Nonlinear Dyn (2021) 105:3849–3858
https://doi.org/10.1007/s11071-021-06673-z

ORIGINAL PAPER

Comment on "Bilinear Bäcklund transformation, soliton and periodic wave solutions for a (3+1)-dimensional variable-coefficient generalized shallow water wave equation" (Nonlinear Dyn. 87, 2529, 2017)

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Received: 5 May 2021 / Accepted: 25 June 2021 / Published online: 23 August 2021

Abstract In this Comment, several enhancements on the results in the paper "Bilinear Bäcklund transformation, soliton and periodic wave solutions for a (3+1)-dimensional variable-coefficient generalized shallow water wave equation" (Nonlinear Dyn. 87, 2529, 2017) are described. With respect to the stream under a pressure surface in the water, for the same equation, using the Hirota method and symbolic computation, we are able to build a set of the bilinear forms, two sets of the bilinear auto-Bäcklund transformations along with some analytic solutions, as well as a set of the similarity reductions. Beyond those in the paper (Nonlinear Dyn. 87, 2529, 2017), our results are dependent on the variable coefficients in the equation, while those coefficients respectively represent the perturbed effects, dispersion and nonlinearity.

Keywords Fluids · (3+1)-dimensional variable-coefficient generalized shallow water wave equation · Bilinear forms · Bilinear auto-Bäcklund transformations with analytic solutions · Similarity reductions · Hirota method · Symbolic computation

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1 Introduction

Fluids have been actively studied [1–42], e.g., recent investigations on the nonlinear dynamics of certain marine inertial particles [1], triadic-interaction energy transfer in the fluid flow [2], nonlinear vibrations of a fluid-filled circular shell [3], shallow water waves [6,9], incompressible fluids [10,30], motion of a rigid plate in a Newtonian fluid [11], heat-conducting fluids [12], capillary fluids [13], fluids confined in the cylindrical and slit pores [14], shallow water in an open sea [15], oceans in the Solar System [16–18] and liquids with the gas bubbles [26]. Nowadays, such models have been proposed to describe the fluids [8–42], as different Navier–Stokes systems [10,12,13], a fractional Bagley–Torvik system [11], an extended Peng-Robinson system [14], a generalized (2+1)-dimensional dispersive long-wave system [15,38], a variable-coefficient nonlinear dispersive-wave system [16,17], a higher-order Boussinesq-Burgers system [18,19], different extended (2+1)-dimensional coupled Burgers systems [20,34], an Ablowitz–Kaup–Newell–Segur system [21–24], a generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt system [25], a (3+1)-dimensional nonlinear wave equation [26], different (3+1)-dimensional generalized Kadomtsev-Petviashvili-type systems [9,27,36,39,41], a (2+1)-dimensional generalized Caudrey–Dodd–Gibbon–Kotera–Sawada equation [8], a modified Korteweg–de Vries–Calogero–Bogoyavlenskii-Schiff equation [28,29], a (2+1)-

dimensional generalized Boiti–Leon–Manna–Pempinelli equation [30], a (2+1)-dimensional reduced Yu–Toda–Sasa–Fukuyama equation [31], a (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko equation [32], a (2+1)-dimensional generalized Hirota-Satsuma-Ito equation [33], a Sharma-Tasso-Olver-Burgers equation [35], a (3+1)-dimensional generalized Yu-Toda-Sasa-Fukuyama equation [37], and a (3+1)-dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmid equation [40,42].

Currently interesting, Ref. [43], i.e., the paper (Nonlinear Dyn. 87, 2529, 2017), has studied the following (3+1)-dimensional variable-coefficient generalized shallow water wave equation for the stream under a pressure surface in the water:

$$\Upsilon_1(t)u_{yt} + \Upsilon_2(t)u_{xxxy} + \Upsilon_3(t)u_x u_{xy} + \Upsilon_3(t)u_y u_{xx} + \Upsilon_4(t)u_{xz} = 0, \tag{1}$$

with $u(x, y, z, t)$ being the real differentiable function of the variables x, y, z and t, the real functions $\Upsilon_1(t)$ and $\Upsilon_4(t)$ denoting the perturbed effects, $\Upsilon_2(t)$ implying the dispersion, $\Upsilon_3(t)$ representing the nonlinearity, while the subscripts meaning the partial derivatives [43,44]. For Eq. (1), Ref. [43] has obtained certain bilinear form, bilinear Bäcklund transformation, Lax pair, soliton and periodic wave solutions.

Also for Eq. (1), Ref. [45] has worked out certain lump-solution characteristics, Ref. [44] has got certain breathers, Ref. [47] has investigated a Kadomtsev–Petviashvili hierarchy reduction, some soliton and semi-rational solutions, while Ref. [50] has found some nonautonomous lump solutions and an interaction between a lump wave and a kink soliton. Special cases of Eq. (1) have been a (3+1)-dimensional generalized shallow water equation [for $\Upsilon_1(t) = \Upsilon_2(t) = 1$, $\Upsilon_3(t) = -3$ and $\Upsilon_4(t) = -1$] applied in the weather simulations, tidal waves, irrigation and tsunami prediction [45–47] (and references therein), and a (3+1)-dimensional Jimbo-Miwa equation [for $\Upsilon_1(t) = 2$, $\Upsilon_2(t) = 1$, $\Upsilon_3(t) = 3$ and $\Upsilon_4(t) = -3$] modelling some wave phenomena [27,47] (and references therein).

This Comment will be to enhance the issues published in Ref. [43], in order to make them more complete. Results in this Comment, to be seen below, will also be different from those in Refs. [44,45,47,50].

In Sect. 2 of this Comment, making use of symbolic computation [48,49], we will construct a set of the bilinear forms for Eq. (1), which is different from and beyond that in Ref. [43], while in Sect. 3, two sets of the bilinear auto-Bäcklund transformations for Eq. (1), different from and beyond that in Ref. [43]. In addition, using symbolic computation, we will find a set of the similarity reductions for Eq. (1) in Sect. 4. Section 5 will be our conclusions.

2 Bilinear forms for Eq. (1)

Making use of the Hirota method [51] and symbolic computation, we assume that

$$u(x, y, z, t) = \gamma\, [\ln f(x, y, z, t)]_x - \phi(y) - \psi(z) - \alpha(y)\beta(z), \tag{2}$$

with $f(x, y, z, t)$ being a real differentiable function of x, y, z and t, $\phi(y)$, $\psi(z)$, $\alpha(y)$ and $\beta(z)$ denoting the real differentiable functions, while γ meaning a real non-zero constant.

Substituting Assumption (2) into Eq. (1) and integrating the outcome once as for x with the integration function vanishing bring about

$$\begin{aligned}&\Upsilon_1(t)\,(\ln f)_{yt} + \Upsilon_2(t)\,(\ln f)_{xxxy} \\ &+\gamma\Upsilon_3(t)\,(\ln f)_{xx}\,(\ln f)_{xy} \\ &-\Upsilon_3(t)\left[\phi'(y) + \alpha'(y)\beta(z)\right](\ln f)_{xx} \\ &+\Upsilon_4(t)\,(\ln f)_{xz} = 0,\end{aligned} \tag{3}$$

in which $\phi'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\,\phi(y)$ and $\alpha'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\,\alpha(y)$.

On account of the formulae [51]

$$\begin{aligned}2\,(\ln f)_{yt} &= \frac{D_y D_t f\cdot f}{f^2},\\ 2\,(\ln f)_{xxxy} &= \frac{D_x^3 D_y f\cdot f}{f^2} - 3\frac{D_x^2 f\cdot f}{f^2}\frac{D_x D_y f\cdot f}{f^2},\\ 2\,(\ln f)_{xx} &= \frac{D_x^2 f\cdot f}{f^2},\\ 2\,(\ln f)_{xy} &= \frac{D_x D_y f\cdot f}{f^2},\\ 2\,(\ln f)_{xz} &= \frac{D_x D_z f\cdot f}{f^2},\end{aligned}$$

for the sake of transforming Eq. (1) into certain bilinear forms, we choose

$$\gamma = 6\frac{\Upsilon_2(t)}{\Upsilon_3(t)},$$

to obtain

$$\Big\{\Upsilon_1(t)D_yD_t+\Upsilon_2(t)D_x^3D_y \\ -\Upsilon_3(t)\left[\phi'(y)+\alpha'(y)\beta(z)\right]D_x^2 \\ +\Upsilon_4(t)D_xD_z\Big\}\, f\cdot f=0, \tag{4}$$

with D_x, D_y, D_z and D_t given by [51]

$$D_x^{m_1}D_y^{m_2}D_z^{m_3}D_t^{m_4}H(x,y,z,t)\cdot G(x,y,z,t) \\ \equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x_0}\right)^{m_1}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y_0}\right)^{m_2} \\ \left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z_0}\right)^{m_3}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t_0}\right)^{m_4} \\ H(x,y,z,t)\,G(x_0,y_0,z_0,t_0)\Big|_{x_0=x,\,y_0=y,\,z_0=z,\,t_0=t},$$

x_0, y_0, z_0 and t_0 meaning the formal variables, $H(x,y,z,t)$ representing a C^∞ function of x, y, z and t, $G(x_0,y_0,z_0,t_0)$ representing a C^∞ function of x_0, y_0, z_0 and t_0, while m_1, m_2, m_3 and m_4 being the non-negative integers [51].

In short, under the variable-coefficient constraint

$$\Upsilon_3(t)=\Upsilon_0\Upsilon_2(t), \tag{5}$$

choosing $\gamma=\frac{6}{\Upsilon_0}$, we find a set of the bilinear forms for Eq. (1), i.e.,

$$\Big\{\Upsilon_1(t)D_yD_t+\Upsilon_2(t)D_x^3D_y \\ -\Upsilon_0\Upsilon_2(t)\left[\phi'(y)+\alpha'(y)\beta(z)\right]D_x^2 \\ +\Upsilon_4(t)D_xD_z\Big\}\, f\cdot f=0, \tag{6}$$

with Υ_0 implying a real non-zero constant.

With respect to the stream under a pressure surface in the water, under Variable-Coefficient Constraint (5) and Assumption (2), Bilinear Forms (6) for Eq. (1) are dependent on the dispersion coefficient $\Upsilon_2(t)$ as well as perturbed-effect coefficients $\Upsilon_1(t)$ and $\Upsilon_4(t)$ in Eq. (1). Bilinear Forms (6) are different from the one presented in Ref. [43].

Bilinear Forms (6) are useful, e.g., for us to build some bilinear auto-Bäcklund transformations, to be seen below.

3 Bilinear auto-Bäcklund transformations with analytic solutions for Eq. (1)

Based on Bilinear Forms (6), with the Hirota method, motivated by Ref. [27], assuming that $g\,(x,y,z,t)$ be another solution of Bilinear Forms (6), we take into account the expression

$$f^2\Big\{\Big\{\Upsilon_1(t)D_yD_t+\Upsilon_2(t)D_x^3D_y \\ -\Upsilon_0\Upsilon_2(t)\left[\phi'(y)+\alpha'(y)\beta(z)\right]D_x^2 \\ +\Upsilon_4(t)D_xD_z\Big\}g\cdot g\Big\} \\ -g^2\Big\{\Big\{\Upsilon_1(t)D_yD_t+\Upsilon_2(t)D_x^3D_y \\ -\Upsilon_0\Upsilon_2(t)\left[\phi'(y)+\alpha'(y)\beta(z)\right]D_x^2 \\ +\Upsilon_4(t)D_xD_z\Big\}f\cdot f\Big\}, \tag{7}$$

and make use of the exchange formulae [51]

$$G^2\left(D_yD_t\,H\cdot H\right)-H^2\left(D_yD_t\,G\cdot G\right) \\ =2\,D_y\left(D_t\,H\cdot G\right)\cdot(GH), \tag{8a}$$

$$G^2\left(D_x^2\,H\cdot H\right)-H^2\left(D_x^2\,G\cdot G\right) \\ =2\,D_x\left(D_x\,H\cdot G\right)\cdot(GH), \tag{8b}$$

$$G^2\left(D_xD_z\,H\cdot H\right)-H^2\left(D_xD_z\,G\cdot G\right) \\ =2\,D_x\left(D_z\,H\cdot G\right)\cdot(GH), \tag{8c}$$

$$G^2\left(D_x^3D_y\,H\cdot H\right)-H^2\left(D_x^3D_y\,G\cdot G\right) \\ =\frac{3}{2}D_x\left(D_x^2D_y\,H\cdot G\right)\cdot(GH) \\ +\frac{1}{2}D_y\left(D_x^3\,H\cdot G\right)\cdot(GH) \\ -3\,D_x\left(D_xD_y\,H\cdot G\right)\cdot\left(D_x\,H\cdot G\right) \\ -\frac{3}{2}D_x\left(D_x^2\,H\cdot G\right)\cdot\left(D_y\,H\cdot G\right) \\ -\frac{3}{2}D_y\left(D_x^2\,H\cdot G\right)\cdot\left(D_x\,H\cdot G\right), \tag{8d}$$

to get

$$f^2\Big\{\Big\{\Upsilon_1(t)D_yD_t+\Upsilon_2(t)D_x^3D_y \\ -\Upsilon_0\Upsilon_2(t)\left[\phi'(y)+\alpha'(y)\beta(z)\right]D_x^2 \\ +\Upsilon_4(t)D_xD_z\Big\}g\cdot g\Big\} \\ -g^2\Big\{\Big\{\Upsilon_1(t)D_yD_t+\Upsilon_2(t)D_x^3D_y \\ -\Upsilon_0\Upsilon_2(t)\left[\phi'(y)+\alpha'(y)\beta(z)\right]D_x^2$$

$$
\begin{aligned}
&+\Upsilon_4(t) D_x D_z \Big\} f \cdot f \Big\} \\
&= \Upsilon_1(t) \Big[f^2 \left(D_y D_t g \cdot g\right) - g^2 \left(D_y D_t f \cdot f\right) \Big] \\
&\quad + \Upsilon_2(t) \Big[f^2 \left(D_x^3 D_y g \cdot g\right) \\
&\quad - g^2 \left(D_x^3 D_y f \cdot f\right) \Big] \\
&\quad - \Upsilon_0 \Upsilon_2(t) \left[\phi'(y) + \alpha'(y)\beta(z)\right] \\
&\quad \Big[f^2 \left(D_x^2 g \cdot g\right) - g^2 \left(D_x^2 f \cdot f\right) \Big] \\
&\quad + \Upsilon_4(t) \Big[f^2 \left(D_x D_z g \cdot g\right) \\
&\quad - g^2 \left(D_x D_z f \cdot f\right) \Big] \\
&= 2\, \Upsilon_1(t)\, D_y \left(D_t\, g \cdot f\right) \cdot (fg) \\
&\quad - 2\, \Upsilon_0 \Upsilon_2(t) \left[\phi'(y) + \alpha'(y)\beta(z)\right] \\
&\quad D_x \left(D_x\, g \cdot f\right) \cdot (fg) \\
&\quad + \Upsilon_2(t) \Big[\frac{3}{2}\, D_x \left(D_x^2 D_y\, g \cdot f\right) \cdot (fg) \\
&\quad + \frac{1}{2}\, D_y \left(D_x^3\, g \cdot f\right) \cdot (fg) \\
&\quad - \frac{3}{2}\, D_y \left(D_x^2\, g \cdot f\right) \cdot \left(D_x\, g \cdot f\right) \\
&\quad - 3\, D_x \left(D_x D_y\, g \cdot f\right) \cdot \left(D_x\, g \cdot f\right) \\
&\quad - \frac{3}{2}\, D_x \left(D_x^2\, g \cdot f\right) \cdot \left(D_y\, g \cdot f\right) \Big] \\
&\quad + 2\, \Upsilon_4(t)\, D_x \left(D_z\, g \cdot f\right) \cdot (fg) \\
&= \frac{1}{2}\, D_x \Big\{ \Big\{ 3\, \Upsilon_2(t) D_x^2 D_y - 4\, \Upsilon_0 \Upsilon_2(t) \\
&\quad \left[\phi'(y) + \alpha'(y)\beta(z)\right] D_x + 4\, \Upsilon_4(t) D_z \Big\}\, g \cdot f \Big\} \cdot (fg) \\
&\quad + \frac{1}{2}\, D_y \Big\{ \Big[4\, \Upsilon_1(t) D_t + \Upsilon_2(t) D_x^3 \Big]\, g \cdot f \Big\} \cdot (fg) \\
&\quad - 3\, \Upsilon_2(t) D_x \left(D_x D_y\, g \cdot f\right) \cdot \left(D_x\, g \cdot f\right) \\
&\quad - \frac{3}{2}\, \Upsilon_2(t) D_x \left(D_x^2\, g \cdot f\right) \cdot \left(D_y\, g \cdot f\right) \\
&\quad - \frac{3}{2}\, \Upsilon_2(t) D_y \left(D_x^2\, g \cdot f\right) \cdot \left(D_x\, g \cdot f\right) \quad . \qquad (9)
\end{aligned}
$$

Making the assumptions that

$$
\begin{aligned}
&D_x \Big\{ \Big\{ 3\, \Upsilon_2(t) D_x^2 D_y - 4\, \Upsilon_0 \Upsilon_2(t) \\
&\left[\phi'(y) + \alpha'(y)\beta(z)\right] D_x \\
&+ 4\, \Upsilon_4(t) D_z \Big\}\, g \cdot f \Big\} \cdot (fg) = 0, \qquad (10a)
\end{aligned}
$$

$$D_y \Big\{ \Big[4\, \Upsilon_1(t) D_t + \Upsilon_2(t) D_x^3 \Big]\, g \cdot f \Big\} \cdot (fg) = 0, \qquad (10b)$$

$$D_x \left(D_x D_y\, g \cdot f\right) \cdot \left(D_x\, g \cdot f\right) = 0, \qquad (10c)$$

$$D_x \left(D_x^2\, g \cdot f\right) \cdot \left(D_y\, g \cdot f\right) = 0, \qquad (10d)$$

$$D_y \left(D_x^2\, g \cdot f\right) \cdot \left(D_x\, g \cdot f\right) = 0, \qquad (10e)$$

we can obtain two sets of the bilinear auto-Bäcklund transformations for Eq. (1) in the following:

Set 1: $D_x^2\, g \cdot f \neq 0$ and $D_x D_y\, g \cdot f \neq 0$

We work out the following set of the bilinear auto-Bäcklund transformations for Eq. (1):

$$
\begin{aligned}
u(x, y, z, t) &= \frac{6}{\Upsilon_0} \left[\ln f(x, y, z, t)\right]_x \\
&\quad - \phi(y) - \psi(z) - \alpha(y)\beta(z), \qquad (11a) \\
v(x, y, z, t) &= \frac{6}{\Upsilon_0} \left[\ln g(x, y, z, t)\right]_x \\
&\quad - \phi(y) - \psi(z) - \alpha(y)\beta(z), \qquad (11b)
\end{aligned}
$$

$$
\begin{aligned}
&\Big\{ 3\, \Upsilon_2(t) D_x^2 D_y - 4\, \Upsilon_0 \Upsilon_2(t) \left[\phi'(y) + \alpha'(y)\beta(z)\right] D_x \\
&\quad + 4\, \Upsilon_4(t) D_z \Big\}\, g \cdot f = 0, \qquad (11c)
\end{aligned}
$$

$$\Big[4\, \Upsilon_1(t) D_t + \Upsilon_2(t) D_x^3 \Big]\, g \cdot f = 0, \qquad (11d)$$

$$D_x D_y\, g \cdot f = \lambda_1(t)\, D_x\, g \cdot f, \qquad (11e)$$

$$D_x^2\, g \cdot f = \lambda_2(t)\, D_y\, g \cdot f, \qquad (11f)$$

$$D_x^2\, g \cdot f = \lambda_3(t)\, D_x\, g \cdot f, \qquad (11g)$$

in which $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$ denote the real non-zero differentiable functions of t, while $v\ (x, y, z, t)$ represents another solution of Eq. (1).

With respect to the stream under a pressure surface in the water, under Variable-Coefficient Constraint (5), mutually consistent (as seen right below), Bilinear Auto-Bäcklund Transformations (11) for Eq. (1) are dependent on the dispersion coefficient $\Upsilon_2(t)$ as well as perturbed-effect coefficients $\Upsilon_1(t)$ and $\Upsilon_4(t)$ in Eq. (1). Bilinear Auto-Bäcklund Transformations (11) are different from the one given in Ref. [43].

Bilinear Auto-Bäcklund Transformations (11) denote a system of the equations which connects a set of the solutions of Eq. (1) to another set of the solutions of Eq. (1) itself. Hence, we might, in principle at least, be capable of progressively finding more and more complicated solutions of Eq. (1).

Next, for the mutual consistency, or explicit solvability in relation to f and g, we will construct some analytic solutions for Eq. (1) via Bilinear Auto-Bäcklund Transformations (11).

Under the variable-coefficient constraint

$$\Upsilon_4(t) = \mu_0\,\Upsilon_2(t), \tag{12}$$

with the choices of

$$\begin{aligned}
&\lambda_1(t) = \sigma_2\,\Upsilon_2(t),\\
&\lambda_2(t) = \frac{\sigma_1^2}{\sigma_2}\,\Upsilon_2(t),\\
&\lambda_3(t) = \sigma_1\,\Upsilon_2(t),\\
&\phi(y) = \zeta_1\,y + \zeta_2,\\
&\alpha(y) = \zeta_3\,y + \zeta_4,\\
&\beta(z) = \beta \quad \text{only},
\end{aligned} \tag{13}$$

we construct out the following analytic solutions for Eq. (1) via Bilinear Auto-Bäcklund Transformations (11):

$$u(x,y,z,t) = \frac{6}{\Upsilon_0}\,[\ln f(x,y,z,t)]_x - (\zeta_1\,y + \zeta_2) - \psi(z) - \beta\,(\zeta_3\,y + \zeta_4)\,, \tag{14a}$$

$$v(x,y,z,t) = \frac{6}{\Upsilon_0}\,[\ln g(x,y,z,t)]_x - (\zeta_1\,y + \zeta_2) - \psi(z) - \beta\,(\zeta_3\,y + \zeta_4)\,, \tag{14b}$$

$$f(x,y,z,t) = 1, \tag{14c}$$

$$g(x,y,z,t) = 1 + \epsilon_1 \exp\left\{\sigma_1\,x + \sigma_2\,y - \frac{\sigma_1\,[3\,\sigma_1\,\sigma_2 + 4\,\Upsilon_0\,(\zeta_1 + \beta\zeta_3)]}{4\,\mu_0} z - \frac{\sigma_1^3}{4}\int \frac{\Upsilon_2(t)}{\Upsilon_1(t)}\,\mathrm{d}t\right\}, \tag{14d}$$

with μ_0, σ_1, σ_2 and ϵ_1 as the real non-zero constants, while ζ_1, ζ_2, ζ_3 and ζ_4 as the real constants.

With respect to the stream under a pressure surface in the water, under Variable-Coefficient Constraints (5) and (12), Analytic Solutions (14) are dependent on the dispersion coefficient $\Upsilon_2(t)$ as well as perturbed-effect coefficient $\Upsilon_1(t)$ in Eq. (1).

Set 2: $D_x^2\,g \cdot f = 0$ and $D_x D_y\,g \cdot f = 0$

Similarly, we find the second set of the bilinear auto-Bäcklund transformations for Eq. (1):

$$u(x,y,z,t) = \frac{6}{\Upsilon_0}\,[\ln f(x,y,z,t)]_x - \phi(y) - \psi(z) - \alpha(y)\beta(z), \tag{15a}$$

$$v(x,y,z,t) = \frac{6}{\Upsilon_0}\,[\ln g(x,y,z,t)]_x - \phi(y) - \psi(z) - \alpha(y)\beta(z), \tag{15b}$$

$$\left\{3\,\Upsilon_2(t) D_x^2 D_y - 4\,\Upsilon_0\Upsilon_2(t)\left[\phi'(y) + \alpha'(y)\beta(z)\right] D_x + 4\,\Upsilon_4(t) D_z\right\} g \cdot f = 0, \tag{15c}$$

$$\left[4\,\Upsilon_1(t) D_t + \Upsilon_2(t) D_x^3\right] g \cdot f = 0, \tag{15d}$$

$$D_x D_y\,g \cdot f = 0, \tag{15e}$$

$$D_x^2\,g \cdot f = 0\ . \tag{15f}$$

With respect to the stream under a pressure surface in the water, under Variable-Coefficient Constraint (5), Bilinear Auto-Bäcklund Transformations (15) rely on the dispersion coefficient $\Upsilon_2(t)$ as well as perturbed-effect coefficients $\Upsilon_1(t)$ and $\Upsilon_4(t)$ in Eq. (1). Bilinear Auto-Bäcklund Transformations (15) are different from the one reported in Ref. [43].

Bilinear Auto-Bäcklund Transformations (15) denote a system of the equations which connects a set of the solutions of Eq. (1) to another set of the solutions of Eq. (1) itself. Hence, we might, in principle at least, be capable of progressively finding more and more complicated solutions of Eq. (1).

Next, for the mutual consistency, or explicit solvability in relation to f and g, we will construct some analytic solutions for Eq. (1) via Bilinear Auto-Bäcklund Transformations (15).

Via Bilinear Auto-Bäcklund Transformations (15), under Variable-Coefficient Constraints (5) and (12), with the choices of

$$\begin{aligned}
&\phi(y) = \zeta_1\,y + \zeta_2,\\
&\alpha(y) = \zeta_3\,y + \zeta_4,\\
&\beta(z) = \beta \quad \text{only},
\end{aligned} \tag{16}$$

we also construct out the following analytic solutions for Eq. (1):

$$u(x,y,z,t) = \frac{6}{\Upsilon_0}\,[\ln f(x,y,z,t)]_x$$

$$- (\zeta_1 \, y + \zeta_2) - \psi(z) - \beta \, (\zeta_3 \, y + \zeta_4) \, , \tag{17a}$$

$$v(x, y, z, t) = \frac{6}{\Upsilon_0} \left[\ln g(x, y, z, t)\right]_x - (\zeta_1 \, y + \zeta_2) - \psi(z) - \beta \, (\zeta_3 \, y + \zeta_4) \, , \tag{17b}$$

$$f(x, y, z, t) = 1 - \epsilon_2 \exp\left\{\sigma_1 \, x + \sigma_2 \, y - \frac{\sigma_1 \left[3 \, \sigma_1 \, \sigma_2 + 4 \, \Upsilon_0 \, (\zeta_1 + \beta \zeta_3)\right]}{4 \, \mu_0} z - \frac{\sigma_1^3}{4} \int \frac{\Upsilon_2(t)}{\Upsilon_1(t)} \, \mathrm{d}t \right\}, \tag{17c}$$

$$g(x, y, z, t) = 1 + \epsilon_2 \exp\left\{\sigma_1 \, x + \sigma_2 \, y - \frac{\sigma_1 \left[3 \, \sigma_1 \, \sigma_2 + 4 \, \Upsilon_0 \, (\zeta_1 + \beta \zeta_3)\right]}{4 \, \mu_0} z - \frac{\sigma_1^3}{4} \int \frac{\Upsilon_2(t)}{\Upsilon_1(t)} \, \mathrm{d}t \right\}, \tag{17d}$$

with ϵ_2 as the real non-zero constant.

With respect to the stream under a pressure surface in the water, under Variable-Coefficient Constraints (5) and (12), Analytic Solutions (17) are dependent on the dispersion coefficient $\Upsilon_2(t)$ as well as perturbed-effect coefficient $\Upsilon_1(t)$ in Eq. (1).

4 Similarity reductions for Eq. (1)

To start with, similar to those in Refs. [52–55], the form we assume, i.e.,

$$u(x, y, z, t) = \theta(x, y, z, t) + \kappa(x, y, z, t) p[r(x, y, z, t)], \tag{18}$$

could help us seek certain similarity reductions for Eq. (1), in which $\theta(x, y, z, t)$, $\kappa(x, y, z, t) \neq 0$ and $r(x, y, z, t) \neq 0$ represent the real to-be-determined differentiable functions of x, y, z and t, while $p(r)$ implies a real differentiable function as for r.

Taking into account $r(x, y, z, t) = r(t)$ only, using symbolic computation[1] and substituting Assumption (18) into Eq. (1), we obtain

$$\Upsilon_1(t)\kappa_y r'(t) p' + \Upsilon_3(t) \left(\kappa_x \kappa_{xy} + \kappa_y \kappa_{xx}\right) p^2 + \left[\Upsilon_1(t)\kappa_{yt} + \Upsilon_2(t)\kappa_{xxxy} + \Upsilon_4(t)\kappa_{xz} + \Upsilon_3(t) \left(\kappa_x \theta_{xy} + \kappa_{xy}\theta_x + \kappa_y \theta_{xx} + \kappa_{xx}\theta_y\right)\right] p + \left[\Upsilon_1(t)\theta_{yt} + \Upsilon_2(t)\theta_{xxxy} + \Upsilon_4(t)\theta_{xz} + \Upsilon_3(t) \left(\theta_x \theta_{xy} + \theta_y \theta_{xx}\right)\right] = 0, \tag{19}$$

where $r'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \, r(t)$ and $p' = \frac{\mathrm{d}}{\mathrm{d}r} \, p(r)$.

To represent a real ordinary differential equation (ODE), Eq. (19), for which we require that the ratios of the coefficients of different derivatives and powers of $p(r)$ denote some functions with respect to r only, turns into

$$\Upsilon_1(t)\kappa_y r'(t)\Omega_1(r) = \Upsilon_3(t) \left(\kappa_x \kappa_{xy} + \kappa_y \kappa_{xx}\right), \tag{20a}$$

$$\Upsilon_1(t)\kappa_y r'(t)\Omega_2(r) = \Upsilon_1(t)\kappa_{yt} + \Upsilon_2(t)\kappa_{xxxy} + \Upsilon_4(t)\kappa_{xz} + \Upsilon_3(t) \left(\kappa_x \theta_{xy} + \kappa_{xy}\theta_x + \kappa_y \theta_{xx} + \kappa_{xx}\theta_y\right), \tag{20b}$$

$$\Upsilon_1(t)\kappa_y r'(t)\Omega_3(r) = \Upsilon_1(t)\theta_{yt} + \Upsilon_2(t)\theta_{xxxy} + \Upsilon_4(t)\theta_{xz} + \Upsilon_3(t) \left(\theta_x \theta_{xy} + \theta_y \theta_{xx}\right), \tag{20c}$$

with $\Omega_1(r)$, $\Omega_2(r)$ and $\Omega_3(r)$ as three real to-be-determined differentiable functions of r.

Seeing that the second freedom in Remark 2 in Ref. [52] helps us simplify Eq. (20a) into

$$\kappa(x, y, z, t) = \frac{1}{2}\phi_1^2 x^2 + \phi_2 x + \phi_3(y) + \phi_4(z) + \phi_5(t), \tag{21a}$$

$$r(t) = \phi_1 \int \frac{\Upsilon_3(t)}{\Upsilon_1(t)} \mathrm{d}t, \qquad \Omega_1(r) = 0, \tag{21b}$$

and that based on the first freedom in Remark 2 in Ref. [52], Eq. (20c) leads to

$$\theta(x, y, z, t) = \theta_1 x + \theta_2(z) + \theta_3(t), \quad \Omega_3(r) = 0, \tag{22}$$

we can transform Eq. (20b) into

$$\Omega_2(r) = 0, \tag{23}$$

in which $\phi_1 \neq 0$, ϕ_2 and θ_1 denote the real constants, while $\phi_3(y)$, $\phi_4(z)$, $\phi_5(t)$, $\theta_2(z)$ and $\theta_3(t)$ imply the real differentiable functions.

In short, making use of symbolic computation, we end up with a set of the similarity reductions for Eq. (1), i.e.,

$$u(x, y, z, t) = \theta_1 x + \theta_2(z) + \theta_3(t) + \left[\frac{1}{2}\phi_1^2 x^2 + \phi_2 x + \phi_3(y) + \phi_4(z) + \phi_5(t)\right] p[r(t)], \tag{24a}$$

[1] More on the symbolic computation can be seen, e.g., in Refs. [56–62].

$$r(t) = \phi_1 \int \frac{\Upsilon_3(t)}{\Upsilon_1(t)} \mathrm{d}t, \tag{24b}$$

$$p' + p^2 = 0, \tag{24c}$$

in which ODE (24c) represents a known ODE, the properties of which can be found, e.g., in Ref. [63], and some non-trivial solutions of which can be written as

$$p(r) = \frac{1}{r + \eta}, \tag{25}$$

with η as a real constant.

With respect to the stream under a pressure surface in the water, Similarity Reductions (24) are dependent on the perturbed-effect coefficient $\Upsilon_1(t)$ as well as nonlinearity coefficient $\Upsilon_3(t)$ in Eq. (1).

What we can see is that Similarity Reductions (24) transform Eq. (1) into a known ODE, i.e. ODE (24c).

5 Conclusions

Currently interesting, Ref. [43], i.e., the paper (Nonlinear Dyn. 87, 2529, 2017), has investigated Eq. (1), a (3+1)-dimensional variable-coefficient generalized shallow water wave equation.

In this Comment, with respect to the stream under a pressure surface in the water, several enhancements on Ref. [43] for Eq. (1) have been described, with the aid of the Hirota method and symbolic computation, as follows:

- Bilinear Forms (6), under Variable-Coefficient Constraint (5), via Assumption (2);
- Bilinear Auto-Bäcklund Transformations (11), under Variable-Coefficient Constraint (5), with Analytic Solutions (14), under Variable-Coefficient Constraints (5) and (12);
- Bilinear Auto-Bäcklund Transformations (15), under Variable-Coefficient Constraint (5), with Analytic Solutions (17), under Variable-Coefficient Constraints (5) and (12);
- Similarity Reductions (24), to a known ODE, i.e., ODE (24c).

We have known that (A) Bilinear Forms (6) are useful for us to build some bilinear auto-Bäcklund transformations, (B) each of Bilinear Auto-Bäcklund Transformations (11) and Bilinear Auto-Bäcklund Transformations (15), denoting a system of the equations which connects a set of the solutions of Eq. (1) to another set of the solutions of Eq. (1) itself, might lead to more and more complicated solutions of Eq. (1), and (C) Similarity Reductions (24) transform Eq. (1) into a known ODE, i.e. ODE (24c).

Beyond those in Ref. [43], our results have been shown to be dependent on the variable coefficients in Eq. (1), while those coefficients have respectively represented the perturbed effects, dispersion and nonlinearity.

Acknowledgements We express our sincere thanks to the Editors and Reviewers for their valuable comments. This work has been supported by the National Nature Science Foundation of China under Grant No. 11871116 and Fundamental Research Funds for the Central Universities of China under Grant No. 2019XD-A11. XYG also thanks the National Scholarship for Doctoral Students of China and BUPT Innovation and Entrepreneurship Support Program, Beijing University of Posts and Telecommunications.

Data availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

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