



# Analysis of lumps, single-stripe, breather-wave, and two-wave solutions to the generalized perturbed-KdV equation by means of Hirota's bilinear method

Marwan Alquran · Rahaf Alhami

Received: 15 March 2022 / Accepted: 4 May 2022 / Published online: 16 May 2022  
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

**Abstract** In this paper, we implement the Hirota's bilinear method to extract diverse wave profiles to the generalized perturbed-KdV equation when the test function approaches are taken into consideration. Several novel solutions such as lump-soliton, lump-periodic, single-stripe soliton, breather waves, and two-wave solutions are obtained to the proposed model. We conduct some graphical analysis including 2D and 3D plots to show the physical structures of the recovery solutions. On the other hand, this work contains a correction of previous published results for a special case of the perturbed KdV. Moreover, we investigate the significance of the nonlinearity, perturbation, and dispersion parameters being acting on the propagation of the perturbed KdV. Finally, our obtained solutions are verified by inserting them into the governing equation.

**Keywords** Perturbed-KdV equation · Hirota bilinear · Lump soliton · Breather waves

**Mathematics Subject Classification** 26A33 · 35F25 · 35C10

M. Alquran (✉) · R. Alhami  
Department of Mathematics and Statistics, Jordan  
University of Science & Technology, Irbid 22110, Jordan  
e-mail: marwan04@just.edu.jo

R. Alhami  
e-mail: rfalhami20@sci.just.edu.jo

## 1 Introduction

Finding exact solutions of nonlinear equations plays an imperative role in understanding the processes and phenomena of many nonlinear models arising in fluids, dynamics, physical science, and nonlinear optical fibers. In the theory of solitons, different types of solutions such as bell-shaped, kink, cusp, periodic and others are identified by using suggested forms of solutions either in terms of exponential, trigonometric, or hyperbolic functions as in the cases of simplified bilinear method, tanh expansion,  $(G'/G)$  expansion, Riccati expansion, sine-cosine function, sech-csch function, Kudryashov expansion, unified expansion, Lie symmetry, and many other methods ([1–10]).

Recently, new types of solitons are produced by combining the Hirota bilinear method and the Cole-Hopf transformation  $u = a(\ln f)_x$  or  $u = b(\ln f)_{xx}$ , see ([11–20]). If  $f$  is chosen to be a polynomial, then the resulting solution  $u$  is identified as the lump soliton. If  $f$  is the combination of polynomial and sine/cosine, then  $u$  is of periodic-lump type. The breather-soliton waves are obtained if  $f$  is a combination of sine/cosine and the exponential functions. Finally, the two-wave soliton type can be obtained by combining sin-sinh or cos-cosh with the exponential functions.

In this paper, we investigate new features of solitary wave solutions to the generalized perturbed-KdV equation which reads as

$$u_t + \alpha u_x + \beta u u_x + \gamma u_{xxx} = 0, \quad (1.1)$$

where  $u = u(x, t)$  represents the free surface advancement,  $\alpha$  is the perturbation parameter known as the Coriolis effect, and  $\beta, \gamma$  are the nonlinearity and dispersion factors, respectively. The perturbed-KdV model describes the physical mechanism of sound propagation in fluid and appears in the applications of aerodynamics, acoustics, and medical engineering. Special case of (1.1) has been discussed in [21], for  $\beta = \frac{3}{2}$  and  $\gamma = \frac{1}{6}$ . The authors extracted different lumps and breather solutions but upon assigning a wrong choice of the Cole–Hopf transformation  $u = R(\ln f(x, t))_{xx}$  by taking  $R = 2$ .

The motivation of the current work is threefold: First, we derive the correct value of  $R$  that covers, in particular, the case of [21] and, in general, the case of (1.1). Second, we assign different choices of  $f(x, t)$ , to construct new lump-soliton, lump-periodic, single-stripe soliton, breather waves, and two-wave solutions to (1.1). Finally, we investigate the impact of the involved model's parameters on the propagation form of the retrieved solutions to the proposed model.

The paper is organized as follows: Section (2) deals with the construction of Hirota's bilinear form to the perturbed-KdV equation. Then, we derive both lump and periodic-lump solutions in Sect. (3). The single-stripe soliton and breather-wave solutions are extracted in Sects. (4) and (5), respectively. The two-wave solutions are investigated in Sect. (6), and some dynamical aspects are discussed in Sect. (7). Finally, some concluding remarks based on the obtained results are presented in Sect. (8).

## 2 Hirota bilinear form of the perturbed-KdV equation

To find the Hirota's form to (1.1), we apply the simplified bilinear method. First, we start with the following function

$$u(x, t) = e^{sx-rt}. \quad (2.2)$$

Then, we substitute (2.2) in the linear terms of (1.1) to obtain the dispersion relation as

$$r = \alpha s + \gamma s^3. \quad (2.3)$$

The second step is to bring the following function

$$k(x, t) = 1 + C e^{sx - (\alpha s + \gamma s^3)t}, \quad (2.4)$$

and to apply one of the Cole–Hopf transformations. In particular, we consider

$$u(x, t) = R(\ln(k(x, t)))_{xx}. \quad (2.5)$$

To find  $R$ , we insert (2.5) in (1.1) to get that

$$R = \frac{12\gamma}{\beta}. \quad (2.6)$$

As the third step, we update the assumption of the function  $u$  to take the following action:

$$u(x, t) = \psi_{xx}(x, t). \quad (2.7)$$

Substituting (2.7) in (1.1) and simplifying by integration with respect to  $x$ , we reach at the following relation regarding the new function  $\psi$

$$\psi_{xt} + \alpha\psi_{xx} + \frac{\beta}{2}(\psi_{xx})^2 + \gamma\psi_{xxxx} = 0. \quad (2.8)$$

Then, we choose  $\psi$  as

$$\psi(x, t) = \frac{12\gamma}{\beta} \ln(f(x, t)). \quad (2.9)$$

Finally, we insert (2.9) in (2.8) to deduce the following relation:

$$ff_{xt} - f_x f_t + \alpha f f_{xx} - \alpha f_x^2 + \gamma f f_{xxxx} - 4\gamma f_x f_{xxx} + 3\gamma f_{xx}^2 = 0. \quad (2.10)$$

By using Hirota's bilinear operators, (2.10) is written as

$$(D_x D_t + \alpha D_x^2 + \gamma D_x^4) f \cdot f = 0, \quad (2.11)$$

where  $D$  represents the Hirota bilinear operator defined as

$$D_x^l D_t^k f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^l \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^k f(x, t) g(x', t')|_{x'=x, t'=t}, \quad (2.12)$$

and  $f, g \in C^\infty(\mathbb{R}^2)$ .

## 3 Lump-type solutions

In this section, we derive two types of lump solutions to (1.1) by choosing  $f$  to be either quadratic function, or a combination of quadratic function and cosine function.

### 3.1 Lump soliton

To obtain lump soliton, we consider the following assumption

$$f = X^T A X + u_0, \quad (3.13)$$

where  $X = (1, x, t)^T$ ,  $A = (a_{i,j})_{3 \times 3}$  is a symmetric matrix, and  $a_{ij}, u_0$  are real constants to be determined. By expanding (3.13), we get

$$f(x, t) = a_{1,1} + a_{2,1}t + a_{3,1}x + x(a_{1,2} + a_{2,2}t + a_{3,2}x) + t(a_{1,3} + a_{2,3}t + a_{3,3}x) + u_0. \tag{3.14}$$

Next, we insert (3.14) in (2.10) and solve for the unknowns  $a_{ij}, u_0$ . By doing so, we obtain two cases:

**Case I:**

$$a_{2,3} = -\alpha a_{2,2} - \alpha a_{3,3},$$

$$u_0 = \frac{\alpha a_{1,2}^2 + a_{3,1}(a_{1,3} + a_{2,1} + \alpha a_{3,1}) + a_{1,2}(a_{1,3} + a_{2,1} + 2\alpha a_{3,1}) - a_{1,1}(a_{2,2} + a_{3,3})}{a_{2,2} + a_{3,3}},$$

$$a_{3,2} = 0,$$

where  $a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{3,1}$  and  $a_{3,3}$  are free parameters. Accordingly,

$$f = \frac{(a_{1,2} + a_{3,1} + (a_{2,2} + a_{3,3})t)(\alpha a_{1,2} + a_{1,3} + a_{2,1} + \alpha a_{3,1} + (a_{2,2} + a_{3,3})(x - \alpha t))}{a_{2,2} + a_{3,3}}. \tag{3.15}$$

Recalling (2.7), the first lump soliton to (1.1) is

$$u_1(x, t) = -\frac{12\gamma(a_{2,2} + a_{3,3})^2}{\beta(\alpha a_{1,2} + a_{1,3} + a_{2,1} + \alpha a_{3,1} + (a_{2,2} + a_{3,3})(x - \alpha t))^2}. \tag{3.16}$$

**Case II:**

$$a_{2,1} = -\alpha a_{1,2} - a_{1,3} - \alpha a_{3,1},$$

$$a_{2,3} = a_{3,2} = 0,$$

$$a_{3,3} = -a_{2,2}.$$

Thus,

$$f = u_0 + a_{1,1} + (a_{1,2} + a_{3,1})(x - \alpha t), \tag{3.17}$$

with  $a_{1,1}, a_{1,2}, a_{1,3}, a_{2,2}, a_{3,1}$  and  $u_0$  being free parameters. By this case, the second lump soliton is

$$u_2(x, t) = -\frac{12\gamma(a_{1,2} + a_{3,1})^2}{\beta(u_0 + a_{1,1} + (a_{1,2} + a_{3,1})(x - \alpha t))^2}. \tag{3.18}$$

In Fig. 1, we show the physical structure of the first lump soliton (3.16), which is similar in shape to (3.18).

**3.2 Lump-periodic solution**

To construct lump-periodic solution to (1.1),  $f$  is to be chosen as a linear combination of quadratic and cosine functions, i.e.

$$f = X^T AX + \omega \cos(p_1x + p_2t + p_3) + \sigma. \tag{3.19}$$

We substitute (3.19) in (2.10) and look up for the coefficients of different polynomials of  $x, t$  and trigonomet-

ric functions. Then, we set each coefficient to zero and solve the resulting system to get the following output:

$$\omega = \mp \frac{a_{1,2} + a_{3,1}}{p_1},$$

$$p_2 = -\alpha p_1 + \gamma p_1^3,$$

$$a_{3,2} = a_{2,3} = 0,$$

$$a_{3,3} = -a_{2,2},$$

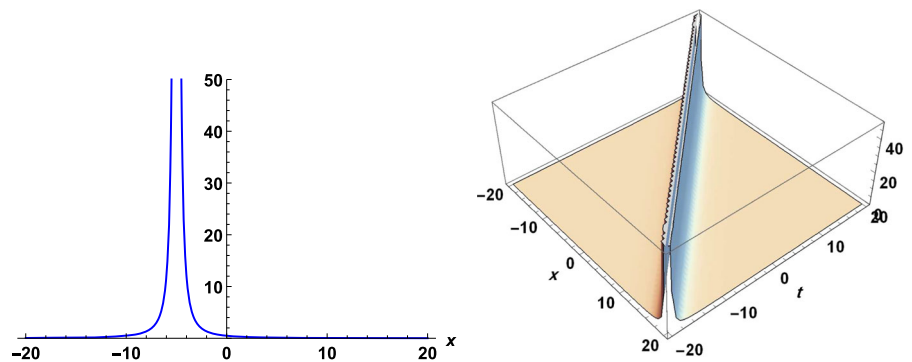
$$a_{1,3} = -\alpha a_{1,2} + 3\gamma p_1^2 a_{1,2} - a_{2,1} - \alpha a_{3,1} + 3\gamma p_1^2 a_{3,1}. \tag{3.20}$$

Let  $\Delta = p_1(x - \alpha t) + \gamma p_1^3 t + p_3$ ,  $\square = \sigma + a_{1,1} + a_{1,2}(x - \alpha t) + a_{3,1}x - \alpha a_{3,1}t$  and  $T = a_{1,2} + a_{3,1}$ . Then,  $f$  has the following form

$$f = \sigma + a_{1,1} + a_{2,1}t + Tx + a_{2,2}xt \pm \frac{T \cos(\Delta)}{p_1} + t((3\gamma p_1^2 - \alpha)T - a_{2,1} - a_{2,2}x). \tag{3.21}$$

Therefore, the lump-periodic solution to (1.1) is

**Fig. 1** 2D and 3D plots of  $u_1(x, t)$  where  $\alpha = \beta = -1, \gamma = 1, a_{2,2} = 2, a_{3,3} = a_{1,2} = a_{1,3} = a_{2,1} = a_{3,1} = 1$



$$u_3(x, t) = \frac{12\gamma T \left( -T(\sin(\Delta) \mp 1)^2 \mp p_1 \cos(\Delta) \left( \square \pm \frac{\cos(\Delta)T}{p_1} + 3\gamma p_1^2 T t \right) \right)}{\beta \left( \square \pm \frac{T \cos(\Delta)}{p_1} + 3\gamma p_1^2 T t \right)^2} \tag{3.22}$$

In Fig. 2, we present the physical structure of the lump-periodic solution (3.22).

### 4 Single-stripe soliton solutions

The approach for finding single-stripe solitons is known as a simplified bilinear method. They are similar to those steps illustrated earlier and given by (2.2)-(2.6). However, it can be derived directly using (2.9) and assume  $f$  as

$$f = 1 + c e^{d_1 x + d_2 t + d_3}, \tag{4.23}$$

where  $d_i, i = 1, 2, 3$  and  $c \neq 0$  are unknown real constants to be determined. Substituting of (4.23) in (2.10) gives that  $d_2 = -\alpha d_1 - \gamma d_1^3$ , where  $d_1$  and  $d_3$  are arbitrary constants. Thus, the single-stripe soliton solution of (1.1) is

$$u_4(x, t) = \frac{12c \gamma e^{(x-\alpha t)d_1 + \gamma t d_1^3 + d_3} d_1^2}{\beta (e^{\gamma t d_1^3} + c e^{(x-\alpha t)d_1 + d_3})^2} \tag{4.24}$$

### 5 Breather-wave solution

To find some families of breather-wave solutions, we consider the following test function

$$f = \epsilon_1 \cos(p_2(x + b_2 t)) + \epsilon_2 e^{p_1(x + b_1 t)} + e^{-p_1(x + b_1 t)}. \tag{5.25}$$

where  $\epsilon_i, p_i, b_i : i = 1, 2$  are real constants to be determined later. Substituting (5.25) in the bilinear form (2.10) and equating the coefficients of exponentials or trigonometric functions to zero, we get the following nonlinear algebraic system:

$$\begin{aligned} 0 &= \alpha p_1^2 \epsilon_1 \epsilon_2 + b_1 p_1^2 \epsilon_1 \epsilon_2 + \gamma p_1^4 \epsilon_1 \epsilon_2 - \alpha p_2^2 \epsilon_1 \epsilon_2 \\ &\quad - b_2 p_2^2 \epsilon_1 \epsilon_2 - 6\gamma p_1^2 p_2^2 \epsilon_1 \epsilon_2 + \gamma p_2^4 \epsilon_1 \epsilon_2, \\ 0 &= \alpha p_1^2 \epsilon_1 + b_1 p_1^2 \epsilon_1 + \gamma p_1^4 \epsilon_1 - \alpha p_2^2 \epsilon_1 - b_2 p_2^2 \epsilon_1 \\ &\quad - 6\gamma p_1^2 p_2^2 \epsilon_1 + \gamma p_2^4 \epsilon_1, \\ 0 &= 2\alpha p_1 p_2 \epsilon_1 \epsilon_2 + b_1 p_1 p_2 \epsilon_1 \epsilon_2 + b_2 p_1 p_2 \epsilon_1 \epsilon_2 \\ &\quad + 4\gamma p_1^3 p_2 \epsilon_1 \epsilon_2 - 4\gamma p_1 p_2^3 \epsilon_1 \epsilon_2, \\ 0 &= -2\alpha p_1 p_2 \epsilon_1 - b_1 p_1 p_2 \epsilon_1 \\ &\quad - b_2 p_1 p_2 \epsilon_1 - 4\gamma p_1^3 p_2 \epsilon_1 + 4\gamma p_1 p_2^3 \epsilon_1, \\ 0 &= 4\alpha p_1^2 \epsilon_2 + 4b_1 p_1^2 \epsilon_2 + 16\gamma p_1^4 \epsilon_2 - \alpha p_2^2 \epsilon_1^2 \\ &\quad - b_2 p_2^2 \epsilon_1^2 + 4\gamma p_2^4 \epsilon_1^2. \end{aligned}$$

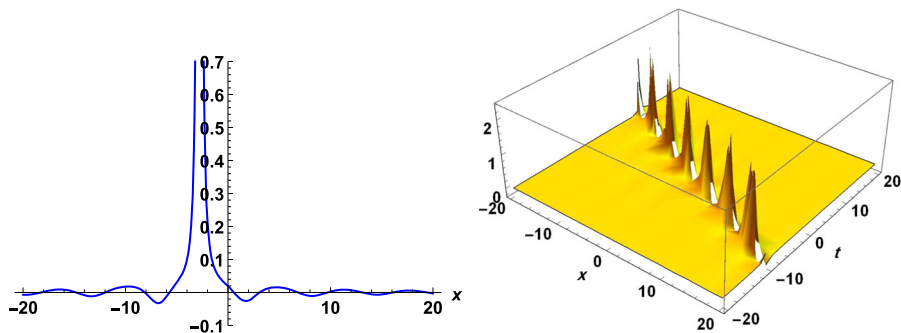
Solving the above system leads to

$$\begin{aligned} b_1 &= -\alpha - \gamma p_1^2 + 3\gamma p_2^2, \\ b_2 &= -\alpha - 3\gamma p_1^2 + \gamma p_2^2, \\ \epsilon_2 &= -\frac{p_2^2 \epsilon_1^2}{4p_1^2}, \end{aligned}$$

with  $\epsilon_i, p_i : i = 1, 2$  being free parameters. Let  $\lambda_1 = p_1(-x + \alpha t + \gamma t p_1^2 - 3\gamma t p_2^2)$  and  $\lambda_2 = p_2(x - \alpha t - 3\gamma t p_1^2 + \gamma t p_2^2)$ , and we get

$$f = \epsilon_1 \cos(\lambda_2) + e^{\lambda_1} - \frac{p_2^2 \epsilon_1^2 e^{-\lambda_1}}{4p_1^2}. \tag{5.26}$$

**Fig. 2** 2D and 3D plots of  $u_3(x, t)$  where  $\alpha = \beta = \gamma = 1$ ,  $a_{1,1} = a_{1,2} = a_{3,1} = \sigma = p_1 = p_3 = 1$



Accordingly, the breather-wave solution to (1.1) is

$$\begin{aligned}
 u_5(x, t) = & (12\gamma(-p_1 e^{\lambda_1} + p_2 \epsilon_1 \sin(\lambda_2) + \frac{p_2^2 \epsilon_1^2 e^{-\lambda_1}}{4p_1})^2 \\
 & + (e^{\lambda_1} + \epsilon_1 \cos(\lambda_2) - \frac{p_2^2 \epsilon_1^2 e^{-\lambda_1}}{4p_1^2})(p_1^2 e^{\lambda_1} + \frac{p_2^2 \epsilon_1}{4} \\
 & (-4 \cos(\lambda_2) - \epsilon_1 e^{-\lambda_1}))) / (\beta(e^{\lambda_1} + \epsilon_1 \cos(\lambda_2) \\
 & - \frac{p_2^2 \epsilon_1^2 e^{-\lambda_1}}{4p_1^2})^2).
 \end{aligned}
 \tag{5.27}$$

In Fig. 3, we present the physical structure of the breather-wave solution (5.27).

### 6 Two-wave solution

To find the two-wave solution to the perturbed-KdV equation, we consider the following test function:

$$\begin{aligned}
 f = & \omega_1 e^{\mu t + x} + \omega_2 e^{-(\mu t + x)} + \omega_3 \sin(c_1 t + x) \\
 & + \omega_4 \sinh(c_2 t + x).
 \end{aligned}
 \tag{6.28}$$

$$\begin{aligned}
 u_6(x, t) = & \frac{12\gamma}{\beta} - \frac{12\gamma(\omega_1 e^{x+\mu t} + \omega_4 \cosh(x - t(2\alpha + 8\gamma + \mu)) + \frac{\omega_4^2 e^{-x-\mu t}}{4\omega_1})^2}{\beta(\omega_1 e^{x+\mu t} + \omega_4 \sinh(x - t(2\alpha + 8\gamma + \mu)) - \frac{\omega_4^2 e^{-x-\mu t}}{4\omega_1})^2}, \\
 = & -\frac{96\gamma\omega_1\omega_4 e^{2t(\alpha+4\gamma)+x}}{\beta(\omega_4 e^{2t(\alpha+4\gamma)} - 2e^{2x}\omega_1)^2}.
 \end{aligned}
 \tag{6.30}$$

To get information about the values of  $\omega_j$  : ( $j = 1, 2, 3, 4$ ),  $c_1$ ,  $c_2$  and  $\mu$ , we substitute (6.28) in Eq. (2.10). Then, we collect the coefficients of same terms and set each to zero to obtain the following system:

$$\begin{aligned}
 0 = & -2\alpha\omega_1\omega_3 - \mu\omega_1\omega_3 - c_1\omega_1\omega_3, \\
 0 = & -4\gamma\omega_1\omega_3 + \mu\omega_1\omega_3 - c_1\omega_1\omega_3,
 \end{aligned}$$

$$\begin{aligned}
 0 = & 2\alpha\omega_1\omega_4 + 8\gamma\omega_1\omega_4 + \mu\omega_1\omega_4 + c_2\omega_1\omega_4, \\
 0 = & 2\alpha\omega_2\omega_3 + \mu\omega_2\omega_3 + c_1\omega_2\omega_3, \\
 0 = & -4\gamma\omega_2\omega_3 + \mu\omega_2\omega_3 - c_1\omega_2\omega_3, \\
 0 = & 2\alpha\omega_2\omega_4 + 8\gamma\omega_2\omega_4 + \mu\omega_2\omega_4 + c_2\omega_2\omega_4, \\
 0 = & -2\alpha\omega_3\omega_4 - c_1\omega_3\omega_4 - c_2\omega_3\omega_4, \\
 0 = & -4\gamma\omega_3\omega_4 - c_1\omega_3\omega_4 + c_2\omega_3\omega_4, \\
 0 = & 4\alpha\omega_1\omega_2 + 16\gamma\omega_1\omega_2 + 4\mu\omega_1\omega_2 - \alpha\omega_4^2 - 4\gamma\omega_4^2 \\
 & - c_2\omega_4^2 - \alpha\omega_3^2 + 4\gamma\omega_3^2 - c_1\omega_3^2.
 \end{aligned}$$

By solving the above system, we retrieve three solution's sets:

**Set(I):**  $\omega_2 = -\frac{\omega_4^2}{4\omega_1}$ ,  $\omega_3 = 0$ ,  $c_2 = -2\alpha - 8\gamma - \mu$ . Then,  $f$  explicitly is

$$\begin{aligned}
 f = & \omega_1 e^{x+\mu t} \\
 & - \frac{\omega_4^2 e^{-x-\mu t}}{4\omega_1} + \omega_4 \sinh(x - t(2\alpha + 8\gamma + \mu)).
 \end{aligned}
 \tag{6.29}$$

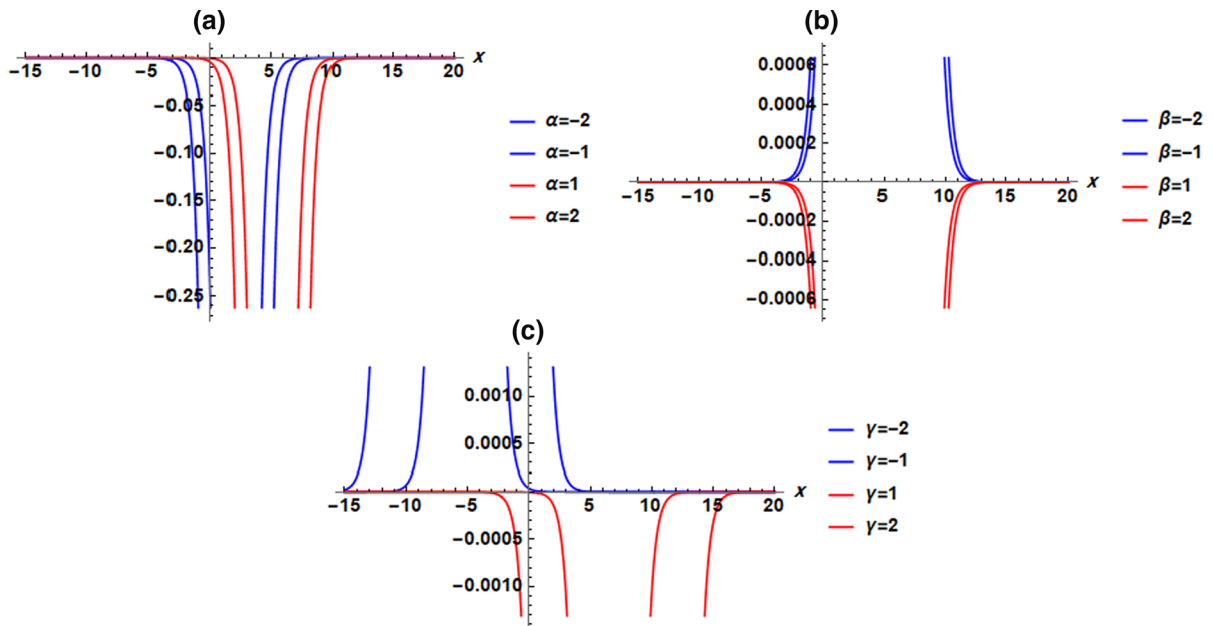
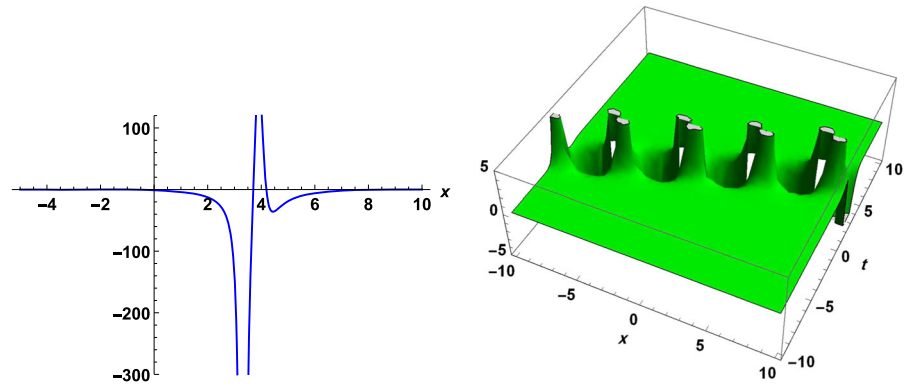
Thus, the sixth recovery solution to (1.1) is

**Set(II):**  $\omega_2 = -\frac{\omega_3^2}{4\omega_1}$ ,  $\omega_4 = 0$ ,  $\mu = -\alpha + 2\gamma$ ,  $c_1 = -\alpha - 2\gamma$ . Then,  $f$  explicitly is

$$f = \omega_1 e^\zeta - \frac{\omega_3^2 e^{-\zeta}}{4\omega_1} + \omega_3 \sin(\eta),
 \tag{6.31}$$

where  $\zeta = x - t(\alpha - 2\gamma)$  and  $\eta = x - t(\alpha + 2\gamma)$ . Hence, the seventh recovery solution to (1.1) is:

**Fig. 3** 2D and 3D plots of  $u_5(x, t)$  where  $\beta = \gamma = -1$ ,  $\alpha = p_1 = p_2 = \epsilon_1 = 1$



**Fig. 4** Propagations of  $u_6(x, t)$  for different values of: **a** The perturbation parameter  $\alpha$  where  $t = \beta = \gamma = \omega_1 = \omega_4 = 1$ . **b** The nonlinearity parameter  $\beta$  where  $t = \alpha = \gamma = \omega_1 = \omega_4 = 1$ . **c** The dispersion parameter  $\gamma$  where  $t = \alpha = \beta = \omega_1 = \omega_4 = 1$

$$u_7(x, t) = \frac{12\gamma(\omega_1 e^\zeta - \omega_3 \sin(\eta) - \frac{\omega_3^2 e^{-\zeta}}{4\omega_1})}{\beta(\omega_1 e^\zeta + \omega_3 \sin(\eta) - \frac{\omega_3^2 e^{-\zeta}}{4\omega_1})} - \frac{12\gamma(\omega_1 e^\zeta + \omega_3 \cos(\eta) + \frac{\omega_3^2 e^{-\zeta}}{4\omega_1})^2}{\beta(\omega_1 e^\zeta + \omega_3 \sin(\eta) - \frac{\omega_3^2 e^{-\zeta}}{4\omega_1})^2},$$

$$= -\frac{96\gamma\omega_1\omega_3 e^{\alpha t + 2\gamma t + x} (4\omega_3\omega_1 e^{\alpha t + 2\gamma t + x} + 4\omega_1^2 e^{4\gamma t + 2x} \cos(\eta) + \omega_3^2 e^{2\alpha t} \cos(\eta))}{\beta (-\omega_3^2 e^{2\alpha t} + 4\omega_3\omega_1 e^{\alpha t + 2\gamma t + x} \sin(\eta) + 4\omega_1^2 e^{4\gamma t + 2x})^2}. \tag{6.32}$$

**Set(III):**  $\omega_1 = 0, \omega_2 = 0, \omega_3 = -\omega_4, c_1 = -\alpha - 2\gamma, c_2 = -\alpha + 2\gamma$ . Then,  $f$  explicitly is:

$$f = -\omega_4 \sin(\eta) + \omega_4 \sinh(\zeta). \tag{6.33}$$

Accordingly, the eighth recovery solution to (1.1) is

$$u_8(x, t) = -\frac{12\gamma(-\omega_4 \cos(\eta) + \omega_4 \cosh(\zeta))^2}{\beta(-\omega_4 \sin(\eta) + \omega_4 \sinh(\zeta))^2}$$

$$+ \frac{12\gamma(\omega_4 \sin(\eta) + \omega_4 \sinh(\zeta))}{\beta(-\omega_4 \sin(\eta) + \omega_4 \sinh(\zeta))} \quad (6.34)$$

## 7 Dynamics of the perturbed KdV

In this section, we study the impact of the perturbation, nonlinearity, and dispersion parameters,  $\alpha$ ,  $\beta$ ,  $\gamma$ , being acting on the propagation of the perturbed KdV. To achieve this goal, we consider the obtained solution depicted earlier as the function  $u_6(x, t)$ . We investigate the physical structures to this function by plotting some curves for different values of the assigned parameters. Figure 4 shows the dynamics of propagating  $u_6$ , and three observations can be drawn:

- The propagation is symmetric when  $\alpha$  changes its sign.
- The propagation has a reflexive relation when  $\beta$  changes its sign.
- The propagation is reflexive due to the sign of  $\gamma$ .

## 8 Conclusion

In this study, we derived the Hirota bilinear form for the generalized perturbed-KdV equation. Then, the Cole–Hopf transformations are used, and different selections of the involved test function are elaborated to retrieve novel types of solitons such as lumps, breather-wave, and multi-wave solutions. Also, the dynamics of the model’s parameters, perturbation, nonlinearity, and dispersion are investigated.

For future work, we aim to extend the use of Hirota’s bilinear methods to study other important nonlinear applications arising in physical and engineering fields and higher-dimensional models.

**Funding** The authors have not disclosed any funding.

**Data availability** Data sharing was not applicable to this article as no datasets were generated or analysed during the current study.

### Declarations

**Conflict of interest** The authors declare that they have no conflict of interest

## References

1. Yong, C., Biao, L., Hong-Qing, Z.: Generalized Riccati equation expansion method and its application to the Bogoyavlenskii’s generalized breaking soliton equation. *Chinese Phys* **12**(9), 940 (2003)
2. Alquran, M., Jaradat, H., Al-Shara, S., Awawdeh, F.: A new simplified bilinear method for the n-soliton solutions for a generalized FmKdV equation with time-dependent variable coefficients. *Int. J. Nonlin. Sci. Numer. Simulat.* **16**(6), 259–269 (2015)
3. Huang, W.H.: A polynomial expansion method and its application in the coupled Zakharov-kuznetsov equations. *Chaos, Solit. & Fractals* **29**(2), 365–371 (2006)
4. Rahman, M.M., Habib, M., Ali, H.S., Miah, M.: The generalized Kudryashov method: a renewed mechanism for performing exact solitary wave solutions of some NLEEs. *J. Mech. Cont. Math. Sci.* **14**(1), 323–339 (2019)
5. Ma, W.X., Chen, M.: Direct search for exact solutions to the nonlinear schrodinger equation. *Appl. Math. Comput.* **215**(8), 2835–2842 (2009)
6. Alquran, M., Sulaiman, T.A., Yusuf, A.: Kink-soliton, singular-kink-soliton and singular-periodic solutions for a new two-mode version of the Burger-Huxley model: applications in nerve fibers and liquid crystals. *Opt. Quant. Electron.* **53**, 227 (2021)
7. Jaradat, I., Alquran, M., Ali, M., Sulaiman, T.A., Yusuf, A., Katatbeh, Q.: New mathematical model governing the propagation of two-wave modes moving in the same direction: classical and fractional potential KdV equation. *Rom. Rep. Phys.* **73**(3), 118 (2021)
8. Jaradat, I., Alquran, M.: Construction of solitary two-wave solutions for a new two-mode version of the Zakharov-Kuznetsov equation. *Mathematics* **8**(7), 1127 (2020)
9. Malik, S., Kumar, S., Biswas, A., et al.: Optical solitons and bifurcation analysis in fiber Bragg gratings with Lie symmetry and Kudryashov’s approach. *Nonlin. Dyn.* **105**, 735–751 (2021)
10. Zhang, Z., Li, B., Chen, J., et al.: Construction of higher-order smooth positons and breather positons via Hirota’s bilinear method. *Nonlin. Dyn.* **105**, 2611–2618 (2021)
11. Ma, W.X., Zhang, L.: Lump solutions with higher-order rational dispersion relations. *Pramana-J. Phys.* **94**, 43 (2020)
12. Wang, X.B., Tian, S.F., Qin, C.Y., Zhang, T.T.: Dynamics of the breathers, rogue waves and solitary waves in the (2+1)-dimensional Ito equation. *Appl. Math. Lett.* **68**, 40–47 (2017)
13. Singh, N., Stepanyants, Y.: Obliquely propagating skew KP lumps. *Wave Mot.* **64**, 92–102 (2016)
14. Ma, W.X.: Lump solutions to the Kadomtsev-Petviashvili equation. *Phys Lett A* **379**(36), 1975–1978 (2015)
15. Ma, W.X., Zhou, Y.: Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. *J Differ Equ* **264**(4), 2633–2659 (2018)
16. Sulaiman, T.A., Yusuf, A., Abdeljabbar, A., Alquran, M.: Dynamics of lump collision phenomena to the (3+1)-dimensional nonlinear evolution equation. *J Geomet Phys* **169**, 104347 (2021)
17. Ali, M.E., Bilkis, F., Paul, G.C., Kumar, D., Naher, H.: Lump, lump-stripe, and breather wave solutions to the (2+1)-

- dimensional Sawada-Kotera equation in fluid mechanics. *Heliyon* **7**(9), e07966 (2021)
18. Alquran, M., Jaradat, H.M., Syam, M.: A modified approach for a reliable study of new nonlinear equation: twomode Korteweg-de Vries-Burgers equation. *Nonlin. Dyn.* **91**(3), 1619–1626 (2018)
  19. Sulaiman, T.A., Yusuf, A., Alquran, M.: Dynamics of optical solitons and nonautonomous complex wave solutions to the nonlinear Schrodinger equation with variable coefficients. *Nonlin. Dyn.* **104**, 639–648 (2021)
  20. Hu, Z., Wang, F., Zhao, Y., et al.: Nonautonomous lump waves of a (3+1)-dimensional Kudryashov-Sinelshchikov equation with variable coefficients in bubbly liquids. *Nonlin. Dyn.* **104**, 4367–4378 (2021)
  21. Rizvi, S., Seadawy, A.R., Ashraf, F., Younis, M., Iqbal, H., Baleanu, D.: Lump and interaction solutions of a geophysical Korteweg-de Vries equation. *Results Phys* **19**, 103661 (2020)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.