



Multi-soliton solutions of the N -component nonlinear Schrödinger equations via Riemann–Hilbert approach

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Received: 9 April 2021 / Accepted: 5 July 2021 / Published online: 19 July 2021
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Abstract In this paper, we utilize the Riemann–Hilbert approach to discuss multi-soliton solutions of the N -component nonlinear Schrödinger equations. Firstly, by transformed Lax pair, we construct the matrix-valued functions $P_{1,2}$ that satisfy the analyticity and normalization and the corresponding jump matrix can be determined. Then, in the reflectionless case, we get the multi-soliton solutions q_l ($l = 1, \dots, N$) of the N -component nonlinear Schrödinger equations, which are related to the spectral parameter η . Particularly, the 2-soliton solutions q_1, q_2 , and q_3 of the three-component nonlinear Schrödinger equations are given and the corresponding 2-soliton diagrams are drawn.

Keywords N -component NLS equations · Lax pair · Riemann–Hilbert approach · Multi-soliton solutions

Mathematics Subject Classification 35Q51 · 35Q15 · 37K10

The work is supported by the National Natural Science Foundation of China (Grant No. 11971297).

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1 Introduction

Riemann–Hilbert (RH) problem is the 21st question that Hilbert mentioned at the International Congress of Mathematicians in Paris [1]. It belongs to the scope of boundary value problem of matrix-valued functions on the complex plane. Usually, the RH problem is defined as follows [2]. Assume that Σ is a directed path on the complex plane \mathcal{C} , $\Sigma^0 = \Sigma \setminus \{\text{self-intersection of } \Sigma\}$. Suppose that there is a smooth map on Σ^0

$$G(z) : \Sigma^0 \rightarrow ML(n, \mathcal{C}).$$

Then, (Σ, G) determines a RH problem, i.e., looking for a $n \times n$ matrix $P(z)$, which satisfies

- $P(z)$ is analytical in $\mathcal{C} \setminus \Sigma$;
- $P(z)$ satisfies the following jump condition

$$P_+(z) = P_-(z)G(z), z \in \Sigma;$$

- $P(z) \rightarrow I, z \rightarrow \infty$.

In fact, the RH problem is a boundary value problem of matrix value function on complex plane, and we can convert it into integral equations. Because RH problem is a problem on complex plane, the biggest advantage of RH approach is to transform the problem solvable on complex plane. For example, some definite integrals are difficult to integrate in the real domain, but can be solved when treated as complex integrals.

It is known that the methods by which the solutions of the integral equations can be obtained include inverse scattering transformation [3], Darboux transformation

[4], symmetry reduction method [5], Hirota bilinear method [6], Lax pair nonlinear method [7], Wronskian technique [8], and so on. RH approach is a direct and simple method to solve the soliton equations. The problem can be solved by Lax pair of integrable systems and analysis of the spectral function. One can construct the RH problem similar to the above description, and then get the soliton solutions of the original equations.

Now RH approach has been developed into a powerful analytical tool to solve problems in a large class of pure and applied mathematics, which can be widely applied to initial boundary value problem [9–16], asymptotic of orthogonal polynomials [17], Bäcklund transformation [18,19], and long-time asymptotics [20–22]. Afterward, It is found that the RH approach can be used to obtain the solutions of integral equations by inverse scattering theory [23–30]. In recent years, Wazwaz solved multiple soliton solutions of the equations [31–34]. Then, RH approach has been widely used to get multi-soliton solutions of multi-dimensional equations [35–39].

In this paper, we mainly discuss multi-soliton solutions of the N -component nonlinear Schrödinger (NLS) equations by RH approach. The N -component NLS equations [40] take the form

$$iq_{lt} + \frac{1}{2}q_{lxx} + \sum_{l=1}^N |q_l|^2 q_l = 0, \quad l = 1, \dots, N, \quad (1.1)$$

where $q_l = q_l(t, x)$ ($l = 1, \dots, N$) are functions and the subscripts mean the partial derivatives.

The organization of this paper is given as follows. In Sect. 2, we transform Lax pair to construct the RH problem for the N -component NLS equations. In Sect. 3, the multi-soliton solutions for the N -component NLS equations are obtained, which are relevant to the spectral parameter. Then, the 2-soliton solutions of the three-component NLS equations are given and the corresponding 2-soliton graphs are drawn. In Sect. 4, we give the conclusion.

2 Riemann–Hilbert problem

Based on Eq. (1.1), we have the Lax pair

$$\begin{cases} \Phi_x + i\eta\sigma\Phi = iQ\Phi, \\ \Phi_t + i\eta^2\sigma\Phi = [i\eta Q + \frac{1}{2}(i\sigma Q^2 - \sigma Q_x)]\Phi, \end{cases} \quad (2.1)$$

where

$$Q = \begin{pmatrix} 0 & q_1^* & q_2^* & \dots & q_N^* \\ q_1 & 0 & 0 & \dots & 0 \\ q_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_N & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (2.2)$$

$$\sigma = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

η is the spectral parameter. Then, we get the Jost solution of Lax pair (2.1) with asymptotic form by

$$\Phi \sim e^{-i\eta\sigma x - i\eta^2\sigma t}, \quad |x| \rightarrow \infty. \quad (2.3)$$

In order to facilitate calculation, we define a matrix function $\Psi = \Psi(x, t; \eta)$. Letting

$$\Phi = \Psi e^{-i\eta\sigma x - i\eta^2\sigma t}, \quad (2.4)$$

then we have

$$\Psi \rightarrow I, \quad |x| \rightarrow \infty. \quad (2.5)$$

The Lax pair (2.1) can be rewritten as

$$\begin{cases} \Psi_x + i\eta[\sigma, \Psi] = U_1\Psi, \\ \Psi_t + i\eta^2[\sigma, \Psi] = U_2\Psi, \end{cases} \quad (2.6)$$

where $U_1 = iQ$ and $U_2 = i\eta Q + \frac{1}{2}(i\sigma Q^2 - \sigma Q_x)$. Then, the Volterra integral equations can be expressed as

$$\begin{aligned} \Psi_1(t, x; \eta) &= I + \int_{-\infty}^x e^{-i\eta(x-x')\sigma} U_1 \Psi_1 e^{i\eta(x-x')\sigma} dx', \\ \Psi_2(t, x; \eta) &= I - \int_x^{+\infty} e^{-i\eta(x-x')\sigma} U_1 \Psi_2 e^{i\eta(x-x')\sigma} dx'. \end{aligned} \quad (2.7)$$

By calculation, we can know

$$e^{-i\eta(x-x')\sigma} U_1 e^{i\eta(x-x')\sigma} = \begin{pmatrix} 0 & iq_1^* e^{2i\eta(x-x')} & \dots & iq_N^* e^{2i\eta(x-x')} \\ iq_1 e^{-2i\eta(x-x')} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ iq_N e^{-2i\eta(x-x')} & 0 & \dots & 0 \end{pmatrix}. \quad (2.8)$$

Let $\Psi_1 = ([\Psi_1]_1, [\Psi_1]_2, \dots, [\Psi_1]_{N+1})$ and $\Psi_2 = ([\Psi_2]_1, [\Psi_2]_2, \dots, [\Psi_2]_{N+1})$. Then, it can be obtained

that $[\Psi_1]_1$ is analytic in \mathcal{C}^- , $[\Psi_1]_2, \dots, [\Psi_1]_{N+1}$ are analytic in \mathcal{C}^+ . $[\Psi_2]_1$ is analytic in \mathcal{C}^+ , $[\Psi_2]_2, \dots, [\Psi_2]_{N+1}$ are analytic in \mathcal{C}^- . We can rewrite $\Psi_{1,2}$ as follows

$$\begin{aligned} \Psi_1 &= ([\Psi_1]_1, [\Psi_1]_2, \dots, [\Psi_1]_{N+1}) \\ &= (\Psi_1^-, \Psi_1^+, \dots, \Psi_1^+), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \Psi_2 &= ([\Psi_2]_1, [\Psi_2]_2, \dots, [\Psi_2]_{N+1}) \\ &= (\Psi_2^+, \Psi_2^-, \dots, \Psi_2^-). \end{aligned} \tag{2.10}$$

Based on the properties of $\Psi_{1,2}$ and $\text{tr } Q=0$, we can know that $\det \Psi_{1,2}$ are independent for all x . By the asymptotic conditions $\Psi_{1,2} \rightarrow I$ as $|x| \rightarrow \infty$, we know that

$$\det \Psi_{1,2} = 1. \tag{2.11}$$

Therefore, $\Psi_{1,2}$ are linearly related by a spectral matrix $S(\eta) = (s_{kj}(\eta))_{(N+1) \times (N+1)}$, which can be expressed as

$$\Psi_1 E = \Psi_2 E S(\eta), \quad E = e^{-i\eta\sigma x}. \tag{2.12}$$

Then, we have

$$\det S(\eta) = 1. \tag{2.13}$$

Taking the inverse of both sides of Eq. (2.12), we can obtain

$$\Psi_1^{-1} = E S(\eta)^{-1} E^{-1} \Psi_2^{-1}. \tag{2.14}$$

Applying Eq. (2.14) and the analytic properties of column vectors of $\Psi_{1,2}$, we can get the analytic properties of $\Psi_{1,2}^{-1}$, that is

$$\Psi_1^{-1} = \begin{pmatrix} (\Psi_1^{-1})^1 \\ (\Psi_1^{-1})^2 \\ \vdots \\ (\Psi_1^{-1})^{N+1} \end{pmatrix} = \begin{pmatrix} \hat{\Psi}_1^+ \\ \hat{\Psi}_1^- \\ \vdots \\ \hat{\Psi}_1^- \end{pmatrix}, \tag{2.15}$$

$$\Psi_2^{-1} = \begin{pmatrix} (\Psi_2^{-1})^1 \\ (\Psi_2^{-1})^2 \\ \vdots \\ (\Psi_2^{-1})^{N+1} \end{pmatrix} = \begin{pmatrix} \hat{\Psi}_2^- \\ \hat{\Psi}_2^+ \\ \vdots \\ \hat{\Psi}_2^+ \end{pmatrix}.$$

In order to construct a matrix RH problem, we need to determine two matrix functions

$$\begin{cases} P_1 = ([\Psi_2]_1, [\Psi_1]_2, \dots, [\Psi_1]_{N+1}) \\ \quad = (\Psi_2^+, \Psi_1^+, \dots, \Psi_1^+), \quad \eta \in \mathcal{C}^+, \\ P_2 = \begin{pmatrix} (\Psi_2^{-1})^1 \\ (\Psi_1^{-1})^2 \\ \vdots \\ (\Psi_1^{-1})^{N+1} \end{pmatrix} = \begin{pmatrix} \hat{\Psi}_2^- \\ \hat{\Psi}_1^- \\ \vdots \\ \hat{\Psi}_1^- \end{pmatrix}, \quad \eta \in \mathcal{C}^-. \end{cases} \tag{2.16}$$

Let

$$B_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.17}$$

Then, $P_{1,2}$ can be rewritten as

$$\begin{aligned} P_1 &= \Psi_1 B_1 + \Psi_2 B_2 \\ &= \Psi_1 \begin{pmatrix} r_{11} & 0 & \cdots & 0 \\ e^{-2i\eta x} r_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e^{-2i\eta x} r_{N+1,1} & 0 & \cdots & 1 \end{pmatrix}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} P_2 &= B_1 \Psi_1^{-1} + B_2 \Psi_2^{-1} \\ &= \begin{pmatrix} s_{11} & e^{2i\eta x} s_{12} & \cdots & e^{2i\eta x} s_{1,N+1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \Psi_1^{-1}, \end{aligned} \tag{2.19}$$

where $R(\eta) = S^{-1}(\eta) = (r_{kj}(\eta))_{(N+1) \times (N+1)}$. We can study the asymptotic expansion of P_1

$$P_1 = P_1^{(0)} + \frac{P_1^{(1)}}{\eta} + \frac{P_1^{(2)}}{\eta^2} + o(\eta^{-3}). \tag{2.20}$$

Submitting (2.20) into the first equation of (2.6) and comparing the corresponding coefficients of η , we obtain

$$\begin{aligned} O(\eta^1) : & \quad i[\sigma, P_1^{(0)}] = 0, \\ O(\eta^0) : & \quad P_{1,x}^{(0)} + i[\sigma, P_1^{(1)}] = U_1 P_1^{(0)}. \end{aligned} \tag{2.21}$$

We can get

$$P_1 \rightarrow I, \quad \eta \in \mathcal{C}^+ \rightarrow \infty. \tag{2.22}$$

Similarly,

$$P_2 \rightarrow I, \quad \eta \in \mathcal{C}^- \rightarrow \infty. \tag{2.23}$$

By the above calculation, we propose the RH problem of the N -component NLS equations

- $P_1(t, x; \eta)$ is analytic in \mathcal{C}^+ , $P_2(t, x; \eta)$ is analytic in \mathcal{C}^- .
- $P_2(t, x; \eta) P_1(t, x; \eta) = G(t, x; \eta), \quad \eta \in \mathcal{R},$
- $P_{1,2}(t, x; \eta) \rightarrow I, \quad \text{as } \eta \rightarrow \infty,$

where the jump matrix takes the form

$$\begin{aligned} G(t, x; \eta) &= \begin{pmatrix} 1 & e^{2i\eta x} s_{12} & \cdots & e^{2i\eta x} s_{1,N+1} \\ e^{-2i\eta x} r_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e^{-2i\eta x} r_{N+1,1} & 0 & \cdots & 1 \end{pmatrix}. \end{aligned} \tag{2.24}$$

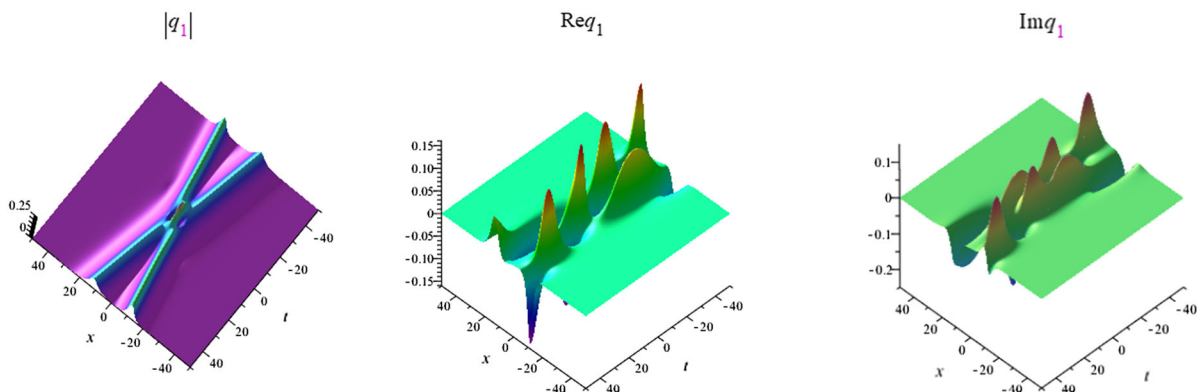


Fig. 1 Evolution plot of the 2-soliton $|q_1|$, $\text{Re}q_1$, and $\text{Im}q_1$

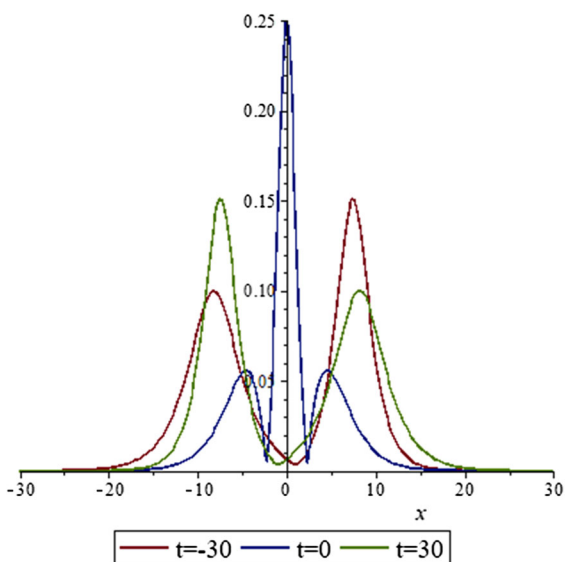


Fig. 2 The soliton along the x -axis with different t

3 Multi-soliton solutions of the N -component NLS equations

3.1 Multi-soliton solutions

In this section, we can get the multi-soliton solutions of Eq. (1.1). Firstly, we should derive the properties of the zeros of $\det P_{1,2}(\eta)$. On the basis of Eqs. (2.18) and (2.19), we can get

$$\begin{aligned} \det P_1(\eta) &= r_{11}(\eta), \quad \eta \in \mathcal{C}^+, \\ \det P_2(\eta) &= s_{11}(\eta), \quad \eta \in \mathcal{C}^-. \end{aligned} \tag{3.1}$$

In other words, zeros of $\det P_1(\eta)$ are zeros of $r_{11}(\eta)$ and zeros of $\det P_2(\eta)$ are zeros of $s_{11}(\eta)$. Obviously,

Q defined by (2.2) is a Hermite matrix, that is

$$Q^\dagger = Q, \tag{3.2}$$

where \dagger means conjugate transpose. Using Eqs. (2.18)–(2.19) and $\Psi_{1,2}^\dagger(\eta^*) = \Psi_{1,2}^{-1}(\eta)$, we can get the following relationship

$$P_2^\dagger(\eta^*) = P_1(\eta). \tag{3.3}$$

Then, it follows from the scattering relation (2.12) that

$$S^\dagger(\eta^*) = S^{-1}(\eta), \quad s_{11}^*(\eta^*) = r_{11}(\eta). \tag{3.4}$$

The points η_j ($j = 1, 2, \dots, N$) are zeros of $\det P_1(\eta)$ in \mathcal{C}^+ , and η_j^* ($j = 1, 2, \dots, N$) are zeros of $\det P_2(\eta)$ in \mathcal{C}^- . According to $\det P_1(\eta_j) = \det P_2^*(\eta_j^*)$, we suppose that nonzero column vectors v_j and nonzero row vectors v_j^* are the solutions of the following linear equations, respectively,

$$\begin{aligned} P_1(\eta_j)v_j(\eta_j) &= 0, \\ v_j^*(\eta_j^*)P_2(\eta_j^*) &= 0. \end{aligned} \tag{3.5}$$

In fact, the scattering data for solving the RH problem (2.24) is composed of the discrete scattering data $\{\eta_j, \eta_j^*, v_j, v_j^*\}$ and the continuous scattering data $\{s_{21}, s_{31}, \dots, s_{N+1,1}\}$. From Eqs. (3.3) to (3.5), we get

$$v_j^* = v_j^\dagger. \tag{3.6}$$

Differentiating Eq. (3.5) with respect to x and considering Lax pair (2.6) and Eq. (3.6), we obtain

$$\begin{cases} v_j = e^{-i\eta_j\sigma x - i\eta_j^2\sigma t} v_{j,0}, \\ v_j^* = v_{j,0}^\dagger e^{i\eta_j^*\sigma x + i\eta_j^{*2}\sigma t}, \end{cases} \tag{3.7}$$

where $v_{j,0}$ are the $(N + 1)$ -dimensional constant column vectors. Then, we can get multi-soliton solutions for

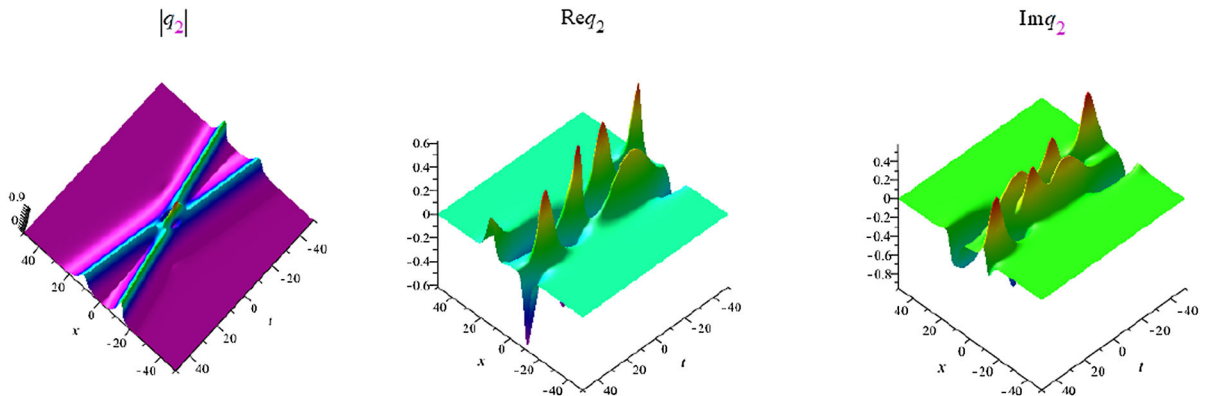


Fig. 3 Evolution plot of the 2-soliton $|q_2|$, $\text{Re}q_2$, and $\text{Im}q_2$

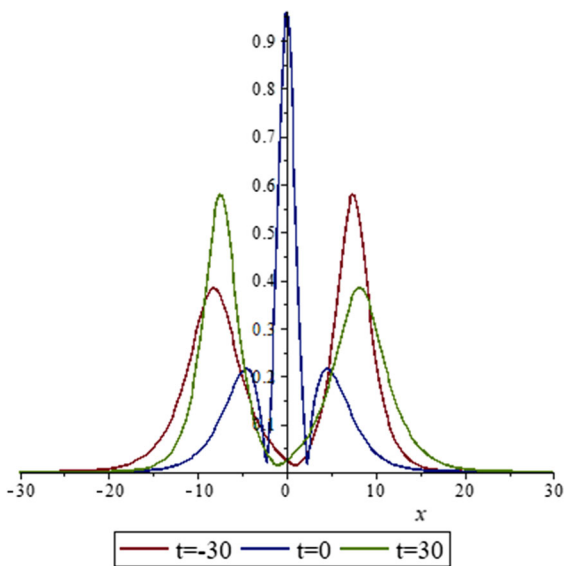


Fig. 4 The soliton along the x -axis with different t

Eq. (1.1) in the reflectionless case. Define the matrix $M = (M_{kj})_{(N+1) \times (N+1)}$

$$M_{kj} = \frac{v_k^* v_j}{\eta_j - \eta_k^*}. \tag{3.8}$$

Based on the canonical normalization condition (2.16), the RH problem has the unique solution

$$P_1(\eta) = I - \sum_{k,j=1}^N \frac{v_k v_j^* (M^{-1})_{kj}}{\eta - \eta_j^*},$$

$$P_2(\eta) = I + \sum_{k,j=1}^N \frac{v_k v_j^* (M^{-1})_{kj}}{\eta - \eta_k}. \tag{3.9}$$

Then, we can expand P_1 as follows

$$P_1 = P_1^{(0)} + \frac{P_1^{(1)}}{\eta} + \frac{P_1^{(2)}}{\eta^2} + o(\eta^{-3}). \tag{3.10}$$

Putting the asymptotic expansion (3.10) into Eq. (2.6), we have

$$i[\sigma, P_1^{(1)}] = iQ. \tag{3.11}$$

Hence, the potential functions q_l ($l = 1, 2, \dots, N$) can be expressed as

$$q_l = 2(P_1^{(1)})_{l+1,1}, \quad l = 1, 2, \dots, N, \tag{3.12}$$

where $(P_1^{(1)})_{l+1,1}$ ($l = 1, 2, \dots, N$) are the $(l + 1, 1)$ entry of matrix $P_1^{(1)}$, which can be obtained from Eq. (3.9), that is

$$P_1^{(1)} = - \sum_{k,j=1}^N v_k v_j^* (M^{-1})_{kj}. \tag{3.13}$$

Substituting Eq. (3.7) into Eq. (3.13), a general multi-soliton solutions for the N -component NLS equations can be shown as

$$q_l = -2 \sum_{k,j=1}^N v_{k,l+1} v_{j,1}^* (M^{-1})_{kj}, \quad l = 1, 2, \dots, N, \tag{3.14}$$

where $v_k = (v_{k1}, v_{k2}, \dots, v_{k,N+1})^T$ and $v_j^* = (v_{j1}^*, v_{j2}^*, \dots, v_{j,N+1}^*)$ ($k, j = 1, 2, \dots, N$) are defined by Eq. (3.7).

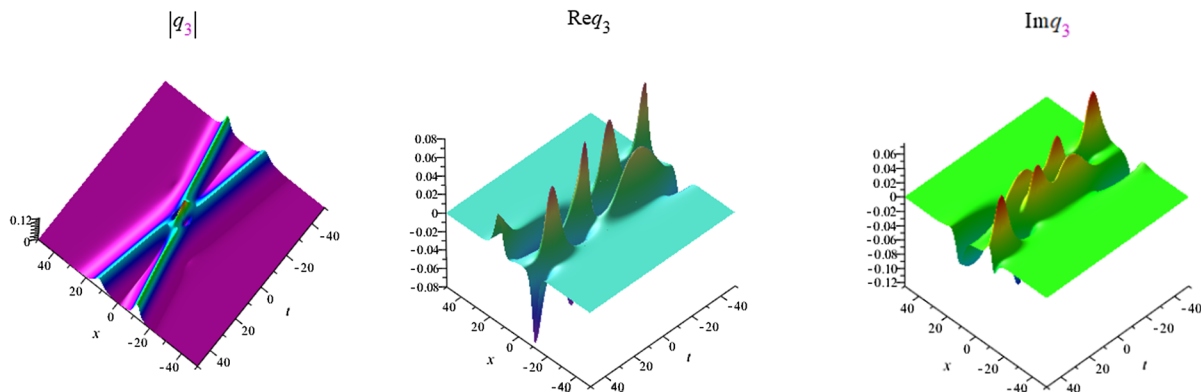


Fig. 5 Evolution plot of the 2-soliton $|q_3|$, $\text{Re}q_3$, and $\text{Im}q_3$

3.2 2-soliton solutions

Particularly, we can study the case of $N = 3$. The three-component NLS equations can be written as

$$iq_{lt} + \frac{1}{2}q_{lxx} + \sum_{l=1}^3 |q_l|^2 q_l = 0, \quad l = 1, 2, 3. \quad (3.15)$$

We can obtain the soliton solutions for this particular case. Assuming $v_{1,0} = (\alpha_1, \beta_1, \gamma_1, \epsilon_1)^T$, $v_{2,0} = (\alpha_2, \beta_2, \gamma_2, \epsilon_2)^T$, and letting $\xi_1 = -i\eta_1 x - i\eta_1^2 t$, $\xi_2 = -i\eta_2 x - i\eta_2^2 t$, $\eta_1 = a_1 + ib_1$, $\eta_2 = a_2 + ib_2$. We can get 2-soliton solutions as follows

$$q_1 = \frac{-2(\beta_1 \alpha_1^* e^{\xi_1 - \xi_1^*} m_{22} - \beta_1 \alpha_2^* e^{\xi_1 - \xi_2^*} m_{12} - \beta_2 \alpha_1^* e^{\xi_2 - \xi_1^*} m_{21} + \beta_2 \alpha_2^* e^{\xi_2 - \xi_2^*} m_{11})}{m_{11} m_{22} - m_{12} m_{21}}, \quad (3.16)$$

$$q_2 = \frac{-2(\gamma_1 \alpha_1^* e^{\xi_1 - \xi_1^*} m_{22} - \gamma_1 \alpha_2^* e^{\xi_1 - \xi_2^*} m_{12} - \gamma_2 \alpha_1^* e^{\xi_2 - \xi_1^*} m_{21} + \gamma_2 \alpha_2^* e^{\xi_2 - \xi_2^*} m_{11})}{m_{11} m_{22} - m_{12} m_{21}}, \quad (3.17)$$

$$q_3 = \frac{-2(\epsilon_1 \alpha_1^* e^{\xi_1 - \xi_1^*} m_{22} - \epsilon_1 \alpha_2^* e^{\xi_1 - \xi_2^*} m_{12} - \epsilon_2 \alpha_1^* e^{\xi_2 - \xi_1^*} m_{21} + \epsilon_2 \alpha_2^* e^{\xi_2 - \xi_2^*} m_{11})}{m_{11} m_{22} - m_{12} m_{21}}, \quad (3.18)$$

where

$$m_{11} = \frac{\alpha_1^* \alpha_1 e^{-\xi_1 - \xi_1^*} + \beta_1^* \beta_1 e^{\xi_1 + \xi_1^*} + \gamma_1^* \gamma_1 e^{\xi_1 + \xi_1^*} + \epsilon_1^* \epsilon_1 e^{\xi_1 + \xi_1^*}}{\eta_1 - \eta_1^*}, \quad (3.19)$$

$$m_{12} = \frac{\alpha_1^* \alpha_2 e^{-\xi_1 - \xi_2^*} + \beta_1^* \beta_2 e^{\xi_1 + \xi_2^*} + \gamma_1^* \gamma_2 e^{\xi_1 + \xi_2^*} + \epsilon_1^* \epsilon_2 e^{\xi_1 + \xi_2^*}}{\eta_2 - \eta_1^*}, \quad (3.20)$$

$$m_{21} = \frac{\alpha_2^* \alpha_1 e^{-\xi_2 - \xi_1^*} + \beta_2^* \beta_1 e^{\xi_2 + \xi_1^*} + \gamma_2^* \gamma_1 e^{\xi_2 + \xi_1^*} + \epsilon_2^* \epsilon_1 e^{\xi_2 + \xi_1^*}}{\eta_1 - \eta_2^*}, \quad (3.21)$$

$$m_{22} = \frac{\alpha_2^* \alpha_2 e^{-\xi_2 - \xi_2^*} + \beta_2^* \beta_2 e^{\xi_2 + \xi_2^*} + \gamma_2^* \gamma_2 e^{\xi_2 + \xi_2^*} + \epsilon_2^* \epsilon_2 e^{\xi_2 + \xi_2^*}}{\eta_2 - \eta_2^*}. \quad (3.22)$$

By selecting appropriate parameter values $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = \frac{1}{4}$, $\gamma_1 = \gamma_2 = \frac{\sqrt{59}}{8}$, $\epsilon_1 = \epsilon_2 = \frac{1}{8}$, $a_1 = -0.1$, $b_1 = 0.2$, $a_2 = 0.2$, and $b_2 = 0.3$, three-dimensional plots and x -curves of solutions are shown in Figs. 1, 2, 3, 4, 5, and 6.

4 Conclusion

In general, we investigate the multi-soliton solutions of the N -component NLS equations via RH approach.

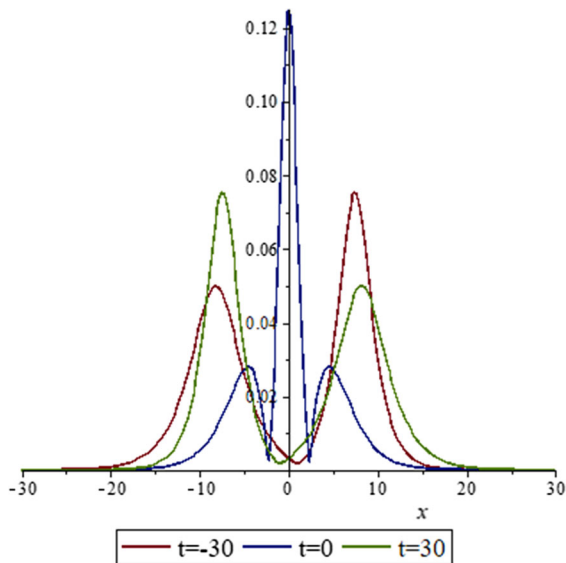


Fig. 6 The soliton along the x -axis with different t

By the Volterra equations, the corresponding analytical properties can be obtained. Then, we define P_1 and P_2 to construct the RH problem. In reflectionless case, making full use of the symmetric relation of the potential matrix and giving the zero point relation of the determinant of two analytic matrix functions in the problem, one can construct the multi-soliton solutions. For the multi-component NLS equations, it is more complicated than standard NLS equations. The multi-soliton solutions of N -dimensional NLS equations via the RH approach have not been well studied before, and therefore we take the factor of multi-component into consideration.

RH approach can be used not only to solve the initial boundary value problem of integrable systems, but also to analyze the solution of long-time behavior, quantum field theory and statistical model, orthogonal polynomial theory, and random matrix theory. In addition, it can be used in plasma physics, ocean engineering, atmospheric sciences, Bose–Einstein condensate, nonlinear optics and so on. The Riemann–Hilbert approach can be applied to many equations of similar types. In future, we can innovate and improve the method and apply it to many more complex equations.

References

- Hilbert, D.: Mathematics problem. *Gott. Nachr.* **3**, 253–297 (1990)
- Faddeev, L.D., Takhtajan, L.A.: *Hamiltonian Methods in the Theory of Solitons*. Springer-Verlag, Berlin (1987)
- Zhou, G.Q., Huang, N.N.: An N -soliton solution to the DNLS equation based on revised inverse scattering transform. *J. Phys. A-Math. Theor.* **40**(45), 13607 (2007)
- Zhaqilao, Z., Yong, C., Li, Z.B.: Darboux transformation and multi-soliton solutions for some soliton equations. *Chaos Soliton Fractal* **41**(2), 661–670 (2009)
- Mei, J.Q., Zhang, H.Q.: Symmetry reductions and explicit solutions of a $(3+1)$ -dimensional PDE. *Appl. Math. Comput.* **211**(2), 347–353 (2009)
- Wazwaz, A.M.: The Hirota's bilinear method and the tanh-coth method for multiple-soliton solutions of the Sawada–Kotera–Kadomtsev–Petviashvili equation. *Appl. Math. Comput.* **200**(1), 160–166 (2008)
- Zen, F.P., Elim, H.I.: Lax pair formulation and multi-soliton solution of the integrable vector nonlinear Schrödinger equation. *Physics* (1999)
- Nimmo, J.J.C., Freeman, N.C.: A method of obtaining the N -soliton solution of the Boussinesq equation in terms of a Wronskian. *Phys. Lett. A* **95**(1), 4–6 (1983)
- Bikbaev, R.F.: Soliton generation for initial-boundary-value problems. *Phys. Rev. Lett.* **68**, 3117–3120 (1992)
- Jenkins, R., Liu, J., Perry, P., Sulem, C.: Soliton resolution for the derivative nonlinear Schrödinger equation. *Commun. Math. Phys.* **363**, 1003–1049 (2018)
- Xu, J., Fan, E.G.: The three-wave equation on the half-line. *Phys. Lett. A* **378**, 26–33 (2014)
- Hu, B.B., Zhang, L., Zhang, N.: On the Riemann–Hilbert problem for the mixed Chen–Lee–Liu derivative nonlinear Schrödinger equation. *J. Comput. Appl. Math.* **390**, 113393 (2021)
- Hu, B.B., Zhang, L., Xia, T.C.: On the Riemann–Hilbert problem of a generalized derivative nonlinear Schrödinger equation. *Commun. Theor. Phys.* **73**, 015002 (2021)
- Monvel, A., Shepelsky, B.D.C.: Riemann–Hilbert approach for the Camassa–Holm equation on the line. *Comptes Rendus Mathématique* **343**, 627–632 (2006)
- Wei, H.Y., Xia, T.C.: Constructing variable coefficient nonlinear integrable coupling super AKNS hierarchy and its self-consistent sources. *Math. Method. Appl. Sci.* **41**, 6883–6894 (2018)
- Hu, B.B., Zhang, L., Xia, T.C., Zhang, N.: On the Riemann–Hilbert problem of the Kundu equation. *Appl. Math. Comput.* **381**, 125262 (2020)
- Wong, R.: Asymptotics of orthogonal polynomials via the Riemann–Hilbert approach. *Acta Math. Sci.* **29**, 1005–1034 (2009)
- Liu, L.C., Tian, B., Qin, B.: Bäcklund transformation, superposition formulae and N -soliton solutions for the perturbed Korteweg–de Vries equation. *Commun. Nonlinear Sci.* **17**, 2394–2402 (2012)
- Zhang, Y., Chen, D.Y.: Bäcklund transformation and soliton solutions for the shallow water waves equation. *Chaos Soliton Fractal* **20**, 343–351 (2004)
- Wang, D.S., Wang, X.L.: Long-time asymptotics and the bright N -soliton solutions of the Kundu–Eckhaus equation via the Riemann–Hilbert approach. *Nonlinear Anal.-Real.* **41**, 334–361 (2018)

21. Geng, X.G., Wang, K.D., Chen, M.M.: Long-time asymptotics for the spin-1 Gross–Pitaevskii equation. *Commun. Math. Phys.* **382**, 1–27 (2021)
22. Cheng, Q.Y., Fan, E.G.: Long-time asymptotics for a mixed nonlinear Schrödinger equation with the Schwartz initial data. *J. Math. Anal. Appl.* **489**, 124188 (2020)
23. Geng, X.G., Wu, J.P.: Riemann–Hilbert approach and N-soliton solutions for a generalized Sasa–Satsuma equation. *Wave Motion* **60**, 62–72 (2016)
24. Ma, W.X.: The inverse scattering transform and soliton solutions of a combined modified Korteweg–de Vries equation. *J. Math. Anal. Appl.* **471**, 796–811 (2019)
25. Chen, X.T., Zhang, Y., Liang, J.L., Wang, R.: The N-soliton solutions for the matrix modified Korteweg–de Vries equation via the Riemann–Hilbert approach. *Eur. Phys. J. Plus* **135**, 574 (2020)
26. Wu, J.P.: Integrability aspects and multi-soliton solutions of a new coupled Gerdjikov–Ivanov derivative nonlinear Schrödinger equation. *Nonlinear Dyn.* **96**, 789–800 (2019)
27. Xu, T., Chen, Y.: Riemann–Hilbert approach of the coupled nonisospectral Gross–Pitaevskii system and its multi-component generalization. *Commun. Nonlinear Sci.* **57**, 276–289 (2017)
28. Li, J., Xia, T.C.: N-soliton solutions for the nonlocal Fokas–Lenells equation via RHP. *Appl. Math. Lett.* **98**, 113 (2020)
29. Ma, W.X.: Riemann–Hilbert problems and N N math-Container loading Mathjax-soliton solutions for a coupled mKdV system. *J. Geom. Phys.* **132**, 45–54 (2018)
30. Wen, L., Zhang, N., Fan, E.G.: N-soliton solution of the Kundu-type equation via Riemann–Hilbert approach. *Acta Math. Sci.* **40**, 113–126 (2020)
31. Wazwaz, A.M.: The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations. *Appl. Math. Comput.* **184**, 1002–1014 (2007)
32. Wazwaz, A.M.: Multiple soliton solutions and multiple singular soliton solutions of the modified KdV equation with first-order correction. *Phys. Scr.* **82**, 055006 (2010)
33. Wazwaz, A.M.: Multiple soliton solutions and multiple complex soliton solutions for two distinct Boussinesq equations. *Nonlinear Dyn.* **85**, 731–737 (2016)
34. Wazwaz, A.M.: Two-mode fifth-order KdV equations: necessary conditions for multiple-soliton solutions to exist. *Nonlinear Dyn.* **87**, 1685–1691 (2017)
35. Kang, Z.Z., Xia, T.C., Ma, W.X.: Riemann–Hilbert approach and N-soliton solution for an eighth-order nonlinear Schrödinger equation in an optical fiber. *Adv. Differ. Equ.* **188**, 1–14 (2019)
36. Yang, J.J., Tian, S.F., Peng, W.Q., Zhang, T.T.: The N-coupled higher-order nonlinear Schrödinger equation: Riemann–Hilbert problem and multi-soliton solutions. *Math. Methods Appl. Sci.* **43**, 1–15 (2019)
37. Li, P.R.: The solvability and explicit solutions of singular integral-differential equations of non-normal type via Riemann–Hilbert problem. *J. Comput. Appl. Math.* **374**, 112759 (2020)
38. Li, J., Xia, T.C.: A Riemann–Hilbert approach to the Kundu–nonlinear Schrödinger equation and its multi-component generalization. *J. Math. Anal. Appl.* **500**, 125109 (2021)
39. Wei, H.Y., Fan, E.G., Guo, H.D.: Riemann–Hilbert approach and nonlinear dynamics of the coupled higher-order nonlinear Schrödinger equation in the birefringent or two-mode fiber. *Nonlinear Dyn.* **43**, 1–12 (2021)
40. Ling, L.M., Zhao, L.C., Guo, B.L.: Darboux transformation and multi-dark soliton for N-component nonlinear Schrödinger equations. *Nonlinearity* **28**, 3243–3261 (2015)

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