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Finite-time stabilization via output feedback for high-order planar systems subjected to an asymmetric output constraint

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Abstract This article addresses the problem of finite-time stabilization via output feedback for high-order planar systems subjected to an asymmetric output constraint. By delicately exploring the features of nonlinearities and utilizing skillful manipulations of signum functions, a new *fraction*-type asymmetric barrier Lyapunov function and a distinctive non-smooth state observer are developed. On the basis of the proposed barrier Lyapunov function along with the state observer, the celebrated adding a power integrator technique is elegantly renovated to develop a novel approach by which a continuous output feedback finite-time stabilizer is constructed in a systematic fashion while ensuring the fulfillment of a pre-specified asymmetric output constraint. The presented scheme is a unification approach able to achieve the finite-time stabilization via output feedback for systems subjected to or free from output constraints simultaneously, without needing to change both the controller and observer structures. A numerical is provided to illustrate the superiority of the developed method.

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1 Introduction

Without doubt, the asymptotic or finite-time stabilization for high-order nonlinear systems has been extensively known as a challenging problem in the nonlinear control field, primarily due to the system inherent nonlinearities as well as the lack of controllability/observability or the non-existence of the Jacobian linearization around the origin [1,2]. A technological breakthrough was made by Qian and Lin in [1,3] where the celebrated strategy named adding a power integrator technique was developed to derive solutions to the stabilization issue for high-order nonlinear systems. The underlying philosophy behind the adding a power integrator technique is the mechanism of feedback domination, which not only contributes to a distinctive perspective in dealing with the deep-seated obstacles stemming from system inherent nonlinearities but also potentially provides insights, for output feedback design, into constructing a state observer without relying on the separation principle, thereby stimulating considerable elegant results dedicated to the stabilization problem of high-order nonlinear systems; see, e.g., [4–17] and the references therein.

When a more ambitious objective, namely the stabilization subjected to some pre-specified output constraints, is pursued further for safety reasons and/or performance specifications [18–24], in the literature relatively less progress has been made toward high-order nonlinear systems [25–30], where [28–30] are particularly concerned with finite-time stabilization. The major remedy proposed in [28–30] for coping with the finite-time stabilization problem of high-order nonlinear systems subjected to output constraints is to introduce/employ a proper barrier Lyapunov function (BLF) (for the definition, see [30,31]) together with the decisive requirement of full-state availability. When full-state measurements are unavailable, the schemes in [28–30] are no longer applicable, and a new method using output feedback design surely deserves further investigation/development. Notably, tackling this issue is a nontrivial and challenging task essentially due to the deficiency of constructive/explicit designs of BLFs and state observers in handling competently the output feedback finite-time stabilization subjected to asymmetric output constraints. As a matter of fact, even for a two-dimensional case, the problem of how to synthesize an output feedback finite-time stabilizer for high-order nonlinear systems subjected to asymmetric output constraints remains unknown and largely open in the literature.

Motivated by the results reviewed above, in this paper we focus our attention on the problem of finite-time stabilization via output feedback for high-order planar systems subjected to an asymmetric output constraint described by

$$\begin{aligned} \dot{x}_1 &= x_2^p + \phi_1(x_1, t) \\ \dot{x}_2 &= \theta(x, t)u + \phi_2(x, t) \\ y &= x_1 \end{aligned} \tag{1}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ denote the system state, control input, and output, respectively, and $p \in \mathbb{R}^+_{odd} := \{r \in \mathbb{R} \mid r = r_1/r_2$ with r_1 and r_2 being positive odd integers$\}$. For $i = 1, 2$, the nonlinearity $\phi_i : \mathbb{R}^i \times \mathbb{R}^+$ and unknown parameter $\theta : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ are continuous. The initial state is represented by $x(t_0) \in \mathbb{R}^2$ with $t_0 \in \mathbb{R}^+$ being the initial time, and the asymmetric output constraint is formulated as $-\kappa_l < y(t) < \kappa_u$ for all $t \geq t_0$ where κ_l and κ_u are pre-specified positive real constants. Because appropriate conditions on system uncertainties and nonlinearities are essentially necessary for output feedback stabilization [4,9,14], we impose the assumptions below on system (1).

Assumption 1 The unknown parameter $\theta(x, t)$ is uniformly bounded in the sense that there exist two positive smooth functions $\underline{\theta} : \mathbb{R} \to (0, \infty)$ and $\overline{\theta} : \mathbb{R} \to (0, \infty)$ such that

$$\underline{\theta}(x_1) \leq \theta(x, t) \leq \overline{\theta}(x_1)$$

for all $(x, t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Assumption 2 The nonlinearities $\phi_1(x_1, t)$ and $\phi_2(x, t)$ satisfy locally homogeneous growth conditions; i.e., there exist a negative real constant σ and two nonnegative smooth functions $\overline{\phi}_i : \mathbb{R} \to \mathbb{R}^+$ for $i = 1, 2$ such that

$$\begin{aligned} |\phi_1(x_1, t)| &\leq \overline{\phi}_1(x_1)|x_1|^{\frac{m_1+\sigma}{m_1}} \\ |\phi_2(x, t)| &\leq \overline{\phi}_2(x_1)\left(|x_1|^{\frac{m_2+\sigma}{m_1}} + |x_2|^{\frac{m_2+\sigma}{m_2}}\right) \end{aligned}$$

for all $(x, t) \in \mathbb{R}^2 \times \mathbb{R}^+$, where $m_1 = 1$, $m_2 = (m_1 + \sigma)/p > 0$ and $m_2 + \sigma > 0$.

Based on the above assumptions, a new *fraction*-type asymmetric BLF acting a delicate treatment of asymmetric constraints is first designed by exploiting and using the intrinsic characteristics of the system nonlinearities $\phi_i(\cdot)$'s. Next, a continuous state feedback controller working as an intermediate design is synthesized by elaborately renovating the adding a power integrator technique through an artful implantation of the developed *fraction*-type asymmetric BLF together with exquisite manipulations of signum functions. Furthermore, a one-dimensional non-smooth state observer furnished with a state-dependent gain is organized by carefully pondering the nonlinearities inherent in system (1). By skillfully integrating the observer with the state feedback controller and befittingly setting the observer gain, a continuous output feedback finite-time stabilizer can be explicitly constructed for system (1), thereby fulfilling the finite-time stabilization task and meanwhile preventing the violation of the constraint on the output.

As the first work successfully achieving the finite-time stabilization via output feedback for high-order planar nonlinear systems subjected to asymmetric output constraints, this paper offers the following appealing novelties/innovations. First, the proposed *fraction*-type asymmetric BLF is significantly distinguished from the common *tangent*-type [19,20,22,23] and *logarithm*-type [21,28,29,31] BLFs in the aspect that the construction of the *fraction*-type asymmetric BLF in this paper fully takes into account and elegantly subsumes the intrinsic characteristics of the system nonlinearities $\phi_i(\cdot)$'s, thus offering the feasibility/utility of the designed *fraction*-type BLF to system (1) and some remarkable features (also, see Remark 4). Second, the presented scheme is a unification methodology by which one is able to achieve simultaneously the finite-time stabilization by output feedback for system (1) subjected to or free from asymmetric output constraints, without needing to change the controller and observer structures. That is to say, when the output constraint is purposely assigned to be infinity (i.e., considering the scenario of no constraint), the proposed strategy will directly evolve into the one applicable to dealing with the *pure* task of output feedback stabilization for system (1) without output constraints (also, refer to Remark 5).

Notation: The notations utilized in this paper are listed below. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ represents the n-dimensional Euclidean space, $\mathbb{R}^+$ is the set of nonnegative real numbers, and $\mathbb{R}^+_{odd} = \{r \in \mathbb{R} \,|\, r = r_1/r_2 \text{ with } r_1 \text{ and } r_2 \text{ being positive odd integers}\}$. Suppose that c_1 is a nonnegative real constant, c_i for $i = 2, 3$ are two positive real constants, and $\mathbb{U} \subset \mathbb{R}^n$ is an open connected set (i.e., a domain); we let $\lceil z \rfloor^{c_1} = |z|^{c_1} sign(z)$ for all $z \in \mathbb{R}$ with $\lceil z \rfloor^0 = 1$ if $z = 0$, where $sign(\cdot)$ being the standard signum function, $\mathbb{M}_i(c_2, c_3) = \{(z_1, \ldots, z_i)^T \in \mathbb{R}^i \mid -c_2 < z_1 < c_3\} \subset \mathbb{R}^i$ for $i = 1, \ldots, n$, and $\partial\mathbb{U}$ be the boundary of $\mathbb{U}$.

2 Preliminaries and technical lemmas

Because the finite-time convergence can be secured only by non-Lipschitz continuous systems [13,32], we first recall the definition and relevant lemma of a BLF in regard to a continuous time-varying nonlinear system.

Definition 1 ([25]) Consider a time-varying nonlinear system described by

$$\dot{\eta} = f(\eta, t), \quad \eta(t_0) \in \mathbb{R}^n, \quad t_0 \in \mathbb{R}^+ \tag{2}$$

where $\eta = (\eta_1, \ldots, \eta_n)^T \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ is continuous. Suppose that $\mathbb{U} \subset \mathbb{R}^n$ satisfying $0 \in \mathbb{U}$ is open connected (i.e., $\mathbb{U} \subset \mathbb{R}^n$ is a domain), and $W : \mathbb{U} \to \mathbb{R}$ is positive definite and continuously differentiable. If $W(\eta)$ satisfies the two conditions:

(i) $W(\eta) \to \infty$ as $\eta \to \partial\mathbb{U}$
(ii) $W(\eta(t)) \le l$ for all $t \ge t_0$ with $l \in \mathbb{R}^+$ and for every complete solution[1] $\eta(t)$ of system (2) starting from the initial state $\eta(t_0) \in \mathbb{U}$

then $W(\eta)$ is called a BLF of system (2).

Remark 1 Definition 1 is an extension of the one in [21,31]. In fact, the definition of a BLF presented in [21,31] is with respect to time-invariant (autonomous) systems. In accordance with similar notions, Definition 1 is established here so as to provide a comprehensive/self-contained definition of a BLF when considering a general time-varying (non-autonomous) nonlinear system (2) which is continuous and not necessary to satisfy the Lipschitz condition.

Remark 2 Suppose that in the design process later there exists a continuously differentiable and positive definite function $W : \mathbb{D} \subset \mathbb{R}^i \to \mathbb{R}$ with $i < n$ and $\mathbb{D}$ being an open connected set, which exactly fulfills (i) of Definition 1 but depends only on partial states of system (1). Then, according to Definition 1, such a function $W(\cdot)$ will be referred to, with an abuse of terminology, as a BLF of system (1).

Lemma 1 *([25]) Consider system (2) and two positive real constants κ_l and κ_u. If there exist two continuously differentiable functions $W_1 : \mathbb{M}_1(\kappa_l, \kappa_u) \to \mathbb{R}$ and $W_2 : \mathbb{M}_n(\kappa_l, \kappa_u) \to \mathbb{R}$, where $W_1(\eta_1)$ is positive definite and $W_2(\eta)$ is nonnegative, such that*

(i) $W_1(\eta_1) \to \infty$ as $\eta_1 \to \partial\mathbb{M}_1(\kappa_l, \kappa_u)$
(ii) When setting $W(\eta) = W_1(\eta_1) + W_2(\eta)$, there hold $W(\eta) \to \infty$ as $\|(\eta_2, \ldots, \eta_n)\| \to \infty$ with any fixed $\eta_1 \in \mathbb{M}_1(\kappa_l, \kappa_u)$, and

$$\frac{\partial W(\eta)}{\partial \eta} f(\eta, t) \le 0$$

for all $(\eta, t) \in \mathbb{M}_n(\kappa_l, \kappa_u) \times \mathbb{R}^+$

then every solution[2] $\eta(t)$ of system (2) starting from $\eta(t_0) \in \mathbb{M}_n(\kappa_l, \kappa_u)$ is defined on $[t_0, \infty)$ and fulfills $\eta(t) \in \mathbb{M}_n(\kappa_l, \kappa_u)$ for all $t \ge t_0$.

[1] That is, $\eta(t)$ is a maximal solution (for the definition, see [33]) defined on $[t_0, \infty)$.

[2] For any $\eta(t_0) \in \mathbb{M}_n(\kappa_l, \kappa_u)$, the solution of system (2) is in general not unique because $f(\eta, t)$ is only continuous [33].

We next introduce five lemmas, which play a crucial role in deriving the main results. The proof of the first three lemmas can be found in [2,5,17], whereas the last two lemmas are new and proved correspondingly.

Lemma 2 *([2]) Let $c_1 \geq 0$ and $c_i > 0$ for $i = 2, 3, 4$ be real constants. For any $z_1, z_2 \in \mathbb{R}$, there holds*

$$\begin{aligned} c_1|z_1|^{c_3}|z_2|^{c_4} &\leq c_2|z_1|^{c_3+c_4} + \frac{c_4}{(c_3+c_4)}\left(\frac{c_3}{c_3+c_4}\right)^{\frac{c_3}{c_4}} \\ &\quad\times c_2^{-\frac{c_3}{c_4}} c_1^{\frac{(c_3+c_4)}{c_4}} |z_2|^{c_3+c_4}. \end{aligned}$$

Lemma 3 *([5,17]) Let $c > 0$ be a real constant. For any $z_i \in \mathbb{R}$ with $i = 1, \dots, n$, there holds*

$$(|z_1| + \cdots + |z_n|)^c \leq \alpha \left(|z_1|^c + \cdots + |z_n|^c\right)$$

where $\alpha = n^{c-1}$ if $c \geq 1$ and $\alpha = 1$ if $c < 1$.

Lemma 4 *([17]) Let $c_1 \geq 1$ and $c_2 > 0$ be real constants. For any $z_1, z_2 \in \mathbb{R}$, there holds*

$$\left|\lceil z_1\rceil^{\frac{c_2}{c_1}} - \lceil z_2\rceil^{\frac{c_2}{c_1}}\right| \leq 2^{1-\frac{1}{c_1}} \left|\lceil z_1\rceil^{c_2} - \lceil z_2\rceil^{c_2}\right|^{\frac{1}{c_1}}.$$

Lemma 5 *Let $c_1 \in (0, 1)$ and $c_2 \in [0, 1]$. For any $z \in \mathbb{R}$, there holds*

$$\lceil z\rceil^{c_1} + \lceil 1-z\rceil^{c_1} + c_2^2|z|^{1+c_1} \geq (2^{c_1}-1)c_2^{1-c_1}.$$

Proof Let $\varrho_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, 3$ be three functions respectively defined as

$$\begin{aligned} \varrho_1(z) &= \lceil z\rceil^{c_1} + \lceil 1-z\rceil^{c_1} + c_2^2|z|^{1+c_1} \\ \varrho_2(z) &= \varrho_1(z) - c_2^2|z|^{1+c_1} \\ \varrho_3(z) &= \left(\varrho_2(z) - \lceil z\rceil^{c_1}\right)|z|^{1-c_1} + z. \end{aligned}$$

Because $\varrho_2(z)$ is differentiable on $(-1, 0) \cup (0, 1)$ with

$$\frac{d\varrho_2(z)}{dz} = c_1|z|^{c_1-1} - c_1|1-z|^{c_1-1}$$

for all $z \in (-1, 0) \cup (0, 1)$, using the mean value theorem and the relation $\varrho_1(z) \geq \varrho_2(z)$ for all $z \in \mathbb{R}$, one has

$$\varrho_1(z) \geq \min\{\varrho_2(-1), \varrho_2(1)\} \geq \left(2^{c_1}-1\right)c_2^{1-c_1} \tag{3}$$

for all $z \in [-1, 1]$. Next, it follows from Lemma 2 that

$$\varrho_1(z) \geq \varrho_3^{\frac{1+c_1}{2}}(z)c_2^{1-c_1} \tag{4}$$

for all $z \in (-\infty, -1) \cup (1, \infty)$. Based on the fact that $z^{c_1} - 1 < (z-1)c_1$ for all $z \in (0, 1) \cup (1, \infty)$ [34], one can easily show that

$$\frac{d\varrho_3(z)}{dz} = 1 + \left(1 - \frac{1}{z}\right)^{c_1-1}\left(\frac{1}{z} - 1 - \frac{c_1}{z}\right) < 0$$

for all $z \in (-\infty, -1) \cup (1, \infty)$. This together with (4) leads to

$$\begin{aligned} \varrho_1(z) &\geq \left(\min\left\{\varrho_3(-1), \lim_{z\to\infty}\varrho_3(z)\right\}\right)^{\frac{1+c_1}{2}} c_2^{1-c_1} \\ &\geq \left(2^{c_1}-1\right)c_2^{1-c_1} \end{aligned} \tag{5}$$

for all $z \in (-\infty, -1) \cup (1, \infty)$. Combining (3) and (5) completes the proof. □

Lemma 6 *Let $c_1 \in (0, 1)$, $c_2 \in (0, \infty)$ and $t_0 \in [0, \infty)$. If $\varphi : [t_0, \infty) \to [0, \infty)$ is a continuous function satisfying $\varphi(t_0) > 0$ such that*

$$\varphi(t) - \varphi(t_0) \leq -c_2 \int_{t_0}^{t} \varphi^{c_1}(s)\, ds \tag{6}$$

for all $t \in [t_0, \infty)$, then there holds

$$\varphi(t) \leq \left[\varphi^{1-c_1}(t_0) - c_2(1-c_1)(t-t_0)\right]^{\frac{1}{1-c_1}}$$

for all $t \in [t_0, T)$ with

$$T = t_0 + \frac{\varphi^{1-c_1}(t_0)}{c_2(1-c_1)}.$$

Proof Define the set

$$\begin{aligned} \mathbb{H} &= \{t \mid t \in (t_0, T] \text{ such that } \varpi(t) < \varphi(t)\} \\ &\subseteq (t_0, T] \end{aligned}$$

and let $\varpi : [t_0, T] \to \mathbb{R}$ be of the form

$$\varpi(t) = \left(\varphi^{1-c_1}(t_0) - c_2(1-c_1)(t-t_0)\right)^{\frac{1}{1-c_1}}$$

which is clearly continuous. Because $\varpi(t_0) = \varphi(t_0) > 0$, $\varpi(T) = 0$ and $\varpi(t) > 0$ for all $t \in [t_0, T)$, it suffices to prove that $\mathbb{H}$ is empty. To this end, assume that there exists $t_1 \in \mathbb{H}$ (i.e., $\varpi(t_1) < \varphi(t_1)$) and consider the set

$$\mathbb{K} = \{t \mid t \in (t_0, t_1) \text{ such that } \varpi(s) < \varphi(s) \text{ for all } s \in (t, t_1]\} \subseteq (t_0, t_1).$$

Letting $t_2 = \inf(\mathbb{K})$, one has $\varpi(t_2) = \varphi(t_2)$ and $0 < \varpi(t) < \varphi(t)$ for all $t \in (t_2, t_1]$ due to the continuity of $\varphi(\cdot)$ and $\varpi(\cdot)$; this implies that

$$0 < \varpi(t) \leq \varphi(t) \tag{7}$$

for all $t \in [t_2, t_1]$. Now, since

$$\varpi(t) - \varpi(t_0) = -c_2 \int_{t_0}^{t} \varpi^{c_1}(s)\, \mathrm{d}s$$

for all $t \in [t_0, T]$, it is easy to derive from (6) that

$$\int_{t_0}^{t} \varphi^{c_1}(s)\, \mathrm{d}s < \int_{t_0}^{t} \varpi^{c_1}(s)\, \mathrm{d}s < 0$$

for all $t \in [t_2 + \varepsilon, t_1]$ with $\varepsilon > 0$; this also implies that there exists $t_3 \in [t_2 + \varepsilon, t_1] \subset [t_2, t_1]$ such that $\varphi(t_3) < \varpi(t_3)$ and therefore leads to a contradiction to (7). The proof is completed. □

Remark 3 It is worth pointing out that if $c_2 \in (-\infty, 0)$, Lemma 6 is exactly a special case of the so-called Bihari's inequality [35]. Due to the generalization of the condition (6) with including a positive real constant $c_2 \in (0, \infty)$, Lemma 6 can be technically viewed as an extension or counterpart of the Bihari's inequality. Notably, from Lemma 6 we have $\varphi(T) = 0$; if, in addition, $\varphi(t)$ is non-increasing, one further has $\varphi(t) = 0$ for all $t \in [T, \infty)$. Such an important consequence will be utilized in proving finite-time convergence in consideration of an asymmetric output constraint.

3 Main results

This section is dedicated to designing a continuous output feedback finite-time controller that stabilizes system (1) while guaranteeing the fulfillment of the asymmetric output constraint $-\kappa_l < y(t) < \kappa_u$ for all $t \geq t_0$ where κ_l and κ_u are pre-specified positive real constants. Specifically, the design begins with organizing a *fraction*-type asymmetric BLF by exploiting and using the intrinsic characteristics of the nonlinearities $\phi_i(\cdot)$'s. The synthesis of a continuous state feedback controller is then performed by elegantly renovating the adding a power integrator technique through an artful implantation of the designed BLF together with subtle manipulations of signum functions. Further, by elaborately pondering the nonlinearities inherent in system (1), the construction of a one-dimensional non-smooth state observer furnished with a state-dependent gain is presented so as to achieve output feedback design. Derived by a delicate integration of the non-smooth observer and the state feedback controller, along with a suitable choice of the observer gain, a continuous output feedback stabilizer is finally proposed for system (1) which is able to effectively ensure the fulfillment of the asymmetric output constraint.

3.1 Design of a new *fraction*-type asymmetric BLF

Considering the asymmetric output constraint $-\kappa_l < y(t) < \kappa_u$ for all $t \geq t_0$ with two pre-specified positive real constants κ_l and κ_u, and taking the intrinsic characteristics of the system nonlinearities $\phi_i(\cdot)$'s depicted by Assumption 2, we pick

$$\omega \geq \mu \geq \max\{m_1, m_2\} \tag{8}$$

with μ being an auxiliary factor, and design a *fraction*-type function $V_F : \mathbb{M}_1(\kappa_l, \kappa_u) \to \mathbb{R}$ as follows

$$V_F(x_1) = \frac{\kappa_u^{2\omega-\sigma} \kappa_l^{2\omega-\sigma} |x_1|^{2\omega-\sigma}}{(2\omega - \sigma)\,(\kappa_u - x_1)^{2\omega-\sigma} (\kappa_l + x_1)^{2\omega-\sigma}} \tag{9}$$

which is obviously positive definite on $\mathbb{M}_1(\kappa_l, \kappa_u)$ and fulfills $V_F(x_1) \to \infty$ as $x_1 \to \partial\mathbb{M}_1(\kappa_l, \kappa_u)$. By a direct calculation, it is easy to verify that

$$\frac{\partial V_F(x_1)}{\partial x_1} = \lambda(x_1) \lceil x_1 \rceil^{2\omega-\sigma-1} \tag{10}$$

for all $x_1 \in \mathbb{M}_1(\kappa_l, \kappa_u)$, where $\lceil x_1 \rceil^{2\omega-\sigma-1}$ is continuous on $\mathbb{M}_1(\kappa_l, \kappa_u)$ and $\lambda : \mathbb{M}_1(\kappa_l, \kappa_u) \to (0, \infty)$ is of the form

$$\lambda(x_1) = \frac{\kappa_u^{2\omega-\sigma}\kappa_l^{2\omega-\sigma}\left(x_1^2+\kappa_u\kappa_l\right)}{(\kappa_u-x_1)^{2\omega-\sigma+1}(\kappa_l+x_1)^{2\omega-\sigma+1}} \qquad (11)$$

which is smooth on $\mathbb{M}_1(\kappa_l,\kappa_u)$; thus, $V_F(x_1)$ is continuously differentiable on $\mathbb{M}_1(\kappa_l,\kappa_u)$ and forms a *fraction*-type asymmetric BLF of system (1) according to Definition 1 (also Remark 2). We call $V_F(x_1)$ an *asymmetric* BLF due to the asymmetry of $V_F(x_1)$ arising from the difference between κ_l and κ_u. As we shall show later, $V_F(x_1)$ acts as a key constituent/brick in renovating the adding a power integrator technique for coping with the output constraints.

Remark 4 Two distinctive traits of the designed *fraction*-type asymmetric BLF $V_F(x_1)$ are emphasized as follows.

(i) The design of the presented $V_F(x_1)$ is directly related to the system nonlinearities $\phi_i(\cdot)$'s, thereby providing the applicability to system (1), when considering asymmetric output constraints. Specifically, with the help of Assumption 2, the intrinsic characteristics of system nonlinearities $\phi_i(\cdot)$'s is subtly extracted and equivalently kept in the parameters m_1, m_2, p and σ. By fully taking into account m_1, m_2, p and σ (i.e., the feature of the nonlinearities $\phi_i(\cdot)$'s) in constructing of $V_F(x_1)$, the resultant $V_F(x_1)$ has the specific fraction structure, which drastically differs from the common *logarithm*-type [21,28,29,31] and *tangent*-type [19,20,22,23] BLFs, and offers for system (1) a delicate treatment of asymmetric constraints.

(ii) The designed $V_F(x_1)$ is a powerful tool making our scheme capable of simultaneously dealing with constrained and unconstrained cases. More precisely, considering $\kappa_l = \kappa_u = \kappa$ with $\kappa \to \infty$ (i.e., the circumstance of no constraint requirement on the output), one has

$$\lim_{\kappa\to\infty} V_F(x_1) = \lim_{\kappa\to\infty} \frac{\kappa^{4\omega-2\sigma}|x_1|^{2\omega-\sigma}}{(2\omega-\sigma)\,(\kappa^2-x_1^2)^{2\omega-\sigma}} = \frac{1}{(2\omega-\sigma)}|x_1|^{2\omega-\sigma}$$

which is exactly the regular function used in [2,6,7,17] for tackling the pure state or output feedback stabilization of high-order nonlinear systems without output constraints. In other words, the designed $V_F(x_1)$ enjoys the appealing property that when there is no constraint on the output, that is, $\kappa_l = \kappa_u = \kappa \to \infty$, it will molt into the one widely adopted for pure stabilization without constraint requirements; hence, $V_F(x_1)$ can be thought of as a technical lever assisting us in unifying and achieving simultaneously the design of continuous output feedback finite-time stabilizers for system (1) subjected to or free from output constraints, without needing to change the resultant controller and observer structures.

3.2 Design of a continuous output feedback finite-time stabilizer

On the basis of the designed $V_F(x_1)$, we are now ready to present the design of a continuous output feedback finite-time stabilizer for system (1), which also ensures the achievement of the asymmetric output constraint specified in advance.

Theorem 1 *Suppose that Assumptions 1 and 2 hold. Then, there exists a continuous output feedback finite-time stabilizer of the following form*

$$\dot{\nu} = -G(x_1)\left(\lceil \nu + L(x_1)\rfloor^{\frac{m_1+\sigma}{\mu}} + \phi_1(x_1,t)\right)$$
$$u(x) = u(x_1,\hat{x}_2) \;\; with \;\; \hat{x}_2 = \lceil \nu + L(x_1)\rfloor^{\frac{m_2}{\mu}} \qquad (12)$$

where $L:\mathbb{R}\to\mathbb{R}$ is a continuously differentiable observer gain satisfying $L(0)=0$ and $\partial L(x_1)/\partial x_1 = G(x_1) \ge 1$ for all $x_1 \in \mathbb{M}_1(\kappa_l,\kappa_u)$ under which every trajectory $(x(t),\nu(t))$ of system (1) starting from $(x(t_0),\nu(t_0)) \in \mathbb{M}_2(\kappa_l,\kappa_u)\times\mathbb{R}$ is defined on $[t_0,\infty)$, converges to the origin in finite time, and fulfills the constraint $-\kappa_l < y(t) = x_1(t) < \kappa_u$ for all $t \ge t_0$ where κ_l and κ_u are pre-specified positive real constants.

Proof The proof as well as the design methodology is divided into four parts as follows.

Part I—Design of a state feedback controller

For a start, define $\xi_1(x_1) = \lceil x_1\rfloor^{\mu}$ and $V_1 : \mathbb{M}_1(\kappa_l,\kappa_u) \to \mathbb{R}$ to be in the form of $V_1(x_1) = V_F(x_1)$ where μ and $V_F(x_1)$ are described by (8) and (9), respectively. Obviously, $V_1(x_1)$ is positive definite and continuously differentiable on $\mathbb{M}_1(\kappa_l,\kappa_u)$, and by (10) and Assumption 2 one has

$$\dot{V}_1(x_1) := \frac{\partial V_1(x_1)}{\partial x_1}\dot{x}_1$$

$$\leq \lambda(x_1)\lceil\xi_1(x_1)\rceil^{\frac{2\omega-\sigma-1}{\mu}} x_2^{*p}(x_1) + \lambda(x_1)\lceil\xi_1(x_1)\rceil^{\frac{2\omega-\sigma-1}{\mu}} \left(x_2^p - x_2^{*p}(x_1)\right) + \lambda(x_1)\overline{\phi}_1(x_1)|\xi_1(x_1)|^{\frac{2\omega}{\mu}} \quad (13)$$

for all[3] $(x,t) \in \mathbb{M}_2(\kappa_l,\kappa_u) \times \mathbb{R}^+$, where $x_2^* : \mathbb{M}_1(\kappa_l,\kappa_u) \to \mathbb{R}$ is a continuous virtual controller and $\lambda(x_1) > 0$ for all $x_1 \in \mathbb{M}_1(\kappa_l,\kappa_u)$ is given by (11). By selecting the virtual controller

$$x_2^*(x_1) = -\beta_1(x_1)\lceil\xi_1(x_1)\rceil^{\frac{m_2}{\mu}} \quad (14)$$

with $\beta_1 : \mathbb{M}_1(\kappa_l,\kappa_u) \to (0,\infty)$ being a smooth function of the form

$$\beta_1(x_1) = \left(\frac{2+\lambda(x_1)\overline{\phi}_1(x_1)}{\lambda(x_1)}\right)^{\frac{1}{p}}$$

it follows from (13) that

$$\begin{aligned}\dot{V}_1(x_1) &\leq -\lambda(x_1)\left(\beta_1^p(x_1)+\overline{\phi}_1(x_1)\right)|\xi_1(x_1)|^{\frac{2\omega}{\mu}} + \lambda(x_1)\lceil\xi_1(x_1)\rceil^{\frac{2\omega-\sigma-1}{\mu}} \left(x_2^p - x_2^{*p}(x_1)\right) \\ &= -2|\xi_1(x_1)|^{\frac{2\omega}{\mu}} + \lambda(x_1)\lceil\xi_1(x_1)\rceil^{\frac{2\omega-\sigma-1}{\mu}} \times \left(x_2^p - x_2^{*p}(x_1)\right) \quad (15)\end{aligned}$$

for all $(x,t) \in \mathbb{M}_2(\kappa_l,\kappa_u) \times \mathbb{R}^+$.

To continue with the design, let $\xi_2(x) = \lceil x_2\rceil^{\mu/m_2} - \lceil x_2^*(x_1)\rceil^{\mu/m_2}$ and $V_2 : \mathbb{M}_2(\kappa_l,\kappa_u) \to \mathbb{R}$ be defined as $V_2(x) = V_1(x_1) + W(x)$ where $x_2^*(x_1)$ is given by (14) and $W : \mathbb{M}_2(\kappa_l,\kappa_u) \to \mathbb{R}$ has the following form

$$W(x) = \int_{x_2^*(x_1)}^{x_2} \left\lceil \lceil s\rceil^{\frac{\mu}{m_2}} - \lceil x_2^*(x_1)\rceil^{\frac{\mu}{m_2}} \right\rceil^{\frac{2\omega-\sigma-m_2}{\mu}} \mathrm{d}s.$$

Using an almost same argument stated in [7,17], one can deduce that $V_2(x)$ is positive definite and continuously differentiable on $\mathbb{M}_2(\kappa_l,\kappa_u)$, and $W(x)$ satisfies

$$\begin{aligned}\frac{\partial W(x)}{\partial x_1} &= -\left(\frac{2\omega-\sigma-m_2}{\mu}\right)\frac{\partial\lceil x_2^*(x_1)\rceil^{\frac{\mu}{m_2}}}{\partial x_1} \\ &\quad\times \int_{x_2^*(x_1)}^{x_2} \left|\lceil s\rceil^{\frac{\mu}{m_2}} - \lceil x_2^*(x_1)\rceil^{\frac{\mu}{m_2}}\right|^{\frac{2\omega-\sigma-m_2-\mu}{\mu}} \mathrm{d}s \\ \frac{\partial W(x)}{\partial x_2} &= \left\lceil \lceil x_2\rceil^{\frac{\mu}{m_2}} - \lceil x_2^*(x_1)\rceil^{\frac{\mu}{m_2}}\right\rceil^{\frac{2\omega-\sigma-m_2}{\mu}} \\ &= \lceil\xi_2(x)\rceil^{\frac{2\omega-\sigma-m_2}{\mu}}\end{aligned}$$

for all $x \in \mathbb{M}_2(\kappa_l,\kappa_u)$, where $\partial W(x)/\partial x_1$ has the property below

$$\begin{aligned}\left|\frac{\partial W(x)}{\partial x_1}\right| &\leq \left(\frac{2\omega-\sigma-m_2}{\mu}\right)\left|\frac{\partial\lceil x_2^*(x_1)\rceil^{\frac{\mu}{m_2}}}{\partial x_1}\right| \times 2^{\frac{\mu-m_2}{\mu}}|\xi_2(x)|^{\frac{2\omega-\sigma-\mu}{\mu}} \\ &\leq \psi_1(x_1)\,|\xi_1(x_1)|^{\frac{\mu-m_1}{\mu}}\,|\xi_2(x)|^{\frac{2\omega-\sigma-\mu}{\mu}}\end{aligned}$$

for all $x \in \mathbb{M}_2(\kappa_l,\kappa_u)$ with $\psi_1 : \mathbb{M}_1(\kappa_l,\kappa_u) \to \mathbb{R}^+$ being a smooth function[4]. By these relations, the inequality (15) and Assumption 2, the time derivative of $V_2(x_1)$ along system (1) is

$$\begin{aligned}\dot{V}_2(x) &:= \frac{\partial V_1(x_1)}{\partial x_1}\dot{x}_1 + \frac{\partial W(x)}{\partial x_1}\dot{x}_1 + \frac{\partial W(x)}{\partial x_2}\dot{x}_2 \\ &\leq -2|\xi_1(x_1)|^{\frac{2\omega}{\mu}} + \theta(x,t)\lceil\xi_2(x)\rceil^{\frac{2\omega-\sigma-m_2}{\mu}}u \\ &\quad+ \lambda(x_1)\lceil\xi_1(x_1)\rceil^{\frac{2\omega-\sigma-1}{\mu}}\left(x_2^p - x_2^{*p}(x_1)\right) \\ &\quad+ |\xi_2(x)|^{\frac{2\omega-\sigma-m_2}{\mu}}\overline{\phi}_2(x_1)\left(|x_1|^{\frac{m_2+\sigma}{m_1}} + |x_2|^{\frac{m_2+\sigma}{m2}}\right) \\ &\quad+ \psi_1(x_1)\,|\xi_1(x_1)|^{\frac{\mu-m_1}{\mu}}\,|\xi_2(x)|^{\frac{2\omega-\sigma-\mu}{\mu}} \\ &\quad\times\left(|x_2|^p + \overline{\phi}_1(x_1)|x_1|^{\frac{m_1+\sigma}{m_1}}\right) \quad (16)\end{aligned}$$

for all $(x,t) \in \mathbb{M}_2(\kappa_l,\kappa_u) \times \mathbb{R}^+$. Before moving on to designing the controller u, we estimate the last three terms on the right-hand side of (16).

First, owing to the inequality

$$\left(x_2^p - x_2^{*p}(x_1)\right)$$

[3] $\dot{V}_1(x_1) := (\partial V_1(x_1)/\partial x_1)\dot{x}_1$ includes the variable x_2 and the function $\phi_1(x_1,t)$; thus, it is directly related to $(x,t) \in \mathbb{M}_2(\kappa_l,\kappa_u) \times \mathbb{R}^+$.

[4] We adapt the fact that any real-valued continuous function has a nonnegative smooth upper bound function (see, e.g., [36, Theorem 6.21, p. 136]).

$$\leq \left| \left\lceil \lceil x_2 \rfloor^{\frac{\mu}{m_2}} \right\rfloor^{\frac{m_1+\sigma}{\mu}} - \left\lceil \lceil x_2^*(x_1) \rfloor^{\frac{\mu}{m_2}} \right\rfloor^{\frac{m_1+\sigma}{\mu}} \right|$$

for all $x \in \mathbb{M}_2(\kappa_l, \kappa_u)$ and $(m_1 + \sigma)/\mu < 1$, it follows from Lemmas 2 and 4 that

$$\begin{aligned}
&\lambda(x_1) \lceil \xi_1(x_1) \rfloor^{\frac{2\omega-\sigma-1}{\mu}} \left(x_2^p - x_2^{*p}(x_1)\right) \\
&\leq \lambda(x_1) |\xi_1(x_1)|^{\frac{2\omega-\sigma-1}{\mu}} 2^{1-\frac{m_1+\sigma}{\mu}} |\xi_2(x)|^{\frac{m_1+\sigma}{\mu}} \\
&\leq \frac{1}{3} |\xi_1(x_1)|^{\frac{2\omega}{\mu}} + \psi_2(x_1) |\xi_2(x)|^{\frac{2\omega}{\mu}} \qquad (17)
\end{aligned}$$

for all $x \in \mathbb{M}_2(\kappa_l, \kappa_u)$ where $\psi_2 : \mathbb{M}_1(\kappa_l, \kappa_u) \to \mathbb{R}^+$ is a smooth function.

Second, using the fact $m_2/\mu \leq 1$ and Lemma 4, one has the inequality

$$|x_2| \leq 2^{1-\frac{m_2}{\mu}} |\xi_2(x)|^{\frac{m_2}{\mu}} + \beta_1(x_1) |\xi_1(x_1)|^{\frac{m_2}{\mu}} \qquad (18)$$

for all $x \in \mathbb{M}_2(\kappa_l, \kappa_u)$. With this in mind, one can verify by using Lemmas 2 and 3 that there exists a smooth function $\hat{\psi}_2 : \mathbb{M}_1(\kappa_l, \kappa_u) \to \mathbb{R}^+$ such that

$$\begin{aligned}
&|\xi_2(x)|^{\frac{2\omega-\sigma-m_2}{\mu}} \overline{\phi}_2(x_1) \left(|x_1|^{\frac{m_2+\sigma}{m_1}} + |x_2|^{\frac{m_2+\sigma}{m_2}} \right) \\
&\leq \overline{\phi}_2(x_1) |\xi_1(x_1)|^{\frac{m_2+\sigma}{\mu}} |\xi_2(x)|^{\frac{2\omega-\sigma-m_2}{\mu}} \\
&\quad + 2^{\frac{(\mu-m_2)(m_2+\sigma)}{\mu m_2}} \left(2^{\frac{m_2+\sigma}{m_2}-1} + 1 \right) \overline{\phi}_2(x_1) |\xi_2(x)|^{\frac{2\omega}{\mu}} \\
&\quad + \left(2^{\frac{m_2+\sigma}{m_2}-1} + 1 \right) \overline{\phi}_2(x_1) \beta_1^{\frac{m_2+\sigma}{m_2}}(x_1) \\
&\quad \times |\xi_1(x_1)|^{\frac{m_2+\sigma}{\mu}} |\xi_2(x)|^{\frac{2\omega-\sigma-m_2}{\mu}} \\
&\leq \frac{1}{3} |\xi_1(x_1)|^{\frac{2\omega}{\mu}} + \hat{\psi}_2(x_1) |\xi_2(x)|^{\frac{2\omega}{\mu}} \qquad (19)
\end{aligned}$$

for all $(x, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R}^+$.

Similarly to deriving (19), using Lemma 2–4 one can obtain the following

$$\begin{aligned}
&\psi_1(x_1) |\xi_1(x_1)|^{\frac{\mu-m_1}{\mu}} |\xi_2(x)|^{\frac{2\omega-\sigma-\mu}{\mu}} \\
&\quad \times \left(|x_2|^p + \overline{\phi}_1(x_1) |x_1|^{\frac{m_1+\sigma}{m_1}} \right) \\
&\leq \left(2^{p-1} + 1\right) 2^{p-\frac{m_2 p}{\mu}} \psi_1(x_1) |\xi_1(x_1)|^{\frac{\mu-m_1}{\mu}} \\
&\quad |\xi_2(x)|^{\frac{2\omega+1-\mu}{\mu}} \\
&\quad + \left(2^{p-1} + 1\right) \beta_1^p(x_1) \psi_1(x_1) |\xi_1(x_1)|^{\frac{\mu+\sigma}{\mu}} \\
&\quad |\xi_2(x)|^{\frac{2\omega-\sigma-\mu}{\mu}} \\
&\quad + \psi_1(x_1) \overline{\phi}_1(x_1) |\xi_1(x_1)|^{\frac{\mu+\sigma}{\mu}} \\
&\quad |\xi_2(x)|^{\frac{2\omega-\sigma-\mu}{\mu}} \\
&\leq \frac{1}{3} |\xi_1(x_1)|^{\frac{2\omega}{\mu}} + \tilde{\psi}_2(x_1) |\xi_2(x)|^{\frac{2\omega}{\mu}} \qquad (20)
\end{aligned}$$

for all $(x, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R}^+$ where $\tilde{\psi}_2 : \mathbb{M}_1(\kappa_l, \kappa_u) \to \mathbb{R}^+$ is a smooth function.

Applying these estimations (17), (19) and (20) to (16) gives

$$\begin{aligned}
\dot{V}_2(x) &\leq -|\xi_1(x_1)|^{\frac{2\omega}{\mu}} + \theta(x, t) \lceil \xi_2(x) \rfloor^{\frac{2\omega-\sigma-m_2}{\mu}} u \\
&\quad + \left(\psi_2(x_1) + \hat{\psi}_2(x_1) + \tilde{\psi}_2(x_1) \right) |\xi_2(x)|^{\frac{2\omega}{\mu}}
\end{aligned} \qquad (21)$$

for all $(x, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R}^+$. Clearly, designing the continuous state feedback controller

$$u(x) = -\beta_2(x_1) \lceil \xi_2(x) \rfloor^{\frac{m_2+\sigma}{\mu}} \qquad (22)$$

with $\beta_2 : \mathbb{M}_1(\kappa_l, \kappa_u) \to (0, \infty)$ being a smooth function taking the form

$$\beta_2(x_1) = \frac{1 + \psi_2(x_1) + \hat{\psi}_2(x_1) + \tilde{\psi}_2(x_1)}{\underline{\theta}(x_1)}$$

and using Assumption 1, one readily obtains

$$\begin{aligned}
\dot{V}_2(x) &\leq -|\xi_1(x_1)|^{\frac{2\omega}{\mu}} - \theta(x, t) \beta_2(x_1) |\xi_2(x)|^{\frac{2\omega}{\mu}} \\
&\quad + \left(\psi_2(x_1) + \hat{\psi}_2(x_1) + \tilde{\psi}_2(x_1) \right) |\xi_2(x)|^{\frac{2\omega}{\mu}} \\
&= -|\xi_1(x_1)|^{\frac{2\omega}{\mu}} - |\xi_2(x)|^{\frac{2\omega}{\mu}} \qquad (23)
\end{aligned}$$

for all $(x, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R}^+$. Note that, with the help of Lemma 4 it is not difficult to see that

$$\begin{aligned}
&V_2(x) \\
&= V_1(x_1) + \int_{x_2^*(x_1)}^{x_2} \left\lceil \lceil s \rfloor^{\frac{\mu}{m_2}} - \lceil x_2^*(x_1) \rfloor^{\frac{\mu}{m_2}} \right\rfloor^{\frac{2\omega-\sigma-m_2}{\mu}} ds \\
&\geq V_1(x_1) + \varepsilon_1 \left| x_2 - x_2^*(x_1) \right|^{\frac{2\omega-\sigma}{m_2}} \qquad (24)
\end{aligned}$$

for all $x \in \mathbb{M}_2(\kappa_l, \kappa_u)$ with ε_1 being a positive real constant; this implies that, with any fixed $x_1 \in \mathbb{M}_1(\kappa_l, \kappa_u)$, $V_2(x) \to \infty$ as $|x_2| \to \infty$. Hence, if

$x = (x_1, x_2)^T \in \mathbb{R}^2$ are completely available, by Lemma 1 one knows that when $x(t_0) \in \mathbb{M}_2(\kappa_l, \kappa_u)$, the continuous state feedback controller (22) successfully achieves the requirement of the asymmetric output constraint $-\kappa_l < y(t) < \kappa_u$ for all $t \geq t_0$. Because of the infeasibility/limitation on the full state measurement, a state observer is quite imperative for feedback design, as depicted in the next part.

Part II—Design of a one-dimensional observer

In order to efficiently perform the output feedback design, a one-dimensional non-smooth state observer is constructed as follows

$$\dot{\nu} = -G(x_1)\left(\lceil \nu + L(x_1)\rceil^{\frac{m_1+\sigma}{\mu}} + \phi_1(x_1, t)\right) \quad (25)$$

where $L : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable observer gain function with $L(0) = 0$ and $\partial L(x_1)/\partial x_1 = G(x_1) \geq 1$ for all $x_1 \in \mathbb{M}_1(\kappa_l, \kappa_u)$; this gain function will be appropriately assigned later. Having the state variable ν, the observer (25) is devoted to estimating the unmeasurable state x_2 by providing $\hat{x}_2 = \lceil \nu + L(x_1)\rceil^{m_2/\mu}$. Consider the estimation error $e = \lceil x_2\rceil^{\mu/m_2} - \lceil \hat{x}_2\rceil^{\mu/m_2}$. It follows from (25) that

$$\begin{aligned}\dot{e}(x, \nu) = {} & \frac{\mu}{m_2}|x_2|^{\frac{\mu}{m_2}-1}(\theta(x, t)u + \phi_2(x, t)) \\ & - G(x_1)\left(x_2^p - \left\lceil \lceil x_2\rceil^{\frac{\mu}{m_2}} - e\right\rceil^{\frac{m_1+\sigma}{\mu}}\right)\end{aligned}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$. Choosing $V_3 : \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \to \mathbb{R}$ as

$$V_3(x, \nu) = \left(\frac{\mu}{2\omega - \sigma - m_1 + \mu}\right)|e(x, \nu)|^{\frac{2\omega-\sigma-m_1+\mu}{\mu}}$$

which is nonnegative and continuously differentiable on $\mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R}$, we have

$$\begin{aligned}&\dot{V}_3(x, \nu) \\ &= \frac{\mu}{m_2}\lceil e(x, \nu)\rceil^{\frac{2\omega-\sigma-m_1}{\mu}} \\ &\quad |x_2|^{\frac{\mu}{m_2}-1}(\theta(x, t)u + \phi_2(x, t)) \\ &\quad - \lceil e(x, \nu)\rceil^{\frac{2\omega-\sigma-m_1}{\mu}} G(x_1) \\ &\quad \times \left(x_2^p - \left\lceil \lceil x_2\rceil^{\frac{\mu}{m_2}} - e(x, \nu)\right\rceil^{\frac{m_1+\sigma}{\mu}}\right)\end{aligned}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$. Because $(m_1 + \sigma)/\mu < 1$ and $G(x_1) \geq 1$, by Lemma 5 one can derive

$$\begin{aligned}&\lceil e(x, \nu)\rceil^{\frac{2\omega-\sigma-m_1}{\mu}}\left(x_2^p - \left\lceil \lceil x_2\rceil^{\frac{\mu}{m_2}} - e(x, \nu)\right\rceil^{\frac{m_1+\sigma}{\mu}}\right) \\ &\geq -G^{-\frac{3}{2}}(x_1)|e(x, \nu)|^{\frac{2\omega-\sigma-m_1-\mu}{\mu}}|x_2|^{\frac{m_1+\sigma+\mu}{m_2}} \\ &\quad + \psi_3 G^{-\frac{3}{4}}(x_1)|e(x, \nu)|^{\frac{2\omega}{\mu}} \quad (26)\end{aligned}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$, in which $\psi_3 = 2^{(m_1+\sigma)/\mu} - 1 > 0$. Using (26), we further have

$$\begin{aligned}&\dot{V}_3(x, \nu) \\ &\leq G^{-\frac{1}{2}}(x_1)|e(x, \nu)|^{\frac{2\omega-\sigma-m_1-\mu}{\mu}}|x_2|^{\frac{m_1+\sigma+\mu}{m_2}} \\ &\quad - \psi_3 G^{\frac{1}{4}}(x_1)|e(x, \nu)|^{\frac{2\omega}{\mu}} + \frac{\mu}{m_2}\lceil e(x, \nu)\rceil^{\frac{2\omega-\sigma-m_1}{\mu}} \\ &\quad \times |x_2|^{\frac{\mu}{m_2}-1}(\theta(x, t)u + \phi_2(x, t)) \quad (27)\end{aligned}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$. Remarkably, applying Lemma (3) to (18) yields

$$|x_2|^{\frac{\tau}{m_2}} \leq \varepsilon_2\left(2^{\frac{(\mu-m_2)\tau}{\mu m_2}}|\xi_2(x)|^{\frac{\tau}{\mu}} + \beta_1^{\frac{\tau}{m_2}}(x_1)|\xi_1(x_1)|^{\frac{\tau}{\mu}}\right)$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$ and $\tau \in \{\mu + \sigma, m_1 + \mu + \sigma\}$ with $\varepsilon_2 = 2^{\tau/m_2 - 1} + 1$. Keeping this in mind and using Lemma 2 along with Assumptions 1 and 2 , one obtains

$$\begin{aligned}&\frac{\mu}{m_2}\lceil e(x, \nu)\rceil^{\frac{2\omega-\sigma-m_1}{\mu}}|x_2|^{\frac{\mu}{m_2}-1}(\theta(x, t)u + \phi_2(x, t)) \\ &\leq \frac{\mu}{m_2}\left(\overline{d}(x_1) + 2\overline{\phi}_2(x_1)\right)|e(x, \nu)|^{\frac{2\omega-\sigma-m_1}{\mu}} \\ &\quad \times \left(|u|^{\frac{\mu+\sigma}{m_2+\sigma}} + |\xi_1(x_1)|^{\frac{\mu+\sigma}{\mu}} + |x_2|^{\frac{\mu+\sigma}{m_2}}\right) \\ &\leq \psi_4(x_1)|e(x, \nu)|^{\frac{2\omega}{\mu}} \\ &\quad + \hat{\psi}_4(x_1)|e(x, \nu)|^{\frac{2\omega-\sigma-\mu}{\mu}}|u|^{\frac{\mu+\sigma}{m_2+\sigma}} \\ &\quad + \frac{1}{2}|\xi_1(x_1)|^{\frac{2\omega}{\mu}} + \frac{1}{6}|\xi_2(x)|^{\frac{2\omega}{\mu}} \quad (28)\end{aligned}$$

and

$$\begin{aligned}&|e(x, \nu)|^{\frac{2\omega-\sigma-m_1-\mu}{\mu}}|x_2|^{\frac{m_1+\sigma+\mu}{m_2}} \\ &\leq |e(x, \nu)|^{\frac{2\omega}{\mu}} + \psi_5(x_1)|\xi_1(x_1)|^{\frac{2\omega}{\mu}} + \hat{\psi}_5|\xi_2(x)|^{\frac{2\omega}{\mu}} \quad (29)\end{aligned}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$, where $\psi_4, \hat{\psi}_4, \psi_5 : \mathbb{M}_1(\kappa_l, \kappa_u) \to \mathbb{R}^+$ are smooth functions and $\hat{\psi}_5$ is a positive real constant. Using (28) and (29), we obtain from (27) that

$$\begin{aligned}\dot{V}_3(x,\nu) \le &- \psi_3 G^{\frac{1}{4}}(x_1)|e(x,\nu)|^{\frac{2\omega}{\mu}} \\ &+ \left(\frac{1}{2} + G^{-\frac{1}{2}}(x_1)\psi_5(x_1)\right)|\xi_1(x_1)|^{\frac{2\omega}{\mu}} \\ &+ \left(\frac{1}{6} + G^{-\frac{1}{2}}(x_1)\hat{\psi}_5\right)|\xi_2(x)|^{\frac{2\omega}{\mu}} \\ &+ \left(\psi_4(x_1) + G^{-\frac{1}{2}}(x_1)\right)|e(x,\nu)|^{\frac{2\omega}{\mu}} \\ &+ \hat{\psi}_4(x_1)|e(x,\nu)|^{\frac{2\omega-\sigma-\mu}{\mu}}|u|^{\frac{\mu+\sigma}{m_2+\sigma}} \end{aligned} \tag{30}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$.

Part III—Selection of the observer gain $L(x_1)$

In the spirit of the certainty equivalence principle, we replace the unmeasurable state x_2 by the estimation $\hat{x}_2 = \lceil \nu + L(x_1)\rceil^{m_2/\mu}$ generated by the observer (25) so that the implementable continuous output feedback controller can be obtained as below

$$\begin{aligned} u(x_1, \hat{x}_2) &= -\beta_2(x_1)\lceil \xi_2(x_1, \hat{x}_2)\rceil^{\frac{m_2+\sigma}{\mu}} \\ &= -\beta_2(x_1)\left\lceil \lceil \hat{x}_2\rceil^{\frac{\mu}{m_2}} - \lceil x_2^*(x_1)\rceil^{\frac{\mu}{m_2}}\right\rceil^{\frac{m_2+\sigma}{\mu}}. \end{aligned} \tag{31}$$

With this controller, it can be verified by using Lemmas 2 and 3 that the last term on the right-hand side of (30) satisfies

$$\begin{aligned} &\hat{\psi}_4(x_1)|e(x,\nu)|^{\frac{2\omega-\sigma-\mu}{\mu}}|u(x_1,\hat{x}_2)|^{\frac{\mu+\sigma}{m_2+\sigma}} \\ &\le \hat{\psi}_4(x_1)|e(x,\nu)|^{\frac{2\omega-\sigma-\mu}{\mu}}\beta_2^{\frac{\mu+\sigma}{m_2+\sigma}}(x_1) \\ &\quad\times \left(2^{\frac{\mu+\sigma}{\mu}-1}+1\right)\left(|\xi_2(x)|^{\frac{\mu+\sigma}{\mu}} + |e(x,\nu)|^{\frac{\mu+\sigma}{\mu}}\right) \\ &\le \frac{1}{6}|\xi_2(x)|^{\frac{2\omega}{\mu}} + \psi_6(x_1)|e(x,\nu)|^{\frac{2\omega}{\mu}} \end{aligned}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$, in which $\psi_6 : \mathbb{M}_1(\kappa_l, \kappa_u) \to \mathbb{R}^+$ is a smooth function. Thus, (30) becomes

$$\begin{aligned}\dot{V}_3(x,\nu) \le &\left(\frac{1}{2} + G^{-\frac{1}{2}}(x_1)\psi_5(x_1)\right)|\xi_1(x_1)|^{\frac{2\omega}{\mu}} \\ &+ \left(\frac{1}{3} + G^{-\frac{1}{2}}(x_1)\hat{\psi}_5\right)|\xi_2(x)|^{\frac{2\omega}{\mu}} \\ &+ \left(\psi_4(x_1) + \psi_6(x_1) + G^{-\frac{1}{2}}(x_1)\right)|e(x,\nu)|^{\frac{2\omega}{\mu}} \\ &- \psi_3 G^{\frac{1}{4}}(x_1)|e(x,\nu)|^{\frac{2\omega}{\mu}} \end{aligned} \tag{32}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$. Additionally, applying the controller (31) to (21), instead of $u(x)$ defined by (22), and utilizing a similar analysis in deriving (17), we obtain

$$\begin{aligned}&\dot{V}_2(x) \\ &\le -|\xi_1(x_1)|^{\frac{2\omega}{\mu}} - |\xi_2(x)|^{\frac{2\omega}{\mu}} + \overline{\theta}(x_1)\beta_2(x_1)|\xi_2|^{\frac{2\omega-\sigma-m_2}{\mu}} \\ &\quad\times \left|\lceil \xi_2(x)\rceil^{\frac{m_2+\sigma}{\mu}} - \lceil \xi_2(x) - e(x,\nu)\rceil^{\frac{m_2+\sigma}{\mu}}\right| \\ &\le -|\xi_1(x_1)|^{\frac{2\omega}{\mu}} - \frac{5}{6}|\xi_2(x)|^{\frac{2\omega}{\mu}} + \psi_7(x_1)|e(x,\nu)|^{\frac{2\omega}{\mu}} \end{aligned} \tag{33}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$, where $\psi_7 : \mathbb{M}_1(\kappa_l, \kappa_u) \to \mathbb{R}^+$ is a smooth function. At present, we choose $V : \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \to \mathbb{R}$ as $V(x, \nu) = V_2(x) + V_3(x, \mu)$ which is surely continuously differentiable and positive definite on $\mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R}$. From (32) and (33), it is clear that

$$\begin{aligned}\dot{V}(x,\nu) \le &-\left(\frac{1}{2} - G^{-\frac{1}{2}}(x_1)\psi_5(x_1)\right)|\xi_1(x_1)|^{\frac{2\omega}{\mu}} \\ &- \left(\frac{1}{2} - G^{-\frac{1}{2}}(x_1)\hat{\psi}_5\right)|\xi_2(x)|^{\frac{2\omega}{\mu}} \\ &- \left(\psi_3 G^{\frac{1}{4}}(x_1) - \psi_4(x_1) - \psi_6(x_1)\right. \\ &\left. -\psi_7(x_1) - 1\right)|e(x,\nu)|^{\frac{2\omega}{\mu}} \end{aligned} \tag{34}$$

for all $(x, \nu, t) \in \mathbb{M}_2(\kappa_l, \kappa_u) \times \mathbb{R} \times \mathbb{R}^+$. Observing (34) and letting $\mathcal{X} = (x, \nu) \in \mathbb{R}^3$, one can directly verify that the selection of the observer gain $L(x_1)$ with $G(x_1) = \partial L(x_1)/\partial x_1$ complying with

$$G^{\frac{1}{2}}(x_1) \ge \max\left\{1, 4\psi_5(x_1), 4\hat{\psi}_5\right\}$$

$$\psi_3 G^{\frac{1}{4}}(x_1) \ge \psi_4(x_1) + \psi_6(x_1) + \psi_7(x_1) + \frac{5}{4}$$

results in

$$\begin{aligned}\dot{V}(\mathcal{X}) &\le -\frac{1}{4}\left(|\xi_1(x_1)|^{\frac{2\omega}{\mu}} + |\xi_2(x)|^{\frac{2\omega}{\mu}} + |e(\mathcal{X})|^{\frac{2\omega}{\mu}}\right) \\ &=: -U(\mathcal{X}) \end{aligned} \tag{35}$$

for all $(\mathcal{X}, t) \in \mathbb{M}_3(\kappa_l, \kappa_u) \times \mathbb{R}^+$, in which $U(\mathcal{X}) := 1/4(|\xi_1(x_1)|^{2\omega/\mu} + |\xi_2(x)|^{2\omega/\mu} + |e(\mathcal{X})|^{2\omega/\mu}$. Remarkably, $U(\mathcal{X})$ is positive definite and continuous on $\mathbb{M}_3(\kappa_l, \kappa_u)$. In addition, from (24), we also have

$$\begin{aligned} V(\mathcal{X}) &= V_1(x_1) + \int_{x_2^*(x_1)}^{x_2} \left\lceil \lceil s \rceil^{\frac{\mu}{m_2}} - \lceil x_2^*(x_1) \rceil^{\frac{\mu}{m_2}} \right\rceil^{\frac{2\omega-\sigma-m_2}{\mu}} ds \\ &\quad + \left(\frac{\mu}{2\omega - \sigma - m_1 + \mu} \right) |e(\mathcal{X})|^{\frac{2\omega-\sigma-m_1+\mu}{\mu}} \\ &\geq V_1(x_1) + \varepsilon_1 \left| x_2 - x_2^*(x_1) \right|^{\frac{2\omega-\sigma}{m_2}} \\ &\quad + \left(\frac{\mu}{2\omega - \sigma - m_1 + \mu} \right) \\ &\quad \times \left| \lceil x_2 \rceil^{\frac{\mu}{m_2}} - \nu - L(x_1) \right|^{\frac{2\omega-\sigma-m_1+\mu}{\mu}} \end{aligned} \tag{36}$$

for all $\mathcal{X} \in \mathbb{M}_3(\kappa_l, \kappa_u)$; this directly gives, with any fixed $x_1 \in \mathbb{M}_1(\kappa_l, \kappa_u)$, $V(\mathcal{X}) \to \infty$ as $\|(x_2, \nu)\| \to \infty$. Hence, according to (35) and Lemma 1, one knows that when $\mathcal{X}(t_0) \in \mathbb{M}_3(\kappa_l, \kappa_u)$, every solution $\mathcal{X}(t) = (x(t), \nu(t))$ of system (1) under the controller (31) is defined on $[t_0, \infty)$ and fulfills $-\kappa_l < y(t) = x_1(t) < \kappa_u$ for all $t \geq t_0$.

Part IV—Analysis of the finite-time convergence

In what follows, we shall prove the finite-time convergence of system (1) under the controller (31). For this purpose, we consider the circumstance with $\mathcal{X}(t_0) \in \mathbb{M}_3(\kappa_l, \kappa_u)$. From (35) and (36), it follows readily that $V(\mathcal{X}(t))$ is non-increasing on $[t_0, \infty)$ and one also has

$$\begin{aligned} 0 &\leq V_1(x_1(t)) + \varepsilon_1 \left| x_2(t) - x_2^*(x_1(t)) \right|^{\frac{2\omega-\sigma}{m_2}} \\ &\quad + \left(\frac{\mu}{2\omega - \sigma - m_1 + \mu} \right) \\ &\quad \times \left| \lceil x_2(t) \rceil^{\frac{\mu}{m_2}} - \nu(t) - L(x_1(t)) \right|^{\frac{2\omega-\sigma-m_1+\mu}{\mu}} \\ &\leq V(\mathcal{X}(t_0)) < \infty \end{aligned}$$

for all $t \in [t_0, \infty)$, and thus $\mathcal{X}(t)$ is uniformly bounded on $[t_0, \infty)$. With these in mind, a tedious but straightforward analysis verifies directly that $|\xi_1(x_1(t))|^{2\omega/\mu}$, $|\xi_2(x(t))|^{2\omega/\mu}$ and $|e(\mathcal{X}(t))|^{2\omega/\mu}$ are uniformly continuous on $[t_0, \infty)$, and the fact $\lim_{t\to\infty} V(\mathcal{X}(t)) = \varepsilon_2 \leq V(\mathcal{X}(t_0))$ for some real constant $\varepsilon_2 \geq 0$; this leads to

$$\lim_{t\to\infty} \int_{t_0}^{t} |\xi_1(x_1(s))|^{\frac{2\omega}{\mu}} \, ds < \infty$$

$$\lim_{t\to\infty} \int_{t_0}^{t} |\xi_2(x(s))|^{\frac{2\omega}{\mu}} \, ds < \infty$$

$$\lim_{t\to\infty} \int_{t_0}^{t} |e(\mathcal{X}(s))|^{\frac{2\omega}{\mu}} \, ds < \infty.$$

Consequently, using Barbalat's lemma, one can conclude that when $\mathcal{X}(t_0) \in \mathbb{M}_3(\kappa_l, \kappa_u)$, $|\xi_1(x_1(t))| \to 0$, $|\xi_2(x(t))| \to 0$ and $|e(\mathcal{X}(t))| \to 0$ as $t \to \infty$; thus, by the fact $L(0) = 0$ and the definitions of $\xi_1(x_1(t)), \xi_2(x(t))$ and $e(\mathcal{X}(t))$, it follows immediately that $\mathcal{X}(t) \to 0$ as $t \to 0$. Now, in view of the continuity and positiveness of $V(\mathcal{X})$, there exists an open connected set $\mathbb{S} = \{\mathcal{X} \in \mathbb{M}_3(\kappa_l, \kappa_u) \mid V(\mathcal{X}) \leq \varepsilon_3\} \subseteq \mathbb{M}_3(\kappa_l, \kappa_u)$ for some real constant $\varepsilon_3 > 0$ such that $-\kappa_l/2 < x_1 < \kappa_u/2$ and $-1 < e(\mathcal{X}) < 1$ for all $\mathcal{X} \in \mathbb{S}$, which as well as Lemma 3 gives

$$\begin{aligned} &\dot{V}(\mathcal{X}) + 2^{4\omega-3} V^{\frac{2\omega}{2\omega-\sigma}}(\mathcal{X}) \\ &\quad \leq -\frac{1}{8} \left(|\xi_1(x_1)|^{\frac{2\omega}{\mu}} + |\xi_2(x)|^{\frac{2\omega}{\mu}} + |e(\mathcal{X})|^{\frac{2\omega}{\mu}} \right) \leq 0 \end{aligned} \tag{37}$$

for all $(\mathcal{X}, t) \in \mathbb{S} \times \mathbb{R}^+$. In light of the construction of $\mathbb{S}$, there exists $T^* \in [t_0, \infty)$ such that $\mathcal{X}(t) \in \mathbb{S}$ for all $t \geq T^*$ whenever $\mathcal{X}(t_0) \in \mathbb{M}_3(\kappa_l, \kappa_u)$; hence, from (37) we have

$$\dot{V}(\mathcal{X}(t)) \leq -2^{4\omega-3} V^{\frac{2\omega}{2\omega-\sigma}}(\mathcal{X}(t)) \tag{38}$$

for all $t \geq T^*$. If $V(\mathcal{X}(T^*)) = 0$, observing (38) and noting the positiveness of $V(\mathcal{X})$, one obtains $\mathcal{X}(t) = 0$ for all $t \geq T^*$. In the case when $V(\mathcal{X}(T^*)) \neq 0$, it can deduced from (38) that

$$\begin{aligned} &V(\mathcal{X}(t)) - V(\mathcal{X}(T^*)) \\ &\quad \leq \int_{T^*}^{t} -2^{4\omega-3} V^{\frac{2\omega}{2\omega-\sigma}}(\mathcal{X}(s)) \, ds \end{aligned}$$

for all $t \geq T^*$, which by Lemma 6 also results in $\mathcal{X}(t) = 0$ for all $t \geq T^{**}$ for some $T^{**} \in (T^*, \infty)$. Combining these two case, one can conclude that when $\mathcal{X}(t_0) \in \mathbb{M}_3(\kappa_l, \kappa_u)$, every trajectory $\mathcal{X}(t) = (x(t), \nu(t))$ of system (1) under the controller (31) converges to the origin in finite time. □

Remark 5 The proof of Theorem 1 explicitly presents a constructive approach to synthesizing a continuous output feedback finite-time stabilizer for system (1) and fulfilling the requirement of the asymmetric output constraint specified in advance. The philosophy and idea behind the development of this approach is to skillfully renovate the adding a power integrator technique based upon the interactive collaboration of the presented *fraction*-type asymmetric BLF (9) and the reduced-order non-smooth observer (25). An attractive trait of the proposed approach is the capability/feasibility of simultaneously coping with, in a unification fashion, the problem of output feedback finite-time stabilization for system (1) subjected to or free from output constraints. More precisely, when the output constraint is deliberately assigned to be infinity, i.e., $\kappa_l = \kappa_u = \kappa$ with $\kappa \to \infty$ (considering the scenario of no constraint), it follows immediately from Remark 5 that $V(\mathcal{X}) = V_2(x) + V_3(\mathcal{X})$ naturally molts into $V_\infty(\mathcal{X})$ having the following structure

$$
\begin{aligned}
V_\infty(\mathcal{X}) = & \frac{1}{(2\omega-\sigma)}|x_1|^{2\omega-\sigma} \\
& + \int_{x_2^*(x_1)}^{x_2} \left\lceil \lceil s\rfloor^{\frac{\mu}{m_2}} - \lceil x_2^*(x_1)\rfloor^{\frac{\mu}{m_2}} \right\rfloor^{\frac{2\omega-\sigma-m_2}{\mu}} \mathrm{d}s \\
& + \left(\frac{\mu}{2\omega-\sigma-m_1+\mu}\right)|e(\mathcal{X})|^{\frac{2\omega-\sigma-m_1+\mu}{\mu}}
\end{aligned}
$$

which is defined on $\mathbb{R}^3$ and is obviously continuously differentiable and positive definite on $\mathbb{R}^3$. Moreover, by using the schemes almost similar to those mentioned in [7,17], it can be further verified that $V_\infty(\mathcal{X})$ is proper; that is, the preimage of any compact set in $\mathbb{R}^+$ under $V_\infty(\mathcal{X})$ is compact. When there is no constraint requirement, adopting $V_\infty(\mathcal{X})$ in the convergence analysis one can immediately show by following the same procedure described in the proof of Theorem 1 that the output feedback controller (12) remains usable and straightly acts a global finite-time stabilizer for system (1), without needing of changing the controller and observer structures. Hence, when the asymmetric output constraint is intentionally assigned to be infinity so as to take into account the scenario of no constraint imposed on the output, the presented method will directly become a *pure* stabilization scheme under which the synthesized output feedback controller as well as the corresponding state observer secures the same structures as (12) and behaves effectively as a global finite-time stabilizer for system (1). This discloses explicitly that the presented approach is a unification methodology by which one is capable of performing simultaneously the design of a continuous output feedback finite-time stabilizer for system (1) subjected to or free from output constraints.

4 An illustrative example

In order to demonstrate the superiority and effectiveness of the proposed scheme, now we consider a planar system as below

$$
\begin{aligned}
\dot{x}_1 &= x_2^3 + \sin(2t)\cos(6x_1)\ln(1+x_1^2) \\
\dot{x}_2 &= \theta(x,t)u + \cos(5x_2+t)\sin(x_1)
\end{aligned}
\tag{39}
$$

where $\theta(x,t) = 1 + 0.3\cos(2x_1x_2 + 0.5t)$. System (39) is structurally identical to system (1) with $p = 3$, $\phi_1(x_1,t) = \sin(2t)\cos(6x_1)\ln(1+x_1^2)$ and $\phi_2(x,t) = \cos(5x_2+t)\sin(x_1)$. It is clear that Assumption 1 is fulfilled with $\underline{\theta}(x_1) = 0.7$ and $\overline{\theta}(x_1) = 1.3$. Simply choosing $m_1 = 1$ and $\sigma = -1/5$, one has $m_2 = 4/15$ and $\omega = \mu = 1$. By using the mean value theorem, it is easy to verify that

$$
\begin{aligned}
&\sin(2t)\cos(6x_1)\ln(1+x_1^2) \le 1.5|x_1|^{\frac{4}{5}} \\
&\cos(5x_2+t)\sin(x_1) \le |x_1|^{\frac{1}{15}}
\end{aligned}
$$

for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$; hence, Assumption 2 is satisfied with $\overline{\phi}_1(x_1) = 1.5$ and $\overline{\phi}_2(x_1) = 1$. Following the procedure given by the proof of Theorem 1 we now consider

$$
V_1(x_1) = \frac{5\kappa_u^{\frac{11}{5}}\kappa_l^{\frac{11}{5}}|x_1|^{\frac{11}{5}}}{11(\kappa_u - x_1)^{\frac{11}{5}}(\kappa_l + x_1)^{\frac{11}{5}}}
$$

and select $x_2^*(x_1) = -\beta_1(x_1)\lceil\xi_1(x_1)\rfloor^{4/15}$ with $\beta_1(x_1) = \left(1.5 + 2\lambda^{-1}(x_1)\right)^{1/3}$. A simple calculation leads to

$$
\dot{V}_1(x_1) \le -2\xi_1^2(x_1) + \lambda(x_1)\lceil\xi_1(x_1)\rfloor^{\frac{6}{5}}\left(x_2^3 - x_2^{*3}(x_1)\right)
$$

for all $(x,t) \in \mathbb{M}_2(\kappa_l,\kappa_u) \times \mathbb{R}^+$, where the function $\lambda(x_1)$ takes the form below

$$
\lambda(x_1) = \frac{\kappa_u^{\frac{11}{5}}\kappa_l^{\frac{11}{5}}\left(x_1^2 + \kappa_u\kappa_l\right)}{(\kappa_u - x_1)^{\frac{16}{5}}(\kappa_l + x_1)^{\frac{16}{5}}}.
$$

To proceed with the controller synthesis, we next take

$$V_2(x) = V_1(x_1) + \int_{x_2^*(x_1)}^{x_2} \left\lceil \lceil s \rceil^{\frac{15}{4}} - \lceil x_2^*(x_1) \rceil^{\frac{15}{4}} \right\rceil^{\frac{29}{15}} \mathrm{d}s$$

for which the time derivative along system (39) is

$$\begin{aligned}\dot V_2(x) &\le -2\xi_1^2(x_1) + \theta(x,t)\lceil \xi_2(x)\rceil^{\frac{29}{15}} u \\ &+ \lambda(x_1)\lceil \xi_1(x_1)\rceil^{\frac{6}{5}} \left(x_2^3 - x_2^{*3}(x_1)\right) \\ &+ |\xi_1(x_1)|^{\frac{1}{15}} |\xi_2(x)|^{\frac{29}{15}} - 1.94\Upsilon(x_1)\dot x_1 \\ &\times \int_{x_2^*(x_1)}^{x_2} \left| \lceil s \rceil^{\frac{15}{4}} - \lceil x_2^*(x_1) \rceil^{\frac{15}{4}} \right|^{\frac{14}{15}} \mathrm{d}s\end{aligned}$$

for all $(x,t) \in \mathbb{M}_2(\kappa_l,\kappa_u) \times \mathbb{R}^+$, where $\Upsilon(x_1) = -(1.5 + 2\lambda^{-1}(x_1))^{1/4}(1.5\lambda^2(x_1) + 2\lambda(x_1) - 2.5x_1(\partial\lambda(x_1)/\partial x_1))\ \lambda^{-2}(x_1)$. Similarly to deducing (17), (19) and (20), we employ Lemmas 2–4 to deduce the following two estimations

$$\begin{aligned}&\lambda(x_1)\lceil \xi_1(x_1)\rceil^{\frac{6}{5}} \left(x_2^3 - x_2^{*3}(x_1)\right) + |\xi_1(x_1)|^{\frac{1}{15}} |\xi_2(x)|^{\frac{29}{15}} \\ &\le \frac{2}{3}\xi_1^2(x_1) + \left(\psi_2(x_1) + \hat\psi_2(x_1)\right)\xi_2^2(x) \\ &\quad - 1.94\Upsilon(x_1)\dot x_1 \int_{x_2^*(x_1)}^{x_2} \left| \lceil s \rceil^{\frac{15}{4}} - \lceil x_2^*(x_1) \rceil^{\frac{15}{4}} \right|^{\frac{14}{15}} \mathrm{d}s \\ &\le \frac{1}{3}\xi_1^2(x_1) + \tilde\psi_2(x_1)\xi_2^2(x)\end{aligned}$$

for all $(x,t) \in \mathbb{M}_2(\kappa_l,\kappa_u) \times \mathbb{R}^+$, where $\psi_2(x_1) = 1.37\ \lambda^{5/2}(x_1)$, $\hat\psi_2(x_1) = 0.89$ and $\tilde\psi_2(x_1) = ((2.51\beta_1^3(x_1) + 3.76)^{5/3} + 3.22)(1 + \Upsilon^2(x_1))^{5/6}$. In addition, considering $V_3(\mathcal{X}) = 5/11|e(\mathcal{X})|^{11/5}$, one has

$$\begin{aligned}\dot V_3(\mathcal{X}) &\le -0.74G^{\frac{1}{4}}(x_1)|e(\mathcal{X})|^2 + G^{-\frac{1}{2}}(x_1)|e(\mathcal{X})|^{\frac{1}{5}}|x_2|^{\frac{27}{4}} \\ &\quad + 3.75\lceil e(\mathcal{X})\rceil^{\frac{6}{5}}|x_2|^{\frac{11}{4}}\ (\theta(x,t)u + \phi_2(x,t))\end{aligned}$$

for all $(\mathcal{X},t) \in \mathbb{M}_3(\kappa_l,\kappa_u) \times \mathbb{R}^+$. Again, using Lemmas 2 and 3 , one can easily obtain

$$\begin{aligned}&G^{-\frac{1}{2}}(x_1)|e(\mathcal{X})|^{\frac{1}{5}}|x_2|^{\frac{27}{4}} \\ &\quad \le \psi_5(x_1)\xi_1^2(x_1) + \hat\psi_5\xi_2^2(x) + e^2(\mathcal{X}) \\ &3.75\lceil e(\mathcal{X})\rceil^{\frac{6}{5}}|x_2|^{\frac{11}{4}}\ (\theta(x,t)u + \phi_2(x,t)) \\ &\quad \le \frac{1}{2}\xi_1^2(x_1) + \frac{1}{3}\xi_2^2(x) + (\psi_4(x_1) + \psi_6(x_1))\, e^2(\mathcal{X})\end{aligned}$$

for all $(\mathcal{X},t) \in \mathbb{M}_3(\kappa_l,\kappa_u) \times \mathbb{R}^+$, in which $\psi_5(x_1) = 1.97\beta_1^{15/2}(x_1)$, $\hat\psi_5 = 1.97$, and $\psi_4(x_1) + \psi_6(x_1) = 7.234 + 2\beta_1^3(x_1)$; moreover, using Lemmas 2 and 4 , one also has

$$\begin{aligned}&\theta(x,t)\beta_2(x_1)\lceil \xi_2(x)\rceil^{\frac{29}{15}} \left| \lceil \xi_2(x)\rceil^{\frac{1}{15}} - \lceil \xi_2(x) - e(\mathcal{X})\rceil^{\frac{1}{15}} \right| \\ &\le \frac{1}{6}\xi_2^2(x) + \psi_7(x_1)e^2(\mathcal{X})\end{aligned}$$

for all $(\mathcal{X},t) \in \mathbb{M}_3(\kappa_l,\kappa_u) \times \mathbb{R}^+$, where $\psi_7(x_1) = 3.132\ \beta_2^2(x_1)$ with $\beta_2(x_1) = (1 + \psi_2(x_1) + \hat\psi_2(x_1) + \tilde\psi_2(x_1))/0.7$. Therefore, we choose $L(x_1) = 671.28x_1$ and directly design the output feedback finite-time controller and the observer as described by (31) and (25), respectively, such that

$$\begin{aligned}\dot V(\mathcal{X}) &= \dot V_2(x) + \dot V_3(\mathcal{X}) \\ &\le -\frac{1}{4}\left(\xi_1^2(x_1) + \xi_2^2(x) + e^2(\mathcal{X})\right)\end{aligned}$$

for all $(\mathcal{X},t) \in \mathbb{M}_3(\kappa_l,\kappa_u) \times \mathbb{R}^+$. For demonstration, the initial time and the initial state in the simulations are set to be $t_0 = 0$ and $(x_1(0), x_2(0), \nu(0)) = (-1, -2.2, 0)$, respectively.

From the simulation results shown in Figs. 1, 2 and 3, it can be found that the designed output feedback controller not only finite-time stabilizes system (39) but also successfully ensures the fulfillment of the asymmetric output constraint $-1.5 = -\kappa_l < y(t) = x_1(t) < \kappa_u = 1$ for all $t \ge 0$. Furthermore, in the scenario when the output constraint is purposely assigned to be quite large (e.g., $\kappa_u = \kappa_l = 100$) in order to simulate the scenario of almost no constraint on the output $y(t) = x_1(t)$, the designed output feedback controller as well as the associated observer is still valid for finite-time stabilizing system (39), without needing to change the controller and observer structures; this also demonstrates the unification of our approach in performing simultaneously the construction of a continuous output feedback finite-time stabilizer subjected to or free from output constraints.

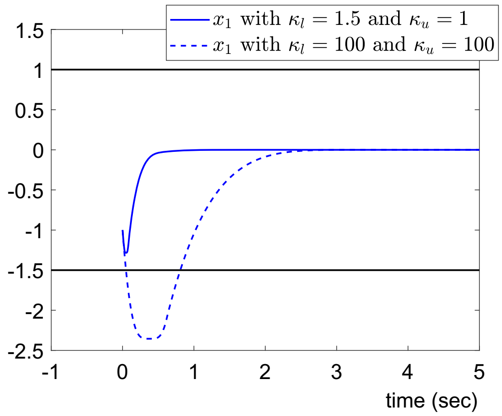

Fig. 1 Timing responses of x_1 with two different κ_l and κ_u

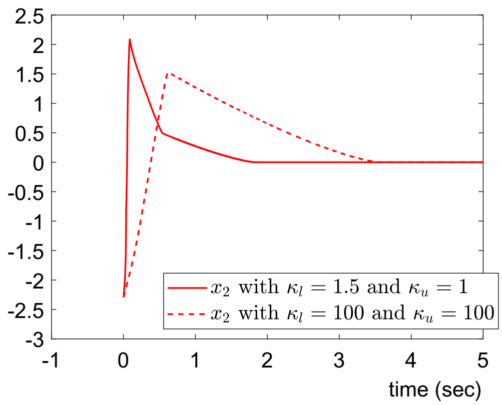

Fig. 2 Timing responses of x_2 with two different κ_l and κ_u

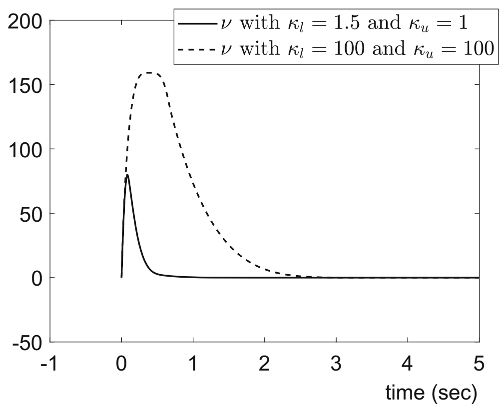

Fig. 3 Timing responses of ν with two different κ_l and κ_u

5 Conclusion

We have presented a solution to the problem of output feedback finite-time stabilization for a significant class of high-order planar systems subjected to asymmetric output constraints. A novel design methodology was proposed by skillfully renovating the technique of adding a power integrator with the subtle implantation of a new developing fraction-type asymmetric barrier Lyapunov function as well as a delicate non-smooth state observer. With full extraction and utilization of the characteristics of system nonlinearities, the proposed scheme enjoys an appealing and attractive property that it enables us to straightly unify and achieve simultaneously the design of a continuous output feedback finite-time stabilizer for systems subjected to or free from asymmetric output constraints, without needing to change the controller and observer structures.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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