



# The influence of the first integrals and the rolling resistance model on tippe top inversion

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**Abstract** In this paper, we analyze the effect which the choice of a friction model has on tippe top inversion in the case where the resulting action of all dissipative forces is described not only by the force applied at the contact point, but also by the additional rolling resistance torque. We show that the possibility or impossibility of tippe top inversion depends on the existence of specific integrals of the motion of the system. In this paper, we consider an example of the law of rolling resistance by which the area integral is preserved in the system. We examine in detail the case where the action of all dissipative forces reduces to the horizontal rolling resistance torque. This model describes fast rotations of the top between two horizontal smooth planes. For this case, we find permanent rotations of the system and analyze their linear stability. The stability analysis suggests that no tippe top inversion is possible under fast rotations between two planes.

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## 1 Introduction

In this paper, we analyze the dynamics of the tippe top on a smooth plane in the presence of forces and torques of rolling resistance. Tippe top inversion has attracted the attention of researchers for the last decades [6–9, 12–14, 20–24]. Reference [8] discusses the development of a spherical prototype of the top which is capable of performing various modes of motion (in particular, complete or partial inversion) by changing the mass-geometric characteristics. Many studies from the last century [9, 13, 14, 20–22] and from the early part of this century [6, 7, 12, 23, 24] gave mathematical explanations of tippe top inversion. They investigated the stability of steady-state (dissipation-free) solutions, analyzed parameter values, and found cases where tippe top inversion is possible.

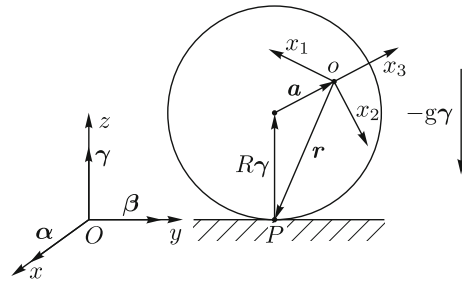
A detailed analysis of the dynamics of an axisymmetric top is possible since the system has the Jellett integral. In his “Treatise on the Theory of Friction” [11], Jellett pointed out that some quantity (named later after him) in the problem of the motion of a body of revolution on a plane remains unchanged when adding an arbitrary friction force applied at the point of contact. There exist various generalizations and analogs of the

Jellett integral. In this paper, we do not claim to provide their complete description, but only give an example of the existence of an analog of Jellett's integral in non-holonomic systems. In particular, [1] presents a number of generalizations of Jellett's integral in the problems of bodies of revolution rolling over a sphere: for a ball with a displaced center of mass and for a dynamically symmetric ball with a balanced gyrostat.

Most studies of tippe top dynamics use the classical sliding friction model, which is proportional to the velocity of the point of contact of the tippe top with the plane. However, various friction models have been proposed recently for a more accurate analysis which provides not only a qualitative, but also a quantitative description of the system dynamics. For example, a comparative analysis of the most frequently used friction models is made in [19]. Reference [3] presents a discussion of some friction models frequently used to capture the motion of a ball on a turntable. This paper examines in detail the rolling resistance model and shows that it adequately describes the ball not only qualitatively, but also quantitatively (a similar study of the dynamics of balls and disks was made in [18]). The authors of [15] present a model of viscous rolling friction which gives a fairly accurate description of the rolling motion of spherical bodies on a horizontal plane. Also, this friction model explains qualitatively some dynamical effects, in particular, retrograde motion of a rolling disk at the final stage [4].

In this paper, we address the problem of the influence of the friction model on tippe top inversion. In particular, we examine the situation where the resultant action of all dissipative forces is described not only by the force applied at the point of contact, but also by an additional rolling resistance torque. It turns out that depending on the chosen friction model, the system admits different first integrals: the Jellett integral, the Lagrange integral or the area integral. As our investigations show, the integral that is preserved in the system has a direct influence on the possibility of top inversion. In this paper, we show that preservation of the area integral prevents top inversion. As an example of this phenomenon, we investigate the motion of the top between two smooth horizontal planes in the presence of horizontal rolling resistance torque.

In this work, we also carry out a qualitative analysis of tippe top dynamics in the case where the action of dissipative forces reduces to the horizontal rolling resistance torque. This resistance model describes fast rota-



**Fig. 1** Schematic model of a spherical tippe top

tions of the tippe top relative to the vertical axis between two parallel horizontal smooth planes. We show that in this friction model, no tippe top inversion is observed.

## 2 Equations of motion and conservation laws

### 2.1 Formulation of the problem

Consider the motion of a heavy unbalanced ball of radius  $R$  and mass  $m$  with axisymmetric mass distribution which rolls with slipping on a horizontal plane under the action of gravity (Fig. 1). The system is acted upon by different resistance forces, which depend on the type of coating of the contacting surfaces, air resistance etc. As is well-known, this system of forces generally reduces to the resultant of resistance forces,  $\mathbf{F}$ , and to the resistance torque  $\mathbf{M}_f$ . We assume that in this case, the motion of the ball is subject to the following assumptions:

- the ball contacts the surface at one point  $P$ ;
- the resultant of resistance forces,  $\mathbf{F}$ , is applied to the point of contact;
- the ball is acted upon by the principal rolling resistance torque  $\mathbf{M}_f$ , which includes torque  $\mathbf{r} \times \mathbf{F}$ , but may not be equal to it in the general case.

To describe the motion of the ball, we introduce two coordinate systems:

- a fixed (inertial) coordinate system  $Oxyz$  with origin on the supporting plane and with the axis  $Oz$  directed vertically upward.
- a moving coordinate system  $ox_1x_2x_3$  attached to the ball, with origin at the center of mass of the system and with the axis  $ox_3$  directed along the symmetry axis of the ball.

In what follows, unless otherwise specified, all vectors will be referred to the moving axes  $ox_1x_2x_3$ .

We assume that the center of mass of the system is displaced relative to the geometric center of the ball along its symmetry axis by distance  $a$  and is given by the vector  $\mathbf{a} = (0, 0, a)$ .

Let us denote the projections of the unit vectors directed along the fixed axes  $Oxyz$  onto the axes of the moving coordinate system  $ox_1x_2x_3$  as follows:

$$\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \alpha_3), \quad \boldsymbol{\beta}=(\beta_1, \beta_2, \beta_3), \quad \boldsymbol{\gamma}=(\gamma_1, \gamma_2, \gamma_3).$$

The orthogonal matrix  $\mathbf{Q} \in SO(3)$  whose columns are the coordinates of the vectors  $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$  specifies the orientation of the body in space.

*Remark* By definition, the vectors  $\boldsymbol{\alpha}, \boldsymbol{\beta}$  and  $\boldsymbol{\gamma}$  satisfy the relations

$$\begin{aligned} (\boldsymbol{\alpha}, \boldsymbol{\alpha}) &= 1, & (\boldsymbol{\beta}, \boldsymbol{\beta}) &= 1, & (\boldsymbol{\gamma}, \boldsymbol{\gamma}) &= 1, \\ (\boldsymbol{\alpha}, \boldsymbol{\beta}) &= 0, & (\boldsymbol{\beta}, \boldsymbol{\gamma}) &= 0, & (\boldsymbol{\gamma}, \boldsymbol{\alpha}) &= 0. \end{aligned}$$

Let  $\mathbf{v}$  be the velocity of the center of mass of the ball, and let  $\boldsymbol{\omega}$  be its angular velocity, both defined in the coordinate system  $ox_1x_2x_3$ . Then, the evolution of the orientation and of the position of the ball is given by the kinematic relations

$$\begin{aligned} \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \boldsymbol{\omega}, & \dot{\boldsymbol{\alpha}} &= \boldsymbol{\alpha} \times \boldsymbol{\omega}, & \dot{\boldsymbol{\beta}} &= \boldsymbol{\beta} \times \boldsymbol{\omega}, \\ \dot{x} &= (\mathbf{v}, \boldsymbol{\alpha}), & \dot{y} &= (\mathbf{v}, \boldsymbol{\beta}), \end{aligned} \tag{1}$$

where  $x$  and  $y$  are the coordinates of the center of mass  $o$  of the ball in the fixed coordinate system  $Oxyz$ . The coordinate  $z$  of the center of mass is uniquely defined from the condition that the ball move without loss of contact with the plane

$$z + (\mathbf{r}, \boldsymbol{\gamma}) = 0, \tag{2}$$

where the radius vector of the contact point  $\mathbf{r}$  in the axes  $ox_1x_2x_3$  can be represented in the form

$$\mathbf{r} = -R\boldsymbol{\gamma} - \mathbf{a}. \tag{3}$$

### 2.2 Equations of motion

Differentiating Eq. (2) taking (3) and the relation  $\dot{z} = (\mathbf{v}, \boldsymbol{\gamma})$  into account, we obtain a (holonomic) constraint equation in the form

$$f = (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}, \boldsymbol{\gamma}) = (\mathbf{v}_p, \boldsymbol{\gamma}) = 0, \tag{4}$$

where  $\mathbf{v}_p$  is the velocity of the ball at the point of contact with the plane.

We write the Newton–Euler equations for changing the linear and angular momenta of the ball in the form

$$m\dot{\mathbf{v}} + m\boldsymbol{\omega} \times \mathbf{v} = N \frac{\partial f}{\partial \mathbf{v}} + \mathbf{F} - mg\boldsymbol{\gamma}, \tag{5}$$

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = N \frac{\partial f}{\partial \boldsymbol{\omega}} + \mathbf{M}_f,$$

where  $\mathbf{I} = \text{diag}(i_1, i_1, i_3)$  is the central tensor of inertia of the ball,  $g$  is the free-fall acceleration,  $\mathbf{F}$  is the friction force applied at the point of contact,  $\mathbf{M}_f$  is the rolling resistance torque,  $N \frac{\partial f}{\partial \mathbf{v}}$  and  $N \frac{\partial f}{\partial \boldsymbol{\omega}}$  are the force and the torque of the reaction of the supporting plane, respectively. We recall that the rolling resistance torque  $\mathbf{M}_f$  includes friction torque  $\mathbf{r} \times \mathbf{F}$ , but is not equal to it in the general case.

Let us express  $N$  from the first equation of (5) and from the derivative of the constraint equation (4) with respect to time  $\frac{df}{dt} = 0$

$$N = m(\mathbf{v}, \boldsymbol{\gamma})' - (\mathbf{F}, \boldsymbol{\gamma}) + mg. \tag{6}$$

If the surfaces of the ball and the plane are homogeneous (but not necessarily isotropic), then the force  $\mathbf{F}$  and the torque  $\mathbf{M}_f$  depend only on the variables  $(\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\gamma})$ . In this case, the system of equations describing the change of variables  $(\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\gamma})$  closes in itself and takes the form

$$\begin{aligned} m\dot{\mathbf{v}} + \boldsymbol{\omega} \times m\mathbf{v} + m(\boldsymbol{\omega}, \mathbf{r} \times \boldsymbol{\gamma})' \boldsymbol{\gamma} &= \mathbf{F}_h, \\ \mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} + m((\boldsymbol{\omega}, (\mathbf{r} \times \boldsymbol{\gamma})') - g)\mathbf{r} \times \boldsymbol{\gamma} &= \\ &= \mathbf{M}_f - \mathbf{r} \times \mathbf{F}_v, \end{aligned} \tag{7}$$

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0,$$

where  $\mathbf{J} = \mathbf{I} + m(\mathbf{r} \times \boldsymbol{\gamma}) \otimes (\mathbf{r} \times \boldsymbol{\gamma})^1$ ,  $\mathbf{F}_h = \mathbf{F} - (\mathbf{F}, \boldsymbol{\gamma})\boldsymbol{\gamma}$  is the horizontal component of force  $\mathbf{F}$ , and  $\mathbf{F}_v = (\mathbf{F}, \boldsymbol{\gamma})\boldsymbol{\gamma}$  is its vertical component.

In addition, these equations must be restricted to the submanifold given by the constraint equation (4) and the geometric relation

$$(\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}, \boldsymbol{\gamma}) = 0, \quad \boldsymbol{\gamma}^2 = 1. \tag{8}$$

Since these functions are first integrals of the system (7), this restriction is satisfied uniquely.

*Remark* In principle, one can consider the trajectories of the system (7) on other level sets of the integrals (8), but they have no explicit physical interpretation.

Thus, Eqs. (1) and (7) completely describe the motion of the tippe top on a smooth plane with friction.

<sup>1</sup> Symbol  $\otimes$  denotes tensor multiplication of vectors, which associates the matrix  $\mathbf{A}$  with components  $A_{i,j} = a_i b_j$  to the pair of vectors  $\mathbf{a}, \mathbf{b}$ .

### 2.3 Law of resistance and additional integral of motion

In [16], we addressed the problem of the existence of additional integrals for different laws of rolling resistance and different mass distributions of spherical tops. As shown in [16], different integrals can remain unchanged for different versions of the resistance laws. For systems with an arbitrary resistance law, the following proposition holds.

**Proposition 1** *The system (7), which describes the rolling of the ball with axisymmetric mass distribution on the plane, admits an additional motion integral, linear in  $\omega$ , for an arbitrary rolling resistance force  $F$  under the following restrictions on the rolling resistance torque  $M_f$ :*

1°. *if  $(M_f, r) = 0$ , then the Jellett integral is preserved*

$$G = -(\mathbf{J}\omega, r) = \text{const};$$

2°. *if  $(M_f, e_3) = 0$ , then the Lagrange integral is preserved*

$$F = i_3\omega_3 = \text{const};$$

3°. *if  $(M_f, \gamma) = 0$ , then the area integral is preserved*

$$C = (\mathbf{J}\omega, \gamma) = \text{const}.$$

The purpose of this paper is to illustrate the dependence of the possibility of top inversion on the preservation of a specific integral of motion.

In the classical model explaining the top inversion [7, 11, 12, 24, 25], the rolling resistance is described only by the friction force applied at the point of contact. As is well-known, in this case, the Jellett integral is preserved and top inversion is possible regardless of the specific type of friction force satisfying the condition of energy dissipation. The proof of top inversion is based on the stability analysis of steady rotations depending on the system parameters and the value of the first integral of motion. As shown in [12, 24], the existence and stability of steady rotations is completely determined by the integral of motion and does not depend on the specific type of friction force.

In this paper, we investigate the dynamics of a top on a plane in the case where the rolling resistance preserves the area integral. We show that the existence of the area integral prevents top inversion. To do so, we consider a specific law of rolling resistance in which the

projection of the total resistance torque onto the vertical is zero, i.e.,  $(M_f, \gamma) = 0$ . In other words, there is no resistance to the spinning (twisting) of the ball.

Using the approaches [12] or [24], one can show that the stability of steady rotations in the case at hand is also determined only by the integral of motion. Consequently, the results obtained for the friction model under consideration also hold for an arbitrary law of rolling resistance that preserves the area integral.

We investigate the simplest friction model which satisfies the condition  $(M_f, \gamma) = 0$ , i.e., preserves the area integral, in the form

$$F = 0, \quad M_f = -\mu_r \omega_h, \tag{9}$$

where  $\omega_h = \omega - (\omega, \gamma)\gamma$  is the horizontal component of the angular velocity and  $\mu_r$  is the coefficient of rolling friction.

Physically, the model (9) features, for example, the case of fast rotations of a ball with slipping between two horizontal planes in the presence of friction. As was shown by Contensou [10], in the case of fast rotations, the contact point experiences viscous friction. In this case, the friction force and the friction torque have the form

$$F = -2\mu v_0 \ll 1, \\ M_f = -2\mu R^2 \omega_h - 2\mu v_0 \times a \approx -2\mu R^2 \omega_h, \tag{10}$$

where  $v_0 = v - \omega \times a$  is the velocity of the geometric center of the top. In the case of fast rotations, the condition  $|v_0| \ll |\omega|R$  is satisfied. As a result, the friction force and the friction torque (10) coincide with sufficient accuracy with (9).

*Remark* The law of friction (9) may be applied in examining the motion of other bodies on a plane, but it will not have the above-mentioned physical interpretation, but can be considered only from a mathematical point of view.

### 3 Reduction in the equations of motion

We now turn to a study of the dynamics of the ball using the friction model (9).

For further analysis of the equations, we introduce dimensionless variables in the following form:

$$t \rightarrow \sqrt{\frac{R}{g}} t, \quad r \rightarrow Rr, \quad \omega \rightarrow \sqrt{\frac{g}{R}} \omega, \quad \mathbf{I} \rightarrow mR^2 \mathbf{I}, \\ a \rightarrow Ra, \quad \mathcal{E} \rightarrow mgR\mathcal{E}.$$

Such a change of variables is equivalent to

$$m = 1, \quad R = 1, \quad g = 1.$$

In the case under consideration, the equations of motion for  $\omega, \gamma$  decouple from the complete system and have the form

$$\begin{aligned} \mathbf{J}\dot{\omega} + \omega \times \mathbf{I}\omega + ((\omega, \mathbf{r} \times (\gamma \times \omega)) - 1)\mathbf{r} \times \gamma &= \\ = \mu_r \gamma \times (\omega \times \gamma), & \quad (11) \\ \dot{\gamma} + \omega \times \gamma = 0. \end{aligned}$$

Due to the existence of a pair of integrals, the phase space of the system (11) is foliated by four-dimensional invariant submanifolds

$$\mathcal{M}_C^4 = \{(\omega, \gamma) \mid (\gamma, \gamma) = 1, (\mathbf{J}\omega, \gamma) = C\}.$$

The analysis of the flow on the submanifolds  $\mathcal{M}_C^4$  becomes simpler since the system (11) has the symmetry field

$$\hat{u} = \omega_1 \frac{\partial}{\partial \omega_2} - \omega_2 \frac{\partial}{\partial \omega_1} + \gamma_1 \frac{\partial}{\partial \gamma_2} - \gamma_2 \frac{\partial}{\partial \gamma_1},$$

which defines rotations about the symmetry axis of the ball  $ox_3$ . This symmetry makes it possible to perform reduction (reduce the order of the system) in the invariant submanifolds  $\mathcal{M}_C^4$ . For this, we proceed as follows. Let us choose variables  $\gamma_3, K_1, K_2, \varphi$  which parameterize  $\mathcal{M}_C^4$  so that three of them are first integrals of the symmetry field [5]:

$$\begin{aligned} \gamma_3 = \gamma_3, \quad K_1 = i_3 \omega_3, \\ K_2 = \frac{1}{k}(\gamma_1 \omega_2 - \gamma_2 \omega_1), \quad \varphi = \arctan \frac{\gamma_2}{\gamma_1}, \end{aligned}$$

where  $k = \sqrt{\frac{1-\gamma_3^2}{i_1+a^2(1-\gamma_3^2)}}$ . The inverse transformation has the form

$$\begin{aligned} \gamma_1 = \sqrt{1-\gamma_3^2} \cos \varphi, \quad \gamma_2 = \sqrt{1-\gamma_3^2} \sin \varphi, \\ \omega_1 = \frac{(C-\gamma_3 K_1)\gamma_1 - i_1 k K_2 \gamma_2}{i_1(1-\gamma_3^2)} \\ \omega_2 = \frac{(C-\gamma_3 K_1)\gamma_2 + i_1 k K_2 \gamma_1}{i_1(1-\gamma_3^2)}, \quad \omega_3 = \frac{K_1}{i_3}. \end{aligned}$$

The evolution of the new variables on the level set  $\mathcal{M}_C^4$  is governed by the equations

$$\begin{aligned} \dot{\gamma}_3 = k K_2, \quad \dot{K}_1 = -\frac{\mu_r}{i_1} (K_1 \tilde{k} - \gamma_3 C), \\ \dot{K}_2 = -\frac{k(C-\gamma_3 K_1)(C\gamma_3 - K_1)}{i_1(1-\gamma_3^2)^2} - ka - \frac{\mu_r K_2 k^2}{1-\gamma_3^2}, \end{aligned} \quad (12)$$

$$\dot{\varphi} = \frac{\dot{\gamma}_1 \gamma_2 - \dot{\gamma}_2 \gamma_1}{\gamma_1^2 + \gamma_2^2} = \frac{K_1}{i_3} - \frac{\gamma_3(C - K_1 \gamma_3)}{i_1(1 - \gamma_3^2)}, \quad (13)$$

where we have introduced the notation  $\tilde{k} = (i_1 - (i_1 - i_3)\gamma_3^2)/i_3$ . Due to the special choice of variables, the first three equations decouple and form a closed reduced system. It is this system that we will investigate in what follows.

The energy of the system (12) in the presence of rolling resistance forces is not preserved and, up to constant terms, has the form

$$\mathcal{E} = \frac{(C - K_1 \gamma_3)^2}{2i_1(1 - \gamma_3^2)} + \frac{K_1^2}{2i_3} + \frac{K_2^2}{2} + a\gamma_3.$$

#### 4 Permanent rotations and their stability

Permanent rotations of the system under consideration correspond to motion of the ball with a constant inclination angle of the symmetry axis relative to the vertical, i.e., when  $\gamma_3 = \text{const}$ . It is obvious that during such motions there should be no dissipation:  $\dot{\mathcal{E}} = 0$ . Permanent rotations are given by the equations

$$\dot{\gamma}_3 = 0, \quad \dot{\gamma}_3 = 0. \quad (14)$$

Solving Eqs. (14), we find the following partial solutions to the system:

1. Two one-parameter families of fixed points

$$\begin{aligned} \sigma_u : \gamma_3 = 1, \quad K_1 = C, \quad K_2 = 0, \\ \sigma_l : \gamma_3 = -1, \quad K_1 = -C, \quad K_2 = 0. \end{aligned} \quad (15)$$

These families correspond to vertical rotations of the ball where the center of mass is above the geometric center ( $\sigma_u$ ) or below it ( $\sigma_l$ ).

2. A one-parameter family of periodic solutions

$$\sigma_0 : \gamma_3 = -\frac{a}{c_1^2(i_1 - i_3)}, \quad K_1 = \frac{ai_3}{c_1(i_1 - i_3)}, \quad K_2 = 0, \quad (16)$$

where  $c_1 \in (-\infty, -c_0) \cup (c_0, \infty)$ ,  $c_0 = \sqrt{a/|i_1 - i_3|}$ , is the parameter of the family. This family of periodic solutions corresponds to rotations of the ball where its symmetry axis deviates from the vertical by the angle  $\vartheta = \arccos \gamma_3$ .

The initial variables  $\omega, \gamma$  and the value of the integral  $C$  are parameterized through  $c_1$  as follows:

$$\omega_1 = c_1 p \cos \varphi, \quad \omega_2 = c_1 p \sin \varphi, \quad \omega_3 = \frac{a}{c_1(i_1 - i_3)},$$

$$\gamma_1 = -p \cos \varphi, \quad \gamma_2 = -p \sin \varphi, \quad \gamma_3 = -\frac{a}{c_1^2(i_1 - i_3)},$$

$$C = -c_1 i_1 + \frac{a^2}{c_1^3(i_1 - i_3)},$$

where  $p = \sqrt{1 - \gamma_3^2}$ .

Using this parameterization and the obvious inequality  $|\gamma_3| \leq 1$ , we can define the critical value of the integral  $C$  at which permanent rotations arise or disappear:

$$C^* = C|_{c_1=c_0} = \frac{i_3 \sqrt{a}}{\sqrt{|i_1 - i_3|}}.$$

### 4.1 Stability analysis

Let us analyze the linear stability of partial solutions  $\sigma_u$  and  $\sigma_l$ . These solutions are families of fixed points of the complete system (11). To investigate their stability, we represent the system of differential equations (11) as

$$\dot{\mathbf{q}} = \mathbf{f}_q(\mathbf{q}),$$

where  $\mathbf{q} = (\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3)$ ,  $\mathbf{f}_q(\mathbf{q})$  is the vector whose components are functions of  $\mathbf{q}$ . Let us linearize the system (11) near the solutions  $\sigma_u, \sigma_l$

$$\tilde{\dot{\mathbf{q}}} = \mathbf{L}_q \tilde{\mathbf{q}}, \quad \mathbf{L}_q = \left. \frac{\partial \mathbf{f}_q(\mathbf{q})}{\partial \mathbf{q}} \right|_{\mathbf{q}=\mathbf{q}_{u,l}},$$

where  $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}_{u,l}$ , and  $\mathbf{q}_u$  and  $\mathbf{q}_l$  are the partial solutions  $\sigma_u$  and  $\sigma_l$  of the system (11), respectively.

The characteristic equation of the linearized system

$$\det(\mathbf{L}_q - \lambda \mathbf{E}) = 0$$

with eigenvalues  $\lambda$  ( $\mathbf{E}$  being an  $6 \times 6$  identity matrix) is an equation of degree 6 in  $\lambda$  of the form

$$P_6(\lambda) = \lambda^2 P_4(\lambda) = \lambda^2(a_0 \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4) = 0.$$

Two zero eigenvalues of the linearized system correspond to the geometric integral  $\mathcal{Y}^2 = 1$  and to the area integral  $C$  (which is the parameter of the family).

To investigate the stability problem, we use the Routh–Hurwitz criterion for definition of the sign of the real part of the roots of algebraic equations.

As is well-known [17], the real parts of all roots of the equation are negative in the case where all diagonal minors of the Hurwitz matrix are positive

$$\Delta_1 = a_1, \quad \Delta_2 = a_1 a_2 - a_0 a_3,$$

$$\Delta_3 = a_3 \Delta_2 - a_1^2 a_4, \quad \Delta_4 = a_4 \Delta_3$$

under the condition that the coefficient with the highest degree,  $a_0 > 0$ , is positive.

1. *The partial solution  $\sigma_l$  corresponding to the lower vertical rotation.* In the variables  $\mathbf{q}$ , this solution is parameterized as follows:

$$\mathbf{q}_l : \quad \begin{aligned} \gamma_1 &= 0, & \gamma_2 &= 0, & \gamma_3 &= -1, \\ \omega_1 &= 0, & \omega_2 &= 0, & \omega_3 &= \frac{C}{i_3}. \end{aligned}$$

The coefficient with the highest degree and the diagonal minors of the Hurwitz matrix which corresponds to the polynomial  $P_4(\lambda)$  have the form

$$\begin{aligned} a_0 &= i_1^2, \\ \Delta_1 &= 2i_1 \mu_r, \\ \Delta_2 &= 2i_1 \mu_r \left( \frac{C^2}{i_3^2} (3i_1^2 - 3i_1 i_3 + i_3^2) + ai_1 + \mu_r^2 \right), \\ \Delta_3 &= 4i_1 \mu_r^2 \left( \frac{C^2}{i_3^2} (2i_1 - i_3)^2 + \mu_r^2 \right) \left( a - \frac{C^2}{i_3^2} (i_1 - i_3) \right), \\ \Delta_4 &= 4i_1 \mu_r^2 \left( \frac{C^2}{i_3^2} (2i_1 - i_3)^2 + \mu_r^2 \right) \left( a - \frac{C^2}{i_3^2} (i_1 - i_3) \right)^3. \end{aligned}$$

It is easy to see that the values of  $a_0, \Delta_1, \Delta_2$  are positive when  $\mu_r > 0$ , and the values of  $\Delta_3$  and  $\Delta_4$  are positive under the condition

$$\begin{aligned} |C| &< C^* \quad \text{for } i_1 - i_3 > 0, \\ |C| &> -C^* \quad \text{for } i_1 - i_3 < 0. \end{aligned}$$

Thus, when  $i_1 < i_3$ , the lower rotation  $\sigma_l$  is always stable, and when  $i_1 > i_3$ , it is stable only if the area integral has values  $|C| < C^*$ .

2. *The partial solution  $\sigma_u$  corresponding to the upper vertical rotation.* In the variables  $\mathbf{q}$ , this solution is parameterized as follows:

$$\mathbf{q}_u : \quad \begin{aligned} \gamma_1 &= 0, & \gamma_2 &= 0, & \gamma_3 &= 1, \\ \omega_1 &= 0, & \omega_2 &= 0, & \omega_3 &= \frac{C}{i_3}. \end{aligned}$$

The coefficient with the highest degree and the diagonal minors of the Hurwitz matrix which corresponds to the polynomial  $P_4(\lambda)$  have the form

$$\begin{aligned} a_0 &= i_1^2, \\ \Delta_1 &= 2i_1 \mu_r, \\ \Delta_2 &= 2i_1 \mu_r \left( \frac{C^2}{i_3^2} (3i_1^2 - 3i_1 i_3 + i_3^2) - ai_1 + \mu_r^2 \right), \\ \Delta_3 &= -4i_1 \mu_r^2 \left( \frac{C^2}{i_3^2} (2i_1 - i_3)^2 + \mu_r^2 \right) \left( a - \frac{C^2}{i_3^2} (i_3 - i_1) \right), \\ \Delta_4 &= -4i_1 \mu_r^2 \left( \frac{C^2}{i_3^2} (2i_1 - i_3)^2 + \mu_r^2 \right) \left( a - \frac{C^2}{i_3^2} (i_3 - i_1) \right)^3. \end{aligned}$$

The values of  $a_0, \Delta_1, \Delta_2$  are always positive when  $\mu_r > 0$ , and the values of  $\Delta_3$  and  $\Delta_4$  are positive under the condition

$$\begin{aligned} |C| &> C^* \text{ for } i_3 - i_1 > 0, \\ |C| &< -C^* \text{ for } i_3 - i_1 < 0. \end{aligned}$$

Consequently, when  $i_3 < i_1$ , the upper rotation  $\sigma_u$  is always unstable, and when  $i_3 > i_1$ , it is stable only for sufficiently large values of the area integral,  $|C| > C^*$ .

3. *The partial solution  $\sigma_0$  corresponding to permanent rotation.* We now investigate the stability of permanent rotations  $\sigma_0$ . These rotations are periodic solutions of the complete system (11) and correspond to fixed points of the reduced system (12).

To analyze the orbital stability of rotations  $\sigma_0$ , we examine the reduced system (12). To do so, we represent the system of differential equations (12) as

$$\dot{\xi} = \mathbf{f}_\xi(\xi), \tag{17}$$

where  $\xi = (\gamma_3, K_1, K_2)$ ,  $\mathbf{f}_\xi(\xi)$  is the vector whose components are functions of  $\xi$ . Let  $\xi_0$  be a partial solution to (17) that corresponds to permanent rotations  $\sigma_0$ .

We linearize the system (17) near the partial solution  $\xi_0$  and obtain a system of the form

$$\dot{\tilde{\xi}} = \mathbf{L}_\xi \tilde{\xi}, \quad \mathbf{L}_\xi = \left. \frac{\partial \mathbf{f}_\xi(\xi)}{\partial \xi} \right|_{\xi=\xi_0},$$

where  $\tilde{\xi} = \xi - \xi_0$ ,  $\mathbf{L}_\xi$  is a linearization matrix.

The characteristic equation of the linearized system

$$\det(\mathbf{L}_\xi - \lambda \mathbf{E}) = 0$$

with eigenvalues  $\lambda$  is an equation of degree 3 in  $\lambda$

$$P_3(\lambda) = a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0.$$

The coefficients of the characteristic equation have the form

$$\begin{aligned} a_0 &= -i_1 i_3 c_2 (i_1 - i_3) (i_1 + a^2 (1 - c_2^2)), \\ a_1 &= -\mu_r c_2 (i_1 - i_3) \left( i_1 i_3 (1 + c_2^2) \right. \\ &\quad \left. + (i_1^2 + i_1 a^2 - a^2 c_2^2 (i_1 - i_3)) (1 - c_2^2) \right), \\ a_2 &= -\mu_r^2 c_2 (i_1 - i_3) (i_1 - c_2^2 (i_1 - i_3)) \\ &\quad + a i_3 (i_1^2 + c_2^2 (i_1 - i_3)) (3i_1 + i_3), \\ a_3 &= a \mu_r (i_1 - i_3) (1 - c_2^2) (i_1 + 3c_2^2 (i_1 - i_3)), \end{aligned}$$

where  $c_2 = -\frac{a}{c_1^2 (i_1 - i_3)}$  is the value of  $\gamma_3$  for the partial solution  $\sigma_0$  (16).

In accordance with the Routh–Hurwitz criterion, the conditions for negativeness of the real parts of the roots of the characteristic equation have the form

$$a_0 > 0, \quad a_1 > 0, \quad a_1 a_2 - a_0 a_3 > 0, \quad a_3 > 0.$$

It is easy to show that all inequalities are satisfied under the condition  $i_1 > i_3$ .

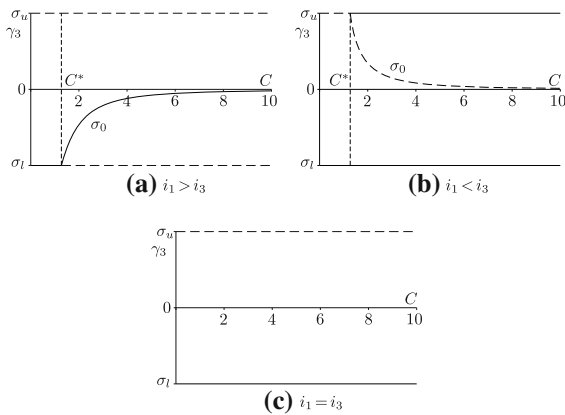
Thus, permanent rotations  $\sigma_0$  are stable only for  $i_1 > i_3$  over the entire interval of their existence.

The result of linear stability analysis of the solution (15), (16) for different values of  $i_1$  and  $i_3$  is presented in Fig. 2. This figure shows families of periodic solutions  $\sigma_u, \sigma_l$  and  $\sigma_0$  on the plane  $(C, \gamma_3)$ . The solid lines represent stable permanent solutions, and the dashed lines are unstable ones. As is evident from the analysis of eigenvalues and from the figure, there are three different cases:

1.  $i_1 > i_3$  (Fig. 2a). In this case, the upper vertical rotation  $\sigma_u$  is always unstable. The lower vertical rotation  $\sigma_l$  is stable for small absolute values of the integral,  $|C| < C^*$ . Permanent rotations  $\sigma_0$  exist for  $|C| > C^*$  and are always stable.
2.  $i_1 < i_3$  (Fig. 2b). In this case, the lower vertical rotation  $\sigma_l$  is always stable. The upper vertical rotation  $\sigma_u$  is stable for sufficiently large absolute values of the integral,  $|C| > C^*$ . Permanent rotations  $\sigma_0$  exist for  $|C| > C^*$  and are always unstable.
3.  $i_3 = i_1$  (Fig. 2c). In this case, the upper vertical rotation  $\sigma_u$  is always unstable, and the lower vertical rotation  $\sigma_l$  is always stable. In this case, there exist no permanent rotations.

As shown in Fig. 2, the situation where on the fixed level set of the integral  $C$  the lower vertical rotation is unstable and the upper one is stable is not observed in this model. Thus, in the friction model (9), a complete *tippe top inversion is impossible under any initial conditions*. As we will see below, this is also confirmed by the analysis of the dependence of the energy of permanent rotations on the area integral.

Nevertheless, the case  $i_1 < i_3$  admits a partial *tippe top inversion* when the *tippe top* tends to permanent rotation at a constant inclination angle of the axis, with a small deviation from the lower vertical rotation. However, in this case, the center of mass of the *tippe top* always lies below the center of the ball. As the initial energy of the *tippe top* increases, the critical inclination angle of the *tippe top* tends to  $\frac{\pi}{2}$  ( $\gamma_3 \rightarrow 0$ ).



**Fig. 2** Stable and unstable rotations of the tippe top versus  $C$  for different ratios of  $i_1$  and  $i_3$  for the parameters  $a = 0.29$ ,  $i_3 = 0.51$ ,  $\mu_r = 1$  and **a**  $i_1 = 0.55$ , **b**  $i_1 = 0.46$ , **c**  $i_1 = 0.51$

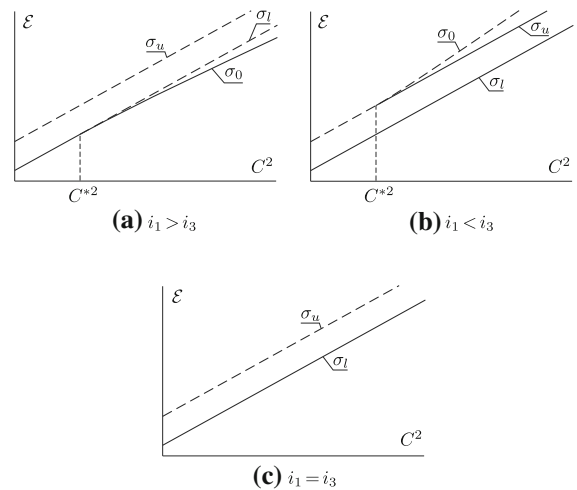
**5 Qualitative dynamics analysis**

A global qualitative analysis of the dynamics of the system can be carried out, for example, using a modified Routh theory by analogy with the study of the classical model of the tippe top [12]. The above partial solutions which correspond to permanent and vertical rotations  $\sigma_0$ ,  $\sigma_u$  and  $\sigma_l$  can be represented in the generalized Smale diagram on the plane  $(C^2, \mathcal{E})$ , where the values of  $\mathcal{E}$  correspond to the magnitude of the initial energy of the system for a given value of the integral  $C$ . Fixing the level set of the integral  $C$  in the diagram and defining the initial level of energy, we can keep track of the dynamics of the system under energy dissipation.

Figure 3 shows generalized Smale diagrams corresponding to different ratios of the moments of inertia of the tippe top. The solid lines correspond to stable steady motions, and the dashed lines to unstable ones. On the fixed level set of the integral  $C$ , all trajectories of the system tend to stable solutions due to dissipation.

It follows from the analysis of the diagram that depending on the system parameters and initial conditions, the following behavior of the tippe top can be observed:

1.  $i_1 > i_3$  (Fig. 3a). When the value of the integral is  $|C| < C^*$ , almost all trajectories tend to lower vertical rotations  $\sigma_l$ , and when  $|C| > C^*$ , they tend to permanent rotations  $\sigma_0$ .
2.  $i_1 < i_3$  (Fig. 3b). When the value of the integral is  $|C| < C^*$ , almost all trajectories tend to lower vertical rotations  $\sigma_l$ . When  $|C| > C^*$ , the trajectories



**Fig. 3** Generalized Smale diagrams of the system (12) for  $a = 0.29$ ,  $i_3 = 0.51$  and **a**  $i_1 = 0.55$ , **b**  $i_1 = 0.46$ , **c**  $i_1 = 0.51$

tend either to lower ( $\sigma_l$ ) or to upper ( $\sigma_u$ ) vertical rotations. We note that the investigation of the domains of attraction of either of the stable solutions is a topic in its own right and goes beyond the scope of this paper. This problem can be solved, for example, by constructing charts of dynamical regimes on the plane of initial conditions.

3.  $i_3 = i_1$  (Fig. 3c). In this case, almost all trajectories tend to lower vertical rotations  $\sigma_l$ .

Thus, no tippe top inversion is possible for the resistance model considered.

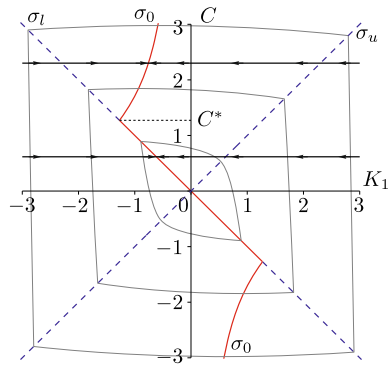
We illustrate the dynamics of the system by constructing a projection of the phase flow onto the plane  $(K_1, C)$ . This projection is convenient for distinguishing between trajectories that are attracted to various periodic solutions  $\sigma_u$  and  $\sigma_l$ , but start at the same point in the Smale diagram (Fig. 3).

In addition, this plane is convenient when it comes to comparing the dynamics of the system with the case without friction. In the dissipation-free case, in the space of first integrals  $(K_1, C, \mathcal{E})$ , one usually constructs a surface of regular precessions [2] which, with friction added, becomes the boundary of the region of possible motions of the system (in this space).

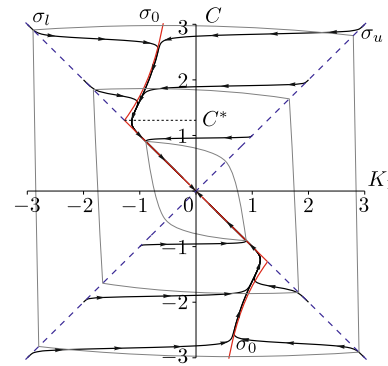
Figure 4 shows projections of the phase trajectories onto the plane  $(K_1, C)$  for the case  $i_1 > i_3$ , when a partial tippe top inversion is observed.

As above, the solid lines indicate stable solutions and the dashed lines represent unstable ones. The thin solid lines correspond to levels of constant energy  $\mathcal{E} =$





**Fig. 4** Projections of the phase trajectories of the system (12) onto the plane  $(K_1, C)$  for the case  $i_1 > i_3$  with the parameters  $a = 0.29, i_1 = 0.55, i_3 = 0.51, \mu_r = 1$



**Fig. 5** Projections of the phase trajectories of the system (18) onto the plane  $(K_1, C)$  for the case  $i_1 > i_3$  with the parameters  $a = 0.29, i_1 = 0.55, i_3 = 0.51, \mu_r = 1, \mu_s = 0.001$

const. The arrows denote the directions of the system trajectories.

As can be seen from the figure, on the fixed level set of the integral  $C$ , all trajectories tend to stable solutions (permanent  $\sigma_0$  or lower vertical  $\sigma_l$  rotations).

### 5.1 Motion with spinning friction torque

To conclude, we briefly examine the case where the additional integral  $C$  ceases to exist by incorporating into the system spinning friction torque in the form

$$M_s = -\mu_s(\omega, \gamma)\gamma,$$

where  $\mu_s$  is the coefficient of spinning friction. We assume that the spinning friction is much smaller than the rolling friction, i.e.,  $\mu_s \ll \mu_r$ .

The equations of motion of (12) involving rolling and spinning friction torques take the form

$$\begin{aligned} \dot{\gamma}_3 &= kK_3, \\ \dot{K}_1 &= -\frac{\mu_r}{i_1} (K_1 \tilde{k} - \gamma_3 C) - \frac{\mu_s \gamma_3}{i_1 i_3} (K_1 \gamma_3 (i_1 - i_3) + C i_3), \\ \dot{K}_2 &= -\frac{k(C - \gamma_3 K_1)(C \gamma_3 - K_1)}{i_1 (1 - \gamma_3^2)^2} - ka - \frac{\mu_r K_2 k^2}{1 - \gamma_3^2}, \\ \dot{C} &= -\frac{\mu_s}{i_1 i_3} (K_1 \gamma_3 (i_1 - i_3) + C i_3). \end{aligned} \tag{18}$$

As an example, we consider the effect of adding small spinning friction on the global dynamics of the system in the case of a partial tippe top inversion.

In this case (under the condition  $\mu_s \ll \mu_r$ ), on times  $t < T_0$ , the area integral  $C$  can be taken to be approximately constant. The characteristic time  $T_0$  is determined by the friction coefficient  $\mu_r$ . The dynamics on

times  $t < T_0$  is the tendency of the tippe top to permanent rotations  $\sigma_0$ . In this case, a partial inversion (elevation of the symmetry axis) of the tippe top is observed.

On times  $t \gg T_0$ , an adiabatic change occurs in the value of the integral  $C$ . The dynamics of the tippe top is characterized by slow motions of the system first along the families of permanent rotations  $\sigma_0$  (when  $|C| > C^*$ ) and then along the lower vertical rotations (when  $|C| < C^*$ ). The tippe top comes to a stop at the lower equilibrium point.

As an illustration, we present projections of the phase trajectories of the system (18) onto the plane  $(K_1, C)$  (Fig. 5). As is seen from the figure, all trajectories tend to permanent rotations  $\sigma_0$  at initial  $|C| > C^*$ , then run along this solution and, when  $|C| < C^*$ , evolve onto the solution corresponding to the lower vertical rotation  $\sigma_l$ . At initial  $|C| < C^*$ , the trajectories immediately tend to the solution  $\sigma_l$ .

### 6 Conclusion

In this paper, we have shown that depending on which of the first integrals is preserved, the tippe top can exhibit different behavior when it rolls on a plane.

Within the framework of the friction model (9), the area integral is preserved and no tippe top inversion is possible. Thus, no complete top inversion occurs under rotations of the top between two horizontal smooth planes, regardless of the mass and geometric parameters of the top. Following [12, 24], one can show that, when the area integral is preserved, top inversion is impossible for any law of rolling resistance satisfying

the condition of energy dissipation. An explicit proof of this fact can be the subject of a separate study.

Preliminary research has shown that in the model of rolling resistance in which the Lagrange integral is preserved, no tippe top inversion is observed either. An example of such a law of rolling resistance is isotropic friction where the projection of the total rolling resistance torque onto the axis of symmetry of the top is zero ( $\mathbf{M}_f, \mathbf{e}_3$ ) = 0 (in this case the Lagrange integral remains unchanged). Interestingly, when the Jellett integral (which is a linear combination of the Lagrange integral and the area integral) is preserved, inversion does occur.

Further research can be aimed at a more mathematically rigorous formulation and proof of the properties of the global dynamics of the system. It would also be interesting to experimentally verify the result to the effect that no top inversion is observed under rotations of the ball between two horizontal planes.

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#### Compliance with ethical standards

**Conflict of interest** The authors declare that they have no conflict of interest.

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