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Linear superposition formula of solutions for the extended (3+1)-dimensional shallow water wave equation

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Abstract Active researches on the water waves have been done, and water waves are essentially complex waves controlled by gravity field and surface tension. Using the Hirota bilinear method, two bilinear auto-Bäcklund transformations of the extended (3+1)-dimensional shallow water wave equation are derived explicitly. The hyperbolic cosine-function solution and cosine-function solution are obtained by means of bilinear auto-Bäcklund transformations. Five linear superposition formulas of this equation are given and proved. All the results depend on the coefficients of the equation and the linear superposition relationship. Thereafter, we perform a numerical simulation to trace and study the dynamical behaviors of the linear superposition solutions via their three-dimensional profiles using symbolic calculation system Mathematica codes.

Keywords Hirota bilinear method · Bilinear auto-Bäcklund transformations · Homoclinic test method · Different types of superposition solutions

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1 Introduction

Shallow water wave equations have been considered as the models in which the depth of the water is much smaller than the wave length of the disturbance of the free surface [1,2]. Shallow water wave equations are one of the important models of nonlinear evolution equations (NLEEs), which are widely used in mathematical physics [3,4]. For instance, the acoustic problem of wave propagation in discontinuous media [3]. The horizontal velocity and the height away from the equilibrium position of water waves depend on the dispersion power of sea water waves [4]. In order to better understand the physical mechanism of natural phenomena described by NLEEs [5–8], it is particularly important to analyze the analytical solutions of NLEEs [9–11]. There exist many significant methods to find the analytical solutions of NLEEs, including tanh function and the sine-cosine method [12], variable-coefficient three-wave approach [13], Darboux transformation [14–16], Bäcklund transformation [17,18], bilinear neural network method [19,20], multiple exp-function method [21,22], Lie group method [23–27], Hirota bilinear method [28–30] and many others.

A new extended (3+1)-dimensional shallow water wave equation [31] is introduced by Wazwaz as follows:

$$\begin{aligned} & u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} \\ & \quad + \lambda_1 u_{xx} + \lambda_2 u_{yy} + \lambda_3 u_{xy} + \lambda_4 u_{yz} = 0, \end{aligned} \tag{1}$$

where $u = u(x, y, z, t)$ is a function of the three scaled spatial variables x, y, z and the temporal variable t, with $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are constants. Equation (1) is used to simulate the dynamic behaviors of water wave propagation in oceanography and atmospheric science. It is proved that the extended terms do not destroy the integrability of Eq. (1). Painlevé analysis was performed on Eq. (1) and the compatibility conditions were checked. Multiple soliton solutions and lump solutions of Eq. (1) are formally derived. The effects caused by the extended terms were obvious on the dispersion relations and the phase shifts as well [31]. Special cases of Eq. (1) have been investigated as follows:

(1) Setting $\partial_y = \partial_x, \psi = -u_x, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ in Eq. (1) gives the famous Korteweg-de Vries (KdV) equation [32]

$$\psi_t + \psi_{xxx} + 6\psi\psi_x = 0. \tag{2}$$

KdV equation describing the long waves in shallow water under the gravity, waves in a nonlinear lattice, ion-acoustic and magneto-acoustic waves in a plasma [32].

(2) When we restrict u to being z-independent and $\lambda_4 = 0$. Equation (1) has been reduced to the extended (2+1)-dimensional shallow water wave equation with constant coefficients [31]

$$\begin{aligned} &u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} + \lambda_1 u_{xx} \\ &\quad + \lambda_2 u_{yy} + \lambda_3 u_{xy} = 0. \end{aligned} \tag{3}$$

Equation (3) is used to simulate the dynamic behaviors of water wave propagation in oceanography and atmospheric science. The integrability of Eq. (3) is studied by Painlevé analysis method, and multiple soliton solutions and lump solutions are obtained [31].

(3) When we restrict u to being z-independent and $\lambda_2 = \lambda_4 = 0$. Equation (1) has been reduced to the (2+1)-dimensional extended shallow water wave equation [33]

$$\begin{aligned} &u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} \\ &+\lambda_1 u_{xx} + \lambda_3 u_{xy} = 0. \end{aligned} \tag{4}$$

Equation (4) simulates the nonlinear waves in shallow water and the (2+1)-dimensional interaction of the Riemann wave propagating along the y-axis and a long wave propagating along the x-axis in plasma physics and weakly dispersive media. By applying the long wave limit method to the N-soliton solutions, the multiple lump solutions of Eq. (4) are gained [33].

(4) When we restrict u to being z-independent and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Equation (1) has been reduced to the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation [34]

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} = 0. \tag{5}$$

Equation (5) describes the interaction of a Riemann wave propagating along the y-axis and a long wave propagating along the x-axis in a fluid. Some exact solutions of Eq. (5) are obtained, including kinky periodic solitary-wave solutions, periodic soliton solutions and kink solutions [34].

Many studies have revealed that nonlinear waves will exhibit more complex and fascinating dynamic characters, as the spatial dimension of system increases [35–38]. For linear systems, different linear superposition forms produce generalized solutions of linear problems. Linear superposition has an impact on the applications of nonlinear models in the real world, but the principle of linear superposition can be applied to some specific nonlinear models [39]. Specifically, it is transformed into bilinear form, and its characteristics are used to study the linear superposition solutions. In several areas of applied science and ocean engineering, investigations of superposition solutions have been played a vital role for demonstrating wave character of nonlinear problems [40]. The innovation of this paper lies in the construction of several formulas of new types of superposition solutions, which are proved to be valid under superposition relations. The trajectories and dynamic evolution of some superimposed solutions are analyzed in detail. The analysis of its properties is closer to the real physical phenomena in complex environment.

The paper is synchronized in the following manner: Section 2 deals with the bilinear form in order to obtain two bilinear auto-Bäcklund transformations of Eq. (1). The hyperbolic cosine-function solution and cosine-function solution are obtained by means of bilinear auto-Bäcklund transformations. In Sect. 3, we give and prove three superposition solutions for Eq. (1), including exponential function superposition solutions, trigonometric function superposition solutions, hybrid solution among trigonometric functions and exponential functions. In Sect. 4, we study the superposition formulas of two kinds of function product solutions, which are exponential function product type superposition solutions and trigonometric function product type superposition solutions. The derived results are studied with the aid of graphics. At last, Sect. 5 ends with the concluding remarks of the findings.

2 Bilinear auto-Bäcklund transformations

To study the bilinear auto-Bäcklund transformations, we need to get the bilinear form for Eq. (1) at first. Under the dependent variable transformation

$$u=-2(\ln f)_x+u_0(z,t), \tag{6}$$

where f is a real function of x, y, z and t, $u_0(z,t)$ is an undetermined function of z and t. Equation (1) has been converted into the following bilinear form

$$\begin{aligned}&(D_yD_t+D_x^3D_y+\lambda_1D_x^2+\lambda_2D_y^2\\&\quad+\lambda_3D_xD_y+\lambda_4D_yD_z)f\cdot f=0,\end{aligned} \tag{7}$$

with $R(x,y,z,t)$, $F(x,y,z,t)$ are real functions with respect to variables x, y, z and t, where D_x, D_y, D_z, D_t are the bilinear operators defined by Hirota [41]

$$\begin{aligned}&D_x^{n_1}D_y^{n_2}D_z^{n_3}D_t^{n_4}R(x,y,z,t)\cdot F(x,y,z,t)\\&\quad=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^{n_1}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^{n_2}\left(\frac{\partial}{\partial z}\right.\\&\qquad\left.-\frac{\partial}{\partial z'}\right)^{n_3}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^{n_4}\\&R(x,y,z,t)F(x',y',z',t')|_{x'=x,y'=y,z'=z,t'=t},\end{aligned} \tag{8}$$

with n_1, n_2, n_3 and n_4 being the non-negative integers. Suppose there is another solution g to bilinear form (7)

$$\begin{aligned}&(D_yD_t+D_x^3D_y+\lambda_1D_x^2+\lambda_2D_y^2\\&\quad+\lambda_3D_xD_y+\lambda_4D_yD_z)g\cdot g=0,\end{aligned} \tag{9}$$

where g is a real function of x, y, z and t. In order to search for certain bilinear auto-Bäcklund transformations between the solutions f and g of bilinear form (7) for Eq. (1), consider the following form

$$\begin{aligned}P&=[(D_yD_t+D_x^3D_y+\lambda_1D_x^2+\lambda_2D_y^2\\&\quad+\lambda_3D_xD_y+\lambda_4D_yD_z)f\cdot f]g^2\\&\quad-f^2[(D_yD_t+D_x^3D_y+\lambda_1D_x^2+\lambda_2D_y^2\\&\quad+\lambda_3D_xD_y+\lambda_4D_yD_z)g\cdot g].\end{aligned} \tag{10}$$

We use the exchange identities of the following Hirota bilinear operators [41]

$$\begin{aligned}&(D_yD_tf\cdot f)g^2-f^2(D_yD_tg\cdot g)\\&\quad=2D_y(D_tf\cdot g)\cdot(fg)\\&\quad=2D_t(D_yf\cdot g)\cdot(fg),\\&(D_xD_yf\cdot f)g^2-f^2(D_xD_yg\cdot g)\\&\quad=2D_x(D_yf\cdot g)\cdot(fg)\\&\quad=2D_y(D_xf\cdot g)\cdot(fg),\\&(D_yD_zf\cdot f)g^2-f^2(D_yD_zg\cdot g)\\&\quad=2D_y(D_zf\cdot g)\cdot(fg)\\&\quad=2D_z(D_yf\cdot g)\cdot(fg),\\&(D_x^2f\cdot f)g^2-f^2(D_x^2g\cdot g)\\&\quad=2D_x(D_xf\cdot g)\cdot(fg),\\&(D_y^2f\cdot f)g^2-f^2(D_y^2g\cdot g)\\&\quad=2D_y(D_yf\cdot g)\cdot(fg),\end{aligned} \tag{11}$$

with

$$\begin{aligned}&(D_x^3D_yf\cdot f)g^2-f^2(D_x^3D_yg\cdot g)\\&\quad=3D_x[(D_x^2D_yf\cdot g)\cdot(fg)]\\&\qquad-D_y[(D_x^3f\cdot g)\cdot(fg)]\\&\qquad-3D_x[(D_x^2f\cdot g)\cdot(D_yf\cdot g)]\\&\qquad-3D_y[(D_x^2f\cdot g)\cdot(D_xf\cdot g)],\end{aligned} \tag{12}$$

and

$$\begin{aligned}&(D_x^3D_yf\cdot f)g^2-f^2(D_x^3D_yg\cdot g)\\&\quad=2D_y[(D_x^3f\cdot g)\cdot(fg)]\\&\qquad-6D_x[(D_xD_yf\cdot g)\cdot(D_xf\cdot g)].\end{aligned} \tag{13}$$

Selecting different exchange identities for the Hirota bilinear operator, we get two different types of bilinear auto-Bäcklund transformations and soliton solutions for Eq. (1) as follows:

Case I Substituting expressions (11) and (12) into Eq. (10) and assuming that

$$D_x^2f\cdot g=\rho_1fg, \tag{14}$$

we derive that

$$\begin{aligned}P_1&=D_x\{[(3D_x^2D_y+3\rho_1D_y+2\lambda_1D_x\\&\qquad+2\lambda_3D_y)f\cdot g]\cdot(fg)\}\\&\quad+D_y\{[(2D_t-D_x^3+3\rho_1D_x+2\lambda_2D_y\\&\qquad+2\lambda_4D_z)f\cdot g]\cdot(fg)\},\end{aligned} \tag{15}$$

with ρ_1 is the real constant. Taking $P_1=0$, the decoupling of Eq. (15) gives rise to an alternative bilinear auto-Bäcklund transformation for Eq. (1) as

$$(D_x^2-\rho_1)f\cdot g=0,\ (3D_x^2D_y+3\rho_1D_y+2\lambda_1D_x+2\lambda_3D_y)f\cdot g=0,\quad (16)$$
$$(2D_t-D_x^3+3\rho_1D_x+2\lambda_2D_y+2\lambda_4D_z)f\cdot g=0.$$

We select $f=1$ as a solution for bilinear form (7) and solve bilinear auto-Bäcklund transformation (16) to obtain the following equations

$$g_{xx}-\rho_1 g=0,\ 3g_{xxy}+(3\rho_1+2\lambda_3)g_y+2\lambda_1 g_x=0,$$
$$2g_t-g_{xxx}+3\rho_1 g_x+2\lambda_2 g_y+2\lambda_4 g_z=0.\quad (17)$$

Assuming that $g=\cosh(a_1x+b_1y+c_1z+d_1t)$ and solving Eq. (17), we get the parameters relationship in solution g as follows:

$$\rho_1=a_1^2,\ d_1=-(a_1^3+\lambda_2b_1+\lambda_4c_1),\ \lambda_3=\frac{-3a_1^2b_1-\lambda_1a_1}{b_1},\ b_1\neq 0,\quad (18)$$

where a_1, b_1, c_1 and d_1 are the real constants. Thus, the corresponding hyperbolic cosine-function solution for Eq. (1) is

$$u_1=-\frac{2a_1\sinh[a_1x+b_1y+c_1z-(a_1^3+\lambda_2b_1+\lambda_4c_1)t]}{\cosh[a_1x+b_1y+c_1z-(a_1^3+\lambda_2b_1+\lambda_4c_1)t]}+u_0(z,t).\quad (19)$$

Case II Substituting expressions (11) and (13) into Eq. (10) and supposing that

$$D_xD_yf\cdot g=\rho_2fg,\quad (20)$$

we can derive

$$P_2=2D_x\{[(3\rho_2D_x+\lambda_1D_x+\lambda_3D_y)f\cdot g]\cdot(fg)\}+2D_y\{[(D_t+D_x^3+\lambda_2D_y+\lambda_4D_z)f\cdot g]\cdot(fg)\},\quad (21)$$

where ρ_2 is the real constant. Taking $P_2=0$, it is concluded that the second bilinear auto-Bäcklund transformation associated with Eq. (1) can be constructed as

$$(D_xD_y-\rho_2)f\cdot g=0,\ (3\rho_2D_x+\lambda_1D_x+\lambda_3D_y)f\cdot g=0,\quad (22)$$
$$(D_t+D_x^3+\lambda_2D_y+\lambda_4D_z)f\cdot g=0.$$

Taking $f=1$ as a solution for bilinear form (7) and solving bilinear auto-Bäcklund transformation (22), we get

$$g_{xy}-\rho_2g=0,\ (3\rho_2+\lambda_1)g_x+\lambda_3g_y=0,\ g_t+g_{xxx}+\lambda_2g_y+\lambda_4g_z=0.\quad (23)$$

Assuming that $g=\cos(a_2x+b_2y+c_2z+d_2t)$ and solving Eq. (23), we obtain the parameters relationship in solution g as follows:

$$a_2=-\frac{\rho_2}{b_2},\ c_2=-\frac{(\rho_2^3+b_2^3d_2+\lambda_2b_2^4)}{\lambda_4b_2^3},\ \lambda_3=\frac{\rho_2(\lambda_1+3\rho_2)}{b_2^2},\ b_2\lambda_4\neq 0.\quad (24)$$

Thus, the corresponding cosine-function solution for Eq. (1) is

$$u_2=\frac{2a_2\sin\left[a_2x+b_2y-\frac{(\rho_2^3+b_2^3d_2+\lambda_2b_2^4)}{\lambda_4b_2^3}z+d_2t\right]}{\cos\left[a_2x+b_2y-\frac{(\rho_2^3+b_2^3d_2+\lambda_2b_2^4)}{\lambda_4b_2^3}z+d_2t\right]}+u_0(z,t).\quad (25)$$

Bilinear auto-Bäcklund transformation is an effective algorithm for solving NLEEs, and it turns the problem of solving equations into pure algebraic operation [42,43]. It can be seen that the two kinds of bilinear auto-Bäcklund transformations (16) and (22) are obtained through different exchange identities, and the process of applying bilinear auto-Bäcklund transformation to solve the analytical solutions is the same. With the help of two kinds of bilinear auto-Bäcklund transformations, the hyperbolic cosine-function solution (19) and cosine-function solution (25) are obtained by assuming different forms of solutions. Similarly, it can be assumed that there are different forms of solutions, and then the parameters relationship can be given by using bilinear auto-Bäcklund transformation. Therefore, the method provides an effective idea for solving the analytical solutions of various NLEEs.

3 Linear superposition formula of solutions

It is well known that for the physical systems frequently characterized by NLEEs, there is no linear superposition formula of solutions. At present, the corresponding nonlinear superposition formula is given by means of Bäcklund transformation method [44]. Based on the Hirota bilinear method, bilinear neural network framework expands to more than one hidden layer to construct test functions [45]. By using the symbolic computation software Maple, periodic-type I, II, and III solutions of the new (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation are obtained [45]. Starting from the potential forms constructed by the master-symmetry approach, the special decompositions and some linear superpositions of the BKP hierarchy and the dispersionless BKP hierarchy are analyzed [46]. The above methods provide some feasible ideas for constructing the linear superposition formula of solutions.

3.1 Linear superposition formula of exponential function type

Firstly, we assume the new test functions of multiple exponential functions, hyperbolic cosine functions and hyperbolic sine functions.

$$
\begin{aligned}
f_{A1} &= \sum_{i=1}^{N} k_i \cosh(\theta_i),\ f_{A2} = \sum_{i=1}^{N} k_i \sinh(\theta_i),\\
f_{A3} &= \sum_{i=1}^{N} k_i \exp(\theta_i),\ \theta_i = \alpha_i x + \beta_i y + \gamma_i z + \omega_i t,
\end{aligned}
\tag{26}
$$

where $k_i, \alpha_i, \beta_i, \gamma_i, \omega_i$ are all the real constants, N is a positive integer.

Theorem *Assuming that the exponential test functions f_{A1}, f_{A2} and f_{A3} (26) are solutions of bilinear form (7), one of the following four linear superposition relationships must be satisfied:*

$$
\begin{aligned}
&Case\ 1.1: \beta_i = -\frac{\lambda_1}{3\alpha_i},\\
&\omega_i = \frac{\lambda_1\lambda_2 - 3\alpha_i(\alpha_i^3 + \lambda_3\alpha_i + \lambda_4\gamma_i)}{3\alpha_i},\ \alpha_i \neq 0;
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&Case\ 1.2: \alpha_i = -\frac{\lambda_1}{3\beta_i},\\
&\omega_i = \frac{\lambda_1^3 + 9\lambda_1\lambda_3\beta_i^2 - 27\beta_i^3(\lambda_2\beta_i + \lambda_4\gamma_i)}{27\beta_i^3},\ \beta_i \neq 0;
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&Case\ 1.3: \beta_i = -\frac{\lambda_1}{3\alpha_i},\\
&\gamma_i = \frac{\lambda_1\lambda_2 - 3\alpha_i(\alpha_i^3 + \lambda_3\alpha_i + \omega_i)}{3\lambda_4\alpha_i},\ \lambda_4\alpha_i \neq 0;
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&Case\ 1.4: \alpha_i = -\frac{\lambda_1}{3\beta_i},\\
&\gamma_i = \frac{\lambda_1^3 + 9\lambda_1\lambda_3\beta_i^2 - 27\beta_i^3(\lambda_2\beta_i + \omega_i)}{27\lambda_4\beta_i^3},\ \lambda_4\beta_i \neq 0.
\end{aligned}
\tag{30}
$$

Proof By employing the properties of D-operator, substituting the exponential test functions (26) and the linear superposition relationship (27) into bilinear form (7), we have

$$
\begin{aligned}
&(D_yD_t + D_x^3D_y + \lambda_1D_x^2 + \lambda_2D_y^2\\
&\quad + \lambda_3D_xD_y + \lambda_4D_yD_z)f_{A1}\cdot f_{A1} = 0,\\
&(D_yD_t + D_x^3D_y + \lambda_1D_x^2 + \lambda_2D_y^2 + \lambda_3D_xD_y\\
&\quad + \lambda_4D_yD_z)f_{A2}\cdot f_{A2} = 0,\\
&(D_yD_t + D_x^3D_y + \lambda_1D_x^2 + \lambda_2D_y^2\\
&\quad + \lambda_3D_xD_y + \lambda_4D_yD_z)f_{A3}\cdot f_{A3} = 0,
\end{aligned}
\tag{31}
$$

then the exponential test functions f_{A1}, f_{A2} and f_{A3} (26) are solutions of bilinear form (7). Similarly, the linear superposition relationships (28), (29) and (30) are true. Therefore, these linear superposition relationships are sufficient for the existence of solutions in bilinear form (7). One can directly prove this theorem. □

Substituting relational formula f_{A1} (26) and the linear superposition relationship (30) into transformation (6), the N-order hyperbolic cosine function superposition solutions of Eq. (1) can be obtained.

$$
\begin{aligned}
&u_{A1} = -2(\ln f_{A1})_x + u_0(z,t),\\
&f_{A1} = \sum_{i=1}^{N} k_i \cosh\Bigg[-\frac{\lambda_1}{3\beta_i}x + \beta_i y\\
&\quad + \frac{\lambda_1^3 + 9\lambda_1\lambda_3\beta_i^2 - 27\beta_i^3(\lambda_2\beta_i + \omega_i)}{27\lambda_4\beta_i^3}z + \omega_i t\Bigg].
\end{aligned}
\tag{32}
$$

Numerical simulations are performed to illustrate the properties of N-order hyperbolic cosine function superposition solutions through graphical forms. By setting $N = 3$ in N-order hyperbolic cosine function superposition solutions (32), the three-dimensional dynamic graphs of interaction between peaked soliton and two bending kink waves are successfully depicted in Fig. 1. Figure 1 is plotted by considering arbitrary constants as $\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = -1, k_1 = \beta_1 = 1.5, k_2 = 2, k_3 = 1, \beta_2 = -0.3, \beta_3 = -1.5, \omega_1 = -0.5, \omega_2 = -1.2, \omega_3 = 1.8$ and function as $u_0(z,t) = 4\text{sech}(\frac{1}{2}z^2 + \frac{1}{2}t^2)$ in 3-order hyperbolic cosine function superposition solutions (32). Figure 1 shows the interaction phenomenon of splitting into two bending kink waves due to the collision between two kink waves and the peaked soliton. With the increase of parameter x and the decrease of parameter y, two kink waves move along the positive direction of the z axis and collide with the peak soliton. It can be seen from Fig. 1b that the collision leads to the reduction of the peak value of the peaked soliton, and the two kink waves are split into two bending kink waves. When the parameters continue to change, the bending length of one of the bending kink waves gradually increases, and the peak of the peaked soliton cannot be recovered as shown in Fig. 1c. It also means that waves can constructively or destructively interfere under the effect of linear superposition.

Substituting relational formula f_{A3} (26) and the linear superposition relationship (27) into transformation (6), the N-order exponential function superposition solutions of Eq. (1) can be obtained.

$$u_{A3} = -2(\ln f_{A3})_x + u_0(z,t),$$
$$f_{A3} = \sum_{i=1}^{N} k_i \exp\left[\alpha_i x - \frac{\lambda_1}{3\alpha_i} y + \gamma_i z + \frac{\lambda_1\lambda_2 - 3\alpha_i(\alpha_i^3 + \lambda_3\alpha_i + \lambda_4\gamma_i)}{3\alpha_i} t\right]. \tag{33}$$

Figure 2 demonstrates the intensity distribution of the N-order exponential function superposition solutions (33) under the conditions $N = 4$. The corresponding parameters in Fig. 2 are $\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = -1, k_1 = k_2 = 1, k_3 = k_4 = 2, \alpha_1 = -1.3, \alpha_2 = -0.1, \gamma_1 = -1, \gamma_2 = 0.8, \alpha_3 = 1.6, \gamma_3 = -0.3, \alpha_4 = 0.2, \gamma_4 = -0.8, u_0(z,t) = \tanh(z)$. With the increase of parameter t and the decrease of parameter y, three adjacent kink waves advance at a certain angle along the positive direction of z-axis and the negative direction of x-axis. When parameter y and parameter t are equal, three adjacent kink waves collide with one kink wave, resulting in the fission of three kink waves. Then, three kink waves split into two kink waves and the distance is getting farther and farther. This process shows that the deformation of multiple waves cannot be recovered due to their collision, but their amplitude has not changed. Similarly, more kinds of exponential function type solutions can be obtained by means of relational formula (26) and the linear superposition relationships (27), (28), (29), (30), which are omitted here.

N-order hyperbolic cosine function superposition solutions (32) and N-order exponential function superposition solutions (33) are all analytical solutions. Figures 1 and 2 analyze the collision between kink waves and peaked soliton, as well as the collision between multiple kink waves. Kink waves are produced by the collision of water waves, which is very helpful to the study of water wave interaction [47].

3.2 Linear superposition formula of trigonometric function type

In this subsection, we choose the new test functions of multiple cosine functions and sine functions.

$$f_{B1} = \sum_{j=1}^{M} h_j \cos(\phi_j),$$
$$f_{B2} = \sum_{j=1}^{M} h_j \sin(\phi_j), \tag{34}$$
$$\phi_j = m_j x + n_j y + p_j z + q_j t,$$

where h_j, m_j, n_j, p_j, q_j are all the real constants, M is a positive integer.

Remark 3.1 Assuming that the trigonometric test functions f_{B1} and f_{B2} (34) are solutions of bilinear form (7), it is necessary to meet one of the following four constraint conditions:

$$Case\ 2.1: n_j = \frac{\lambda_1}{3m_j},$$
$$q_j = \frac{3m_j(m_j^3 - \lambda_3 m_j - \lambda_4 p_j) - \lambda_1\lambda_2}{3m_j}, m_j \neq 0; \tag{35}$$

$$Case\ 2.2: m_j = \frac{\lambda_1}{3n_j},$$
$$q_j = \frac{\lambda_1^3 - 9\lambda_1\lambda_3 n_j^2 - 27n_j^3(\lambda_2 n_j + \lambda_4 p_j)}{27n_j^3}, n_j \neq 0; \tag{36}$$

$$Case\ 2.3: n_j = \frac{\lambda_1}{3m_j},$$
$$p_j = \frac{3m_j(m_j^3 - \lambda_3 m_j - q_j) - \lambda_1\lambda_2}{3\lambda_4 m_j}, \lambda_4 m_j \neq 0; \tag{37}$$

$$Case\ 2.4: m_j = \frac{\lambda_1}{3n_j},$$
$$p_j = \frac{\lambda_1^3 - 9\lambda_1\lambda_3 n_j^2 - 27n_j^3(\lambda_2 n_j + q_j)}{27\lambda_4 n_j^3}, \lambda_4 n_j \neq 0. \tag{38}$$

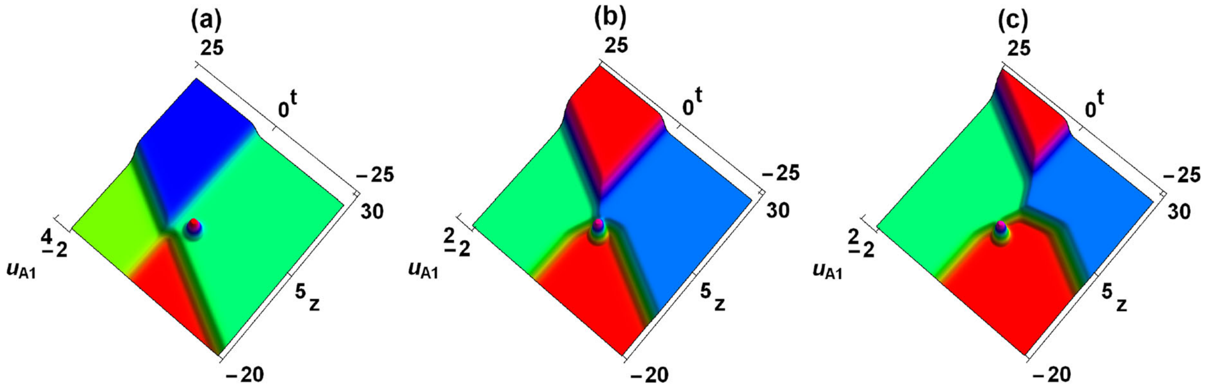

Fig. 1 (Color online) Interaction between the peaked soliton and two kink waves via N-order hyperbolic cosine function superposition solutions (32). **a** $x = 2$, $y = 6$, **b** $x = 6$, $y = 2$, and **c** $x = 10$, $y = -2$

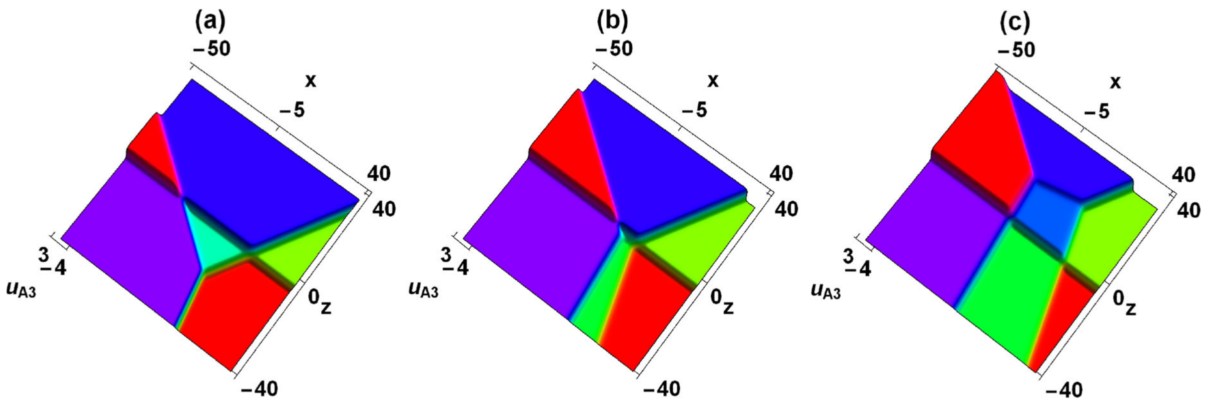

Fig. 2 (Color online) Interaction between three kink waves and one kink wave via N-order exponential function superposition solutions (33). **a** $y = 8$, $t = 0$, **b** $y = t = 4$, and **c** $y = 0$, $t = 8$

Substituting relational formula f_{B1} (34) and the linear superposition relationship (35) into transformation (6), corresponding M-order cosine function superposition solutions of Eq. (1) appear as

$$\begin{aligned} u_{B1} &= -2(\ln f_{B1})_x + u_0(z,t), \\ f_{B1} &= \sum_{j=1}^{M} h_j \cos\left[m_j x + \frac{\lambda_1}{3m_j} y + p_j z \right. \\ &\quad \left. + \frac{3m_j(m_j^3 - \lambda_3 m_j - \lambda_4 p_j) - \lambda_1\lambda_2}{3m_j} t\right]. \end{aligned} \tag{39}$$

Similarly, more kinds of trigonometric function type solutions can be obtained by means of relational formula (34) and the constraint conditions (35), (36), (37), (38), which are omitted here.

3.3 Linear superposition formula of trigonometric function and exponential function type

Select the test functions formed by the combination of three exponential function types (26) and two trigonometric function types (34). Assume that the six superposition solutions of bilinear form (7) are as follows

$$f_{C1} = \sum_{i=1}^{N} k_i \cosh(\theta_i) + \sum_{j=1}^{M} h_j \cos(\phi_j),$$

$$f_{C2} = \sum_{i=1}^{N} k_i \cosh(\theta_i) + \sum_{j=1}^{M} h_j \sin(\phi_j),$$

$$f_{C3} = \sum_{i=1}^{N} k_i \sinh(\theta_i) + \sum_{j=1}^{M} h_j \cos(\phi_j),$$

$$f_{C4} = \sum_{i=1}^{N} k_i \sinh(\theta_i) + \sum_{j=1}^{M} h_j \sin(\phi_j),$$
$$f_{C5} = \sum_{i=1}^{N} k_i \exp(\theta_i) + \sum_{j=1}^{M} h_j \cos(\phi_j),$$
$$f_{C6} = \sum_{i=1}^{N} k_i \exp(\theta_i) + \sum_{j=1}^{M} h_j \sin(\phi_j),$$
$$\theta_i = \alpha_i x + \beta_i y + \gamma_i z + \omega_i t, \phi_j = m_j x + n_j y + p_j z + q_j t, \quad (40)$$

where $k_i, \alpha_i, \beta_i, \gamma_i, \omega_i, h_j, m_j, n_j, p_j, q_j$ are all the real constants, N and M are positive integers.

Remark 3.2 The superposition solutions $f_{Ci}(i = 1, 2, 3, 4, 5, 6)$ (40) are composed of exponential functions f_{A1}, f_{A2}, f_{A3} (26) and trigonometric functions f_{B1}, f_{B2} (34). Then $f_{Ci}(i = 1, 2, 3, 4, 5, 6)$ (40) are solutions of bilinear form (7), which must satisfy the linear superposition relationships (27), (28), (29), (30) and (35), (36), (37), (38). There are 16 kinds of superposition relations, we only choose the following two cases and the rest are omitted.

$$\begin{aligned} &Case\ 3.1: \alpha_i = -\frac{\lambda_1}{3\beta_i}, \\ &\gamma_i = \frac{\lambda_1^3 + 9\lambda_1\lambda_3\beta_i^2 - 27\beta_i^3(\lambda_2\beta_i + \omega_i)}{27\lambda_4\beta_i^3}, \lambda_4\beta_i \neq 0, \\ &n_j = \frac{\lambda_1}{3m_j}, \\ &q_j = \frac{3m_j(m_j^3 - \lambda_3 m_j - \lambda_4 p_j) - \lambda_1\lambda_2}{3m_j}, m_j \neq 0; \end{aligned} \quad (41)$$

$$\begin{aligned} &Case\ 3.2: \beta_i = -\frac{\lambda_1}{3\alpha_i}, \\ &\omega_i = \frac{\lambda_1\lambda_2 - 3\alpha_i(\alpha_i^3 + \lambda_3\alpha_i + \lambda_4\gamma_i)}{3\alpha_i}, \alpha_i \neq 0, \\ &m_j = \frac{\lambda_1}{3n_j}, \\ &p_j = \frac{\lambda_1^3 - 9\lambda_1\lambda_3 n_j^2 - 27n_j^3(\lambda_2 n_j + q_j)}{27\lambda_4 n_j^3}, \lambda_4 n_j \neq 0. \end{aligned} \quad (42)$$

Substituting relational formula f_{C1} (40) and the linear superposition relationship (41) into transformation (6), corresponding hybrid solution between N-order hyperbolic cosine functions and M-order cosine functions of Eq. (1) appear as

$$\begin{aligned} u_{C1} &= -2(\ln f_{C1})_x + u_0(z,t), \\ f_{C1} &= \sum_{i=1}^{N} k_i \cosh\left[-\frac{\lambda_1}{3\beta_i}x + \beta_i y \right. \\ &\quad \left. + \frac{\lambda_1^3 + 9\lambda_1\lambda_3\beta_i^2 - 27\beta_i^3(\lambda_2\beta_i + \omega_i)}{27\lambda_4\beta_i^3}z + \omega_i t\right] \\ &\quad + \sum_{j=1}^{M} h_j \cos\left[m_j x + \frac{\lambda_1}{3m_j}y + p_j z \right. \\ &\quad \left. + \frac{3m_j(m_j^3 - \lambda_3 m_j - \lambda_4 p_j) - \lambda_1\lambda_2}{3m_j}t\right]. \end{aligned} \quad (43)$$

By setting $N = M = 2$ and considering the parameters $\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = -1, k_1 = k_2 = 1, h_1 = 2, h_2 = -3, \beta_1 = 0.27, \omega_1 = -1.4, \beta_2 = 1.6, \omega_2 = 0.6, m_1 = -0.5, p_1 = -0.2, m_2 = 1.4, p_1 = 0.2, u_0(z,t) = 0.1$ in hybrid solution between N-order hyperbolic cosine functions and M-order cosine functions (43). The localized characteristics and energy distribution of interaction between two rogue waves and two kink waves are shown clearly in Fig. 3. The rogue wave consists of an upward peak and a downward valley. As parameters y and t change, two rogue waves generated by the collision of two kink waves move at a certain angle along the negative direction of x axis and the positive direction of z axis. It can be seen from Fig. 3b that the amplitude of two rogue waves decreases with the change of parameters y and t. Subsequently, the shape of two rogue waves and two kink waves does not change and the amplitude changes as shown in Fig. 3c.

Substituting auxiliary function f_{C5} (40) and the linear superposition relationship (42) into transformation (6), corresponding hybrid solution among N-order exponential functions and M-order cosine functions of Eq. (1) appear as

$$\begin{aligned} u_{C5} &= -2(\ln f_{C5})_x + u_0(z,t), \\ f_{C5} &= \sum_{i=1}^{N} k_i \exp\left[\alpha_i x - \frac{\lambda_1}{3\alpha_i}y + \gamma_i z \right. \\ &\quad \left. + \frac{\lambda_1\lambda_2 - 3\alpha_i(\alpha_i^3 + \lambda_3\alpha_i + \lambda_4\gamma_i)}{3\alpha_i}t\right] \\ &\quad + \sum_{j=1}^{M} h_j \cos\left[\frac{\lambda_1}{3n_j}x + n_j y \right. \\ &\quad \left. + \frac{\lambda_1^3 - 9\lambda_1\lambda_3 n_j^2 - 27n_j^3(\lambda_2 n_j + q_j)}{27\lambda_4 n_j^3}z + q_j t\right]. \end{aligned} \quad (44)$$

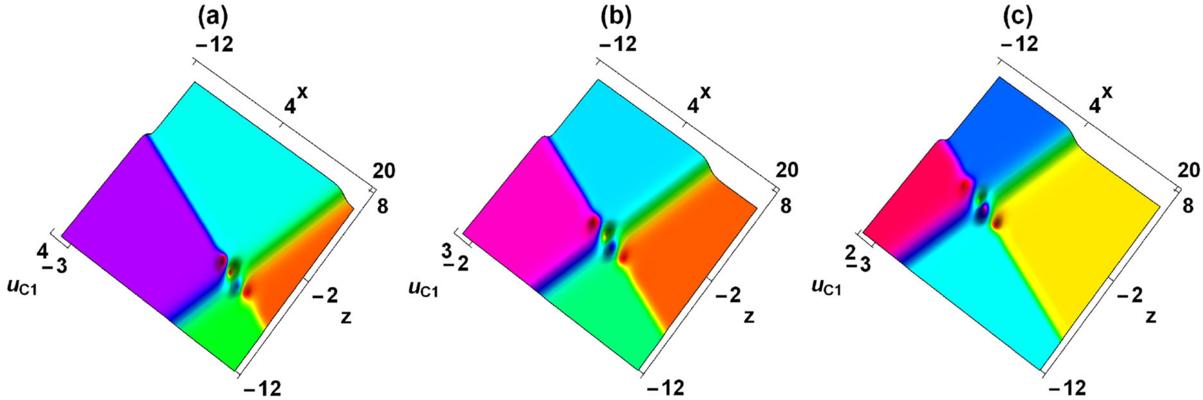

Fig. 3 (Color online) Interaction between two rogue waves and two kink waves via hybrid solution between N-order hyperbolic cosine functions and M-order cosine functions (43). **a** y = 10, t = −2, **b** y = 6, t = 2, and **c** y = 2, t = 6

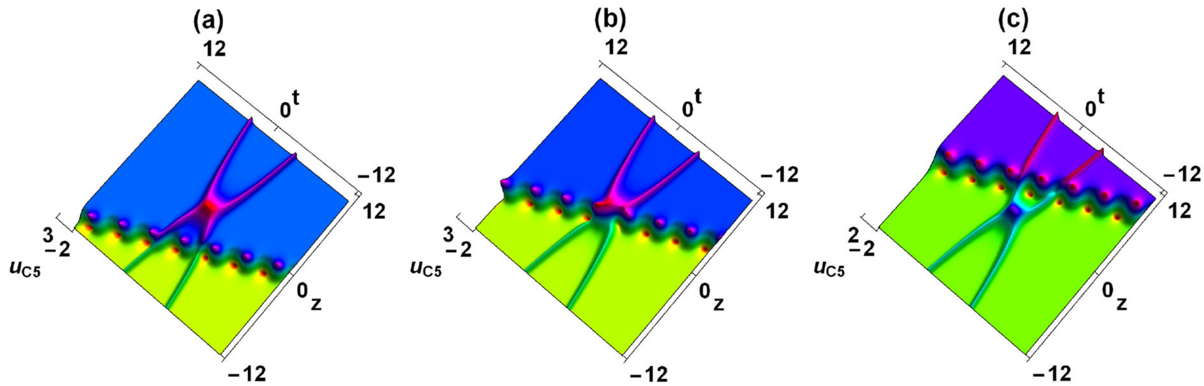

Fig. 4 (Color online) Interaction between breather wave and two bell-shaped waves via hybrid solution among N-order exponential functions and M-order cosine functions (44). **a** $x = -4$, y = 12, **b** $x = y = 4$, and **c** x = 12, $y = -4$

Derived result of hybrid solution among N-order exponential functions and M-order cosine functions (44) at $N = 2$, $M = 1$ and $\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = -1, k_1 = 1, k_2 = h_1 = 1.5, \alpha_1 = 0.5, \alpha_2 = -0.5, \gamma_1 = -2, \gamma_2 = 2, n_1 = 0.3, q_1 = 1.5, u_0(z,t) = \mathrm{sech}(t^2 + z) + \mathrm{sech}(t^2 - z)$ reveals interaction between breather wave and two bell-shaped waves profile. It is obvious from Fig. 4 that the breather wave is composed of two adjacent humps with periodicity on both sides of the horizontal plane, two bell-shaped waves intersect to form a bright soliton. As the parameters x and y change, the breather wave moves along the positive direction of the z-axis, and its amplitude decreases after colliding with the bright soliton.

Hybrid solution between N-order hyperbolic cosine functions and M-order cosine functions (43), hybrid solution among N-order exponential functions and M-order cosine functions (44) are all analytical solutions. Figures 3 and 4 analyze the interaction between rogue waves and kink waves, as well as the interaction between breather wave and bell-shaped waves. The mechanism of rogue waves can be regarded as the high-amplitude waves generated by the collision of multi-solitons [48,49]. It rises from an approximately constant background plane before reaching the maximum amplitude, and then gradually drops back the initial background plane [50]. Since the center of gravity of water waves fluctuates up and down, the results show that periodic waveform, the constructed bell-shaped waves and breather waves can help better understand the hydrodynamics of water waves in ocean engineering [51].

4 Linear superposition formula of function product type

This section mainly introduces two kinds of function product superposition theorems, including exponential function product superposition solutions and trigonometric function product superposition solutions.

4.1 Linear superposition formula of exponential function product type

In this subsection, we choose the new test functions which are composed of the product of exponential functions, hyperbolic cosine functions and hyperbolic sine functions.

$$\begin{aligned} f_{D1} &= \sum_{i=1}^{N} k_i \cosh(\eta_i)\cosh(\varphi_i), \\ f_{D2} &= \sum_{i=1}^{N} k_i \sinh(\eta_i)\sinh(\varphi_i), \\ f_{D3} &= \sum_{i=1}^{N} k_i \exp(\eta_i)\exp(\varphi_i), \\ f_{D4} &= \sum_{i=1}^{N} k_i \cosh(\eta_i)\sinh(\varphi_i), \\ f_{D5} &= \sum_{i=1}^{N} k_i \cosh(\eta_i)\exp(\varphi_i), \\ f_{D6} &= \sum_{i=1}^{N} k_i \sinh(\eta_i)\exp(\varphi_i), \\ \eta_i &= a_i x + b_i y + c_i z + d_i t, \\ \varphi_i &= e_i x + g_i y + r_i z + s_i t, \end{aligned} \tag{45}$$

where $k_i, a_i, b_i, c_i, d_i, e_i, g_i, r_i, s_i$ are all the real constants, N is a positive integer.

Remark 4.1 Assuming that the exponential product test functions f_{D1}, f_{D2}, f_{D3}, f_{D4}, f_{D5} and f_{D6} (45) are solutions of bilinear form (7), one of the following four linear superposition relationships must be satisfied:

$$\begin{aligned} &Case\ 4.1: b_i = \frac{\lambda_1 a_i}{3(e_i^2 - a_i^2)}, \\ &d_i = \frac{\lambda_1\lambda_2 a_i}{3(a_i^2 - e_i^2)} - a_i(a_i^2 + 3e_i^2 + \lambda_3) \\ &\quad -\lambda_4 c_i,\ a_i^2 \neq e_i^2, \\ &g_i = \frac{\lambda_1 e_i}{3(a_i^2 - e_i^2)}, \\ &s_i = \frac{\lambda_1\lambda_2 e_i}{3(e_i^2 - a_i^2)} - e_i(e_i^2 + 3a_i^2 + \lambda_3) - \lambda_4 r_i; \end{aligned} \tag{46}$$

$$\begin{aligned} &Case\ 4.2: a_i = \frac{\lambda_1 b_i}{3(g_i^2 - b_i^2)}, \\ &c_i = \frac{\lambda_1^3(b_i^3 + 3b_i g_i^2)}{27\lambda_4(b_i^2 - g_i^2)^3} - \frac{\lambda_2 b_i + d_i}{\lambda_4} \\ &\quad + \frac{\lambda_1\lambda_3 b_i}{3\lambda_4(b_i^2 - g_i^2)},\ b_i^2 \neq g_i^2, \\ &e_i = \frac{\lambda_1 g_i}{3(b_i^2 - g_i^2)}, \\ &r_i = \frac{\lambda_1^3(g_i^3 + 3g_i b_i^2)}{27\lambda_4(g_i^2 - b_i^2)^3} - \frac{\lambda_2 g_i + s_i}{\lambda_4} \\ &\quad + \frac{\lambda_1\lambda_3 g_i}{3\lambda_4(g_i^2 - b_i^2)},\ \lambda_4 \neq 0; \end{aligned} \tag{47}$$

$$\begin{aligned} &Case\ 4.3: b_i = \frac{\lambda_1 a_i}{3(e_i^2 - a_i^2)}, \\ &c_i = \frac{\lambda_1\lambda_2 a_i}{3\lambda_4(a_i^2 - e_i^2)} \\ &\quad - \frac{a_i(a_i^2 + 3e_i^2 + \lambda_3) + d_i}{\lambda_4},\ a_i^2 \neq e_i^2, \\ &g_i = \frac{\lambda_1 e_i}{3(a_i^2 - e_i^2)}, \\ &r_i = \frac{\lambda_1\lambda_2 e_i}{3\lambda_4(e_i^2 - a_i^2)} \\ &\quad - \frac{e_i(e_i^2 + 3a_i^2 + \lambda_3) + s_i}{\lambda_4},\ \lambda_4 \neq 0; \end{aligned} \tag{48}$$

$$\begin{aligned} &Case\ 4.4: a_i = \frac{\lambda_1 b_i}{3(g_i^2 - b_i^2)}, \\ &d_i = \frac{\lambda_1^3(b_i^3 + 3b_i g_i^2)}{27(b_i^2 - g_i^2)^3} - \lambda_2 b_i - \lambda_4 c_i \\ &\quad + \frac{\lambda_1\lambda_3 b_i}{3(b_i^2 - g_i^2)},\ b_i^2 \neq g_i^2, \\ &e_i = \frac{\lambda_1 g_i}{3(b_i^2 - g_i^2)}, \\ &s_i = \frac{\lambda_1^3(g_i^3 + 3g_i b_i^2)}{27(g_i^2 - b_i^2)^3} - \lambda_2 g_i - \lambda_4 r_i + \frac{\lambda_1\lambda_3 g_i}{3(g_i^2 - b_i^2)}. \end{aligned} \tag{49}$$

Substituting auxiliary function f_{D1} (45) and the linear superposition relationship (46) into transformation (6), corresponding N-order cosh×cosh function solutions of Eq. (1) appear as

$$\begin{aligned} u_{D1} &= -2(\ln f_{D1})_x + u_0(z,t),\ f_{D1} \\ &= \sum_{i=1}^{N} k_i \cosh(\eta_i)\cosh(\varphi_i), \end{aligned}$$

$$\eta_i = a_i x + \frac{\lambda_1 a_i}{3(e_i^2 - a_i^2)} y + c_i z + \left[\frac{\lambda_1\lambda_2 a_i}{3(a_i^2 - e_i^2)} - a_i(a_i^2 + 3e_i^2 + \lambda_3) - \lambda_4 c_i\right] t,$$

$$\varphi_i = e_i x + \frac{\lambda_1 e_i}{3(a_i^2 - e_i^2)} y + r_i z + \left[\frac{\lambda_1\lambda_2 e_i}{3(e_i^2 - a_i^2)} - e_i(e_i^2 + 3a_i^2 + \lambda_3) - \lambda_4 r_i\right] t. \quad (50)$$

For the N-order cosh×cosh function solutions (50) with $N = 2$, the three-dimensional dynamic graphs of 2-order cosh×cosh function solutions are successfully depicted in Fig. 5. Figure 5 is plotted by taking arbitrary parameters as $\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = -1, k_1 = k_2 = 1, a_1 = -0.7, a_2 = 0.9, c_1 = 0.5, c_2 = r_1 = 2, e_1 = -0.3, e_2 = -0.7, r_2 = -1.2, u_0(z,t) = 0$. With the increase of parameter y and the decrease of parameter t, four bending kink waves collide with each other and their shapes change. Figure 5b shows that four bending kink waves are fused into two kink waves under extrusion. Then, four bending kink waves split under mutual collision and the amplitude did not change in the whole process. Using relational formulas (45) and the linear superposition relationships (47), (48), (49), we can obtain another exponential product superposition solutions for Eq. (1). Here, we omit them.

4.2 Linear superposition formula of trigonometric function product type

Here, we assume the new test functions which are composed of the product of cosine functions and sine functions.

$$f_{E1} = \sum_{j=1}^{M} h_j \cos(\theta_j)\cos(\phi_j),$$
$$f_{E2} = \sum_{j=1}^{M} h_j \sin(\theta_j)\sin(\phi_j), \quad (51)$$
$$f_{E3} = \sum_{j=1}^{M} h_j \cos(\theta_j)\sin(\phi_j),$$
$$\theta_j = \alpha_j x + \beta_j y + \gamma_j z + \omega_j t, \phi_j = m_j x + n_j y + p_j z + q_j t,$$

where $h_j, \alpha_j, \beta_j, \gamma_j, \omega_j, m_j, n_j, p_j, q_j$ are all the real constants, M is a positive integer.

Remark 4.2 Assuming that the trigonometric product test functions f_{E1}, f_{E2} and f_{E3} (51) are solutions of bilinear form (7), it is necessary to meet one of the following four constraint conditions:

$$\mathit{Case}\ 5.1: \beta_j = \frac{\lambda_1\alpha_j}{3(\alpha_j^2 - m_j^2)}, \omega_j = \frac{\lambda_1\lambda_2\alpha_j}{3(m_j^2 - \alpha_j^2)} + \alpha_j(\alpha_j^2 + 3m_j^2 - \lambda_3) - \lambda_4\gamma_j, \alpha_j^2 \neq m_j^2,$$
$$n_j = \frac{\lambda_1 m_j}{3(m_j^2 - \alpha_j^2)}, q_j = \frac{\lambda_1\lambda_2 m_j}{3(\alpha_j^2 - m_j^2)} + m_j(m_j^2 + 3\alpha_j^2 - \lambda_3) - \lambda_4 p_j. \quad (52)$$

$$\mathit{Case}\ 5.2: \alpha_j = \frac{\lambda_1\beta_j}{3(\beta_j^2 - n_j^2)}, \gamma_j = \frac{\lambda_1^3(\beta_j^3 + 3\beta_j n_j^2)}{27\lambda_4(\beta_j^2 - n_j^2)^3} - \frac{\lambda_2\beta_j + \omega_j}{\lambda_4} + \frac{\lambda_1\lambda_3\beta_j}{3\lambda_4(n_j^2 - \beta_j^2)}, n_j^2 \neq \beta_j^2$$
$$m_j = \frac{\lambda_1 n_j}{3(n_j^2 - \beta_j^2)}, p_j = \frac{\lambda_1^3(n_j^3 + 3n_j\beta_j^2)}{27\lambda_4(n_j^2 - \beta_j^2)^3} - \frac{\lambda_2 n_j + q_j}{\lambda_4} + \frac{\lambda_1\lambda_3 n_j}{3\lambda_4(\beta_j^2 - n_j^2)}, \lambda_4 \neq 0; \quad (53)$$

$$\mathit{Case}\ 5.3: \beta_j = \frac{\lambda_1\alpha_j}{3(\alpha_j^2 - m_j^2)}, \gamma_j = \frac{\lambda_1\lambda_2\alpha_j}{3\lambda_4(m_j^2 - \alpha_j^2)} + \frac{\alpha_j(\alpha_j^2 + 3m_j^2 - \lambda_3) - \omega_j}{\lambda_4}, \alpha_j^2 \neq m_j^2,$$
$$n_j = \frac{\lambda_1 m_j}{3(m_j^2 - \alpha_j^2)}, p_j = \frac{\lambda_1\lambda_2 m_j}{3\lambda_4(\alpha_j^2 - m_j^2)} + \frac{m_j(m_j^2 + 3\alpha_j^2 - \lambda_3) - q_j}{\lambda_4}, \lambda_4 \neq 0; \quad (54)$$

$$\mathit{Case}\ 5.4: \alpha_j = \frac{\lambda_1\beta_j}{3(\beta_j^2 - n_j^2)}, \omega_j = \frac{\lambda_1^3(\beta_j^3 + 3\beta_j n_j^2)}{27(\beta_j^2 - n_j^2)^3} + \frac{\lambda_1\lambda_3\beta_j}{3(n_j^2 - \beta_j^2)} - \lambda_2\beta_j - \lambda_4\gamma_j, n_j^2 \neq \beta_j^2,$$
$$m_j = \frac{\lambda_1 n_j}{3(n_j^2 - \beta_j^2)}, q_j = \frac{\lambda_1^3(n_j^3 + 3n_j\beta_j^2)}{27(n_j^2 - \beta_j^2)^3} + \frac{\lambda_1\lambda_3 n_j}{3(\beta_j^2 - n_j^2)} - \lambda_2 n_j - \lambda_4 p_j. \quad (55)$$

Substituting auxiliary function f_{E1} (51) and the linear superposition relationship (52) into transformation (6), corresponding M-order cos×cos function solutions of Eq. (1) appear as

$$u_{E1} = -2(\ln f_{E1})_x + u_0(z,t), f_{E1} = \sum_{j=1}^{M} h_j \cos(\theta_j)\cos(\phi_j),$$

$$\theta_j = \alpha_j x + \frac{\lambda_1\alpha_j}{3(\alpha_j^2 - m_j^2)} y + \gamma_j z$$

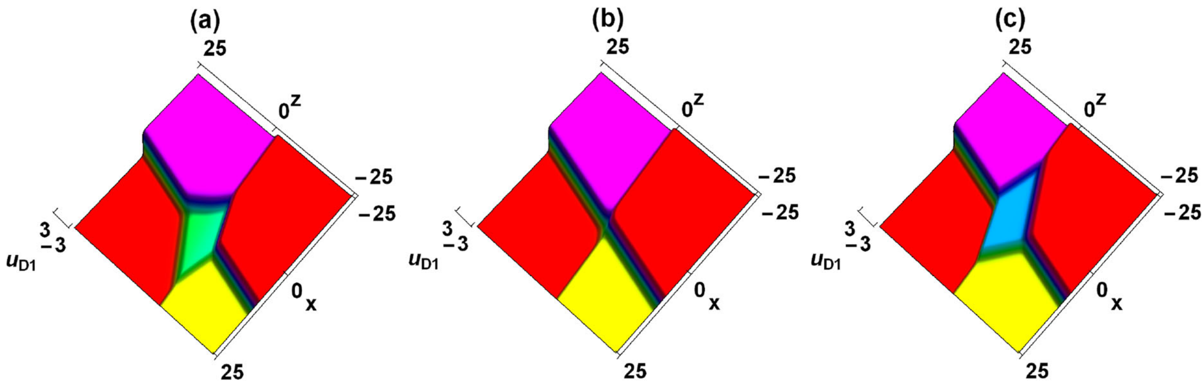

Fig. 5 (Color online) Interaction between four bending kink waves via N-order cosh×cosh function solutions (50). **a** $y = 1$, $t = 7$, **b** $y = 5$, $t = 3$, and **c** $y= 9$, $t= -1$

$$+\left[\frac{\lambda_1\lambda_2\alpha_j}{3(m_j^2-\alpha_j^2)}+\alpha_j(\alpha_j^2+3m_j^2-\lambda_3)-\lambda_4\gamma_j\right]t,$$

$$\phi_j = m_jx+\frac{\lambda_1 m_j}{3(m_j^2-\alpha_j^2)}y+p_jz$$

$$+\left[\frac{\lambda_1\lambda_2 m_j}{3(\alpha_j^2-m_j^2)}+m_j(m_j^2+3\alpha_j^2-\lambda_3)-\lambda_4 p_j\right]t. \tag{56}$$

Using relational formulas (51) and the linear superposition relationships (53), (54), (55), we can obtain another trigonometric product superposition solutions for Eq. (1). Here, we do not propose this part. It is essential to investigate the behavior of waves in nonlinear sciences. Therefore, we analyze some linear superposition solutions given by Eqs. (32), (33), (43), (44) and (50). Through the symbolic calculation system Mathematica and selecting appropriate parameters for numerical simulation, the properties of the evolution profiles of these wave expressions are studied. The results are helpful to the study of shallow water waves and provide a new way to explain the physical properties of nonlinear phenomena.

5 Conclusion

It is generally believed that waves play a pervasive role in nature. The formation and propagation of waves have important applications in water waves, seismic waves, gravitational waves and mechanical waves. With the help of symbolic computation, we have studied the extended (3+1)-dimensional shallow water wave equation in this work. Through the Hirota bilinear method, two kinds of bilinear auto-Bäcklund transformations are given and two different types of solutions are obtained, including the hyperbolic cosine-function solution and cosine-function solution. Through the homoclinic test method, five kinds of linear superposition formulas are given. However, according to the particularity of undetermined coefficients in Eq. (1), this method cannot be applied to all NLEEs. The results obtained by this method have important practical significance for explaining the nonlinear physical phenomena of some important models.

The interactions of different types of superposition solutions are studied by means of three-dimensional diagram. Figure 1 shows the interaction phenomenon of splitting into two bending kink waves due to the collision between two kink waves and the peaked soliton. One can evidently observe from Fig. 2 that the collision between three adjacent kink waves and one kink wave leads to the splitting of three kink waves into two bending kink waves. Figure 3 exhibits the interaction phenomenon of two kink waves colliding with each other to generate two rogue waves. The interaction phenomenon of bright soliton generated by the intersection of breather wave and two bell-shaped waves are found from Fig. 4. The interaction phenomenon of collision and fusion of four bending kink waves as shown in Fig. 5.

Our findings confirm the existence of some possible special linear superposition solutions in nonlinear systems and add the richness of analytical solutions. In the future work, the generalized bilinear form is obtained based on the generalized bilinear differential operators [52], and then some new linear superposition solutions are studied. With the help of the physics-informed neural networks [53] and bilinear neural network method [54], the diversity of analytical solutions can be enriched. The existence of these linear superposition solutions in nonlinear systems provides a new idea for us to analyze nonlinear phenomena. Naturally, we hope that the linear superposition principle can find different types of superposition solutions as much as possible, so as to enrich our understanding of nonlinear systems.

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Data availability All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest The authors declare that there is no conflict of interests regarding the research effort and the publication of this paper.

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