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Sliding mode vibration control of an Euler–Bernoulli beam with unknown external disturbances

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Abstract This paper addresses the stabilization problem of an Euler–Bernoulli beam system subject to an unknown time-varying distributed load and boundary disturbance. Based on Lagrangian–Hamiltonian mechanics, the model of the beam system is derived as a partial differential equation. Based on Lyapunov functions, a sliding surface is designed, on which the system exhibits exponential bounded stability and robustness against the external disturbances. A sliding mode controller which only uses boundary information is further proposed to drive the system to reach the sliding surface in finite time. Numerical simulations are shown to illustrate the validity of the proposed boundary control.

Keywords Boundary control · Distributed parameter system · Sliding mode control · Euler–Bernoulli beam

Abbreviations

L	Length of the beam
m	Mass of the payload
E_{I}	Bending stiffness of the beam
T	Tension of the beam

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ρ	Uniform mass per unit length of the beam
$w(x,t)$	Displacement of the beam at the position x for the time t
$f(x,t)$	Time-varying distributed load on the beam except end point
$u(t)$	Boundary control force at the end of the beam
$d(t)$	Boundary input disturbance force at the end of the beam
C^1	Continuously differentiable function space
R^+	The sets of positive real numbers

1 Introduction

Nowadays, the control of flexible beams has attracted more and more attention, not only because of its wide application in engineering, such as marine risers for oil transportation [1,2], flexible aircraft wings [3] and flexible manipulators for grasping [4,5], but also since it is a theoretical challenge due to the difficulty of the control. Different from lumped parameter systems represented by ordinary differential equations, the flexible beam is modeled as a distributed parameter system, which is related to time and spatial position. In addition, the distributed parameter system has an infinite dimensional state space, which brings the challenge of its control problems for such systems.

In the past few decades, a lot of researches have focused on the controller design and stability analysis of flexible beams [6–8]. There are various control methods for the stabilization of flexible beams such as

back-stepping technique, Lyapunov synthesis methods and so on. The back-stepping technique is a powerful method for designing the control law for the distributed parameter system [9]. Its main idea is transforming the system into a stable and convergent target system through a kernel function, and explaining the existence and well-posedness of the kernel function further. Then, the control gain function of the system can be easily obtained. Some existing results are given in [10–12]. The controllers proposed in these papers are all designed by combining order reduction and the back-stepping method. The difference is that the system is transformed into a coupled heat-like system by using a complex variable in [10,11], while in [12], an alternative transformation is developed, which transforms the fourth-order system into two coupled second-order systems. On the basis of [11,13] develops a fault compensation scheme to deal with certain boundary input faults. However, the controllers designed by the back-stepping method usually require full measurement of the system states rather than the boundary information. It means that the sensor needs to detect all the displacements of the flexible beam in the process of vibration or to obtain the full state information by designing an additional observer, which limits the application of these controllers.

Lyapunov synthesis provides a method to prove the stability of the system without knowing the exact solution of the system, which is used widely in the control field. There are some remarkable results [1,4,14–17] for the flexible beam which just use the information at the boundary of the system. In [1] and [14], the authors apply the adaptive control to deal with unknown system parameters. For the boundary disturbance, a robust controller and a disturbance observer are employed in [1,14] to suppress the effect of the boundary disturbance. Different from [1] and [14,15] puts forward a hybrid backstepping-boundary iterative learning control to stabilize the flexible beam under external disturbances. The prescribed performance technique is applied in [4] and a barrier Lyapunov function is presented in [16] to handle the vibration of the flexible beam. The authors in [17] employ the Lyapunov method to design an output feedback controller and show the existence of a solution to the closed-loop system by using a Galerkin approximation scheme. A backlash-like input nonlinearity and the output constraint problem are considered in [18]. In [19], for the input and output constraint prob-

lems for Euler–Bernoulli beam, a boundary scheme is developed and this method is expanded to the three-dimensional Euler–Bernoulli beam in [20]. In [21], the problem of actuator faults is considered and an adaptive actuator fault-tolerant control scheme is developed.

Sliding mode theory is a nonlinear control technique in essence, which can handle the uncertainty and the external disturbance of the system as the system performance rely on the sliding surface only. Sliding mode control (SMC) has been widely used in many fields including linear systems [22], nonlinear systems [23] and distributed parameter systems [24–26] due to its strong robustness. For the Euler–Bernoulli beam, [27] and [28] propose a boundary feedback controller based on sliding mode control to reject the unknown boundary disturbance and analyze the stability of the closed-loop system. Comparisons of the active disturbance rejection control and the sliding mode control are made in [29].

In this paper, we consider an Euler–Bernoulli model with tension term subject to unknown external disturbances including time-varying distributed load and boundary disturbance. A novel boundary vibration control is proposed to suppress the vibration of the system. The main contributions of the paper are as follows:

(i) The unknown external disturbances are considered systematically.
(ii) A sliding mode controller is developed to suppress unknown external disturbances for the system. Only information at the boundary is used in the proposed controller.
(iii) The exponential bounded stability for the closed-loop system is analyzed using the Lyapunov method.

The structure of the rest of the paper is as follows. The problem statement is given in Sect. 2, which introduces the partial differential equation (PDE) model of the Euler–Bernoulli beam and some necessary preparations. The sliding mode controller is developed in Sect. 3. It is shown that both the exponential bounded stability of the system on the sliding surface and the reach-ability of the sliding surface are achieved by the proposed controller. Section 4 presents the numerical simulations to verify the effectiveness of the control law. The conclusion of the paper is offered in Sect. 5.
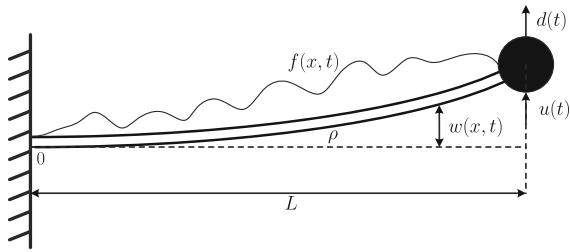
Fig. 1 The top view of a typical beam system

2 Problem statement

In this paper, we consider a beam system that moves in the horizontal direction, whose length can be stretched by the external force. When the diameter/length ratio of beam is very small and the rotary inertia of the beam is neglected, the beam system can be modeled as an Euler–Bernoulli beam [14]. Figure 1 shows an Euler–Bernoulli beam exerted by the unknown time-varying distributed load $f(x, t)$, unknown boundary disturbance $d(t)$ and a control force $u(t)$. The left boundary of the beam is fixed at origin.

2.1 Dynamic model

In order to derive the Euler–Bernoulli beam model, the kinetic energy $E_k(t)$, the potential energy $E_p(t)$ and the non-conservative work $W(t)$ of the system are listed as follows. For clarity, we make the following definition $(\cdot)_x = \partial(\cdot)/\partial x$, $(\cdot)_t = \partial(\cdot)/\partial t$, where x and t represent the independent spatial and temporal variables, respectively, and δ is a variation sign.

$$E_k(t) = \frac{1}{2}\rho \int_0^L [w_t(x, t)]^2 \mathrm{d}x + \frac{1}{2}m[w_t(L, t)]^2,$$

$$E_p(t) = \frac{1}{2}E_\mathrm{I} \int_0^L [w_{xx}(x, t)]^2 \mathrm{d}x + \frac{1}{2}T \int_0^L [w_x(x, t)]^2 \mathrm{d}x,$$

$$\delta W(t) = u(t)\delta w(L, t) + d(t)\delta w(L, t)$$
$$+ \int_0^L f(x, t)\delta w(x, t)\mathrm{d}x.$$

Then, the calculus of variation is used to derive the model of the Euler–Bernoulli beam system

$$\int_{t_1}^{t_2} \delta[E_k(t) - E_p(t) + W(t)]\mathrm{d}t = 0, \tag{1}$$

where t_1 and t_2 are two time instants satisfying $t_1 < t < t_2$.

Through the direct calculations of (1), it yields

$$\rho w_{tt}(x, t) + E_\mathrm{I}w_{xxxx}(x, t) - Tw_{xx}(x, t) = f(x, t),$$
$$\forall (x, t) \in (0, L) \times [0, \infty), \tag{2}$$

and the boundary conditions are given as follows for $\forall t \in [0, \infty)$

$$w(0, t) = w_x(0, t) = w_{xx}(L, t) = 0, \tag{3}$$
$$-E_\mathrm{I}w_{xxx}(L, t) + Tw_x(L, t) = u(t)$$
$$+ d(t) - mw_{tt}(L, t). \tag{4}$$

Assumption 1 For the unknown time-varying distributed load $f(x, t)$ and boundary disturbance $d(t)$, we make the assumption that there exist positive constants $\bar{f} \in R^+$ and $\bar{d} \in R^+$, such that $f(x, t) \leq \bar{f}, \forall(x, t) \in (0, L) \times [0, \infty)$ and $|d(t)| \leq \bar{d}, \forall t \in [0, \infty)$.

2.2 Preliminaries

The following essential lemmas, which will be used to design and analyze the boundary control law, are introduced.

Lemma 1 [30] *Let* $\phi_1, \phi_2 \in C([0, L] \times [0, \infty) : R)$. *Then, the following inequalities hold:*

$$\phi_1(x, t)\phi_2(x, t) \leq \frac{\kappa}{2}\phi_1^2(x, t) + \frac{1}{2\kappa}\phi_2^2(x, t), \tag{5}$$

where κ *is a positive constant.*

Lemma 2 [31] *Let* $\phi \in C^1([0, L] \times [0, \infty) : R)$, *which satisfies* $\phi(0, t) = 0$, *then the following inequality holds for* $\forall t \in [0, \infty)$:

$$\phi^2(x, t) \leq L \int_0^L [\phi_x(x, t)]^2 \mathrm{d}x, \forall x \in [0, L]. \tag{6}$$

Similarly, while $\phi_x(0, t) = 0$, *for* $\forall t \in [0, \infty)$, *the following inequality also holds*

$$\phi_x^2(x, t) \leq L \int_0^L [\phi_{xx}(x, t)]^2 \mathrm{d}x, \forall x \in [0, L]. \tag{7}$$

3 Sliding mode controller design

The control objective is to suppress the vibration of the Euler–Bernoulli beam system (2)–(4) under the unknown time-varying distributed load $f(x, t)$ and boundary disturbance $d(t)$. In this section, the Lyapunov synthesis method and sliding mode control are used to design a boundary controller $u(t)$ at the right boundary of the beam including an analysis of the closed-loop stability of the system and the reach ability of the sliding surface.

Here, a Lyapunov function and a sliding surface are proposed to guarantee that the beam system (2)–(4) has exponentially bounded stability on the sliding surface. The sliding surface is chosen as

$$S = w_t(L, t) + \frac{\beta L}{\alpha} w_x(L, t) = 0. \tag{8}$$

In order to make the system reach the sliding surface in finite time, the sliding mode control law is proposed as follows

$$u(t) = -E_I w_{xxx}(L, t) + T w_x(L, t)$$
$$- \frac{\beta m L}{\alpha} w_{xt}(L, t) - k_1 s(t) - (\bar{d} + k_2)\text{sgn}(s(t)), \tag{9}$$

where k_1, α, β and k_2 are positive constants and $\text{sgn}(\cdot)$ is a sign function defined as

$$\text{sgn}(s) = \begin{cases} 1, & s > 0, \\ 0, & s = 0, \\ -1, & s < 0, \end{cases}$$

and the corresponding sliding mode function is defined as

$$s(t) = w_t(L, t) + \frac{\beta L}{\alpha} w_x(L, t). \tag{10}$$

Remark 1 For the proposed controller (9), the first three terms $-E_I w_{xxx}(L, t)$, $T w_x(L, t)$ and $-\frac{\beta m L}{\alpha} w_{xt}(L, t)$ are introduced mainly to cancel these dynamics on the boundary of the system, the fourth term $-k_1 s(t)$ is to drive the system to quickly reach a small neighborhood of the sliding mode surface, and the last term $(\bar{d} + k_2)\text{sgn}(s(t))$ is to suppress the boundary disturbance $d(t)$ and ensure that the system reaches the sliding mode surface.

Remark 2 From the sliding mode function (10), it can be found that the signs of $w_t(L, t)$ and $w_x(L, t)$ are the same when $|w(L, t)|$ increases and are opposite when $|w(L, t)|$ decreases in normal circumstances. Therefore, the value of the sliding mode function $s(t)$ can reflect the vibration of the system to a certain extent. Furthermore, when $s(t) \equiv 0$ and $|w(L, t)|$ decreases, $|w_t(L, t)|$ and $|w_x(L, t)|$ will always decrease, then $w_x(L, t) = 0$ and $w_t(L, t) = 0$ hold.

Theorem 1 *For the system dynamics described by the governing equation (2) and the boundary conditions (3)–(4), suppose that Assumption 1 holds. Given that the parameters of the controller satisfy the following conditions*

$$\begin{cases} k_1 > 0, k_2 > 0, \\ \beta < \dfrac{\min\left(\alpha\rho, \kappa_1^2 \alpha T\right)}{\kappa_1 \rho L}, \\ \dfrac{\alpha^2 T}{2\beta L} - \dfrac{\beta\rho L}{2} > 0, \\ \dfrac{\beta T}{2} - \dfrac{\kappa_2 \beta L}{2} > 0, \\ \dfrac{\beta\rho}{2} - \dfrac{\kappa_3 \alpha}{2} > 0, \end{cases} \tag{11}$$

where α, β, κ_1, κ_2 and κ_3 are positive constants, the proposed sliding mode control (9) yields that

(i) *there exists a $t_0 = \frac{m}{k_1} \ln\left(\frac{k_1|s(0)|}{k_2} + 1\right)$ such that the system reaches the sliding surface as $t \geq t_0$,*

(ii) *the state of the closed-loop system is exponentially bounded stable on the sliding surface and it will eventually converge to the compact set Ω defined by*

$$\Omega = \left\{ w(x, t) \in R \,\middle|\, \lim_{t \to \infty} |w(x, t)| \leq D, \forall x \in [0, L] \right\},$$

where $D = \sqrt{\frac{2L\varepsilon_f}{\alpha\lambda T(1-\beta_1)}}$, $\beta_1 = \frac{\kappa_1 \beta\rho L}{\min(\alpha\rho, \kappa_1^2 \alpha T)}$, $\varepsilon_f = \left(\frac{\beta L}{2\kappa_2} + \frac{\alpha}{2\kappa_3}\right) L\bar{f}^2$ and $\lambda > 0$.

Proof First, we show that the proposed control law (9) ensures that the states of the system (2) reach the sliding surface in finite time and remain there afterward. Define the following Lyapunov candidate function

$$V_s(t) = \frac{1}{2}ms^2(t). \tag{12}$$

Combining the boundary condition (4) and substituting the control law (9), then the derivative of the Lyapunov function (12) can be written as

$$\dot{V}_s(t) = s(t)\left(mw_{tt}(L, t) + \frac{\beta m L}{\alpha} w_{xt}(L, t)\right)$$
$$= s(t)\left(E_I w_{xxx}(L, t) - T w_x(L, t)\right.$$
$$\left. + u(t) + d(t) + \frac{\beta m L}{\alpha} w_{xt}(L, t)\right)$$

$$= -k_1 s^2(t) - (\bar{d} + k_2)|s(t)| + d(t)s(t). \quad (13)$$

From Assumption 1, the boundary input disturbance is bounded, i.e., $d(t) \leq \bar{d}$. According to Lemma 1, one gets from (13)

$$\dot{V}_s(t) = ms(t)\dot{s}(t) = -k_1 s^2(t) - (\bar{d} - d(t))|s(t)|$$
$$-k_2|s(t)|$$
$$\leq -k_1 s^2(t) - k_2|s(t)|. \quad (14)$$

Obviously, it can be seen that the Lyapunov function (12) decreases all the time until reaching the sliding surface. When $|s(0)| \neq 0$, $|s(t)| \neq 0$ holds before reaching the sliding surface. Therefore, dividing both sides of the inequality (14) by $|s(t)|$ results in

$$m\dot{s}(t)\text{sgn}(s(t)) \leq -k_1 s(t)\text{sgn}(s(t)) - k_2. \quad (15)$$

Multiplying both sides of the above inequality by $e^{\frac{k_1}{m}t}$ and integrating it with respect to time from 0 to t lead to

$$|s(t)| \leq |s(0)|e^{-\frac{k_1}{m}t} + \frac{k_2}{k_1}e^{-\frac{k_1}{m}t} - \frac{k_2}{k_1}. \quad (16)$$

With easy calculation $t_0 = \frac{m}{k_1}\ln\left(\frac{k_1|s(0)|}{k_2} + 1\right)$, when $t \geq t_0$, $|s(t)| = 0$, i.e., the system will reach the sliding surface in finite time.

Next, the system stability on the sliding surface is proved. Consider the Lyapunov candidate function as

$$V_0(t) = V_1(t) + V_2(t), \quad (17)$$

where the energy term $V_1(t)$ and the crossing term $V_2(t)$ are defined as

$$V_1(t) = \frac{\alpha}{2}\rho\int_0^L [w_t(x,t)]^2 \mathrm{d}x + \frac{\alpha}{2}E_\mathrm{I}\int_0^L [w_{xx}(x,t)]^2 \mathrm{d}x$$
$$+\frac{\alpha}{2}T\int_0^L [w_x(x,t)]^2 \mathrm{d}x, \quad (18)$$

$$V_2(t) = \beta\rho\int_0^L x w_t(x,t)w_x(x,t)\mathrm{d}x. \quad (19)$$

According to Lemma 1, we have

$$|V_2(t)| \leq \beta\rho L\int_0^L \left(\frac{\kappa_1}{2}[w_t(x,t)]^2 + \frac{1}{2\kappa_1}[w_x(x,t)]^2\right)\mathrm{d}x$$
$$\leq \beta_1 V_1(t), \quad (20)$$

where

$$\beta_1 = \frac{\kappa_1\beta\rho L}{\min\left(\alpha\rho, \kappa_1^2\alpha T\right)}. \quad (21)$$

Obviously, α, β and κ_1 can be chosen to satisfy $\beta_1 < 1$ such that the Lyapunov candidate function (17) is positive definite.

The derivative of Lyapunov function (17) is

$$\dot{V}_0(t) = \dot{V}_1(t) + \dot{V}_2(t).$$

The derivative of the energy term $\dot{V}_1(t)$ yields

$$\dot{V}_1(t) = A_1(t) + A_2(t) + A_3(t) \quad (22)$$

with

$$A_1(t) = \alpha\rho\int_0^L w_t(x,t)w_{tt}(x,t)\mathrm{d}x,$$

$$A_2(t) = \alpha E_\mathrm{I}\int_0^L w_{xx}(x,t)w_{xxt}(x,t)\mathrm{d}x,$$

$$A_3(t) = \alpha T\int_0^L w_x(x,t)w_{xt}(x,t)\mathrm{d}x. \quad (23)$$

Substituting the governing equation (2) into $A_1(t)$ and integrating the equation by parts yield

$$A_1(t) = \alpha\int_0^L w_t(x,t)\big(-E_\mathrm{I}w_{xxxx}(x,t)$$
$$+Tw_{xx}(x,t) + f(x,t)\big)\mathrm{d}x$$
$$= \alpha\Big[w_t(x,t)\big(-E_\mathrm{I}w_{xxx}(x,t) + Tw_x(x,t)\big)\Big]_0^L$$
$$-\alpha\int_0^L w_{xt}(x,t)\big(-E_\mathrm{I}w_{xxx}(x,t)$$
$$+Tw_x(x,t)\big)\mathrm{d}x$$
$$+\alpha\int_0^L f(x,t)w_t(x,t)\mathrm{d}x. \quad (24)$$

Similarly, integrating $A_2(t)$ by parts yields

$$A_2(t) = \alpha E_\mathrm{I}\Big[w_{xt}(x,t)w_{xx}(x,t)\Big]_0^L$$
$$-\alpha E_\mathrm{I}\int_0^L w_{xt}(x,t)w_{xxx}(x,t)\mathrm{d}x. \quad (25)$$

Substituting (23)–(25) into (22) and combining with the boundary condition (3) give

$$\dot{V}_1(t) = -\alpha E_I w_t(L, t) w_{xxx}(L, t) + \alpha T w_t(L, t) w_x(L, t)$$
$$+ \alpha \int_0^L f(x, t) w_t(x, t) \mathrm{d}x. \tag{26}$$

Substituting (2) into the derivative of crossing item $\dot{V}_2(t)$ leads to

$$\dot{V}_2(t) = \beta \rho \int_0^L x\big(w_{tt}(x, t) w_x(x, t) + w_t(x, t) w_{xt}(x, t)\big) \mathrm{d}x$$
$$= B_1(t) + B_2(t) + B_3(t) + B_4(t), \tag{27}$$

where

$$B_1(t) = -\beta E_I \int_0^L x w_x(x, t) w_{xxxx}(x, t) \mathrm{d}x,$$

$$B_2(t) = \beta T \int_0^L x w_x(x, t) w_{xx}(x, t) \mathrm{d}x,$$

$$B_3(t) = \beta \rho \int_0^L x w_t(x, t) w_{xt}(x, t) \mathrm{d}x,$$

$$B_4(t) = \beta \int_0^L x f(x, t) w_x(L, t) \mathrm{d}x. \tag{28}$$

Integrating $B_1(t)$ by parts twice and combining the boundary condition (3) yield

$$B_1(t) = -\beta E_I L w_x(L, t) w_{xxx}(L, t)$$
$$- \frac{3\beta E_I}{2} \int_0^L [w_{xx}(x, t)]^2 \mathrm{d}x. \tag{29}$$

Similarly calculating $B_2(t)$ yields

$$B_2(t) = \frac{\beta T L}{2} [w_x(L, t)]^2 - \frac{\beta T}{2} \int_0^L [w_x(x, t)]^2 \mathrm{d}x. \tag{30}$$

Integrating the equation $B_3(t)$ by parts and using the boundary condition (3), we have

$$B_3(t) = \frac{\beta \rho L}{2} [w_t(L, t)]^2 - \frac{\beta \rho}{2} \int_0^L [w_t(x, t)]^2 \mathrm{d}x. \tag{31}$$

Combining (28)–(31), we obtain

$$\dot{V}_2(t) = -\beta E_I L w_x(L, t) w_{xxx}(L, t) + \frac{\beta T L}{2} [w_x(L, t)]^2$$
$$+ \frac{\beta \rho L}{2} [w_t(L, t)]^2 - \frac{3\beta E_I}{2} \int_0^L [w_{xx}(x, t)]^2 \mathrm{d}x$$
$$- \frac{\beta T}{2} \int_0^L [w_x(x, t)]^2 \mathrm{d}x - \frac{\beta \rho}{2} \int_0^L [w_t(x, t)]^2 \mathrm{d}x$$
$$+ \beta \int_0^L x f(x, t) w_x(L, t) \mathrm{d}x. \tag{32}$$

Next, substituting (26) and (32) into (17) arrives at

$$\dot{V}_0(t) = -E_I w_{xxx}(L, t)\big(\alpha w_t(L, t) + \beta L w_x(L, t)\big)$$
$$+ \frac{\beta T L}{2} [w_x(L, t)]^2$$
$$+ \alpha T w_x(L, t) w_t(L, t) + \frac{\beta \rho L}{2} [w_t(L, t)]^2$$
$$- \frac{3\beta E_I}{2} \int_0^L [w_{xx}(x,t)]^2 \mathrm{d}x - \frac{\beta T}{2} \int_0^L [w_x(x,t)]^2 \mathrm{d}x$$
$$- \frac{\beta \rho}{2} \int_0^L [w_t(x, t)]^2 \mathrm{d}x + \alpha \int_0^L f(x, t) w_t(x, t) \mathrm{d}x$$
$$+ \beta \int_0^L x f(x, t) w_x(L, t) \mathrm{d}x. \tag{33}$$

Substituting the sliding surface (8) into (33) and combining Lemma 1, we obtain

$$\dot{V}_0(t) = -\frac{\alpha T}{2} \frac{\beta L}{\alpha} [w_x(L, t)]^2 - \frac{\alpha T}{2} \frac{\alpha}{\beta L} [w_t(L, t)]^2$$
$$+ \frac{\beta T L}{2} [w_x(L, t)]^2$$
$$+ \frac{\beta \rho L}{2} [w_t(L, t)]^2$$
$$- \frac{3\beta E_I}{2} \int_0^L [w_{xx}(x, t)]^2 \mathrm{d}x$$
$$- \frac{\beta T}{2} \int_0^L [w_x(x, t)]^2 \mathrm{d}x$$
$$- \frac{\beta \rho}{2} \int_0^L [w_t(x, t)]^2 \mathrm{d}x$$

$$+\alpha \int_0^L f(x,t)w_t(x,t)\mathrm{d}x$$

$$+\beta \int_0^L xf(x,t)w_x(L,t)\mathrm{d}x$$

$$\leq -\left(\frac{\alpha^2 T}{2\beta L} - \frac{\beta\rho L}{2}\right)[w_t(L,t)]^2$$

$$-\frac{3\beta E_\mathrm{I}}{2}\int_0^L [w_{xx}(x,t)]^2\mathrm{d}x$$

$$-\left(\frac{\beta T}{2} - \frac{\kappa_2\beta L}{2}\right)\int_0^L [w_x(x,t)]^2\mathrm{d}x$$

$$-\left(\frac{\beta\rho}{2} - \frac{\kappa_3\alpha}{2}\right)\int_0^L [w_t(x,t)]^2\mathrm{d}x$$

$$+\left(\frac{\beta L}{2\kappa_2} + \frac{\alpha}{2\kappa_3}\right)\int_0^L f^2(x,t)\mathrm{d}x$$

$$\leq -\lambda V_0(t) + \varepsilon_f, \qquad (34)$$

where

$$\varepsilon_f = \left(\frac{\beta L}{2\kappa_2} + \frac{\alpha}{2\kappa_3}\right)L\bar{f}^2,$$

and the constants α, β, κ_1, κ_2 and κ_3 are chosen to satisfy the conditions (11) such that

$$\lambda = \frac{1}{1+\beta_1}\min\left(\frac{3\beta}{\alpha}, \frac{2\sigma_1}{\alpha T}, \frac{2\sigma_2}{\alpha\rho}\right) > 0, \qquad (35)$$

where $\sigma_1 = \frac{\beta T}{2} - \frac{\kappa_2\beta L}{2}$ and $\sigma_2 = \frac{\beta\rho}{2} - \frac{\kappa_3\alpha}{2}$.

Therefore, it follows from (34) that

$$V_0(t) \leq \left(V_0(0) - \frac{\varepsilon_f}{\lambda}\right)e^{-\lambda t} + \frac{\varepsilon_f}{\lambda}. \qquad (36)$$

Using the inequalities (6), (20) and combining with (18), (36), we have

$$\frac{\alpha T}{2L}[w(x,t)]^2 \leq \frac{\alpha T}{2}\int_0^L [w_x(x,t)]^2\mathrm{d}x \leq V_1(t)$$

$$\leq \frac{1}{1-\beta_1}\left(\left(V_0(0) - \frac{\varepsilon_f}{\lambda}\right)e^{-\lambda t} + \frac{\varepsilon_f}{\lambda}\right).$$

The above inequality implies that the system state $w(x,t)$ converges uniformly in x:

$$|w(x,t)| \leq \sqrt{\frac{2L}{\alpha T(1-\beta_1)}\left(\left(V_0(0) - \frac{\varepsilon_f}{\lambda}\right)e^{-\lambda t} + \frac{\varepsilon_f}{\lambda}\right)},$$

$$\forall (x,t) \in [0,L] \times [0,\infty).$$

Furthermore, it shows

$$\lim_{t\to\infty}|w(x,t)| \leq \sqrt{\frac{2L\varepsilon_f}{\alpha\lambda T(1-\beta_1)}}.$$

This completes the proof. $\qquad\square$

Based on the above analysis, the following corollary can be obtained.

Corollary 1 *For the system dynamics described by the governing equation (2) with $f(x,t) = 0$ and the boundary condition (3)–(4), suppose that Assumption 1 holds. Given that the parameters of the controller satisfy the following conditions*

$$\begin{cases} k_1 > 0, k_2 > 0, \\ \beta < \dfrac{\min\left(\alpha\rho, \kappa_1^2\alpha T\right)}{\kappa_1\rho L}, \\ \dfrac{\alpha^2 T}{2\beta L} - \dfrac{\beta\rho L}{2} > 0, \end{cases} \qquad (37)$$

where α, β and κ_1 are positive constants, the proposed sliding mode control (9) yields that

(i) *there exists a $t_0 = \frac{m}{k_1}\ln\left(\frac{k_1|s(0)|}{k_2} + 1\right)$ such that the system reaches the sliding surface as $t \geq t_0$,*

(ii) *the state of the closed-loop system is exponentially stable on the sliding surface.*

Proof The proof follows analogously to the proof of Theorem 1. $\qquad\square$

Remark 3 From the above analysis, it can be seen that the system is exponentially bounded stable on the proposed sliding surface (8) and the convergence rate depends on the selection of parameters α and β, i.e., the selection of the sliding surface. From (8), it is clear that the parameters α and β affect the same item $w_x(L,t)$. When β is fixed, from (11), due to $\beta_1 < 1$, it can be concluded that

$$\lambda \geq \frac{1}{2\alpha}\min\left(3\beta, \frac{2\sigma_1}{T}, \frac{2\sigma_2}{\rho}\right).$$

Obviously, the decrease of the parameter α will result in the increase of the λ; thus, the convergence rate of the system on the sliding surface is enlarged.

Remark 4 Compared with using an isokinetic reaching law (i.e., $s\dot{s} \leq -k_1|s|$) to make the system reach the sliding surface in [29], the exponential reaching law

Table 1 Parameters of the beam

Parameter	Description	Value
E_I	Bending stiffness	$7\ \mathrm{Nm}^2$
L	length of the beam	1 m
T	Tension	10 N
ρ	Uniform mass per unit length	0.1 kg/m
m	Mass of the payload	1 kg

(i.e., $s\dot{s} \leq -k_1 s^2 - k_2|s|$) is used in the proposed control law (9) instead. It will result in that the switching amplitude of the controller is determined by $k_2 + \bar{d}$ rather than $k_1 + \bar{d}$ when the system is stable. In order to guarantee the velocity of reaching the sliding surface, k_1 should be chosen as large as possible. While k_2 can be chosen as a small constant due to the introduction of the exponential item, thus the chattering phenomenon will be improved.

Remark 5 All feedback states in the control law (9) can be obtained relatively more easily. $w(L, t)$ can be measured by a laser displacement sensor at the right boundary of the beam. $w_x(L, t)$ can be sensored by an inclinometer and $w_{xxx}(L, t)$ can be measured by a shear force sensor. Then, the backward difference algorithm can be used to calculate the signals $w_t(L, t)$ and $w_{xt}(L, t)$. It should be noted that all the desired signals can be measured directly or obtained by one derivative with respect to time, thus avoiding the measurement noise caused by high-order difference, i.e., $w_{tt}(L, t)$ and $w_{xtt}(L, t)$.

4 Numerical simulations

In this section, we provide a simulation to verify the effectiveness of the proposed controller (9) by using the finite difference method (FDM) [32] to approximate the solution of the PDE. The space step and the time step are chosen as $\Delta x = 1/9$ m and $\Delta t = 0.0001$ s. The initial conditions of the beam are taken as $w(x, 0) = 0.1x$ and $w_t(x, 0) = 0$. The detailed parameters used in simulation are provided in the Table 1.

In the simulation, the boundary disturbance $d(t)$ is generated by the following equation

$$d(t) = 0.4 + 0.4\sin(\pi t) + 0.8\sin(2\pi t) + 1.6\sin(3\pi t).$$
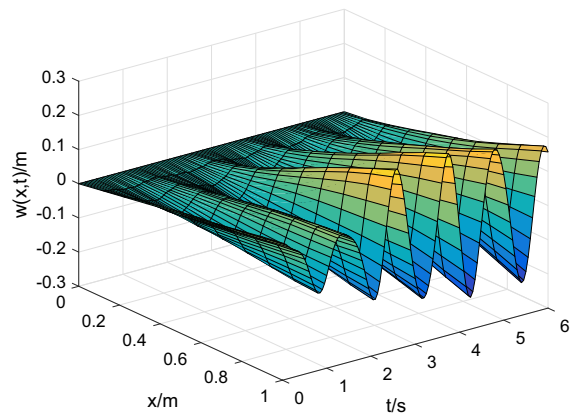
Fig. 2 Displacement of beam without control under external disturbances

and the time-varying distributed load $f(x, t)$ is described as

$$f(x, t) = \frac{[1 + 2\sin(0.1\pi xt) + 3\sin(0.2\pi xt) + 4\sin(0.3\pi xt)]x}{100L}.$$

The dynamic responses of the Euler–Bernoulli beam are examined for the following four cases.

4.1 Without control

The responses of system state for free vibration (i.e., $u(t) = 0$) under external disturbances are shown in Fig. 2. There are significant vibrations along the beam due to the initial condition of the system and external disturbances.
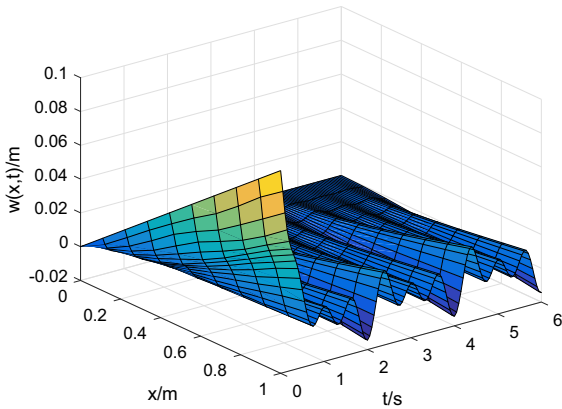
4.2 Boundary control without sliding mode

For comparison, the response of the system state with the controller

$$u(t) = -E_I w_{xxx}(L, t) + T w_x(L, t)$$
$$- \frac{\beta mL}{\alpha} w_{xt}(L, t) - k_1 s(t) \tag{38}$$

without the sliding mode term is shown in Fig. 3. The parameters are chosen as $\alpha = 1.5$, $\beta = 5$ and $k_1 = 10$ in the simulation. It can be seen that the system state still vibrates at the equilibrium position due to the input disturbances, although it converges quickly. Therefore, the boundary control cannot work well with the input disturbances.

Fig. 3 Displacement of beam with the controller (38)
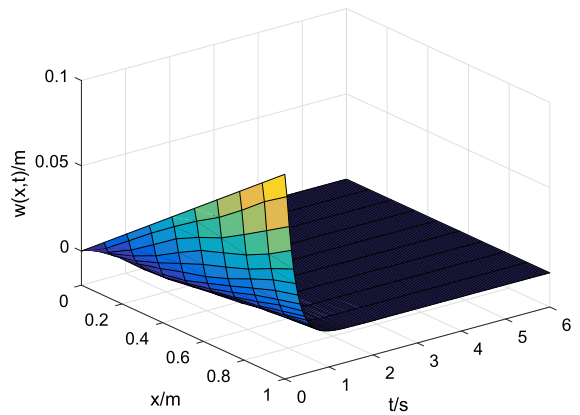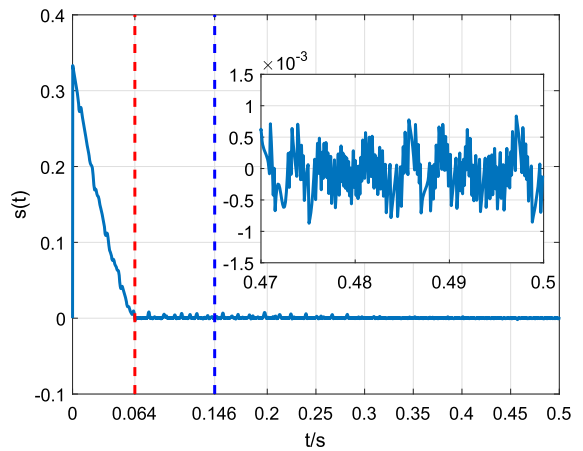

Fig. 4 Displacement of beam with SMC (9)

4.3 Sliding mode boundary control

The responses of the system state with the sliding mode boundary controller (9) under external disturbances are illustrated in Fig. 4, where the design parameters are chosen as

$$\alpha = 1.5, \ \beta = 5, \ k_1 = 10, \ \bar{d} = 4, \ k_2 = 1.$$

The response of the sliding mode function is shown in Fig. 5, in which the red line represents the reaching time of the sliding mode surface, and the blue line represents the time calculated by the equation $\frac{m}{k_1} \ln\left(\frac{k_1|s(0)|}{k_2} + 1\right)$. Obviously, the proposed sliding mode control (9) can drive the system reach to the sliding surface in finite time and remain there afterward, thus the vibration of the beam system can be suppressed within 2 seconds as shown in Fig. 4. In addition, the control input and the boundary displacement of the beam system under different α are shown in Figs. 6 and 7, respectively.
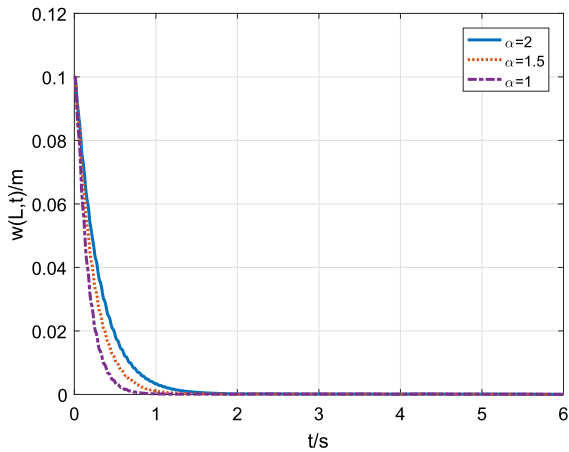
4.4 Boundary control with disturbance observer

For comparison, the response of the system state with the boundary controller proposed in [14]

Fig. 5 The sliding mode function $s(t)$
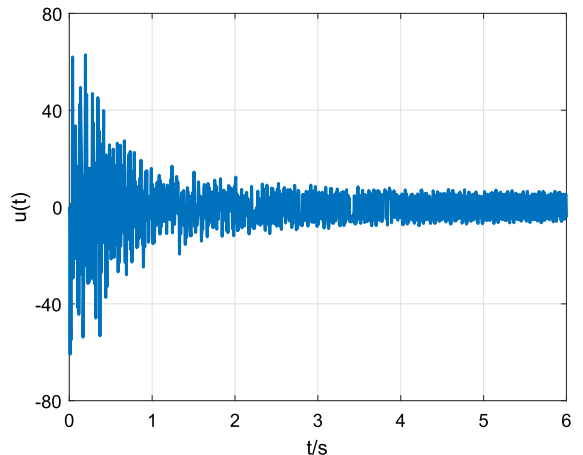
Fig. 6 The control input $u(t)$ of SMC (9)

Fig. 7 Boundary displacement of beam with SMC under different α

Fig. 8 Displacement of beam with controller in [14]

$$u(t) = -E_I w_{xxx}(L, t) + T w_x(L, t) - m[w_x(L, t)$$
$$- w_{xxx}(L, t)]_t - k u_a(t) - \hat{d}(t) \qquad (39)$$

under different external disturbances is illustrated in Fig. 8, where $\hat{d}(t)$ is the estimate of the $d(t)$, k is the control gain, and the auxiliary signal $u_a(t)$ is defined as
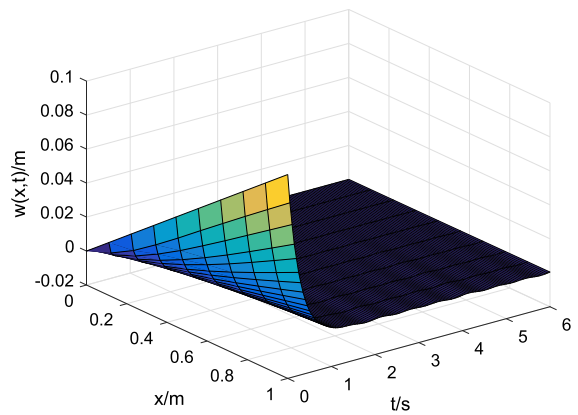
$$u_a(t) = w_t(L, t) - w_{xxx}(L, t) + w_x(L, t).$$

The disturbance observer is designed as

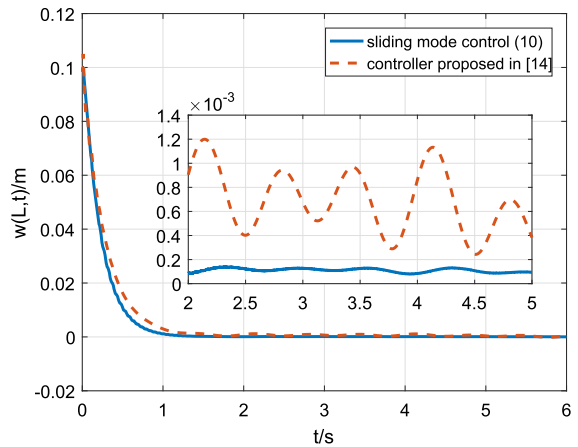$$\dot{\hat{d}}(t) = \gamma u_a(t) - \zeta_d \gamma \hat{d}(t)$$

with two positive constants γ and ζ_d. Figure 8 shows that the boundary control (39) can suppress the vibration of the system. However, as shown in Fig. 9, the proposed sliding mode boundary control (9) leads to better performace.

Fig. 9 The comparison for SMC and boundary controller in [14]

4.5 Discussion

This paper proposes a sliding mode control method for the Euler–Bernoulli system under external disturbances. Compared with [14], the derivative of the state $w_{xxx}(L, t)$ is not required in our control law. Moreover, since the sliding mode control approach is used to suppress the boundary disturbance instead of a disturbance observer, the bound of the convergence of the system state can be smaller; thus, a better control performance will be obtained. As shown in Fig. 9, the

convergence bound of the system state under the proposed controller is in magnitude smaller than that of the controller in [14]. Besides, compared with [18,33] and [19], the proposed controller does not depend on the damping term of the system. Of course, the proposed controller still has some limitations. Firstly, the chattering phenomenon of the system is still inevitable, especially when the external disturbances are large. Secondly, the proposed controller is dependent on system parameters E_I, T and the upper bound of the boundary disturbance $d(t)$. For the above problem, the hyperbolic tangent function can be used to replace the sign function to reduce the chattering issue and the adaptive laws of E_I, T and $\bar{d}$ can be designed to solve the parameter uncertainties, which are also the direction of the future work.

5 Conclusion

In this paper, a sliding mode boundary controller is developed to stabilize an Euler–Bernoulli beam in the presence of unknown disturbances including time-varying distributed load and boundary disturbance. In the control design, a sliding surface based on boundary state information is adopted to handle the impact of external disturbances. Further, the reachability of the sliding surface and the stability of the closed-loop system are analyzed by Lyapunov approach. The numerical simulations validate the theoretical results. Compared with the previous work about boundary control of the Euler–Bernoulli beam system, the proposed controller has following advantages: 1) It can stabilize the beam system with the unknown external disturbances; meanwhile, both the controller and the sliding surface only require boundary state information. 2) The control design and stability analysis are based on partial differential equations, and there is no need to simplify or discretize the beam model.

Data availability All data included in this study are available upon request by contact with the corresponding author.

Conflict of interest The authors declare that they have no conflict of interest concerning the publication of this manuscript.

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