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# Local stability and Hopf bifurcations analysis of the Muthuswamy-Chua-Ginoux system

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**Abstract** The three-dimensional Muthuswamy–Chua–Ginoux (MCG, for short) circuit system based on a thermistor is a generalization of the classical Muthuswamy–Chua circuit differential system. At present, there are only partial numerical simulations for the qualitative analysis of the MCG circuit system. In this work, we study local stability and Hopf bifurcations of the MCG circuit system depending on 8 parameters. The emerging of limit cycles under zero-Hopf bifurcation and Hopf bifurcation is investigated in detail by using the averaging method and the center manifolds theory, respectively. We provide sufficient conditions for a class of the circuit systems to have a prescribed number of limit cycles bifurcating from the zero-Hopf equilibria by making use of the third-order averaging method, as well as the methods of Gröbner basis and real solution classification from symbolic computation. Such algebraic analysis allows one to study the zero-Hopf bifurcation for any other differential system in dimension 3 or higher. After, the classical Hopf bifurcation of the circuit system is analyzed by computing the first three focus quantities near the Hopf equilibria. Some examples and numerical simulations are presented to verify the established theoretical results.

**Keywords** Averaging method · Circuit system · Limit cycles · Symbolic computation · Hopf and zero-Hopf bifurcations

**Mathematics Subject Classification** 34C23 · 34C25 · 34C29

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## 1 Introduction

The invention of electronic circuits has brought a profound impact on human behavior and society. Nowadays, circuits have played an important role in many aspects of social life. More and more complex circuits have been designed in the electronics field. Initially, the electrical engineers believed that all circuits only can be decomposed into three classical passive circuit elements: the resistor, the capacitor and the inductor. In 1971, Chua [1] broke this traditional point and proposed the fourth fundamental circuit element, which is known as memristor. Chua also characterized the properties of memristor. After Chua, memristor seems to be hidden until Hewlett Packard Lab in 2008 observed that memristance phenomenon occurs in a novel $TiO_2$ nanoscale systems [2]. The memristor has gained wide attention since Hewlett Packard Lab's discovery. Over nearly ten years, memristor has important effect in applied science and engineering, specially in chaotic circuits [3–6] and neural networks [7,8].

Since the memristor can provide chaotic signals, the majority of modern chaotic circuits are constructed by it. The circuits based on memristors can exhibit rich complex dynamical behaviors, such as self-excited chaos, hyperchaos and hidden chaotic attractor. In 1833, electromagnetic pioneer Faraday realized that the resistance of a thermistor varies nonlinearly with temperature. Most notably, Chua in [9] showed that the thermistor is functionally equivalent to memristive devices. The authors in [10] constructed a chaotic circuit based on a thermistor, which they named *Muthuswamy–Chua–Ginoux circuit* (MCG circuit). The MCG circuit can be described by the following system of differential equations

$$\begin{aligned} \dot{x} &= k_1 y, \\ \dot{y} &= k_2 \left(x + f\left(y\right) + R\left(z\right) y\right), \\ \dot{z} &= R\left(z\right) y^2 - \epsilon z, \end{aligned} \tag{1}$$

where $k_1 = 1/\alpha$, $k_2 = -1/\eta$, $f(y) = ay + by^3$ and $R(z) = cz^2 + dz + s$. The interest in system (1) basically comes from two main reasons. The first one is because it is a new three-dimensional autonomous dynamical system exhibiting very particular properties such as the transition from torus breakdown to "double spiral-chaos." The second one is because the system should credibly model some important unsolved problem in nature and shed insight on that problem and should exhibit some behavior previously unobserved.

In the qualitative theory of differential equations, one of the main subjects is to find limit cycles of differential equations. In general, the periodic orbits are studied numerically because, usually, their analytical study is very difficult. Here, using the third-order averaging method and the center manifolds theory, we shall study analytically the periodic orbits of the three-dimensional MCG circuit system (1) which bifurcate firstly from a zero-Hopf bifurcation and secondly from a classical Hopf bifurcation.

The zero-Hopf and Hopf bifurcations analysis are powerful tools to find limit cycles, and they have been successfully applied to various concrete models [11–15]. Cândido and Llibre [16] reported that in many cases the periodic solutions that generate (via period-doubling) the chaotic attractor started with a periodic solution coming from a zero-Hopf or a Hopf bifurcation. This helps us to understand the mechanism of chaos in some systems. As far as we know, the study of zero-Hopf bifurcation and Hopf bifurcation in the MCG system has not been considered in the literature. In this work, we have this objective.

The rest of this paper is organized as follows. In Sect. 2, we present the description and equilibrium points of the MCG system. We briefly recall some main tools for proving the main results in Sect. 3, including the averaging method of third order, focus quantities and the theory of Hopf bifurcation. Section 4 is devoted to the study of zero-Hopf bifurcation and number of bifurcating limit cycles. The Hopf bifurcation and the number of bifurcated limit cycles are investigated in Sect. 5. Finally, we make a conclusion and give some future directions.

## 2 Muthuswamy–Chua–Ginoux system: description and equilibrium points

The MCG circuit can be designed as in Fig. 1. In fact, the MCG circuit is a generalization of the *Muthuswamy–Chua circuit* (MC circuit [17]). In terms of application, scholars are extremely concerned about the long-term dynamical behaviors of a circuit, especially the stability and the existence or non-existence of oscillation. From the qualitative theory of differential equations point of view, a nice way to answer these questions is to investigate the qualitative behaviors of circuit systems including the local stability of equilibrium point and limit cycles (oscillation). In [10], the numerical simulations suggest that system (1) can display "2-torus," "limit cycle," "spiral-chaos" and "double spiral-chaos" for appropriately choice of the parameter values of $k_1, k_2, \epsilon, a, b, c, d, s$, see Fig. 2.

*Remark 1* For system (1), we can assume $\epsilon \geq 0$, because $\epsilon < 0$ can be translated into $\epsilon > 0$ by the change of variables $(x, y, z, t) \mapsto (x, y, -z, -t)$.

The next result is about the stability conditions for the equilibrium point of the MCG system (1).

**Theorem 1** *The MCG system (1) has the equilibrium points* $(x, y, z) = (0, 0, z)$ *(here* $z \in \mathbb{R}$*) when* $\epsilon = 0$. *The origin* $(0, 0, 0)$ *is the unique equilibrium point of system (1) when* $\epsilon \neq 0$*, and it is asymptotically stable when one of the following conditions holds:*

$$\begin{aligned} &[0 < \epsilon,\ 0 < k_1,\ k_2 < 0,\ -s < a], \\ &[0 < \epsilon,\ k_1 < 0,\ 0 < k_2,\ a < -s]. \end{aligned} \tag{2}$$

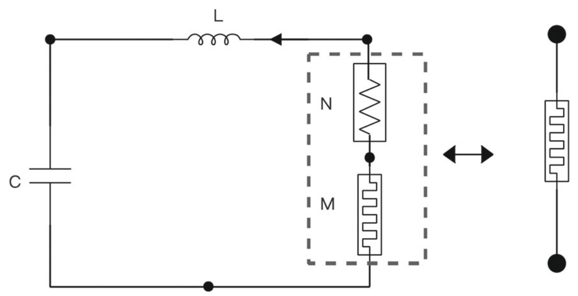



**Fig. 1** Muthuswamy–Chua–Ginoux circuit

*Proof* It is obvious that $(x, y, z) = (0, 0, z)$ for any $z \in \mathbb{R}$ is the equilibrium point of system (1) when $\epsilon = 0$; and $(x, y, z) = (0, 0, 0)$ is the unique equilibrium point of system (1) when $\epsilon \neq 0$. Moreover, the Jacobian matrix of system (1) at $(x, y, z) = (0, 0, 0)$ is

$$\begin{pmatrix} 0 & k_1 & 0 \\ k_2 & k_2(a+s) & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}. \tag{3}$$

The characteristic polynomial of matrix (3) is given by

$$p(\lambda) = (\lambda + \epsilon)\left(\lambda^2 - k_2(a+s)\lambda - k_1 k_2\right). \tag{4}$$

One can easily check that this polynomial has three roots with negative real parts if and only if the conditions in (2) hold. This completes the proof. □

We remark that the investigation in this paper is restricted to the equilibrium point (0, 0, 0). The following example provides a numerical simulation of Theorem 1.

*Example 1* Consider the following parameter vector

$$(a, b, c, d, s, k_1, k_2, \epsilon) = (1, 3, 3, -2, 0.25, 1, -0.01, 0.6).$$

Such values of parameters satisfy the conditions in Theorem 1. Hence, the origin of system (1) is asymptotically stable, see Fig. 3.

## 3 Preliminaries to the study of zero-Hopf and Hopf bifurcations

In this section, we recall the averaging method of third-order and Hopf bifurcation method for proving the main results.

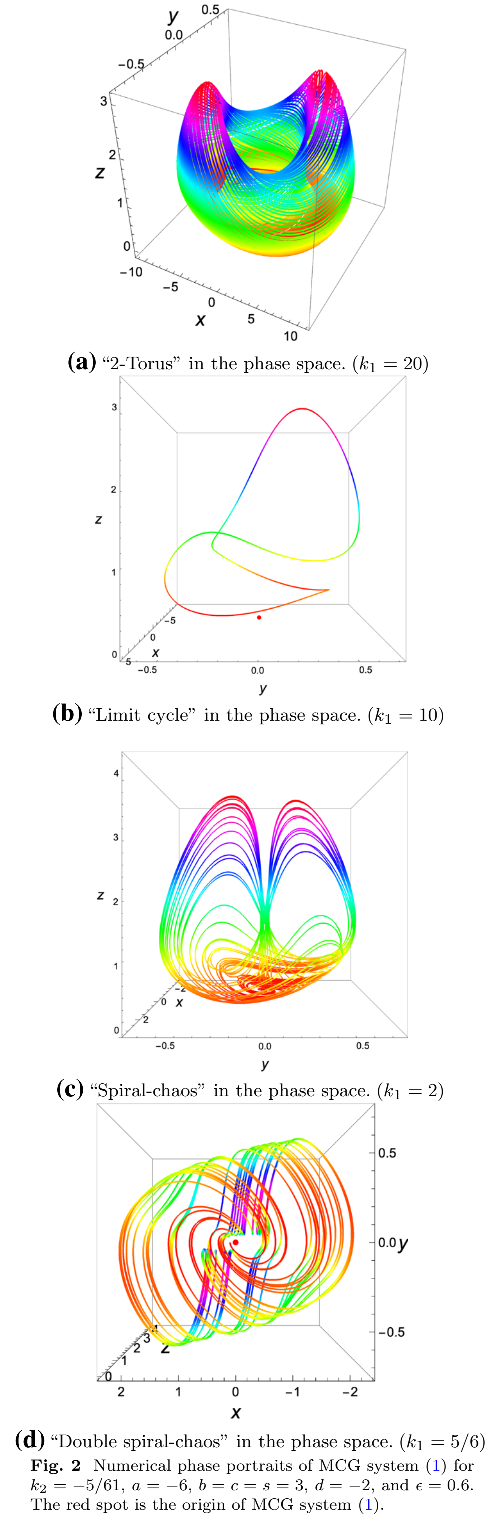
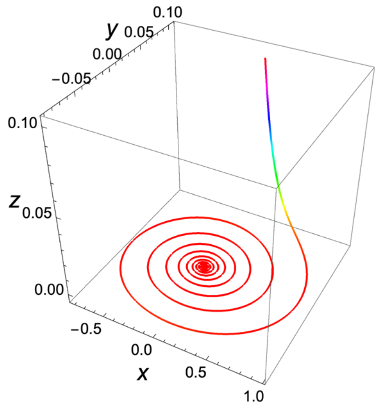
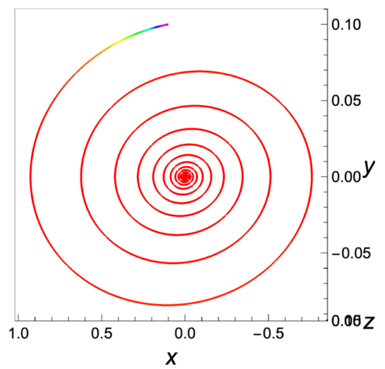



**Fig. 2** Numerical phase portraits of MCG system (1) for $k_2 = -5/61$, $a = -6$, $b = c = s = 3$, $d = -2$, and $\epsilon = 0.6$. The red spot is the origin of MCG system (1) (Color figre online)




**(a)** Local asymptotically stable for the origin.




**(b)** Projection of orbit in the $xy$-plane.

**Fig. 3** Numerical simulations of local phase portrait of system (1) for $(a, b, c, d, s, k_1, k_2, \epsilon) = (1, 3, 3, -2, 0.25, 1, -0.01, 0.6)$

### 3.1 Averaging method of third order

The averaging method for studying periodic solutions up to third order in $\varepsilon$ was developed in [18]. Recently, the averaging method for computing periodic solutions to an arbitrary order in $\varepsilon$ was provided in [19]. An expository account of recent work in this area can be found in [20].

Consider the differential system

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}) + \varepsilon^3 F_3(t, \mathbf{x}) + \varepsilon^4 R(t, \mathbf{x}, \varepsilon), \tag{5}$$

where $F_1, F_2, F_3 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, $T$-periodic in the variable $t$, and $D$ is a bounded open subset of $\mathbb{R}^n$.

To determine the limit cycles of system (5), we define the averaged functions $f_1, f_2, f_3 : D \to \mathbb{R}^n$ as

$$\begin{aligned} f_1(\mathbf{z}) &= \frac{1}{T}\int_0^T F_1(s, \mathbf{z})\mathrm{d}s, \\ f_2(\mathbf{z}) &= \frac{1}{T}\int_0^T [D_{\mathbf{z}}F_1(s, \mathbf{z}) \cdot y_1(s, \mathbf{z}) + F_2(s, \mathbf{z})]\mathrm{d}s, \\ f_3(\mathbf{z}) &= \frac{1}{T}\int_0^T \left[\frac{1}{2}D^2_{\mathbf{z}}F_1(s, \mathbf{z}) \cdot y_1^2(s, \mathbf{z}) + \frac{1}{2}D_{\mathbf{z}}F_1(s, \mathbf{z}) \right. \\ &\qquad \left. \times\, y_2(s, \mathbf{z}) + D_{\mathbf{z}}F_2(s, \mathbf{z}) \cdot y_1(s, \mathbf{z}) + F_3(s, \mathbf{z})\right]\mathrm{d}s, \end{aligned} \tag{6}$$

where

$$\begin{aligned} y_1(s, \mathbf{z}) &= \int_0^s F_1(t, \mathbf{z})\mathrm{d}t, \\ y_2(s, \mathbf{z}) &= 2\int_0^s [D_{\mathbf{z}}F_1(t, \mathbf{z}) \cdot y_1(t, \mathbf{z}) + F_2(t, \mathbf{z})]\mathrm{d}t. \end{aligned}$$

**Theorem 2** *For the differential system (5), we assume the following conditions hold.*

(i) *$F_1(t, \cdot) \in \mathcal{C}^2(D)$, $F_2(t, \cdot) \in \mathcal{C}^1(D)$ for all $t \in \mathbb{R}$, $F_1$, $F_2$, $F_3$, $R$, $D^2_{\mathbf{x}}F_1$ and $D_{\mathbf{x}}F_2$ are locally Lipschitz in the variable $\mathbf{x}$, and $R$ is twice differentiable in $\varepsilon$.*

(ii) *Assume that $f_i = 0$ for $i = 1, 2, \ldots, j-1$ and $f_j \neq 0$ with $j \in \{1, 2, 3\}$ (here, $f_0 = 0$). Suppose that for some $\mathbf{z}^* \in D$ with $f_j(\mathbf{z}^*) = 0$, there exists a bounded open set $V \subset D$ of $\mathbf{z}^*$ such that $f_j(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in \bar{V} \setminus \{\mathbf{z}^*\}$, and that $d_B(f_j(\mathbf{z}), V, 0) \neq 0$, where $d_B(f_j(\mathbf{z}), V, 0) \neq 0$ is the Brouwer degree of $f_j$ at 0 in the set $V$.*

*Then, for $\varepsilon \neq 0$ sufficiently small, there exists a $T$-periodic solution $\varphi(\cdot, \varepsilon)$ of system (5) such that $\varphi(0, \varepsilon) \to \mathbf{z}^*$ when $\varepsilon \to 0$.*

The proof of Theorem 2 can be found in [20]. Remark that, the Brouwer degree of $f_j$ at 0 is given by

$$d_B(f_j(\mathbf{z}), V, 0) = \sum_{\mathbf{z} \in \mathcal{Z}_{f_j}} \operatorname{sign}\left(J_{f_j}(\mathbf{z})\right),$$

where $\mathcal{Z}_{f_j} = \{\mathbf{z} \in V : f_j(\mathbf{z}) = 0\}$. In this case, $J_{f_j}(\mathbf{z}^*) \neq 0$ implies $d_B(f_j(\mathbf{z}), V, 0) \neq 0$. For more properties of the Brouwer degree, we refer to [21].

We also remark that the stability of the limit cycles associated with the simple zero $\mathbf{z}^*$ is controlled by the eigenvalues of the Jacobian of $f_j$ evaluated at $\mathbf{z}^*$. From Lemma 1 of [19], we know that the limit cycle associated with the zero $\mathbf{z}^*$ of $f_3(\mathbf{z})$ when $f_1(\mathbf{z}) = f_2(\mathbf{z}) = 0$ is given by

$$x(t, \mathbf{z}^*, \varepsilon) = \mathbf{z}^* + \varepsilon y_1(t, \mathbf{z}^*) + \varepsilon^2 \frac{y_2(t, \mathbf{z}^*)}{2} + \mathcal{O}(\varepsilon^3). \tag{7}$$

## 3.2 Focus quantities and the theory of Hopf bifurcation

### *3.2.1 Focus quantities*

Consider the following smooth differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{8}$$

Suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and the Jacobian matrix $D\mathbf{f}(\mathbf{0})$ has a pair of purely imaginary eigenvalues $\pm \mathrm{i}\omega, \omega > 0$, and one nonzero. After a linear change in the coordinates and a rescaling of the time variable, system (8) can be written in the form

$$\begin{aligned} \dot{x} &= -y + \widetilde{P}(x, y, z) = P(x, y, z), \\ \dot{y} &= x + \widetilde{Q}(x, y, z) = Q(x, y, z), \\ \dot{z} &= \mu z + \widetilde{S}(x, y, z) = S(x, y, z) \end{aligned} \tag{9}$$

with $\mu \in \mathbb{R} \setminus \{0\}$. Denoted by $X$ the corresponding vector field

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + S\frac{\partial}{\partial z}.$$

Let $U$ be an open set of $\mathbb{R}^3$. A non-locally constant differentiable function $\Phi : U \to \mathbb{R}$ is called a *first integral* of system (9) if it is constant along any solution trajectories of system (9), or equivalently,

$$X\Phi = P\frac{\partial \Phi}{\partial x} + Q\frac{\partial \Phi}{\partial y} + S\frac{\partial \Phi}{\partial z} \equiv 0. \tag{10}$$

The first integral $\Phi$ is a *formal power series* if $\Phi$ is a formal power series in $x$, $y$ and $z$.

The linear eigenspace of $D\mathbf{f}(\mathbf{0})$ corresponding to $\pm \mathrm{i}\omega$ is denoted by $T^{\mathrm{c}}$. From Theorem 5.1 of [22], we have that there exists a locally two-dimensional *invariant manifold* $W^{\mathrm{c}}_{\mathrm{loc}}$ of system (9) that is tangent to $T^{\mathrm{c}}$ at the origin. We say that the invariant manifold $W^{\mathrm{c}}_{\mathrm{loc}}$ is a *center manifold* of system (9).

The origin is a *monodromic singular point* (the trajectories near the origin turn around the origin either in forward or in backward time) for the vector field $X$ restricted to the center manifold $W^{\mathrm{c}}_{\mathrm{loc}}$. For an analytic vector field, a monodromic singular point is either a center or a focus, see [23]. One of the classical problems is to distinguish between a center and a focus, which was called *center problem*. This problem was solved by Poincaré and Lyapunov in $\mathbb{R}^2$, see for instance [24]. The necessary and sufficient conditions for the existence of a center on $W^{\mathrm{c}}_{\mathrm{loc}}$ of system (9) are characterized by the following theorem, whose proof can be found in [25].

**Theorem 3** *For system* (9)*, the following statements are equivalent.*

(i) *The origin of system (9) is a center on a center manifold* $W^c_{loc}$.
(ii) *System (9) in a neighborhood of the origin has a local analytic first integral of the form* $\Phi(x, y, z) = x^2 + y^2 + \cdots$.
(iii) *System (9) in a neighborhood of the origin has a formal first integral of the form* $\Phi(x, y, z) = x^2 + y^2 + \cdots$.

The equivalence of statements (a) and (b) is known as the *Lyapunov Center Theorem*. Theorem 3 tells us that the center problem of system (9) on $W^{\mathrm{c}}_{\mathrm{loc}}$ is transformed to detect whether system (9) has a first integral of the form $\Phi(x, y, z) = x^2 + y^2 + \cdots$ in a neighborhood of the origin.

Let $\widetilde{P}(x, y, z)$, $\widetilde{Q}(x, y, z)$ and $\widetilde{S}(x, y, z)$ be polynomials in system (9). In order to simplify the computation, we apply the complex coordinates

$$(x, y, z) \mapsto \left(\frac{x+y}{2}, -\frac{\mathrm{i}(x-y)}{2}, z\right).$$

Therefore, the complexification of system (9) is given by

$$\begin{aligned} \dot{x} &= \mathrm{i}x + \sum_{i+j+l=2}^{n} a_{ijl} x^i y^j z^l, \\ \dot{y} &= -\mathrm{i}y + \sum_{i+j+l=2}^{n} b_{ijl} x^i y^j z^l, \\ \dot{z} &= \mu z + \sum_{i+j+l=2}^{n} c_{ijl} x^i y^j z^l, \end{aligned} \tag{11}$$

where $b_{jil} = \bar{a}_{ijl}$ and the coefficients $c_{ijl}$ are such that $\sum_{i+j+l=2}^{n} c_{ijl} x^i \bar{x}^j z^l$ is real for all $x \in \mathbb{C}$ and $z \in \mathbb{R}$. Note that system (9) has a first integral of the form $\Phi(x, y, z) = x^2 + y^2 + \cdots$ if and only if system (11) has a first integral of the form

$$H(x, y, z) = xy + \textstyle\sum_{i+j+l=3} v_{i,j,l} x^i y^j z^l. \tag{12}$$

Denoted by $\mathcal{X}$ the vector field associated with (11). Then, we obtain

$$\mathcal{X}H = g_{0,0,0}xy + g_{1,1,0}(xy)^2 + g_{2,2,0}(xy)^3 + \cdots, \tag{13}$$

see [25] for more details. As was done in [25], the coefficient $g_{k,k,0}$ of $(xy)^{k+1}$ in Eq. (13) is called the *kth focus quantity* and also is denoted by $\nu_k$. Obviously, $\nu_0 = 0$. Edneral et al. [25] proved that system (11) has a formal first integral of the form (12) if and only if the focus quantity $\nu_k$ is to vanish for all $k \geq 1$.

### 3.2.2 Basic theory of Hopf bifurcation

Under appropriate perturbations, small amplitude limit cycles may be bifurcated from the Hopf equilibrium point of system (9), which is called *Hopf bifurcation*. For more details about the Hopf bifurcation, we refer to the book [26].

In the polar coordinates $(\theta, \rho)$, the local center manifold $W^c_{\text{loc}}$ can be parametrized by $\theta$ and $\rho$, and $\rho = 0$ corresponds to the origin of system (9). This means that system (9) restricted to $W^c_{\text{loc}}$ becomes a planar differential system. We can define the *Poincaré return map* in a neighborhood of the origin and introduce displacement map $\Pi(\rho) - \rho$, that is,

$$\Pi(\rho) - \rho = l_1\rho + l_2\rho^2 + \cdots .$$

The *mth Lyapunov coefficient* is defined by the coefficient $l_m$.

The next result follows immediately from Theorem 3.1.5 of [24].

**Theorem 4** *For system (9), the following statements hold.*

(i) *The origin of system (9) on $W^c_{loc}$ is a center if and only if all the Lyapunov coefficients vanish.*

(ii) *If $l_1 \neq 0$ or for some $k \in \mathbb{N}$ such that*

$$l_1 = l_2 = \cdots = l_{2k} = 0, \; l_{2k+1} \neq 0, \tag{14}$$

*then the origin of system (9) on $W^c_{loc}$ is stable focus (respectively, unstable focus) if $l_1 < 0$ or (14) holds with $l_{2k+1} < 0$ (respectively, $l_1 > 0$ or (14) holds with $l_{2k+1} > 0$).*

The origin of system (9) is called a *fine focus of order k* if (14) holds for $k \geq 1$. Under appropriate perturbations, Roussarie [27] showed that at most $k$ limit cycles can bifurcate from a fine focus of order $k$ of system (9).

The relation between the focus quantities and the Lyapunov coefficients is algebraic equivalence, that is,

$$\langle \nu_1, \nu_2, \nu_3, \ldots \rangle = \langle l_1, l_2, l_3, \ldots \rangle = \langle l_3, l_5, l_7, \ldots \rangle, \tag{15}$$

see Corollary 6.2.4 in [24]. More concretely, we have the following proposition, see for instance page 263 of [24].

**Proposition 5** *A fine focus of system (9) is of order k if and only if*

$$\nu_1 = \nu_2 = \cdots = \nu_{k-1} = 0, \; \nu_k \neq 0.$$

**Theorem 6** *Let $\Lambda$ be a parameter space of system (8). Suppose that system (8) has a fine focus of order k with $\tau \in \Lambda$. If the linear parts of the focus quantities $\nu_{i_1}, \ldots, \nu_{i_{\ell-1}}$ for $0 < i_1 < \cdots < i_{\ell-1} < k$ (with respect to the expansion of $\nu_j$ about $\tau$ ) are linearly independent, then system (8) has exactly $\ell$ limit cycles, which can bifurcate from a fine focus for parameter value $\tau$.*

For more details about a proof of Theorem 6 see [28, 29].

*Remark 2* From the above relation (15) and Theorem 6, we will use the focus quantities instead of the Lyapunov coefficients to investigate Hopf bifurcations in this paper.

## 4 Zero-Hopf bifurcation and number of bifurcating limit cycles

Recall that an equilibrium point of system (1) is called a *zero-Hopf equilibrium point* if its linear part has a zero eigenvalue $\lambda_1 = 0$ and a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega \neq 0$. The next result characterizes the zero-Hopf equilibrium point of the MCG system.

**Proposition 7** *The origin of system (1) is a zero-Hopf equilibrium point when $\epsilon = 0$, $k_1k_2 = -\omega^2$ and $s = -a$.*

*Proof* If the origin is a zero-Hopf equilibrium point of system (1), then the characteristic polynomial $p(\lambda)$ must be of the form $p(\lambda) = \lambda\left(\lambda^2 + \omega^2\right)$ with $\omega \neq 0$. The desired conditions: $\epsilon = 0$, $k_1k_2 = -\omega^2$ and $s = -a$ follow from equation (4). This completes the proof of the result. □

*Remark 3* We remark that there exists other zero-Hopf equilibrium points of system (1) besides the origin if we consider the equilibrium points $(x, y, z) = (0, 0, z)$ for $z \in \mathbb{R}$ when $\epsilon = 0$. In this paper, we are interested in the number of limit cycles that can bifurcate from the origin in a zero-Hopf bifurcation.

To study the zero-Hopf bifurcation of the MCG system by using the third-order averaging method, we consider the vector $(a, b, c, d, s, k_1, k_2, \epsilon)$ given by

$$a \leftarrow a + \sum_{i=1}^{3} \varepsilon^i a_i,\ b \leftarrow b + \sum_{i=1}^{3} \varepsilon^i b_i,\ c \leftarrow c + \sum_{i=1}^{3} \varepsilon^i c_i,$$
$$d \leftarrow d + \sum_{i=1}^{3} \varepsilon^i d_i,\ s \leftarrow -a,\ k_1 \leftarrow k_1 + \sum_{i=1}^{3} \varepsilon^i \ell_i,$$
$$k_2 \leftarrow -\frac{\omega^2}{k_1} + \sum_{i=1}^{3} \varepsilon^i m_i,\ \epsilon \leftarrow \sum_{i=1}^{3} \varepsilon^i \epsilon_i, \tag{16}$$

where $\varepsilon$ is a small parameter. Applying (16) to system (1) and after some calculations, we obtain the following MCG-like perturbations of system (1)

$$\begin{aligned}
\dot{x} &= k_1 y + \sum_{i=1}^{3} \varepsilon^i \alpha_i y,\\
\dot{y} &= -\frac{\omega^2}{k_1}(by^3 + cyz^2 + dyz + x) + \sum_{i=1}^{3} \varepsilon^i [\beta_{i,1} x\\
&\quad + \beta_{i,2} y + \beta_{i,3} y^3 + y(\beta_{i,4} z^2 + \beta_{i,5} z + \beta_{i,6})],\\
\dot{z} &= y^2(cz^2 + dz - a) + \sum_{i=1}^{3} \varepsilon^i [y^2(\gamma_{i,1} z^2 + \gamma_{i,2} z\\
&\quad + \gamma_{i,3}) - \gamma_{i,4} z],
\end{aligned} \tag{17}$$

where the constants $\alpha_i = \ell_i$, $\beta_{i,j_1}$ for $i = 1, 2, 3$ and $j_1 = 1, 2, \ldots, 6$ are expressions in the variables $k_1, \omega, b_i, c_i, d_i, \ell_i, m_i$, and $\gamma_{i,1} = c_i$, $\gamma_{i,2} = d_i$, $\gamma_{i,3} = -a_i$, $\gamma_{i,4} = \epsilon_i$ are all real numbers. We remark that for the procedure of how system (17) is derived from system (1), we refer the reader to the study of a Chua system in [30].

Our result on the number of limit cycles of system (17) is stated as follows.

**Theorem 8** *The following statements hold for $\varepsilon \neq 0$ sufficiently small,*

(i) *Up to the first-order averaging, system (17) has at most 1 limit cycle bifurcates from the origin, and this number can be reached;*

(ii) *Up to the second-order averaging, system (17) has at most 2 limit cycles bifurcate from the origin, and this number can be reached if one of the eight following conditions holds:*

$$\begin{aligned}
\mathcal{C}_1 &= [R_1 < 0, R_2 < 0, R_3 < 0, R_4 < 0, R_5 \le 0, 0 < R_6],\\
\mathcal{C}_2 &= [R_1 < 0, R_2 < 0, 0 < R_3, R_4 < 0, 0 \le R_5, 0 < R_6],\\
\mathcal{C}_3 &= [R_1 < 0, 0 < R_2, R_3 < 0, 0 < R_4, R_5 \le 0, 0 < R_6],\\
\mathcal{C}_4 &= [R_1 < 0, 0 < R_2, 0 < R_3, 0 < R_4, 0 \le R_5, 0 < R_6],\\
\mathcal{C}_5 &= [0 < R_1, R_2 < 0, R_3 < 0, 0 < R_4, 0 \le R_5, 0 < R_6],\\
\mathcal{C}_6 &= [0 < R_1, R_2 < 0, 0 < R_3, 0 < R_4, R_5 \le 0, 0 < R_6],\\
\mathcal{C}_7 &= [0 < R_1, 0 < R_2, R_3 < 0, R_4 < 0, 0 \le R_5, 0 < R_6],\\
\mathcal{C}_8 &= [0 < R_1, 0 < R_2, 0 < R_3, R_4 < 0, R_5 \le 0, 0 < R_6];
\end{aligned} \tag{18}$$

(iii) *Up to the third-order averaging, system (17) has at most 3 limit cycles bifurcate from the origin, and this number can be reached if one of the four following conditions holds:*

$$\begin{aligned}
\bar{\mathcal{C}}_1 &= [\bar{R}_1 < 0, \bar{R}_2 < 0, 0 < \bar{R}_3, \bar{R}_4 \le 0, \bar{R}_5 \le 0, \bar{R}_6 < 0],\\
\bar{\mathcal{C}}_2 &= [\bar{R}_1 < 0, 0 < \bar{R}_2, 0 < \bar{R}_3, \bar{R}_4 \le 0, 0 \le \bar{R}_5, \bar{R}_6 < 0],\\
\bar{\mathcal{C}}_3 &= [0 < \bar{R}_1, \bar{R}_2 < 0, \bar{R}_3 < 0, 0 \le \bar{R}_4, 0 \le \bar{R}_5, \bar{R}_6 < 0],\\
\bar{\mathcal{C}}_4 &= [0 < \bar{R}_1, 0 < \bar{R}_2, \bar{R}_3 < 0, 0 \le \bar{R}_4, \bar{R}_5 \le 0, \bar{R}_6 < 0],
\end{aligned} \tag{19}$$

*where the expressions of $R_i$ and $\bar{R}_i$ for $i = 1, 2, \ldots, 6$ are given in (32) and (37), respectively.*

*Proof* In order to study the zero-Hopf bifurcation of system (17), we need to write the linear part of system (17) at the origin in its real Jordan normal form, i.e., into the form

$$\begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the new variables defined by $(x, y, z) \mapsto \big(-y/\omega, x/k_1, z\big)$, system (17) becomes

$$\begin{aligned}
\dot{x} &= \omega y - \frac{\omega^2}{k_1^3}\left(cz^2 k_1^2 + dz k_1^2 + bx^2\right)x + \Big(-\frac{k_1 \beta_{1,1} y}{\omega}\\
&\quad + \beta_{1,2} x + \frac{\beta_{1,3} x^3}{k_1^2} + xz^2 \beta_{1,4} + xz\beta_{1,5} + x\beta_{1,6}\Big)\varepsilon\\
&\quad + \Big(-\frac{k_1 \beta_{2,1}}{\omega} y + \beta_{2,2} x + \frac{\beta_{2,3}}{k_1^2} x^3 + \beta_{2,4} xz^2 + \beta_{2,5} xz\\
&\quad + \beta_{2,6} x\Big)\varepsilon^2 + \Big(-\frac{k_1 \beta_{3,1}}{\omega} y + \beta_{3,2} x + \frac{\beta_{3,3}}{k_1^2} x^3 + \beta_{3,4} xz^2\\
&\quad + \beta_{3,5} xz + \beta_{3,6} x\Big)\varepsilon^3,\\
\dot{y} &= -\omega x - \frac{\omega}{k_1}(\alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \alpha_3 \varepsilon^3)x,\\
\dot{z} &= \frac{1}{k_1^2}(z(cz + d) - a)x^2 + \Big(\frac{\gamma_{1,1}}{k_1^2} x^2 z^2 + \frac{\gamma_{1,2}}{k_1}^2 x^2 z\\
&\quad + \frac{\gamma_{1,3}}{k_1}^2 x^2 - \gamma_{1,4} z\Big)\varepsilon + \Big(\frac{\gamma_{2,1}}{k_1^2} x^2 z^2 + \frac{\gamma_{2,2}}{k_1^2} x^2 z
\end{aligned}$$

$$+\frac{\gamma_{2,3}}{k_1^2}x^2-\gamma_{2,4}z\Big)\varepsilon^2+\Big(\frac{\gamma_{3,1}}{k_1^2}x^2z^2+\frac{\gamma_{3,2}}{k_1^2}x^2z+\frac{\gamma_{3,3}}{k_1^2}x^2-\gamma_{3,4}z\Big)\varepsilon^3. \tag{20}$$

Taking the rescaling of variables $(x, y, z) \mapsto (\varepsilon x, \varepsilon y, \varepsilon z)$, we obtain the differential system

$$\begin{aligned}
\dot{x} &= \omega y+\sum_{i=1}^{5}\varepsilon^i h_i(x,y,z),\\
\dot{y} &= -\omega x-\frac{\omega}{k_1}\left(\alpha_1\varepsilon+\alpha_2\varepsilon^2+\alpha_3\varepsilon^3\right)x,\\
\dot{z} &= \frac{1}{k_1^2}\Big[(-\gamma_{1,4}k_1^2z-ax^2)\varepsilon+(dx^2z-\gamma_{2,4}k_1^2z+\gamma_{1,3}x^2)\\
&\quad\times\varepsilon^2+(cx^2z^2+\gamma_{1,2}x^2z-k_1^2\gamma_{3,4}z+\gamma_{2,3}x^2)\varepsilon^3\\
&\quad+(\gamma_{1,1}z^2+\gamma_{2,2}z+\gamma_{3,3})x^2\varepsilon^4+(\gamma_{2,1}z+\gamma_{3,2})x^2z\varepsilon^5\\
&\quad+\gamma_{3,1}x^2z^2\varepsilon^6\Big],
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
h_1(x,y,z) &= -\frac{1}{\omega k_1}\Big(\omega^3dxz-\beta_{1,2}\omega k_1x-\beta_{1,6}\omega k_1x\\
&\quad+k_1^2\beta_{1,1}y\Big),\\
h_2(x,y,z) &= -\frac{1}{\omega k_1^3}\Big(\omega^3bx^3+k_1^4\beta_{2,1}y-\beta_{2,2}k_1^3\omega x\\
&\quad-\beta_{1,5}k_1^3\omega xz+\omega^3ck_1^2xz^2-\beta_{2,6}k_1^3\omega x\Big),\\
h_3(x,y,z) &= \frac{1}{\omega k_1^2}\Big(\beta_{1,3}\omega x^3-k_1^3\beta_{3,1}y+\beta_{3,2}k_1^2\omega x\\
&\quad+\beta_{2,5}k_1^2\omega xz+\beta_{1,4}k_1^2\omega xz^2+\beta_{3,6}k_1^2\omega x\Big),\\
h_4(x,y,z) &= \frac{1}{k_1^2}\left(k_1^2\beta_{2,4}z^2+k_1^2\beta_{3,5}z+\beta_{2,3}x^2\right)x,\\
h_5(x,y,z) &= \frac{\left(k_1^2\beta_{3,4}z^2+\beta_{3,3}x^2\right)x}{k_1^2}.
\end{aligned}$$

Using the cylindrical change of variables $(x, y, z) \mapsto (r\sin\theta, r\cos\theta, z)$, we obtain a differential system $\{\mathrm{d}r/\mathrm{d}t,\ \mathrm{d}\theta/\mathrm{d}t,\ \mathrm{d}z/\mathrm{d}t\}$. Note that $\mathrm{d}\theta/\mathrm{d}t \neq 0$ in a suitable small neighborhood of $(r, z) = (0, 0)$. By taking $\theta$ as the new independent variable of the differential system and carrying out Taylor expansions in the variable $\varepsilon$ around $\varepsilon = 0$, we get

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \sum_{i=1}^{3}\varepsilon^iF_{i,1}(\theta,r,z)+O\left(\varepsilon^4\right),\\
\frac{\mathrm{d}z}{\mathrm{d}\theta} &= \sum_{i=1}^{3}\varepsilon^iF_{i,2}(\theta,r,z)+O\left(\varepsilon^4\right),
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
F_{1,1}(\theta,r,z) &= \frac{\cos\theta}{\omega^2k_1}(-\cos\theta\beta_{1,6}\omega k_1+\sin\theta\alpha_1\omega^2\\
&\quad-\cos\theta\beta_{1,2}\omega k_1+d\omega^3\cos\theta z+k_1^2\beta_{1,1}\sin\theta)r,\\
F_{1,2}(\theta,r,z) &= \frac{1}{2\omega k_1^2}(ar^2\cos(2\theta)+2\gamma_{1,4}zk_1^2+ar^2),
\end{aligned}$$

the expressions of $F_{2,1}(\theta, r, z)$, $F_{3,1}(\theta, r, z)$, $F_{2,2}(\theta, r, z)$ and $F_{3,2}(\theta, r, z)$ are quite long, so we omit them for brevity.

For performing the averaging method, system (22) is written in the normal form (5) with $t = \theta$, $T = 2\pi$, $x = (r, z)$, and it satisfies all the assumptions of Theorem 2. According to Eq. (6), we have

$$f_{1,i}(r,z)=\frac{1}{2\pi}\int_0^{2\pi}F_{1,i}(\theta,r,z)\,\mathrm{d}\theta,\quad i=1,2. \tag{23}$$

A direct computation shows that

$$\begin{aligned}
f_{1,1}(r,z) &= \frac{r}{2\omega k_1}\Big(\mathrm{d}\omega^2z-k_1(\beta_{1,2}+\beta_{1,6})\Big),\\
f_{1,2}(r,z) &= \frac{1}{2\omega k_1^2}\Big(ar^2+2zk_1^2\gamma_{1,4}\Big).
\end{aligned} \tag{24}$$

Note that $r > 0$, the above first-order averaged function $(f_{1,1}(r, z), f_{1,2}(r, z))$ has a unique solution

$$\bar{r}_1=\sqrt{-2\frac{k_1^3\gamma_{1,4}\left(\beta_{1,2}+\beta_{1,6}\right)}{\mathrm{d}\omega^2a}},\quad \bar{z}_1=\frac{k_1\left(\beta_{1,2}+\beta_{1,6}\right)}{d\omega^2},$$

if $k_1\gamma_{1,4}(\beta_{1,2}+\beta_{1,6})da < 0$. The corresponding Jacobian of $(f_{1,1}(r, z), f_{1,2}(r, z))$ at $(\bar{r}_1, \bar{z}_1)$ takes the value

$$\begin{aligned}
J(\bar{r}_1,\bar{z}_1) &= \det\begin{pmatrix}\frac{\partial f_{1,1}}{\partial r} & \frac{\partial f_{1,1}}{\partial z}\\ \frac{\partial f_{1,2}}{\partial r} & \frac{\partial f_{1,2}}{\partial z}\end{pmatrix}\Big|_{(r,z)=(\bar{r}_1,\bar{z}_1)}\\
&= \frac{\left(\beta_{1,2}+\beta_{1,6}\right)\gamma_{1,4}}{\omega^2}\neq 0.
\end{aligned} \tag{25}$$

According to Theorem 2, we conclude that, up to the first-order averaging, system (17) has at most 1 limit cycle, and this number can be reached. This completes the proof of statement (a) of Theorem 8.

In order to consider the second-order averaging, we must use the conditions of $f_{1,1}(r, z) \equiv 0$ and $f_{1,2}(r, z) \equiv 0$. The averaged function $(f_{1,1}(r, z), f_{1,2}(r, z))$ is identically zero if and only if

$$a=0,\ d=0,\ \gamma_{1,4}=0,\ \beta_{1,6}=-\beta_{1,2}. \tag{26}$$

Using the above conditions to update the normal form of averaging (22), we then compute the following expression:

$$\begin{pmatrix} \frac{\partial F_{1,1}}{\partial r} & \frac{\partial F_{1,1}}{\partial z} \\ \frac{\partial F_{1,2}}{\partial r} & \frac{\partial F_{1,2}}{\partial z} \end{pmatrix} \begin{pmatrix} \int_0^\theta F_{1,1}(s,r,z)\mathrm{d}s \\ \int_0^\theta F_{1,2}(s,r,z)\mathrm{d}s \end{pmatrix} + \begin{pmatrix} F_{2,1}(\theta,r,z) \\ F_{2,2}(\theta,r,z) \end{pmatrix}. \tag{27}$$

Computing the integral of equation (27) between 0 and $2\pi$ and dividing by $2\pi$, we obtain the second-order averaged function

$$f_{2,1}(r,z) = \frac{r}{8\omega k_1^3}\Big(4c\omega^2k_1^2z^2 + 3b\omega^2r^2 - 4k_1^3\beta_{1,5}z - 4k_1^3\beta_{2,2} - 4k_1^3\beta_{2,6}\Big), \tag{28}$$

$$f_{2,2}(r,z) = -\frac{1}{2\omega k_1^2}\Big(-2k_1^2\gamma_{2,4}z + \gamma_{1,3}r^2\Big).$$

To analyze the zeros of system (28), we compute the Gröbner basis of the polynomial set

$$\{4c\omega^2k_1^2z^2+3b\omega^2r^2-4k_1^3\beta_{1,5}z - 4k_1^3\beta_{2,2} - 4k_1^3\beta_{2,6}, \\ -2k_1^2\gamma_{2,4}z + \gamma_{1,3}r^2\}$$

with respect to the lexicographic term ordering determined by $z \succ r$. One finds that a Gröbner basis is given by $\mathcal{G}_1 = [g_1, g_2]$, where

$$g_1 = -4k_1^5\beta_{2,2}\gamma_{2,4}^2 - 4k_1^5\beta_{2,6}\gamma_{2,4}^2 + \Big(3b\omega^2k_1^2\gamma_{2,4}^2 - 2k_1^3\beta_{1,5}\gamma_{1,3}\gamma_{2,4}\Big)r^2 + \gamma_{1,3}^2c\omega^2r^4, \tag{29}$$

$$g_2 = 2k_1^2\gamma_{2,4}z - \gamma_{1,3}r^2.$$

Note that $r > 0$, so system (29) has at most 2 suitable solutions. Therefore, system (17), up to second-order averaging, has at most 2 limit cycles bifurcating from the origin. To show this number can be reached, we consider the following Jacobian of $(f_{2,1}(r,z), f_{2,2}(r,z))$

$$J_1(r,z) = \det\begin{pmatrix} \frac{\partial f_{2,1}}{\partial r} & \frac{\partial f_{2,1}}{\partial z} \\ \frac{\partial f_{2,2}}{\partial r} & \frac{\partial f_{2,2}}{\partial z} \end{pmatrix} = \frac{\bar{J}_1(r,z)}{8\omega^2k_1^3}, \tag{30}$$

where

$$\bar{J}_1(r,z) = \Big(8c\omega^2\gamma_{1,3}z + 9\gamma_{2,4}\omega^2b - 4k_1\beta_{1,5}\gamma_{1,3}\Big)r^2 + 4\gamma_{2,4}\omega^2ck_1^2z^2 - 4\gamma_{2,4}k_1^3\beta_{1,5}z - 4\gamma_{2,4}k_1^3\beta_{2,2} - 4\gamma_{2,4}k_1^3\beta_{2,6}.$$

According to Theorem 2 and the above analysis, system (17) has exactly 2 limit cycles bifurcating from the origin if the following semi-algebraic system

$$\Big\{g_1 = g_2 = 0, \quad r > 0, \quad \bar{J}_1(r,z) \neq 0, \quad k_1 \neq 0\Big\} \tag{31}$$

has exactly 2 distinct real solutions. The above semi-algebraic system may be solved by the method of Yang and Xia [31] for real solution classification (implemented as a Maple package DISCOVERER by Xia [32], see also recent improvements in the Maple package RegularChains[SemiAlgebraicSetTools]), or the method of discriminant varieties of Lazard and Rouillier [33] (implemented as a Maple package DV by Moroz and Rouillier [34]).

By using the package of RegularChains in Maple, we obtain the semi-algebraic system (31) has exactly 2 distinct real solutions if and only if one of the eight conditions in (18) holds, where

$$\begin{aligned} &R_1 = c, \quad R_2 = k_1, \quad R_3 = \gamma_{2,4}, \quad R_4 = \beta_{2,2} + \beta_{2,6}, \\ &R_5 = 3b\omega^2\gamma_{2,4} - 2k_1\beta_{1,5}\gamma_{1,3}, \\ &R_6 = 9b^2\gamma_{2,4}^2\omega^4 - 12b\omega^2k_1\beta_{1,5}\gamma_{1,3}\gamma_{2,4} + 16c\omega^2k_1\beta_{2,2}\gamma_{1,3}^2 \\ &\quad + 16c\omega^2k_1\beta_{2,6}\gamma_{1,3}^2 + 4\beta_{1,5}^2\gamma_{1,3}^2k_1^2. \end{aligned} \tag{32}$$

As we see, up to the second-order averaging, there exist many systems expressed like (17) which have exactly 2 limit cycles bifurcating from the origin. In fact, we not only introduce a systematic approach to constructing such systems by symbolic computation methods, but also provide explicit conditions on the parameters satisfying this property. In summary, we conclude that applying the second-order averaging method system (17) has at most 2 limit cycles, and this number can be reached. Hence, statement (b) of Theorem 8 is proved.

To consider the third-order bifurcation of system (17), we must verify that the second-order averaged function $(f_{2,1}(r,z), f_{2,2}(r,z))$ is identically zero. For this, we take

$$b = 0, c = 0, \gamma_{1,3} = 0, \gamma_{2,4} = 0, \beta_{1,5} = 0, \\ \beta_{2,2} = -\beta_{2,6}. \tag{33}$$

Now, update the normal form of averaging (22) by using the conditions (26) and (33). To apply the third-order averaging method, according to Theorem 2, we must know the following expressions.

$$y_{1,1}(\theta,r,z) = \int_0^\theta F_{1,1}(s,r,z)ds = \frac{r}{4\omega^2k_1}\Big(\omega^2\alpha_1 + k_1^2\beta_{1,1}\Big)(1-\cos(2\theta)),$$

$$y_{1,2}(\theta,r,z) = \int_0^\theta F_{1,2}(s,r,z)ds = 0,$$

and

$$
\begin{aligned}
y_{2,1}(\theta,r,z) &= 2\int_0^{\theta}\Big(F_{2,1}(s,r,z)+\frac{\partial F_{1,1}}{\partial r}y_{1,1}(s,r,z)\\
&\quad+\frac{\partial F_{1,1}}{\partial z}y_{1,2}(s,r,z)\Big)\mathrm{d}s\\
&=\frac{r}{32\omega^4k_1^2}\Big(3\alpha_1^2\omega^4\cos(4\theta)+6\alpha_1\beta_{1,1}k_1^2\omega^2\cos(4\theta)\\
&\quad+3k_1^4\beta_{1,1}^2\cos(4\theta)-16\alpha_2k_1\omega^4\cos(2\theta)\\
&\quad+4\alpha_1^2\omega^4\cos(2\theta)-16k_1^3\beta_{2,1}\omega^2\cos(2\theta)\\
&\quad-8\alpha_1\beta_{1,1}k_1^2\omega^2\cos(2\theta)-12k_1^4\beta_{1,1}^2\cos(2\theta)\\
&\quad+16\alpha_2k_1\omega^4-7\alpha_1^2\omega^4+16k_1^3\beta_{2,1}\omega^2\\
&\quad+2\alpha_1\beta_{1,1}k_1^2\omega^2+9k_1^4\beta_{1,1}^2\Big),\\
y_{2,2}(\theta,r,z) &= 2\int_0^{\theta}\Big(F_{2,2}(s,r,z)+\frac{\partial F_{1,2}}{\partial r}y_{1,1}(s,r,z)\\
&\quad+\frac{\partial F_{1,2}}{\partial z}y_{1,2}(s,r,z)\Big)\mathrm{d}s\\
&=0.
\end{aligned}
$$

Now, computing the third-order averaged function, we obtain

$$
\begin{aligned}
f_{3,1}(r,z) &= \frac{1}{2\pi}\int_0^{2\pi}\Big[F_{3,1}(\theta,r,z)+\frac{\partial F_{2,1}}{\partial r}y_{1,1}(\theta,r,z)\\
&\quad+\frac{\partial F_{2,1}}{\partial z}y_{1,2}(\theta,r,z)+\frac{1}{2}\Big(\frac{\partial F_{1,1}}{\partial r}y_{2,1}(\theta,r,z)\\
&\quad+\frac{\partial F_{1,1}}{\partial z}y_{2,2}(\theta,r,z)\Big)+\frac{1}{2}\Big(\frac{\partial^2 F_{1,1}}{\partial r^2}y_{1,1}^2(\theta,r,z)\\
&\quad+\frac{\partial^2 F_{1,1}}{\partial r\partial z}y_{1,1}(\theta,r,z)y_{1,2}(\theta,r,z)\\
&\quad+\frac{\partial^2 F_{1,1}}{\partial z\partial r}y_{1,1}(\theta,r,z)y_{1,2}(\theta,r,z)\\
&\quad+\frac{\partial^2 F_{1,1}}{\partial z^2}y_{1,2}^2(\theta,r,z)\Big)\Big]\mathrm{d}\theta\\
&=-\frac{r}{8\omega k_1^2}\Big(4k_1^2\beta_{1,4}z^2+4k_1^2\beta_{2,5}z+3\beta_{1,3}r^2\\
&\quad+4k_1^2\beta_{3,2}+4k_1^2\beta_{3,6}\Big),\\
f_{3,2}(r,z) &= \frac{1}{2\pi}\int_0^{2\pi}\Big[F_{3,2}(\theta,r,z)+\frac{\partial F_{2,2}}{\partial r}y_{1,1}(\theta,r,z)\\
&\quad+\frac{\partial F_{2,2}}{\partial z}y_{1,2}(\theta,r,z)+\frac{1}{2}\Big(\frac{\partial F_{1,2}}{\partial r}y_{2,1}(\theta,r,z)\\
&\quad+\frac{\partial F_{1,2}}{\partial z}y_{2,2}(\theta,r,z)\Big)+\frac{1}{2}\Big(\frac{\partial^2 F_{1,2}}{\partial r^2}y_{1,1}^2(\theta,r,z)\\
&\quad+\frac{\partial^2 F_{1,2}}{\partial r\partial z}y_{1,1}(\theta,r,z)y_{1,2}(\theta,r,z)+\frac{\partial^2 F_{1,2}}{\partial z\partial r}y_{1,1}(\theta,r,z)\\
&\quad\times y_{1,2}(\theta,r,z)+\frac{\partial^2 F_{1,2}}{\partial z^2}y_{1,2}^2(\theta,r,z)\Big)\Big]d\theta\\
&=-\frac{1}{2\omega k_1^2}\Big(\gamma_{1,2}r^2z+\gamma_{2,3}r^2-2k_1^2\gamma_{3,4}z\Big).
\end{aligned}
$$

To analyze the zeros of $\{f_{3,1}(r,z)=0,\ f_{3,2}(r,z)=0\}$, we compute the Gröbner basis of the polynomial set

$$
\{4k_1^2\beta_{1,4}z^2+4k_1^2\beta_{2,5}z+3\beta_{1,3}r^2+4k_1^2\beta_{3,2}+4k_1^2\beta_{3,6},\ \gamma_{1,2}r^2z+\gamma_{2,3}r^2-2k_1^2\gamma_{3,4}z\}
$$

with respect to the lexicographic term ordering determined by $z\succ r$. One finds that a Gröbner basis is given by $\mathcal{G}_2=[\bar{g}_1,\bar{g}_2]$, where

$$
\begin{aligned}
\bar{g}_1 &= 16k_1^6\gamma_{3,4}^2\beta_{3,2}+16k_1^6\gamma_{3,4}^2\beta_{3,6}+\Big(12k_1^4\beta_{1,3}\gamma_{3,4}^2\\
&\quad+8k_1^4\beta_{2,5}\gamma_{2,3}\gamma_{3,4}-16k_1^4\beta_{3,2}\gamma_{1,2}\gamma_{3,4}\\
&\quad-16k_1^4\beta_{3,6}\gamma_{1,2}\gamma_{3,4}\Big)r^2+\Big(-12k_1^2\beta_{1,3}\gamma_{1,2}\gamma_{3,4}\\
&\quad+4k_1^2\beta_{1,4}\gamma_{2,3}^2-4k_1^2\beta_{2,5}\gamma_{1,2}\gamma_{2,3}+4k_1^2\beta_{3,2}\gamma_{1,2}^2\\
&\quad+4k_1^2\beta_{3,6}\gamma_{1,2}^2\Big)r^4+3\gamma_{1,2}^2\beta_{1,3}r^6,\\
\bar{g}_2 &= 8k_1^4\beta_{3,2}\gamma_{1,2}\gamma_{3,4}+8k_1^4\beta_{3,6}\gamma_{1,2}\gamma_{3,4}+\Big(6k_1^2\beta_{1,3}\gamma_{1,2}\gamma_{3,4}\\
&\quad-4k_1^2\beta_{1,4}\gamma_{2,3}^2+4k_1^2\beta_{2,5}\gamma_{1,2}\gamma_{2,3}-4k_1^2\beta_{3,2}\gamma_{1,2}^2\\
&\quad-4k_1^2\beta_{3,6}\gamma_{1,2}^2\Big)r^2+8k_1^4z\beta_{1,4}\gamma_{2,3}\gamma_{3,4}-3\gamma_{1,2}^2\beta_{1,3}r^4.
\end{aligned}
\tag{34}
$$

Note that $r>0$, so system (34) has at most 3 suitable solutions. Therefore, the averaging method up to third order provides the existence of at most 3 limit cycles of system (17). To show this number can be reached, we consider the following Jacobian of $(f_{3,1}(r,z),\ f_{3,2}(r,z))$

$$
J_2(r,z)=\det(M)=\det\begin{pmatrix}\frac{\partial f_{3,1}}{\partial r} & \frac{\partial f_{3,1}}{\partial z}\\ \frac{\partial f_{3,2}}{\partial r} & \frac{\partial f_{3,2}}{\partial z}\end{pmatrix}=\frac{\bar{J}_2(r,z)}{16\omega^2k_1^4},
\tag{35}
$$

where

$$
\begin{aligned}
\bar{J}_2(r,z) &= 9\gamma_{1,2}\beta_{1,3}r^4+\Big(4\gamma_{1,2}\beta_{3,2}k_1^2-12\gamma_{1,2}\beta_{1,4}k_1^2z^2\\
&\quad+4\gamma_{1,2}\beta_{3,6}k_1^2-4\gamma_{1,2}\beta_{2,5}k_1^2z-18k_1^2\beta_{1,3}\gamma_{3,4}\\
&\quad-16k_1^2\beta_{1,4}\gamma_{2,3}z-8\gamma_{2,3}\beta_{2,5}k_1^2\Big)r^2-8k_1^4\beta_{1,4}\gamma_{3,4}z^2\\
&\quad-8k_1^4\beta_{2,5}\gamma_{3,4}z-8\beta_{3,2}\gamma_{3,4}k_1^4-8\beta_{3,6}\gamma_{3,4}k_1^4.
\end{aligned}
$$

Using a similar argument to the second-order analysis, we know that system (17) has exactly 3 limit cycles bifurcating from the origin if the following semi-algebraic system

$$\left\{\bar{g}_1=\bar{g}_2=0,\quad r>0,\quad \bar{J}_2(r,z)\neq 0,\quad k_1\neq 0\right\} \tag{36}$$

has exactly 3 real solutions. Using the package of RegularChains in Maple, we obtain the semi-algebraic system (36) has exactly 3 distinct real solutions if and only if one of the four conditions in (19) holds, where

$$\begin{aligned}
&\bar{R}_1=\beta_{1,3},\quad \bar{R}_2=\gamma_{3,4},\quad \bar{R}_3=\beta_{3,2}+\beta_{3,6},\\
&\bar{R}_4=3\beta_{1,3}\gamma_{1,2}\gamma_{3,4}-\beta_{1,4}\gamma_{2,3}^2+\beta_{2,5}\gamma_{1,2}\gamma_{2,3}\\
&-\beta_{3,2}\gamma_{1,2}^2-\beta_{3,6}\gamma_{1,2}^2,\\
&\bar{R}_5=-3\beta_{1,3}\beta_{2,5}^2\gamma_{1,2}^2\gamma_{2,3}^2\gamma_{3,4}+18\beta_{1,3}^2\gamma_{1,2}\gamma_{3,4}^2\beta_{1,4}\gamma_{2,3}^2\\
&+24\beta_{1,3}\beta_{3,2}\gamma_{1,2}^4\gamma_{3,4}\beta_{3,6}-20\beta_{2,5}\beta_{3,2}\gamma_{1,2}^4\gamma_{2,3}\beta_{3,6}\\
&+12\beta_{3,6}^2\gamma_{1,2}^4\gamma_{3,4}\beta_{1,3}-3\beta_{1,3}\beta_{1,4}^2\gamma_{2,3}^4\gamma_{3,4}\\
&+9\beta_{1,3}^2\gamma_{1,2}^3\gamma_{3,4}^2\beta_{3,2}+9\beta_{1,3}^2\gamma_{1,2}^3\gamma_{3,4}^2\beta_{3,6}\\
&+12\beta_{3,2}^2\gamma_{1,2}^4\gamma_{3,4}\beta_{1,3}+8\beta_{2,5}^2\beta_{3,2}\gamma_{1,2}^3\gamma_{2,3}^2\\
&+8\beta_{2,5}^2\beta_{3,6}\gamma_{1,2}^3\gamma_{2,3}^2+4\beta_{1,4}\beta_{2,5}^2\gamma_{2,3}^4\gamma_{1,2}\\
&+8\beta_{3,2}^2\gamma_{1,2}^3\beta_{1,4}\gamma_{2,3}^2-10\beta_{3,2}^2\gamma_{1,2}^4\beta_{2,5}\gamma_{2,3}\\
&+8\beta_{3,6}^2\gamma_{1,2}^3\beta_{1,4}\gamma_{2,3}^2-10\beta_{3,6}^2\gamma_{1,2}^4\beta_{2,5}\gamma_{2,3}\\
&+4\beta_{1,4}^2\beta_{3,2}\gamma_{1,2}\gamma_{2,3}^4+4\beta_{1,4}^2\beta_{3,6}\gamma_{1,2}\gamma_{2,3}^4\\
&+16\beta_{1,4}\beta_{3,2}\gamma_{1,2}^3\gamma_{2,3}^2\beta_{3,6}-12\beta_{1,4}\beta_{3,6}\gamma_{1,2}^2\gamma_{2,3}^3\beta_{2,5}\\
&-12\beta_{1,4}\beta_{3,2}\gamma_{1,2}^2\gamma_{2,3}^3\beta_{2,5}+12\beta_{3,2}^2\gamma_{1,2}^5\beta_{3,6}
\end{aligned} \tag{37}$$

$$\begin{aligned}
&-2\beta_{2,5}^3\gamma_{1,2}^2\gamma_{2,3}^3-2\beta_{1,4}^2\beta_{2,5}\gamma_{2,3}^5\\
&+12\beta_{3,6}^2\gamma_{1,2}^5\beta_{3,2}+18\beta_{1,3}\beta_{2,5}\gamma_{1,2}\gamma_{2,3}^3\gamma_{3,4}\beta_{1,4}\\
&+3\beta_{1,3}\beta_{2,5}\gamma_{1,2}^3\gamma_{2,3}\gamma_{3,4}\beta_{3,2}+3\beta_{1,3}\beta_{2,5}\gamma_{1,2}^3\gamma_{2,3}\gamma_{3,4}\beta_{3,6}\\
&-39\beta_{1,4}\beta_{3,2}\gamma_{1,2}^2\gamma_{2,3}^2\gamma_{3,4}\beta_{1,3}-39\beta_{1,4}\beta_{3,6}\gamma_{1,2}^2\gamma_{2,3}^2\gamma_{3,4}\beta_{1,3}\\
&+4\beta_{3,2}^3\gamma_{1,2}^5+4\beta_{3,6}^3\gamma_{1,2}^5,\\
&\bar{R}_6=90\beta_{1,3}^2\beta_{1,4}\beta_{2,5}\gamma_{1,2}\gamma_{2,3}\gamma_{3,4}^2-12\beta_{1,3}\beta_{1,4}^2\beta_{2,5}\gamma_{2,3}^3\gamma_{3,4}\\
&+108\beta_{1,3}^2\beta_{1,4}\beta_{3,2}\gamma_{1,2}^2\gamma_{3,4}^2+108\beta_{1,3}^2\beta_{1,4}\beta_{3,6}\gamma_{1,2}^2\gamma_{3,4}^2\\
&+72\beta_{1,3}\beta_{1,4}\beta_{3,2}^2\gamma_{1,2}^3\gamma_{3,4}+72\beta_{1,3}\beta_{1,4}\beta_{3,6}^2\gamma_{1,2}^3\gamma_{3,4}\\
&-4\beta_{2,5}^4\gamma_{1,2}^2\gamma_{2,3}^2+16\beta_{1,4}^3\beta_{3,2}\gamma_{2,3}^4-4\beta_{1,4}^2\beta_{2,5}^2\gamma_{2,3}^4\\
&+16\beta_{1,4}^3\beta_{3,6}\gamma_{2,3}^4+16\beta_{1,4}\beta_{3,6}^3\gamma_{1,2}^4+16\beta_{1,4}\beta_{3,2}^3\gamma_{1,2}^4\\
&+48\beta_{1,4}\beta_{3,2}^2\beta_{3,6}\gamma_{1,2}^4+48\beta_{1,4}\beta_{3,2}\beta_{3,6}^2\gamma_{1,2}^4\\
&-8\beta_{2,5}^2\beta_{3,2}\beta_{3,6}\gamma_{1,2}^4+32\beta_{1,4}^2\beta_{3,2}^2\gamma_{1,2}^2\gamma_{2,3}^2\\
&+32\beta_{1,4}^2\beta_{3,6}^2\gamma_{1,2}^2\gamma_{2,3}^2+8\beta_{1,4}\beta_{2,5}^3\gamma_{1,2}\gamma_{2,3}^3\\
&+54\beta_{1,3}^3\beta_{1,4}\gamma_{1,2}\gamma_{3,4}^3-9\beta_{1,3}^2\beta_{1,4}^2\gamma_{2,3}^2\gamma_{3,4}^2\\
&-9\beta_{1,3}^2\beta_{2,5}^2\gamma_{1,2}^2\gamma_{3,4}^2+8\beta_{2,5}^3\beta_{3,2}\gamma_{1,2}^3\gamma_{2,3}
\end{aligned}$$

$$\begin{aligned}
&+8\beta_{2,5}^3\beta_{3,6}\gamma_{1,2}^3\gamma_{2,3}-12\beta_{1,3}\beta_{2,5}^2\beta_{3,2}\gamma_{1,2}^3\gamma_{3,4}\\
&-12\beta_{1,3}\beta_{2,5}^2\beta_{3,6}\gamma_{1,2}^3\gamma_{3,4}+8\beta_{1,4}\beta_{2,5}^2\beta_{3,6}\gamma_{1,2}^2\gamma_{2,3}^2\\
&-32\beta_{1,4}\beta_{2,5}\beta_{3,2}^2\gamma_{1,2}^3\gamma_{2,3}-32\beta_{1,4}\beta_{2,5}\beta_{3,6}^2\gamma_{1,2}^3\gamma_{2,3}\\
&-32\beta_{1,4}^2\beta_{2,5}\beta_{3,6}\gamma_{1,2}\gamma_{2,3}^3+64\beta_{1,4}^2\beta_{3,2}\beta_{3,6}\gamma_{1,2}^2\gamma_{2,3}^2\\
&+8\beta_{1,4}\beta_{2,5}^2\beta_{3,2}\gamma_{1,2}^2\gamma_{2,3}^2-32\beta_{1,4}^2\beta_{2,5}\beta_{3,2}\gamma_{1,2}\gamma_{2,3}^3\\
&-12\beta_{1,3}\beta_{2,5}^3\gamma_{1,2}^2\gamma_{2,3}\gamma_{3,4}+12\beta_{1,3}\beta_{1,4}\beta_{2,5}\beta_{3,2}\gamma_{1,2}^2\gamma_{2,3}\gamma_{3,4}\\
&+12\beta_{1,3}\beta_{1,4}\beta_{2,5}\beta_{3,6}\gamma_{1,2}^2\gamma_{2,3}\gamma_{3,4}\\
&-64\beta_{1,4}\beta_{2,5}\beta_{3,2}\beta_{3,6}\gamma_{1,2}^3\gamma_{2,3}\\
&+144\beta_{1,3}\beta_{1,4}\beta_{3,2}\beta_{3,6}\gamma_{1,2}^3\gamma_{3,4}\\
&+48\beta_{1,3}\beta_{1,4}\beta_{2,5}^2\gamma_{1,2}\gamma_{2,3}^2\gamma_{3,4}\\
&-120\beta_{1,3}\beta_{1,4}^2\beta_{3,2}\gamma_{1,2}\gamma_{2,3}^2\gamma_{3,4}\\
&-120\beta_{1,3}\beta_{1,4}^2\beta_{3,6}\gamma_{1,2}\gamma_{2,3}^2\gamma_{3,4}\\
&-4\beta_{2,5}^2\beta_{3,6}^2\gamma_{1,2}^4-4\beta_{2,5}^2\beta_{3,2}^2\gamma_{1,2}^4.
\end{aligned}$$

□

In the following, we provide an example which has exactly 3 limit cycles bifurcating from the origin.

*Example 2* Consider the polynomial differential system

$$\begin{aligned}
\dot{x}&=(1+\varepsilon)y,\\
\dot{y}&=-x+\varepsilon\left(-50y^3+yz^2\right)-\varepsilon^2yz+10\varepsilon^3y,\\
\dot{z}&=-2\varepsilon y^2z+10\varepsilon^2y^2+10\varepsilon^3z.
\end{aligned} \tag{38}$$

The corresponding system (22) associated with system (38) satisfies

$$\begin{aligned}
F_{1,1}(\theta,r,z)&=\frac{1}{2}r\sin(2\theta),\\
F_{2,1}(\theta,r,z)&=-\frac{1}{8}r\big(\sin(4\theta)+2\sin(2\theta)\big),\\
F_{3,1}(\theta,r,z)&=\frac{1}{32}\Big[200\big(\cos(4\theta)+4\cos(2\theta)+3\big)r^3\\
&\quad-16((\cos(2\theta)+1)z^2+16(\cos(2\theta)+1)z+\sin(6\theta)\\
&\quad+4\sin(4\theta)+5\sin(2\theta)-160\cos(2\theta)-160)r\Big],\\
F_{1,2}(\theta,r,z)&=0,\\
F_{2,2}(\theta,r,z)&=0,\\
F_{3,2}(\theta,r,z)&=(\cos(2\theta)z-5\cos(2\theta)+z-5)r^2-10z.
\end{aligned}$$

In order to find the limit cycles of system (38), we must study the real roots of the third-order averaged function

$$f_{3,1}(r,z)=\frac{r}{4}\Big(75r^2-2z^2+2z-20\Big),\quad f_{3,2}(r,z)=r^2z-5r^2-10z. \tag{39}$$

Using the RootFinding[Isolate] command built-in Maple to isolate the real roots of the polynomial system $[75r^2-2z^2+2z-20, r^2z-5r^2-10z]$, we obtain three real solutions:

$$r_1=\left[\frac{146501178860233}{281474976710656},\frac{36625294715059}{70368744177664}\right]\approx 0.52,$$
$$z_1=\left[-\frac{156747130244935}{1125899906842624},-\frac{78373565122467}{562949953421312}\right]\approx -0.14,$$
$$r_2=\left[\frac{194420694065201}{70368744177664},\frac{388841388130409}{140737488355328}\right]\approx 2.76,$$
$$z_2=\left[-\frac{70933937689247}{4398046511104},-\frac{283735750756985}{17592186044416}\right]\approx -16.13,$$
$$r_3=\left[\frac{505394075053437}{140737488355328},\frac{505394075053463}{140737488355328}\right]\approx 3.59,$$
$$z_3=\left[\frac{391738040933555}{17592186044416},\frac{391738040933563}{17592186044416}\right]\approx 22.27.$$

According to (35), we have the determinants of the Jacobian matrix $M$ at the points $(r_1,z_1)$, $(r_2,z_2)$, $(r_3,z_3)$ are $J_2(r_1,z_1)\approx -97.05\neq 0$, $J_2(r_2,z_2)\approx 4686.44\neq 0$ and $J_2(r_3,z_3)\approx 11094.61\neq 0$, respectively. This verifies that system (38) has exactly 3 limit cycles bifurcating from the origin. Now, we shall present the expressions of these 3 limit cycles. The limit cycles $\Lambda_i$ for $i=1,2,3$ of system (22) associated with system (38) and corresponding to the zeros $(r_i,z_i)$ given by (39) can be written as $\{(\widetilde{r}_i(\theta,\varepsilon),\widetilde{z}_i(\theta,\varepsilon)),\theta\in[0,2\pi]\}$, where from (7) we have

$$\Lambda_i:=\begin{pmatrix}\widetilde{r}_i(\theta,\varepsilon)\\ \widetilde{z}_i(\theta,\varepsilon)\end{pmatrix}=\begin{pmatrix}r_i\\ z_i\end{pmatrix}+\varepsilon\begin{pmatrix}y_{1,1}(\theta,r_i,z_i)\\ 0\end{pmatrix}+\varepsilon^2\begin{pmatrix}y_{2,1}(\theta,r_i,z_i)/2\\ 0\end{pmatrix}+\mathcal{O}(\varepsilon^3),\quad i=1,2,3, \tag{40}$$

where

$$y_{1,1}(\theta,r_i,z_i)=-\frac{r_i}{4}\left(\cos(2\theta)-1\right),$$
$$y_{2,1}(\theta,r_i,z_i)=\frac{r_i}{32}\Big(-7+3\cos(4\theta)+4\cos(2\theta)\Big).$$

Moreover, the eigenvalues of the Jacobian matrix $M$ (see (35)) at the points $(r_1,z_1)$, $(r_2,z_2)$, $(r_3,z_3)$ are, respectively, about

$(10.07,-9.64)$, $(266.29,17.60)$, $(462.49,23.99)$.

We have the corresponding limit cycles $\Lambda_1$ is semistable, $\Lambda_2$ and $\Lambda_3$ are unstable. We remark that one can obtain the expressions of the 3 limit cycles of system (38) by going back through the changes of variables: $(x,y,z)\mapsto(r\sin\theta,r\cos\theta,z)$, $(x,y,z)\mapsto(\varepsilon x,\varepsilon y,\varepsilon z)$, and $(x,y,z)\mapsto(-y/\omega,x/k_1,z)$ with $\omega=k_1=1$. Here, we do not provide them for brevity.

Hence, we complete the proof of Theorem 8.

## 5 Hopf bifurcation and number of bifurcating limit cycles

We call an equilibrium point of system (1) a *Hopf equilibrium point* if its linear part has one eigenvalue $\lambda_1\neq 0$ and a pair of purely imaginary eigenvalues $\lambda_{2,3}=\pm i\omega\neq 0$. The next result characterizes the Hopf equilibrium point of the MCG system.

**Proposition 9** *The origin of system (1) is a Hopf equilibrium point when $\epsilon>0$, $k_1k_2=-\omega^2$ and $s=-a$.*

*Proof* Using similar arguments to the proof of Proposition 7, we obtain directly the desired conditions for the origin to be a Hopf equilibrium point. □

In the following, we study the classical Hopf bifurcation of the MCG system. By using an alternative simple method proposed by Edneral et al. [25], we obtain the following result.

**Theorem 10** *Consider the differential system*

$$\dot{x}=\frac{k_1}{\omega}y,\quad \dot{y}=-\frac{\omega}{k_1}\Big(by^3+cyz^2+dyz+x\Big),\quad \dot{z}=\frac{y^2}{\omega}\Big(cz^2+dz-a\Big)-\xi z \tag{41}$$

*with $\xi=\epsilon/\omega$, $ad\neq 0$, and $\epsilon>0$. Let $\mathfrak{p}=(a,b,c,d,\xi,\omega,k_1)$ be the parameter vector, denote the hypersurfaces by $\mathcal{S}_1=\Big\{(a,b,c,d,\xi,\omega,k_1):\big(3\xi^2+8\big)ad-3\xi\omega\big(\xi^2+4\big)b=0\Big\}$ and*

$$\mathcal{S}_2=\Big\{(a,b,c,d,\xi,\omega,k_1):8ad\omega\Big(ad\big(\xi^2+4\big)-6b\xi\omega\Big)+a\xi k_1\Big(\big(\xi^2+4\big)\Big(5\xi^2\big(d^2-ac\big)-8ac\Big)+32d^2\Big)=0\Big\}.$$

*Then, the following statements hold.*

(i) *For $\mathfrak{p}\notin\mathcal{S}_1$, system (41) can have exactly 1 limit cycle bifurcates from the origin.*
(ii) *For $\mathfrak{p}\in\mathcal{S}_1\setminus\mathcal{S}_1\cap\mathcal{S}_2$, system (41) can have exactly 2 limit cycles bifurcate from the origin.*

(iii) *Let $\xi^*$ be the positive zero of equation $\delta(\xi)$ defined in (48). If $\mathfrak{p} \in \mathcal{S}_1 \cap \mathcal{S}_2$ with $\xi \in (\xi^*, +\infty)$, then system (41) can have exactly 3 limit cycles bifurcate from the origin. Moreover, $\xi^*$ is a unique positive zero of equation $\delta(\xi)$ given by (48) and*

$$\xi^* \in \left[\frac{587457651}{536870912}, \frac{1174915303}{1073741824}\right].$$

*Proof* Using the conditions of Proposition 9, we write system (1) as

$$\begin{aligned} \dot{x} &= k_1 y, \\ \dot{y} &= -\frac{\omega^2}{k_1}\left(by^3 + cyz^2 + dyz + x\right), \\ \dot{z} &= y^2\left(cz^2 + dz - a\right) - \epsilon z. \end{aligned} \tag{42}$$

Performing the change of variables

$$x = -\frac{X}{\omega}, \; y = \frac{Y}{k_1}, \; z = Z,$$

system (42) becomes

$$\begin{aligned} \dot{X} &= -\omega Y, \\ \dot{Y} &= \omega X - \frac{\omega^2 Y\left(bY^2 + ck_1^2 Z^2 + dk_1^2 Z\right)}{k_1^3}, \\ \dot{Z} &= \left(cZ^2 + dZ - a\right)\frac{Y^2}{k_1^2} - \epsilon Z. \end{aligned} \tag{43}$$

Doing the scaling $\mathrm{d}t = \mathrm{d}\tau/\omega$, we obtain

$$\begin{aligned} X' &= -Y, \\ Y' &= X - \frac{\omega Y\left(bY^2 + ck_1^2 Z^2 + dk_1^2 Z\right)}{k_1^3}, \\ Z' &= \left(cZ^2 + dZ - a\right)\frac{Y^2}{\omega k_1^2} - \xi Z, \end{aligned} \tag{44}$$

where $\xi = \epsilon/\omega$ and the prime denotes derivative with respect to $\tau$. For convenience, we also denote

$$X \to x, \; Y \to y, \; Z \to z, \; X' \to \dot{x}, \; Y' \to \dot{y}, \; Z' \to \dot{z}.$$

System (44) can be written as

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x - \frac{\omega y\left(by^2 + ck_1^2 z^2 + dk_1^2 z\right)}{k_1^3}, \\ \dot{z} &= \left(cz^2 + dz - a\right)\frac{y^2}{\omega k_1^2} - \xi z, \end{aligned} \tag{45}$$

where $\xi = \epsilon/\omega > 0$. After the complex variables $(x, y, z) \mapsto ((x + y)/2, -(x - y)\mathrm{i}/2, z)$, the complexification of system (45) is

$$\begin{aligned} \dot{x} &= \mathrm{i}x - \frac{\omega z(x-y)(cz+d)}{2k_1} + \frac{b\omega(x-y)^3}{8k_1^3}, \\ \dot{y} &= -\mathrm{i}y + \frac{\omega z(x-y)(cz+d)}{2k_1} - \frac{b\omega(x-y)^3}{8k_1^3}, \\ \dot{z} &= -\xi z + \frac{(a - z(cz+d))(x-y)^2}{4k_1^2\omega}. \end{aligned} \tag{46}$$

Let $H(x, y, z) = xy + \sum_{\sigma+\beta+\gamma\geq 3} \upsilon_{\sigma,\beta,\gamma} x^\sigma y^\beta z^\gamma$ and denote by $\mathfrak{X}$ the vector field associated with system (46). Then, we have

$$\mathfrak{X}(H(x, y, z)) = \sum_{m\geq 2} h_m(x, y, z), \tag{47}$$

where $h_m$ are homogeneous polynomials of degree $m$ in the variables $x$, $y$ and $z$. It is easy to get that $h_2(x, y, z) \equiv 0$. The coefficients $\upsilon_{\sigma,\beta,\gamma}$ of $H(x, y, z)$ can be obtained from the linear equations defined by all the vanishing of the coefficients of $h_m(x, y, z)$ with $\sigma + \beta + \gamma \leq m$.

With the aid of computer algebra system Maple, we get that the first three nonzero focus quantities are as follows:

$$\begin{aligned} \nu_1 &= g_{1,1,0} = \frac{\Upsilon_1(a, b, c, d, \xi, \omega, k_1)}{4\xi\left(\xi^2+4\right)k_1^3}, \\ \nu_2 &= g_{2,2,0} = \frac{\Upsilon_2(a, b, c, d, \xi, \omega, k_1)}{8k_1^6\xi^3\left(\xi^2+4\right)^2\omega}, \\ \nu_3 &= g_{3,3,0} = \frac{\Upsilon_3(a, b, c, d, \xi, \omega, k_1)}{1024k_1^9\xi^5\left(\xi^2+4\right)^4\left(\xi^2+16\right)\left(\xi^2+1\right)\omega^2}, \end{aligned}$$

where

$$\begin{aligned} &\Upsilon_1(a, b, c, d, \xi, \omega, k_1) = \left(3\xi^2+8\right)ad - 3\xi\omega\left(\xi^2+4\right)b, \\ &\Upsilon_2\left(a, b, c, d, \xi, \omega, k_1\right) = 8ad\omega\left(ad\left(\xi^2+4\right) - 6b\xi\omega\right) \\ &+ a\xi k_1\left(\left(\xi^2+4\right)\left(5\xi^2\left(d^2-ac\right) - 8ac\right) + 32d^2\right), \\ &\Upsilon_3\left(a, b, c, d, \xi, \omega, k_1\right) = 663b^3\xi^5\left(\xi^2+1\right)\left(\xi^2+4\right)^4 \\ &\times\left(\xi^2+16\right)\omega^5 - ab^2d\xi^2\left(\xi^2+1\right)\left(\xi^2+4\right) \\ &\times\left(1989\xi^{10} + 51976\xi^8 + 389040\xi^6 + 1088640\xi^4\right. \\ &\left.+ 578560\xi^2 - 589824\right)\omega^4 + a^2d^2b\xi\left(\xi^2+1\right)\left(1989\xi^{12}\right. \\ &+ 56588\xi^{10} + 509296\xi^8 + 1927488\xi^6 + 2585600\xi^4 \\ &\left.- 1089536\xi^2 - 3538944\right)\omega^3 + \left(48ab\xi^2k_1\left(\xi^2+4\right)\right. \end{aligned}$$

$$\times ac(\xi^2+4)(11\xi^8+213\xi^6+640\xi^4+3808\xi^2+2560)$$
$$-3d^2(\xi^2+1)(3\xi^8+64\xi^6+272\xi^4+2816\xi^2+4096)$$
$$-a^3d^3(\xi^2+1)(\xi^2+4)(663\xi^{10}+15220\xi^8+86208\xi^6$$
$$+130304\xi^4-118784\xi^2-327680)\Big)\omega^2+8a^2d\xi k_1$$
$$\times\Big(d^2(\xi^2+1)(9\xi^{10}+856\xi^8+11984\xi^6+87040\xi^4$$
$$+262144\xi^2+262144)-ac(\xi^2+4)(21\xi^{10}+837\xi^8$$
$$+9172\xi^6+50944\xi^4+108032\xi^2+57344)\Big)\omega$$
$$+16\xi^2k_1^2ad(\xi^4+5\xi^2+4)\Big(d^2(35\xi^8+680\xi^6$$
$$+2800\xi^4+7680\xi^2+8192)-ac(105\xi^8+2380\xi^6$$
$$+13152\xi^4+28160\xi^2+24576)\Big).$$

The first focus quantity $\nu_1$ and the second focus quantity $\nu_2$ vanish on the hypersurfaces

$$\mathcal{S}_1=\{(a,b,c,d,\xi,\omega,k_1):\Upsilon_1(a,b,c,d,\xi,\omega,k_1)=0\}$$

and

$$\mathcal{S}_2=\{(a,b,c,d,\xi,\omega,k_1):\Upsilon_2(a,b,c,d,\xi,\omega,k_1)=0\},$$

respectively.

(i) For the parameter vector $\mathfrak{p}\notin\mathcal{S}_1$, the origin of system (42) is a fine focus of order 1. Adding the linear perturbation term in system (42), system (42) has exactly 1 limit cycle, which can bifurcate from the origin. Statement (i) holds.

(ii) The intersection of hypersurfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ is

$$\mathcal{S}_1\cap\mathcal{S}_2=\Big\{(a,b,c,d,,\omega,k_1):b=\frac{ad(3\xi^2+8)}{3\xi(\xi^2+4)\omega},$$
$$c=\frac{d^2(8a\xi\omega(\xi^2+2)+(\xi^2+4)(5(\xi^2+4)\xi^2+32)k_1)}{ak_1(\xi^2+4)^2(5\xi^2+8)}\Big\}.$$

We have $\nu_1\mid_{\mathcal{S}_1}=0$ and $\nu_2\mid_{\mathcal{S}_1\backslash\mathcal{S}_1\cap\mathcal{S}_2}\neq 0$. So, the origin of system (42) is a fine focus of order 2 with the parameters vector $\mathfrak{p}\in\mathcal{S}_1\backslash\mathcal{S}_1\cap\mathcal{S}_2$. Since

$$\frac{d\nu_1}{db}\mid_{\mathcal{S}_1\backslash\mathcal{S}_1\cap\mathcal{S}_2}=-\frac{3\omega}{4k_1^3}\neq 0,$$

by the Theorem 6, the origin of system (42) can exactly bifurcate 2 limit cycles for the parameters vector $\mathfrak{p}\in\mathcal{S}_1\backslash\mathcal{S}_1\cap\mathcal{S}_2$. Statement (ii) is confirmed.

(iii) The third focus quantity $\nu_3$ restricted to the hypersurfaces $\mathcal{S}_1\cap\mathcal{S}_2$ becomes

$$\nu_3\mid_{\nu_1=\nu_2=0}=\nu_3\mid_{\mathcal{S}_1\cap\mathcal{S}_2}$$
$$=\frac{ad^3\widetilde{\Upsilon}_3(\chi)}{32k_1^9\xi^3(\xi^2+4)^5(\xi^2+16)(5\xi^2+8)},$$

where

$$\widetilde{\Upsilon}_3(\chi)=-(\xi^2+4)(380672\xi^4+458752\xi^2+5(35\xi^6+962\xi^4+8256\xi^2+34144)\xi^6+262144)\chi^2$$
$$-8a\xi(44544\xi^2+5(3(\xi^4+41\xi^2+372)\xi^2+4544)\xi^4+32768)\chi-8a^2(25\xi^{10}+390\xi^8+1764\xi^6+2336\xi^4-640\xi^2-2048)$$

with $\chi=k_1/\omega$.

Note that $\widetilde{\Upsilon}_3(\chi)$ is a quadratic equation with respect to $\chi$. Its discriminant is $\Delta=-32a^2\delta(\xi)$ with

$$\begin{aligned}\delta(\xi)&=4375\xi^{24}+205550\xi^{22}+3933700\xi^{20}+41035990\xi^{18}\\&+261993040\xi^{16}+1073396320\xi^{14}+2817146368\xi^{12}\\&+4377710592\xi^{10}+2588950528\xi^8-3643932672\xi^6\\&-8789164032\xi^4-7113539584\xi^2-2147483648.\end{aligned}\tag{48}$$

By command RealRoot in Maple with accuracy $10^{-8}$, we obtain that $\delta(\xi)$ has a unique positive zero

$$\xi^*\in\left[\frac{587457651}{536870912},\frac{1174915303}{1073741824}\right],$$

that is, $\delta(\xi^*)=0$. It is easy to check that $\delta(\xi)>0$ for $\xi\in(\xi^*,+\infty)$. So $\Delta<0$ for $\xi\in(\xi^*,+\infty)$. This implies that $\nu_3\mid_{\nu_1=\nu_2=0}\neq 0$ with $\mathfrak{p}\in\mathcal{S}_1\cap\mathcal{S}_2$ and $\xi\in(\xi^*,+\infty)$. Therefore, the origin of system (42) is a fine focus of order 3 with $\mathfrak{p}\in\mathcal{S}_1\cap\mathcal{S}_2$ and $\xi\in(\xi^*,+\infty)$.

Since

$$\det\left(\frac{\partial(\nu_1,\nu_2)}{\partial(b,c)}\right)\mid_{\mathcal{S}_1\cap\mathcal{S}_2}=\frac{3a^2(5\xi^2+8)}{32k_1^8\xi^2(\xi^2+4)}\neq 0,$$

it follows from Theorem 6 that system (42) has exactly 3 limit cycles that can bifurcate from the origin.

This ends the proof of Theorem 10. □

In the following, we present some examples and numerical simulations to illustrate the obtained theoretical results.

*Remark 4* Under the condition of the existence of Hopf equilibrium point, system (41) and system (1) have essentially the same phase portrait. The computation of focus quantities are quite hard and tedious. As applications of Theorem 10, the following examples are used to make easier the calculation of the focus quantities.

*Example 3* Let

$$\mathfrak{p} = (a, b, c, d, \xi, \omega, k_1) = (-1, 0, 3, -1, 1, 1, 1)\,.$$

Consider the perturbed system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\left(3yz^2 - yz + x\right) - \varepsilon y, \\ \dot{z} &= y^2\left(3z^2 - z - 1\right) - z. \end{aligned} \tag{49}$$

When $\varepsilon = 0$, system (49) becomes the unperturbed system (41) with the parameter vector $\mathfrak{p} \notin \mathcal{S}_1$. For $\varepsilon = 10^{-5}$, the perturbed focus quantities are $\nu_0 = -10^{-5}$ and $\nu_1 = 11/20$. By statement (i) of Theorem 10, the perturbed system (49) has one limit cycle bifurcating from the origin, see (a) of Fig. 4.

*Example 4* Let

$$\mathfrak{p}_\varepsilon = (a, b, c, d, \xi, \omega, k_1) = \Big(-1, 1, 0, -1, 1, \frac{11}{15} - \varepsilon, 1 - \frac{60872\varepsilon}{72567}\Big).$$

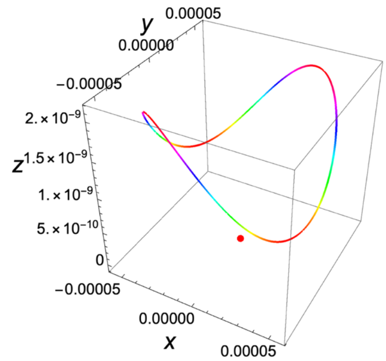
Consider the perturbed system

$$\begin{aligned} \dot{x} &= \frac{5(60872\varepsilon - 72567)}{24189(15\varepsilon - 11)}y, \\ \dot{y} &= -\frac{24189(15\varepsilon - 11)\left(x + y^3 - yz\right)}{5(60872\varepsilon - 72567)} - 10\varepsilon y, \\ \dot{z} &= \frac{15y^2(z-1)}{15\varepsilon - 11} - z. \end{aligned} \tag{50}$$

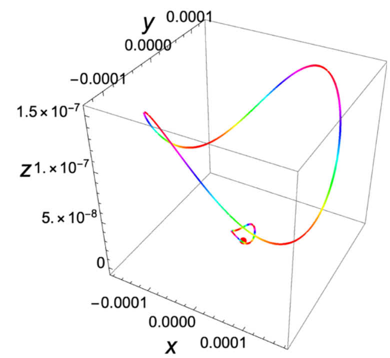
When $\varepsilon = 10^{-6}$, the perturbed focus quantities are $\nu_0 = -10^{-5}$, $\nu_1 \approx 7.50002 \times 10^{-7}$ and $\nu_2 \approx -0.364638$. Using statement (ii) of Theorem 10, the perturbed system (50) can have two limit cycles that bifurcate from the origin, see (b) of Fig. 4.

*Example 5* Let

$$\mathfrak{p}_\varepsilon = (a, b, c, d, \xi, \omega, k_1) = \Big(1, -1, 1, -1, 2 - \frac{203099678278\varepsilon}{973677}, \frac{101552760170\varepsilon}{2921031} + \frac{5}{12}, \frac{2234301906905\varepsilon}{93472992} + \frac{5}{32}\Big).$$




**(a)** One limit cycle by Hopf bifurcation of system (49) .




**(b)** Two limit cycles by Hopf bifurcation of system (50).

**Fig. 4** Numerical simulations of phase portraits for Examples 3, 4 and 5 , respectively. The red spot is the origin. (Color figre online)

Consider the perturbed system

$$\begin{aligned} \dot{x} &= \frac{446860381381\varepsilon + 2921031}{8(81242208136\varepsilon + 973677)}y, \\ \dot{y} &= -\frac{8(81242208136\varepsilon + 973677)\left(x - y\left(y^2 - z^2 + z\right)\right)}{446860381381\varepsilon + 2921031} \\ &\quad - 100000\varepsilon y, \\ \dot{z} &= \frac{11684124y^2\left(z^2 - z - 1\right)}{5(81242208136\varepsilon + 973677)} + \left(\frac{203099678278\varepsilon}{973677} - 2\right)z. \end{aligned} \tag{51}$$

When $\varepsilon = 10^{-10}$, the perturbed focus quantities are $\nu_0 = -10^{-5}$, $\nu_1 \approx 1.342253157 \times 10^{-8}$, $\nu_2 \approx -5.344257264 \times 10^{-7}$ and $\nu_3 \approx 186101.2208$. Applying statement (iii) of Theorem 10, the perturbed system (51) can have three limit cycles that bifurcate from the origin, see (c) of Fig. 4.

## 6 Conclusion

In this paper, we study two kinds of bifurcations for the MCG system (1), i.e., zero-Hopf bifurcation and Hopf bifurcation. Using the averaging method and the Lyapunov coefficient method, we obtained the number of limit cycles for the MCG-like system (17) and the differential system (41), respectively. Moreover, we found that the number of limit cycles that can bifurcate from the origin for the two differential systems (17) and (41) is the same for each order (up to the third order).

We conjecture that the averaging method and the focus quantity method of the same order can produce the same number of limit cycles for the two differential systems (17) and (41), but the relationship between these two methods for the study of limit cycles coming from a singular equilibrium point is still not clear at the present time. We leave this as a future research problem.

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**Declarations**

**Conflict of Interest** The authors declare that they have no conflict of interest.

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