi)$ as $\hat{\mu}(\xi_k)$ and $\hat{v}(\xi)$, respectively. The approximated event-based control policy $\hat{\mu}(\xi_k)$ is applied to the uncertain nonlinear system while using the event triggering rule (21) to track the desired trajectory.

Fig. 1 Flowchart of the proposed method



Remark 6 The values of the sampling frequencies η_1 and η_2 are chosen such that the terms $\|e_T\|^2$ and $\|\hat{e}_T\|^2$ become positive, respectively. The increase in the value of η_1 and η_2 will increase the sampling frequency and the number of the event triggering instants. Furthermore, it improves tracking performance. However, we have to select the sampling frequency such that there is a trade-off between the number of triggering instants and the tracking performance. Similar to relevant literature [38], other parameters are chosen heuristically such that the convergence time of the critic weights and the number of triggering instants are minimum with acceptable tracking performance.

6 Simulation illustration

In this section, two simulation examples are presented to exhibit the efficacy of the proposed event-based robust trajectory tracking scheme. In Example 1, we have considered a linear system with unmatched uncertainty, and in Example 2, the spring-mass-damper system with nonlinear spring constant and unmatched uncertainty is considered.

6.1 Example 1

Consider the following linear unmatched uncertain system [21]

$$\dot{x} = \begin{bmatrix} x_2 \\ -100x_1 - 2x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \Delta f(x), \tag{76}$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the state vector, $u \in \mathbb{R}$ denotes the control input, $\Delta f(x) = l(x)d(x)$ and $l(x) = [1, 0]^\top$. The perturbation $d(x) = 0.5\theta_1 x_1 sin(x_2+\theta_2)$, where the parameters θ_1 and θ_2 are unknown. We consider $\theta_1 \in [-1, 1]$, $\theta_2 \in [-5, 5]$ and the upper bound of the perturbation $d(x)$ is $\lambda_d(x) = |x_1|$. Let $x_0 = [0.6, -0.5]^\top$ be the initial state. The desired trajectory $x_d(t)$ is generated from

$$\dot{x}_d(t) = \begin{bmatrix} x_{d2} \\ -100x_{d1} \end{bmatrix}, \tag{77}$$

where $x_d = [x_{d1}, x_{d2}]^\top \in \mathbb{R}^2$ with the initial condition $x_{d0} = [0.3, -0.3]^\top$. The tracking error e_t is defined as $e_t = x - x_d$, where $e_t = [e_{t1}, e_{t2}]^\top \in \mathbb{R}^2$, and initial condition $e_{to} = x_0 - x_{d0}$. Then an augmented state vector $\xi = [\xi_1, \xi_2, \xi_3, \xi_4]^\top \in \mathbb{R}^4$ is defined and following augmented system is formed

$$\dot{\xi} = \begin{bmatrix} \xi_2 \\ -100\xi_1 - 2(\xi_2 + \xi_4) \\ \xi_4 \\ -100\xi_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(\xi) + L(\xi)d(\xi), \tag{78}$$

where $L(\xi) = [1, 0, 0, 0]^\top$ and $d(\xi) = 0.5\theta_1(\xi_1 + \xi_3)sin((\xi_2 + \xi_4) + \theta_2)$. The initial condition $\xi_0 = [e_{t0}, x_{d0}]^\top = [0.3, -0.2, 0.3, -0.3]^\top$. The upper bound for $\lambda_d(\xi)$ is derived as $\lambda_d(\xi) = |\xi_1 + \xi_3|$. We have obtained $G^+(\xi) = [0, 1, 0, 0]$ so $(I - G(\xi)G^+(\xi))L(\xi) = [1, 0, 0, 0]^\top$. As in (7), the auxiliary system is formulated as

$$\dot{\xi} = \begin{bmatrix} \xi_2 \\ -100\xi_1 - 2(\xi_2 + \xi_4) \\ \xi_4 \\ -100\xi_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(\xi) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v(\xi). \tag{79}$$

Since, $\|G^+(\xi)L(\xi)d(\xi)\| = 0$ we have taken $g_M(\xi) = 0$. Let $R = I_1$, $M = I_1$ and $Q = 500I_2$. For the simulation purpose, consider $\gamma = 0.5$ and $\beta = 0.85$.

Our aim is to develop an event-based robust controller for the system (76) to track the reference trajectory generated by (77). As described in the theoretical analysis, to achieve this design criteria, the augmented system (78) is formed and then the original control problem is transformed to designing an event-based optimal controller for auxiliary system (79). Based on (8), the cost function for (79) can be presented as

$$J(\xi(t)) = \int_t^\infty e^{-0.5(\tau - t)}\{500\|e_t\|^2 + \|u(\xi)\|^2 + 0.72\|v(\xi)\|^2 + 0.72|\xi_1 + \xi_3|^2\}d\tau. \tag{80}$$

The critic network (35) is employed to find the solution of the event-based optimal control problem approximately. We have considered $l = 10$ numbers of hidden layer neurons and the weight vector of the critic network is represented as $\hat{\omega}_c = [\hat{\omega}_{c1}, \ldots, \hat{\omega}_{c10}]^\top$. The activation function for the critic network is selected as $\sigma_c(\xi) = [\xi_1^2, \xi_2^2, \xi_3^2, , \xi_4^2, \xi_1\xi_2, \xi_1\xi_3, \xi_1\xi_4, \xi_2\xi_3, \xi_2\xi_4, \xi_3\xi_4]^\top$. The weights are trained using the tuning rule (49) and the triggering condition (54) is used during the training process. The parameters used during the tuning process are $l_c = 3, l_s = 0.1, (A^2\nabla\sigma_{cM}^2 + B^2G_M^2) = 8$ and $\eta_2 = 0.7$.

To satisfy the PE condition, a small exponentially decreasing probing noise is applied to the control input for the initial 10 seconds of the training process. All the elements of the weight vector are initialized to zero. As shown in Fig. 2, the critic weight vector converges to $\hat{\omega}_c = [0.46, 3.64, 0.01, -0.69, -0.85, -0.32, -0.02, 0.71, -3.89, -1.60]^\top$. During the training process the event-based controller updates 5714 times. On the contrary, under the same design criteria, the time-based controller updates 18000 times.

Then, we used the converged weights to obtain the approximate values of control policies control polices (39) and (40). Now, we select $\theta_1 = -0.3$ and $\theta_2 = 5$ to demonstrate the trajectory tracking ability of the designed control policy $\hat{\mu}(\xi_k)$ and the triggering rule described in (21). We considered $\eta_1 = 0.65$ and $\mathcal{L} = 2.5$. The sampling period is taken as 0.01 second. The tracking performance of the designed controller is displayed in Figs. 3 and 4. The obtained event-based control policy $\hat{\mu}(\xi_k)$ is shown in Fig. 5.

The value of the sampling frequency η_1 is considered as $\eta_1 \in (0, 1)$. Table 1 illustrates the relationship between η_1 and the number of triggering instants N_s. From the table it is clear that as η_1 increases the number of event-triggering instants N_s increases.

The evolution of the triggering condition with $\|e_T\|^2$ and $\|e_k\|^2$ is displayed in Fig. 6. The sampling period is shown in Fig. 7. The minimal intersample time is found to be 0.01 second. That means the infamous Zeno behavior is excluded. Furthermore, Fig. 7 also conveys that only 435 state samples are used during the tracking process. So, the controller is updated only 435 times. Nonetheless, if we use the time-triggering method under the same condition then 1600 samples are required. So, developed event-based tracking control strategy reduces the resources used significantly.

Next, in order to show that the derived controller is robust, we have taken $\theta_1 = 0.4$ and $\theta_2 = -1$. The tracking performance for new value of θ_1 and θ_2 is shown in Figs. 8 and 9. In this scenario, the event-based con-

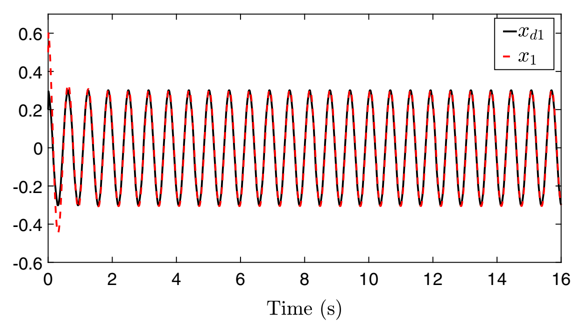


Fig. 3 Tracking performance of x_1 for $\theta_1 = -0.3$ and $\theta_2 = 5$

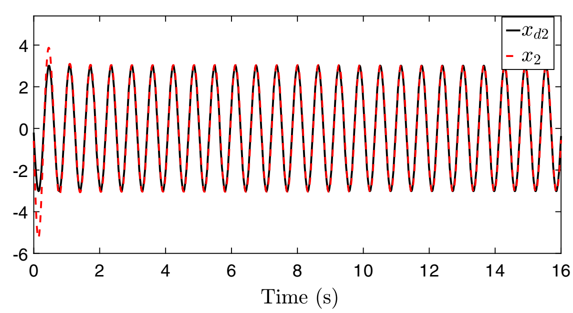


Fig. 4 Tracking performance of x_2 for $\theta_1 = -0.3$ and $\theta_2 = 5$

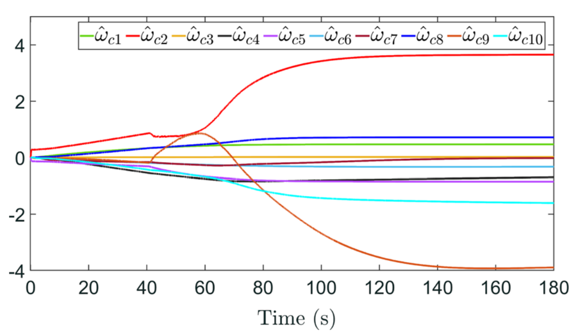


Fig. 2 Convergence process of critic weights

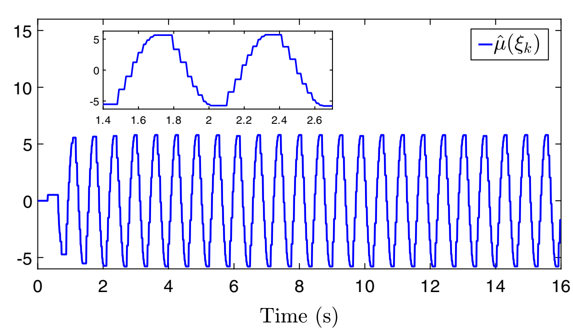


Fig. 5 Event-based control input

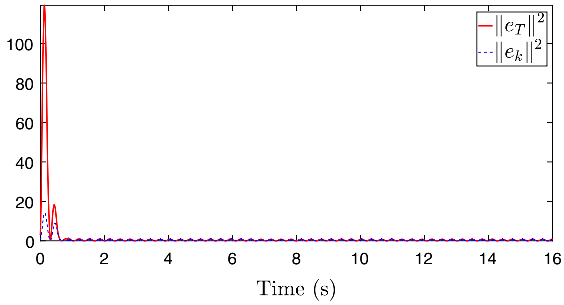


Fig. 6 Evolution of the triggering condition with $\|e_T\|^2$ and $\|e_k\|^2$

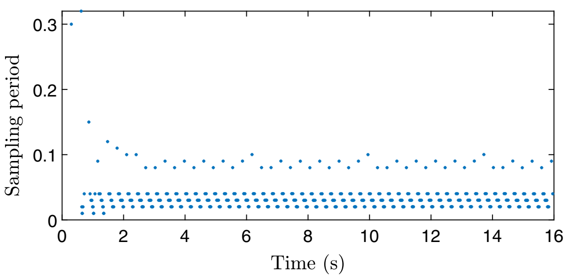


Fig. 7 Triggering instants during the tracking process

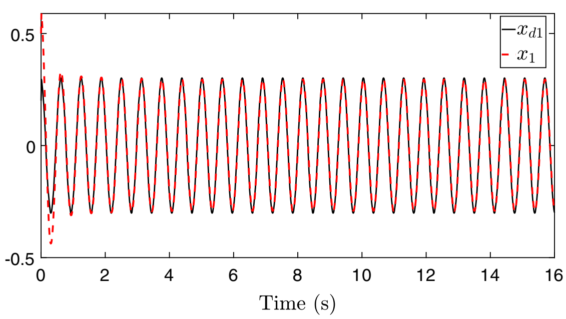


Fig. 8 Tracking performance of x_1 for $\theta_1 = 0.4$ and $\theta_2 = -1$

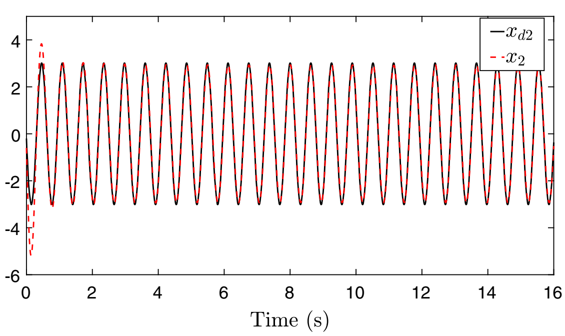


Fig. 9 Tracking performance of x_2 for $\theta_1 = 0.4$ and $\theta_2 = -1$

Table 1 Effect of η_1 on number of triggering instants

Parameters	Case 1	Case 2	Case 3
η_1	0.65	0.75	0.8
N_s	435	777	906

troller updates 468 times only. On the contrary, the conventional time-triggered controller updates 1600 times under the same design criteria.

6.2 Example 2

Consider the spring-mass-damper system [36]

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u, \tag{81}$$

where $x = [x_1, x_2]^\top$, x_1 is the position and x_2 is the velocity, m represents the mass of the object, k denotes the spring constant and c is the damping. Let $m = 1Kg$, $c = 0.5N$.s/m and the spring is nonlinear with the nonlinearity $k(x) = -5x^3$N/m. After adding an unmatched uncertainty $\Delta f(x)$, the system dynamics is obtained as

$$\dot{x} = \begin{bmatrix} x_2 \\ -5x_1^3 - 0.5x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \Delta f(x), \tag{82}$$

where $\Delta f(x) = l(x)d(x)$ and $l(x) = [1, 0]^\top$. The perturbation $d(x) = 0.5\theta_1 x_1 x_2 sin(x_1) cos(x_2+\theta_2)$, where the parameters θ_1 and θ_2 are unknown. We consider $\theta_1 \in [-1, 1]$, $\theta_2 \in [-5, 5]$ and the upper bound of the perturbation $d(x)$ is $\lambda_d(x) = |x_2|$. Let $x_0 = [0.5, 0.2]^\top$ be the initial state. The desired trajectory $x_d(t)$ is generated from

$$\dot{x}_d(t) = \begin{bmatrix} x_{d2} \\ -5x_{d1} \end{bmatrix}, \tag{83}$$

where $x_d = [x_{d1}, x_{d2}]^\top \in \mathbb{R}^2$ with the initial condition $x_{d0} = [0.2, -0.2]^\top$. Then an augmented state vector $\xi = [\xi_1, \xi_2, \xi_3, \xi_4]^\top \in \mathbb{R}^4$ is defined and following augmented system is formed

$$\dot{\xi} = \begin{bmatrix} \xi_2 \\ -5(\xi_1+\xi_3)^3 - 0.5(\xi_2+\xi_4) + 5\xi_3 \\ \xi_4 \\ -5\xi_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(\xi) + L(\xi)d(\xi), \quad (84)$$

where $L(\xi) = [1, 0, 0, 0]^\top$ and $d(\xi) = 0.5\theta_1(\xi_1 + \xi_3)(\xi_2 + \xi_4) sin(\xi_1 + \xi_3) cos((\xi_2 + \xi_4) + \theta_2)$. The initial condition $\xi_0 = [0.3, 0.4, 0.2, -0.2]^\top$. The upper bound for $\lambda_d(\xi)$ is derived as $\lambda_d(\xi) = |\xi_2 + \xi_4|$. The auxiliary system is formulated as

$$\dot{\xi} = \begin{bmatrix} \xi_2 \\ -5(\xi_1+\xi_3)^3 - 0.5(\xi_2+\xi_4) + 5\xi_3 \\ \xi_4 \\ -5\xi_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(\xi) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v(\xi). \quad (85)$$

Since $\|G^+(\xi)L(\xi)d(\xi)\| = 0$, we have taken $g_M(\xi) = 0$. Let $R = I_1$, $M = I_1$ and $Q = 300I_2$. For the simulation purpose, consider $\gamma = 1.2$ and $\beta = 0.9$.

Based on (8), the cost function for (85) can be presented as

$$J(\xi(t)) = \int_t^\infty \mathrm{e}^{-1.2(\tau-t)}\{300\|e_t\|^2 + \|u(\xi)\|^2 + 0.81\|v(\xi)\|^2 + 0.81|\xi_2+\xi_4|^2\}\mathrm{d}\tau. \quad (86)$$

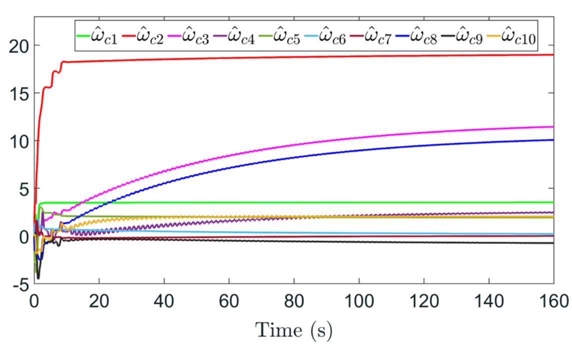


Fig. 10 Convergence process of critic weights

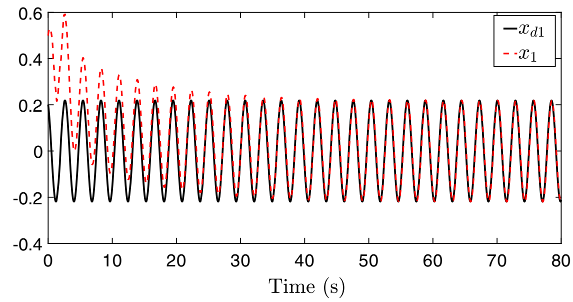


Fig. 11 Tracking performance of x_1 for $\theta_1 = -0.9$ and $\theta_2 = -0.3$

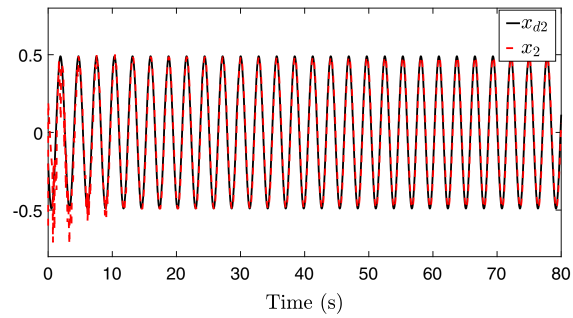


Fig. 12 Tracking performance of x_2 for $\theta_1 = -0.9$ and $\theta_2 = -0.3$

We have considered $l = 10$ numbers of hidden layer neurons and the activation function is chosen as $\sigma_c(\xi) = [\xi_1^2, \xi_1\xi_2, \xi_1\xi_3, \xi_1\xi_4, \xi_2^2, \xi_2\xi_3, \xi_2\xi_4, \xi_3^2, \xi_3\xi_4, \xi_4^2]^\top$. The parameters used during the tuning process are $l_c = 4$, $l_s = 0.5$, $(A^2\nabla\sigma_{cM}^2 + B^2G_M^2) = 8$ and $\eta_2 = 0.7$.

To fulfill the PE criteria, a small exponentially decreasing probing noise is applied to the control input for the initial 10 seconds of the training process. All the elements of the weight vector are initialized to zero. The critic weight vector $\hat{\omega}_c$ converges to [3.53, 19, 11.46, 2.49, 1.93, 0.23, 0.01, 10.07, −0.73, 2.05]$^\top$ as shown in Fig. 10. During the training process the event-based controller updates 8947 times. On the contrary, under the same design criteria, the time-based controller updates 16000 times.

Then, we used the converged weights to obtain the control polices (39) and (40). Now, we select $\theta_1 = -0.9$ and $\theta_2 = -0.3$ to check the trajectory tracking performance of the designed control policy $\hat{\mu}(\xi_k)$ and the triggering rule described in (21). We considered $\eta_1 = 0.7$ and $\mathcal{L} = 10$. The sampling period is taken as 0.01 second. The performance of the designed tracking

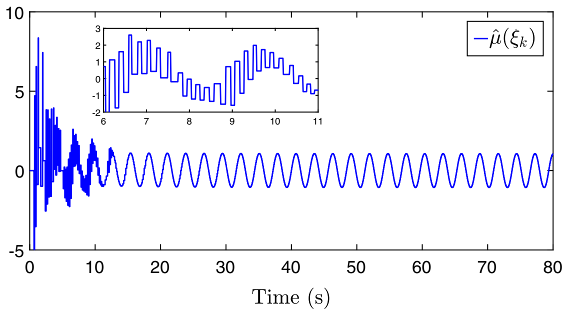


Fig. 13 Event-based control input

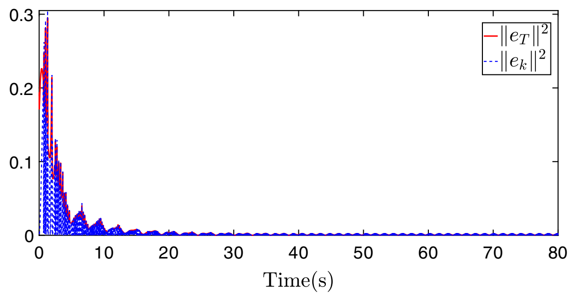


Fig. 14 Evolution of the triggering condition with $\|e_T\|^2$ and $\|e_k\|^2$

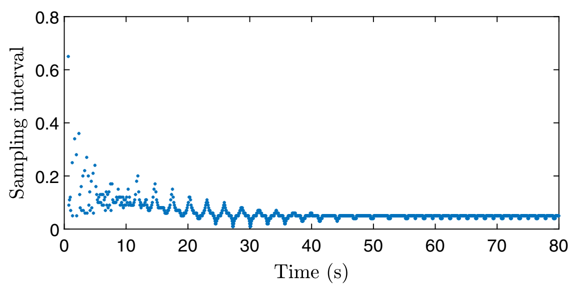


Fig. 15 Triggering instants during the tracking process

controller is displayed in Figs. 11 and 12. The obtained event-based control policy $\hat{\mu}(\xi_k)$ is displayed in Fig. 13.

The evolution of the triggering condition with $\|e_T\|^2$ and $\|e_k\|^2$ is displayed in Fig. 14. The sampling period is shown in Fig. 15. The minimal intersample time is found to be 0.01 second. That means the infamous Zeno behavior is excluded. Furthermore, Fig. 15 also conveys that only 1452 state samples are used during the tracking process. So, the controller is updated 1452 times only. Nonetheless, if we use the time-triggering method under the same condition then 8000 samples are required. So, developed event-based tracking control strategy reduces the resources used significantly.

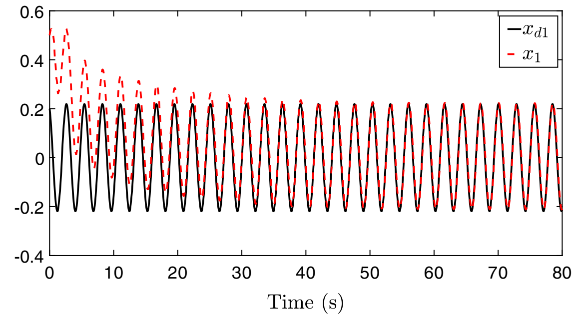


Fig. 16 Tracking performance of x_1 for $\theta_1 = 0.8$ and $\theta_2 = 4$

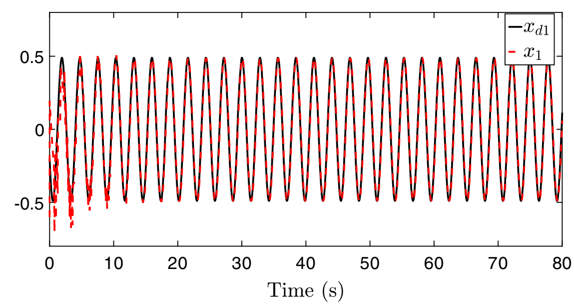


Fig. 17 Tracking performance of x_2 for $\theta_1 = 0.8$ and $\theta_2 = 4$

Next, in order to show that the derived controller is robust, we have taken $\theta_1 = 0.8$ and $\theta_2 = 4$. The tracking performance for new value of θ_1 and θ_2 is shown in Figs. 16 and 17. Here, only 1438 state samples are used during the tracking process. In other words, the event-based controller updates 1438 times only. On the other hand, the conventional time-triggered controller updates 8000 times under the same design criteria.

7 Conclusion

In this work, an event-based robust tracking strategy for an unmatched uncertain system is developed. By forming an auxiliary system and decomposing the unmatched uncertainty, the original control problem is transformed into obtaining an optimal controller for an auxiliary system. The related event-based HJB equation is solved via the ADP approach. The critic weights tuning law is modified to avoid the need for initial stabilizing control at the beginning of the tuning process. In the meantime, a novel event-triggering law is developed, and the uniform ultimate boundedness of the tracking error is verified using the Lyapunov method. The closed-loop auxiliary system's asymptotic stability and the uniform ultimate boundedness of the critic approximation error are assured. Finally, two simulation examples are included to demonstrate the usefulness of the proposed methodology.

The main limitations of the proposed work and future scope are as follows.

1. The work developed in this article needs complete knowledge of system dynamics. However, we may not know the system dynamics completely in many applications.
2. The proposed method is not suitable for time-delay systems. In the future, a tracking controller will be designed for uncertain nonlinear systems with time-delay using event-based ADP approach.

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Availability of data Data sharing not applicable to this paper as no datasets were generated or analyzed during this work.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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