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ORIGINAL PAPER

Bilinear Bäcklund transformation, Lax pair and interactions of nonlinear waves for a generalized (2 + 1)-dimensional nonlinear wave equation in nonlinear optics/fluid mechanics/plasma physics

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Abstract In this paper, outcomes of the study on the Bäcklund transformation, Lax pair, and interactions of nonlinear waves for a generalized (2 + 1)-dimensional nonlinear wave equation in nonlinear optics, fluid mechanics, and plasma physics are presented. Via the Hirota bilinear method, a bilinear Bäcklund transformation is obtained, based on which a Lax pair is constructed. Via the symbolic computation, mixed rogue–solitary and rogue–periodic wave solutions are derived. Interactions between the rogue waves and solitary waves, and interactions between the rogue waves and periodic waves, are studied. It is found that (1) the one rogue wave appears between the two solitary waves and then merges with the two solitary waves; (2) the interaction between the one rogue wave and one periodic wave is periodic; and (3) the periodic lump waves with the amplitudes invariant are depicted. Furthermore, effects of the noise perturbations on the obtained solutions will be investigated.

Keywords Nonlinear optics · Fluid mechanics · Plasma physics · (2 + 1)-dimensional nonlinear wave equation · Bilinear Bäcklund transformation · Lax pair · Interactions of nonlinear waves · Noise perturbations

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1 Introduction

Within the past few years, analysis of the nonlinear evolution equations (NLEEs) has been proven to be a nice methodology to explore the machine learning for advancing fluid mechanics [1–3], an organically functionalized surface with major emphasis on microcavity nonlinear optics [4], generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation for fluid mechanics [5], nonlinear Schrödinger (NLS) equation [6], unsteady flow motion equations for the fluid flow and heat transfer [7], and Zakharov–Kuznetsov equation in plasma physics [8]. Recently, the NLEEs suitable to analyze quartic autocatalysis on the dynamics of water conveying [9], thermophoresis and Brownian motion [10] and bioconvection in the MHD nanofluid flow [11] have been presented.

As the localized waves, rogue waves [12–14], solitons [15,16], lumps [17,18] and breathers [19,20] have been linked to the NLEEs with such relevant experimental observations as those on the solitary waves on a coastal-bridge deck [21], rogue waves in a water wave tank [22] and periodic photonic filters [23].

Correspondingly, researchers have searched the analytic solutions using the Bäcklund transformation (BT) for the dispersive long-wave system [24] and modified Kadomtsev–Petviashvili (KP) system [25], inverse scattering method for a coupled Gerdjikov–Ivanov derivative NLS equation [26], generalized unified method for a KP equation [27], KP hierarchy reduction for the nonlinear evolution equation [28] and B-type KP equation [29], Darboux transformation for the generalized AB system [30] and Gerdjikov–Ivanov equation [31], Hirota bilinear method for a quintic time-dependent coefficient derivative NLS equation [32,33], Bilinear neural network method for a B-type KP equation [34,35], homotopy analysis method for the damped Duffing equation [36] and Airy equation [37]. There have been such numerical methods to solve the NLEEs as the analytic approximate method [38], shooting method [39], Runge–Kutta–Fehlberg method [40], finite difference method [41], hybrid block method [42,43], etc.

Among the NLEEs, Ref. [44] has proposed a (3 + 1)-dimensional Hirota bilinear equation,

$$u_{yt} - u_{xxxy} - 3\left(u_x u_y\right)_x - 3u_{xx} + 3u_{zz} = 0, \quad (1)$$

which admits the similar physical meaning as the Korteweg–de Vries (KdV) equation[1] and describes the nonlinear waves in fluid mechanics, plasma physics and weakly dispersive media [46], where u is a real differentiable function of the independent variables x, y, z and t, and the subscripts denote the partial derivatives.

(2 + 1)-dimensionally, Ref. [47] has studied the lumps for Eq. (1) under $z = y$, and Ref. [48] has extended the one in Ref. [47] to a generalized (2 + 1)-dimensional Hirota bilinear equation,

$$u_{yt} + c_1\left[u_{xxxy} + 3\left(2u_x u_y + u u_{xy}\right) + 3u_{xx}\int_{-\infty}^{x} u_y \mathrm{d}x\right] + c_2 u_{yy} = 0, \quad (2)$$

where c_1 and c_2 represent the real constants and $\int$ is the integral operation.

Reference [49] has further extended Eq. (2) to a generalized (2 + 1)-dimensional nonlinear wave equation,

$$u_{yt} + c_1\left[u_{xxxy} + 3\left(2u_x u_y + u u_{xy}\right) + 3u_{xx}\int_{-\infty}^{x} u_y \, \mathrm{d}x\right] + c_2 u_{yy} + c_3 u_{xx} = 0, \quad (3)$$

for certain nonlinear phenomena in nonlinear optics, fluid mechanics and plasma physics, where c_3 is a real constant. Lump, breather and N-soliton solutions as well as their hybrid ones for Eq. (3) have been constructed via the Hirota bilinear method [49], where N is a positive integer. Lump, lumpoff, and rogue wave solutions for Eq. (3) have been investigated [50].

Via the transformation

$$u = 2\,(\ln f)_{xx}\,, \quad (4)$$

bilinear form for Eq. (3) has been obtained as [49]

$$\left(c_1 D_x^3 D_y + c_2 D_y^2 + c_3 D_x^2 + D_t D_y\right) f \cdot f = 0, \quad (5)$$

where f is a real function of x, y and t, D_x, D_y and D_t are the bilinear derivative operators defined by [51]

$$D_x^{m_1} D_y^{m_2} D_t^{m_3} \gamma(x,y,t)\cdot\beta(x,y,t) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{m_1}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^{m_2} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^{m_3} \gamma(x,y,t)\beta(x',y',t')\Big|_{x'=x,y'=y,t'=t}, \quad (6)$$

with $\gamma(x,y,t)$ and $\beta(x',y',t')$ as the differentiable functions, x', y' and t' as the independent variables and m_1, m_2 and m_3 being the nonnegative integers.

However, to our knowledge, bilinear BT, Lax pair, mixed rogue–solitary and rogue–periodic wave solutions for Eq. (3) have not been reported. In Sect. 2, bilinear BT will be given, based on which the Lax pair for Eq. (3) will be constructed. In Sect. 3, the mixed rogue–solitary wave solutions for Eq. (3) and interactions between the rogue waves and solitary waves will be studied. In Sect. 4, the mixed rogue–periodic wave solutions for Eq. (3) and interactions between the rogue waves and periodic waves will be discussed. In Sect. 5, effect of the noise perturbations on the obtained solutions will be investigated. Our conclusions will be drawn in Sect. 6.

2 Bilinear BT and Lax pair for Eq. (3)

In order to obtain the bilinear BT between the solutions f and g of Bilinear Form (5) for Eq. (3), motivated by the method in Ref. [51], the following expression is introduced:

$$P = \left[\left(c_1 D_x^3 D_y + c_2 D_y^2 + c_3 D_x^2 + D_t D_y\right) g \cdot g\right] f^2 - g^2\left[\left(c_1 D_x^3 D_y + c_2 D_y^2 + c_3 D_x^2 + D_t D_y\right) f \cdot f\right], \quad (7)$$

[1] Under the transformation, $t = -T$, $x = X$, $y = X$, $z = X$ and $-u_x = U$, Eq. (1) has been reduced to the KdV equation,

$$U_T + U_{XXX} - 6UU_X = 0,$$

for the acoustic waves in an anharmonic crystal, hydromagnetic waves in a cold plasma or shallow-water waves, where $U(X, Y)$ denotes the wave height, X and T are the independent variables [45,46].

where g is a real function of x, y and t. Using the following exchange formulas for the Hirota bilinear operators [51]:

$$\begin{aligned}&\left(D_x^2 f\cdot f\right)g^2-f^2\left(D_x^2 g\cdot g\right)\\&\quad=2D_x\left(D_x f\cdot g\right)\cdot gf,\\&\left(D_xD_t f\cdot f\right)g^2-f^2\left(D_xD_t g\cdot g\right)\\&\quad=2D_x\left(D_t f\cdot g\right)\cdot gf,\\&\left(D_x^3D_t f\cdot f\right)g^2-f^2\left(D_x^3D_t g\cdot g\right)\\&\quad=2D_t\left(D_x^3 f\cdot g\right)\cdot gf\\&\qquad-6D_x\left(D_xD_t f\cdot g\right)\cdot\left(D_x f\cdot g\right),\end{aligned}\tag{8}$$

Expression (7) can be transformed into

$$\begin{aligned}P&=\left[\left(c_1D_x^3D_y+c_2D_y^2+c_3D_x^2+D_tD_y\right)g\cdot g\right]f^2\\&\quad-g^2\left[\left(c_1D_x^3D_y+c_2D_y^2+c_3D_x^2+D_tD_y\right)f\cdot f\right]\\&=c_1\left[\left(D_x^3D_yg\cdot g\right)f^2-g^2\left(D_x^3D_yf\cdot f\right)\right]\\&\quad+c_2\left[\left(D_y^2g\cdot g\right)f^2-g^2\left(D_y^2f\cdot f\right)\right]\\&\quad+c_3\left[\left(D_x^2g\cdot g\right)f^2-g^2\left(D_x^2f\cdot f\right)\right]\\&\quad+\left[\left(D_tD_yg\cdot g\right)f^2-g^2\left(D_tD_yf\cdot f\right)\right]\\&=2c_1D_y\left(D_x^3g\cdot f\right)\cdot gf\\&\qquad-6c_1D_x\left(D_xD_yg\cdot f\right)\cdot\left(D_xg\cdot f\right)\\&\quad+2c_2D_y\left(D_yg\cdot f\right)\cdot gf\\&\quad+2c_3D_x\left(D_xg\cdot f\right)\cdot gf+2D_t\left(D_yg\cdot f\right)\cdot gf\\&=2D_y\left[\left(c_1D_x^3+c_2D_y+D_t\right)g\cdot f\right]\cdot gf\\&\quad-2D_x\left(3c_1D_xD_yg\cdot f+c_3gf\right)\cdot\left(D_xg\cdot f\right).\end{aligned}\tag{9}$$

Then, bilinear BT for Eq. (3) is obtained as

$$\begin{aligned}&\left(c_1D_x^3+c_2D_y+D_t\right)g\cdot f=\xi_1 gf,\\&3c_1D_xD_yg\cdot f+c_3gf=\xi_2D_xg\cdot f,\end{aligned}\tag{10}$$

with ξ_1 and ξ_2 as the constants, since $P=0$ under Bilinear BT (10).

Two solutions of Bilinear Form (5) for Eq. (3) are chosen as

$$f=1,\qquad g=1+\epsilon e^{\varrho_1x+\varrho_2y+\varrho_3t},\tag{11}$$

where ϵ, ϱ_1, ϱ_2 and ϱ_3 are all the constants to be determined. With the substitution of Expressions (11) into Bilinear BT (10), the related constraints are obtained, i.e.,

$$\varrho_3=-\varrho_1^3c_1-\varrho_2c_2,\quad \xi_2=3\varrho_2c_1,\quad \xi_1=c_3=0.\tag{12}$$

Soliton solutions for Eq. (3) are derived as

$$u=2\left(\ln g\right)_{xx}=\frac{2\varrho_1^2\epsilon e^{\varrho_1^3c_1t+\varrho_2c_2t+\varrho_1x+\varrho_2y}}{\left(e^{\varrho_1^3c_1t+\varrho_2c_2t}+\epsilon e^{\varrho_1x+\varrho_2y}\right)^2},\tag{13}$$

which are in accord with the solitons given in Ref. [49].

Via Bilinear BT (10), under the transformations $\phi=\frac{g}{f}$ and $v=2\left(\ln f\right)_x$, Lax pair

$$L\phi=0,\qquad M\phi=0,\tag{14}$$

is derived, where

$$\begin{aligned}&L=c_1\partial_{xxx}+3c_1v_x\partial_x+c_2\partial_y+\partial_t-\xi_1,\\&M=3c_1\partial_{xy}+3c_1v_y+c_3-\xi_2\partial_x.\end{aligned}$$

Eq. (3) can be derived via $[L,M]=LM-ML=0$ when $u=v_x$.

3 Mixed rogue–solitary wave solutions for Eq. (3)

Motivated by the method in Ref. [52], mixed rogue–solitary wave solutions for Eq. (3) can be assumed, i.e.,

$$\begin{aligned}f&=\Xi_1^2+\Xi_2^2+\zeta_1e^{r_1x+r_2y+r_3t+r_4}\\&\qquad+\zeta_2e^{-r_1x-r_2y-r_3t-r_4}+\alpha_1,\\\Xi_1&=a_1x+a_2y+a_3t+a_4,\\\Xi_2&=a_5x+a_6y+a_7t+a_8,\end{aligned}\tag{15}$$

where a_{i_1}'s ($i_1=1,2,\ldots,8$), r_1, r_2, r_3, r_4, α_1, ζ_1 and ζ_2 are all the real parameters to be determined.

With the substitution of Expression (15) into Bilinear Form (5), the related constraints are obtained as

Case 1

$$\begin{aligned}&a_2=-\frac{a_1r_2}{r_1},\quad a_3=a_1\left(\frac{-a_6c_2}{a_5}-\frac{3c_1r_1^2}{4}\right),\\&a_4=-\frac{a_5a_8}{a_1},\quad a_7=-a_6c_2-\frac{3}{4}a_5c_1r_1^2,\quad r_2=-\frac{a_6r_1}{a_5},\\&r_3=\frac{a_6c_2r_1}{a_5}-\frac{c_1r_1^3}{4},\quad \zeta_2=\frac{\left(a_1^2+a_5^2\right)^2}{r_1^4\zeta_1},\quad c_3=\frac{3a_6c_1r_1^2}{4a_5},\end{aligned}\tag{16}$$

with $r_1\neq0$, $\zeta_1\neq0$, $a_1\neq0$ and $a_5\neq0$;

Case 2

$$\begin{aligned}&r_2=c_3=0,\quad a_3=-a_2c_2,\quad a_5=-\frac{a_1a_2}{a_6},\\&a_7=-a_6c_2,\quad r_3=-c_1r_1^3,\end{aligned}\tag{17}$$

with $a_6\neq0$;

Case 3

$$\begin{aligned}&r_2=r_3=c_1=c_3=0,\quad a_3=-a_2c_2,\\&a_6=\frac{a_2a_5}{a_1},\quad a_7=-a_6c_2,\quad a_8=-\frac{a_1a_4}{a_5},\end{aligned}\tag{18}$$

with $a_1 \neq 0$ and $a_5 \neq 0$.

Under Constraints (16)–(18), via Expression (15), the mixed rogue–solitary wave solutions can be derived.

As an illustration, some three-dimensional plots to study the interactions between the rogue waves and solitary waves are presented. With the substitution of Constraints (16) into Expression (15), when $\alpha_1 > 0$ and $\zeta_1 > 0$, a class of the positive solutions for Bilinear Form (5) is derived as

$$f = \left(1 + \frac{a_5^2}{a_1^2}\right) a_8^2 + \frac{\left(a_1^2 + a_5^2\right) A_2^2}{4} + \alpha_1 + \frac{\left(a_1^2 + a_5^2\right)^2 e^{-B_1}}{r_1^4 \zeta_1} + \zeta_1 e^{B_1},$$

which yields the mixed rogue–solitary wave solutions through Transformation (4) for Eq. (3),

$$u = 2(\ln f)_{xx} = \frac{2\left[-\left(A_1 A_2 + \frac{1}{r_1^3 \zeta_1} A_1^2 e^{-B_1} - r_1 \zeta_1 e^{B_1}\right)^2 + \frac{1}{r_1^2 \zeta_1} e^{-B_2 - B_3} C_1 \left(A_1 e^{B_2} + r_1^2 \zeta_1 e^{B_3}\right)^2\right]}{C_1^2}, \tag{19}$$

where

$$C_1 = A_3 + \frac{A_1^2}{r_1^4 \zeta_1} e^{-B_1} + \zeta_1 e^{B_1},$$

$$B_1 = r_4 + \frac{a_6 c_2 r_1 t}{a_5} - \frac{1}{4} c_1 r_1^3 t + r_1 x - \frac{a_6 r_1 y}{a_5},$$

$$B_2 = \frac{a_6 r_1 y}{a_5},$$

$$B_3 = r_4 + \frac{a_6 c_2 r_1 t}{a_5} + r_1 x - \frac{1}{4} c_1 r_1^3 t,$$

$$A_1 = a_1^2 + a_5^2, \quad A_2 = \frac{4 a_6 c_2 t + 3 a_5 c_1 r_1^2 t - 4 a_5 x - 4 a_6 y}{2 a_5},$$

$$A_3 = \left(1 + \frac{a_5^2}{a_1^2}\right) a_8^2 + \frac{1}{16 a_5^2} \left(a_1^2 + a_5^2\right) (4 a_6 c_2 t + 3 a_5 c_1 r_1^2 t - 4 a_5 x - 4 a_6 y)^2 + \alpha_1.$$

Figure 1 shows the interaction among the two solitary waves and one rogue wave via Solutions (19) with $t = -5$, $t = 0$ and $t = 5$. The rogue wave spreads together with the two solitary waves. During the interaction, two solitary waves and one rogue wave keep their shapes unchanged.

With the substitution of Constraints (17) into Expression (15), when $\alpha_1 > 0$ and $\zeta_1,\ \zeta_2 > 0$, a class of the positive solutions for Bilinear Form (5) is derived as

$$f = [a_4 + a_1 x + a_2(-c_2 t + y)]^2 + \left[a_8 - \frac{a_1 a_2 x}{a_6} + a_6(-c_2 t + y)\right]^2 + \alpha_1 + \zeta_1 e^{r_4 - c_1 r_1^3 t + r_1 x} + \zeta_2 e^{-r_4 + c_1 r_1^3 t - r_1 x},$$

which yields the mixed rogue–solitary wave solutions through Transformation (4) for Eq. (3),

$$u = 2(\ln f)_{xx} = \frac{2\left[-(A_4 + r_1 C_{21})^2 + \left(A_5^2 + A_6^2 + \alpha_1 + C_{22}\right)\left(A_7 + r_1^2 C_{22}\right)\right]}{\left(A_5^2 + A_6^2 + \alpha_1 + C_{22}\right)^2}, \tag{20}$$

where

$$B_4 = r_4 - c_1 r_1^3 t + r_1 x, \quad C_{21} = \zeta_1 e^{B_4} - \zeta_2 e^{-B_4},$$

$$C_{22} = \zeta_1 e^{B_4} + \zeta_2 e^{-B_4},$$

$$A_4 = \frac{2 a_1 \left[a_4 a_6^2 - a_2 a_6 a_8 + a_1 \left(a_2^2 + a_6^2\right) x\right]}{a_6^2},$$

$$A_5 = a_4 + a_1 x + a_2(-c_2 t + y),$$

$$A_6 = a_8 - \frac{a_1 a_2 x}{a_6} + a_6(-c_2 t + y),$$

$$A_7 = a_1^2 \left(2 + 2\frac{a_2^2}{a_6^2}\right).$$

Figure 2 depicts the interaction among the one rogue wave and two solitary waves via Solutions (20). When $t = -3$, Fig. 2a shows the two solitary waves move along the same direction. As t goes on, it is observed that the one rogue wave appears between the two solitary waves, as shown in Fig. 2b. When $t = 3$, one rogue wave merges with the two solitary waves, as shown in Fig. 2c.

With the substitution of Constraints (18) into Expression (15), when $\alpha_1 > 0$ and $\zeta_1,\ \zeta_2 > 0$, a class of the positive solutions for Bilinear Form (5) is derived as

$$f = a_4^2 \left(1 + \frac{a_1^2}{a_5^2}\right) + \frac{\left(a_1^2 + a_5^2\right)[a_1 x + a_2(-c_2 t + y)]^2}{a_1^2} + \alpha_1 + \zeta_1 e^{r_4 + r_1 x} + \zeta_2 e^{-r_4 - r_1 x},$$

which yields the mixed rogue–solitary wave solutions through Transformation (4) for Eq. (3),

$$u=2\left(\ln f\right)_{xx}=2\frac{-\frac{1}{a_1^2}\left(2A_1A_8+a_1r_1\zeta_1e^{B_5}-a_1r_1\zeta_2e^{-B_5}\right)^2+C_3\left(2A_1+r_1^2\zeta_1e^{B_5}+r_1^2\zeta_2e^{-B_5}\right)}{C_3^2}, \tag{21}$$

where

$$B_5=r_4+r_1x,\quad A_8=a_1x+a_2\left(-c_2t+y\right),$$
$$C_3=a_4^2\left(1+\frac{a_1^2}{a_5^2}\right)+\frac{A_1A_8^2}{a_1^2}+\alpha_1+\zeta_1e^{B_5}+\zeta_2e^{-B_5}.$$

Figure 3 shows that one rogue wave spreads together with the two solitary waves via Solutions (21). Two solitary waves and one rogue wave move from negative y axis to positive y axis, as shown in Fig. 3. During the propagation, two solitary waves and one rogue wave keep their shapes unchanged.

4 Mixed rogue–periodic wave solutions for Eq. (3)

Motivated by the method in Ref. [52], mixed rogue–periodic wave solutions for Eq. (3) can be assumed that

$$f=(b_1x+b_2y+b_3t+b_4)^2+(b_5x+b_6y+b_7t+b_8)^2+\kappa\cos[\iota_1x+\iota_2y+\iota_3t+\iota_4]+b_9, \tag{22}$$

where b_{i_2}'s $(i_2=1,2,\ldots,9)$, κ, ι_1, ι_2, ι_3 and ι_4 are all the real parameters to be determined.

With the substitution of Expression (22) into Bilinear Form (5), the related constraints are obtained as

Case 1

$$b_2=\frac{b_5\iota_2}{\iota_1},\quad b_3=\frac{-b_2^2c_2+b_5^2c_3}{b_2},\quad b_6=-\frac{b_1b_2}{b_5},$$
$$b_7=\frac{b_1\left(b_2^2c_2-b_5^2c_3\right)}{b_2b_5},\quad b_9=-\frac{\iota_1^2\kappa^2}{2\left(b_1^2+b_5^2\right)},$$
$$\iota_3=\frac{-c_3\iota_1^2+c_1\iota_1^3\iota_2-c_2\iota_2^2}{\iota_2},\quad c_1=\frac{2c_3}{3\iota_1\iota_2}, \tag{23}$$

with $b_2\neq0$, $b_5\neq0$, $\iota_1\neq0$ and $\iota_2\neq0$;

Case 2

$$c_3=\iota_2=0,\quad \iota_3=c_1\iota_1^3,\quad b_3=-b_2c_2,$$
$$b_6=-\frac{b_1b_2}{b_5},\quad b_7=\frac{b_1b_2c_2}{b_5}, \tag{24}$$

with $b_5\neq0$;

Case 3

$$c_3=\iota_2=b_1=b_5=0,\quad \iota_3=c_1\iota_1^3,$$
$$b_3=\frac{-b_2^3c_2-b_2b_6^2c_2}{b_2^2+b_6^2},$$
$$b_7=\frac{-b_2^2b_6c_2-b_6^3c_2}{b_2^2+b_6^2}, \tag{25}$$

with $b_2^2+b_6^2\neq0$.

Under Constraints (23)–(25), via Expression (22), the mixed rogue–periodic wave solutions can be obtained.

Interactions between the rogue waves and periodic waves under Constraints (24) and (25) will be studied. With the substitution of Constraints (24) into Expression (22), when $b_9>|\kappa|$, a class of the positive solutions for Bilinear Form (5) is derived as

$$f=b_9+\left[b_4+b_1x+b_2\left(-c_2t+y\right)\right]^2+\left(b_8+\frac{b_1b_2c_2t+b_5^2x-b_1b_2y}{b_5}\right)^2+\kappa\cos\left(x\iota_1+c_1t\iota_1^3+\iota_4\right),$$

which yields the mixed rogue–periodic wave solutions through Transformation (4) for Eq. (3),

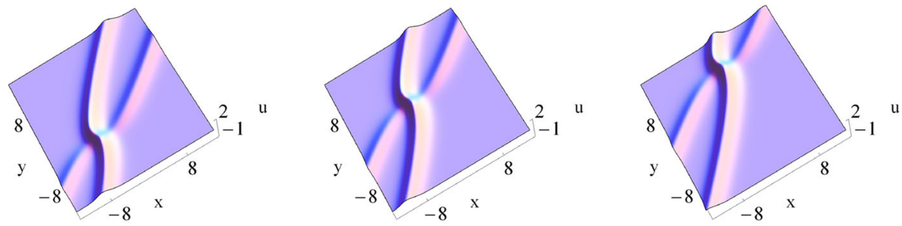
$$u=2\left(\ln f\right)_{xx}=2\frac{\left[W_1+\kappa\cos\left(S_1\right)\right]\left[2W_2-\iota_1^2\kappa\cos\left(S_1\right)\right]-\left[W_3+\iota_1\kappa\sin\left(S_1\right)\right]^2}{\left[W_1+\kappa\cos\left(S_1\right)\right]^2}, \tag{26}$$

where

$$S_1=x\iota_1+c_1t\iota_1^3+\iota_4,$$
$$W_1=b_9+\left[b_4+b_1x+b_2\left(-c_2t+y\right)\right]^2+\left(b_8+\frac{b_1b_2c_2t+b_5^2x-b_1b_2y}{b_5}\right)^2,$$
$$W_2=b_1^2+b_5^2,$$
$$W_3=-2\left[b_1b_4+b_1^2x+b_5\left(b_8+b_5x\right)\right].$$

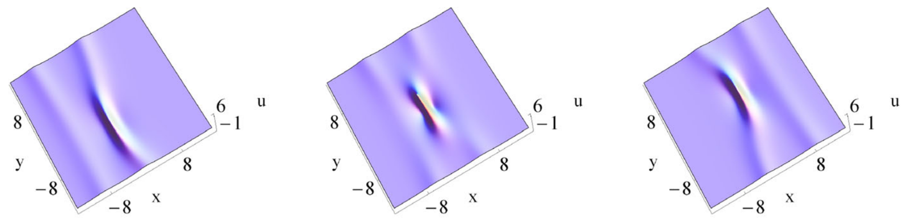
Figure 4 shows the interaction between the one rogue wave and one periodic wave via Solutions (26). The one rogue wave possesses two peaks with different amplitudes. The peak near the positive x axis has the bigger amplitude than the other one, as shown in Fig. 4a. As t goes on, amplitude of the peak near the positive x axis increases and the other peak fades away, as shown in

Fig. 1 Interaction among the one rogue wave and two solitary waves via Solutions (19) with $r_1 = r_4 = 1, a_1 = a_5 = a_6 = \frac{1}{2}, a_8 = 1, \alpha_1 = 1, c_1 = c_2 = 1$ and $\zeta_1 = 4$



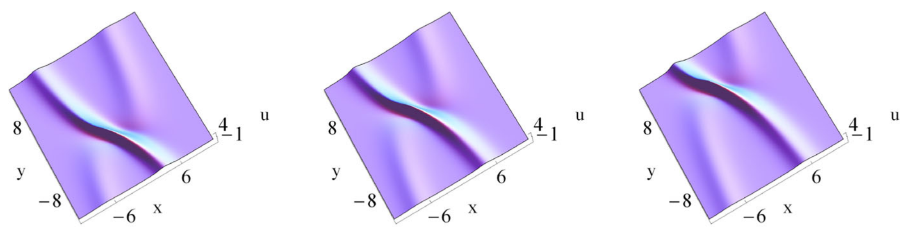
(a) $t = -5$ **(b)** $t = 0$ **(c)** $t = 5$

Fig. 2 Interaction among the one rogue wave and two solitary waves via Solutions (20) with $r_1 = r_4 = 1, a_1 = a_2 = a_4 = a_8 = 1, a_6 = \frac{1}{2}, \alpha_1 = 1, c_1 = c_2 = 1$ and $4\zeta_1 = \zeta_2 = 4$



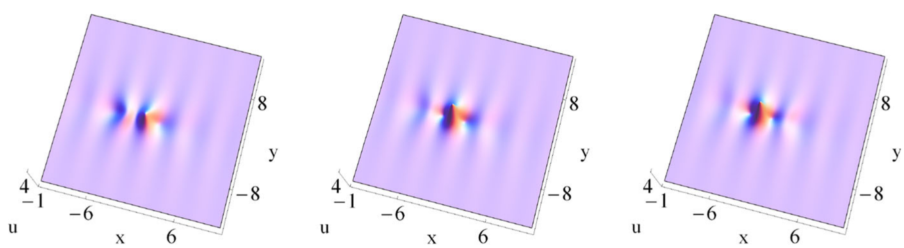
(a) $t = -3$ **(b)** $t = 0$ **(c)** $t = 3$

Fig. 3 Interaction among the one rogue wave and two solitary waves via Solutions (21) with $r_1 = r_4 = 1, a_1 = a_2 = a_4 = 1, a_5 = \frac{2}{3}, \alpha_1 = -2, c_2 = 1$ and $2\zeta_1 = \zeta_2 = 1$

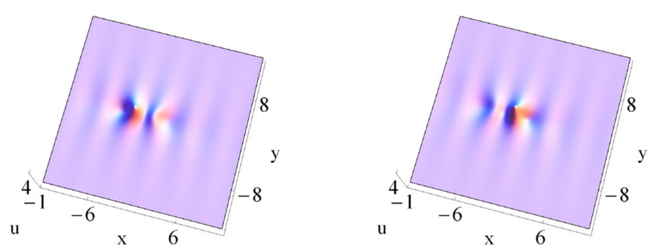


(a) $t = -5$ **(b)** $t = 0$ **(c)** $t = 5$

Fig. 4 Interaction between the one rogue wave and one periodic wave via Solutions (26) with $\iota_1 = \frac{3}{2}, \iota_4 = \frac{1}{2}, 2b_1 = 2b_2 = b_4 = 2b_8 = b_9 = 1, b_5 = \frac{1}{3}, c_1 = 2c_2 = 2$ and $\kappa = \frac{1}{4}$

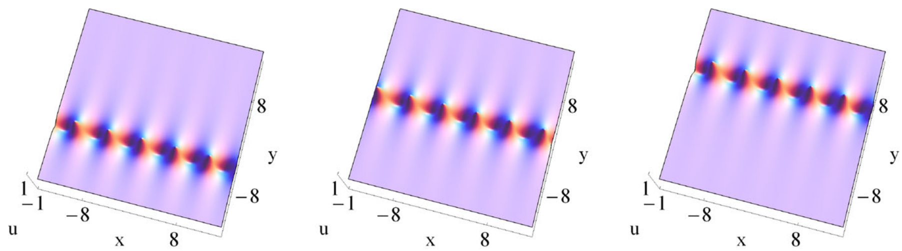


(a) $t = -0.5$ **(b)** $t = -0.25$ **(c)** $t = 0$



(d) $t = 0.25$ **(e)** $t = 0.5$

Fig. 5 Interaction among the one periodic wave and lumps via Solutions (27) with $\iota_1 = \frac{7}{6}$, $\iota_4 = \frac{1}{2}$, $2b_2 = 2b_4 = b_6 = 2b_8 = b_9 = 1$, $c_1 = 2c_2 = 2$ and $\kappa = \frac{1}{4}$



(a) $t = -5$ **(b)** $t = 0$ **(c)** $t = 5$

Fig. 6 The same as Fig. 1a except that $\lambda = 1$

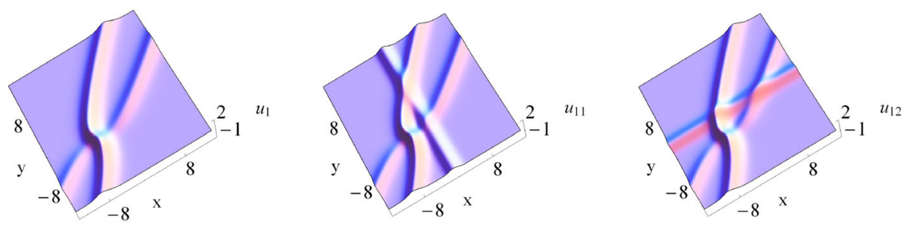


Fig. 1**a** **(a)** u_{11} **(b)** u_{12}

Fig. 4b, c. A new peak near the positive x axis occurs and the amplitude of the other peak decreases, as shown in Fig. 4d. Finally, amplitudes of the two peaks for the rogue wave return to the initial state, as shown in Fig. 4a. Therefore, the interaction between the one rogue wave and one periodic wave is periodic.

With the substitution of Constraints (25) into Expression (22), when $b_9 > |\kappa|$, a class of the positive solutions for Bilinear Form (5) is derived as

$$\begin{aligned} f &= b_9 + [b_4 + b_2(-c_2 t + y)]^2 \\ &\quad + [b_8 + b_6(-c_2 t + y)]^2 + \kappa \cos\left(x\iota_1 + c_1 t\iota_1^3 + \iota_4\right), \end{aligned}$$

which yields the mixed lump-periodic wave solutions through Transformation (4) for Eq. (3),

$$u = 2(\ln f)_{xx} = -2\iota_1^2\kappa \frac{[\kappa + W_4\cos(S_1)]}{[W_4 + \kappa\cos(S_1)]^2}, \tag{27}$$

where

$$\begin{aligned} W_4 &= b_4^2 + b_8^2 + b_9 + b_2^2c_2^2t^2 + b_6^2c_2^2t^2 \\ &\quad - 2b_2^2c_2ty - 2b_6^2c_2ty + b_2^2y^2 + b_6^2y^2 \\ &\quad + 2b_2b_4(-c_2t + y) + 2b_6b_8(-c_2t + y). \end{aligned}$$

Via Solutions (27), for lumps, the central positions are located in

$$\left(\frac{-c_1t\iota_1^3 - \iota_4 + 2\pi\varpi}{\iota_1}, \frac{-b_2b_4 - b_6b_8 + b_2^2c_2t + b_6^2c_2t}{b_2^2 + b_6^2}\right),$$

and

$$\left(\frac{\pi - c_1t\iota_1^3 - \iota_4 + 2\pi\varpi}{\iota_1}, \frac{-b_2b_4 - b_6b_8 + b_2^2c_2t + b_6^2c_2t}{b_2^2 + b_6^2}\right),$$

with the amplitude as

$$\frac{4\left(b_2^2 + b_6^2\right)\left|\left[(b_4b_6 - b_2b_8)^2 + \left(b_2^2 + b_6^2\right)b_9\right]\iota_1^2\kappa\right|}{\left|\left[(b_4b_6 - b_2b_8)^2 + \left(b_2^2 + b_6^2\right)b_9\right]^2 - \left(b_2^2 + b_6^2\right)^2\kappa^2\right|},$$

where ϖ is an integer.

Figure 5 depicts the periodic lump waves with the amplitude invariant. In Fig. 5, it is obvious that the hole is located in $\left(\frac{-54-343t+216\pi\varpi}{126}, \frac{-3+5t}{5}\right)$ and the peak is located in $\left(\frac{-54-343t+108\pi+216\pi\varpi}{126}, \frac{-3+5t}{5}\right)$ with the amplitude as $\frac{1715}{1248}$.

5 Effects of the noise perturbations on Solutions (19), (20) and (27)

Stabilities of the mixed rogue–solitary and rogue–periodic waves will be investigated. Motivated by the method in Refs. [53,54], Mixed Rogue–Solitary Wave Solutions (19) and (20) and Mixed Lump-Periodic Wave Solutions (27) are perturbed by the white noises, and the corresponding expressions are as follows:

$$u_{11} = u_1 + \lambda R(x), \qquad u_{12} = u_1 + \lambda R(y), \tag{28}$$

$$u_{21} = u_2 + \lambda R(x), \qquad u_{22} = u_2 + \lambda R(y), \tag{29}$$

$$u_{31} = u_3 + \lambda R(x), \qquad u_{32} = u_3 + \lambda R(y), \tag{30}$$

where u_1, u_2 and u_3 represent Solutions (19), (20) and (27), respectively, $R(x)$ and $R(y)$ denote the standard normal distribution about x and y, and λ means the noise level.

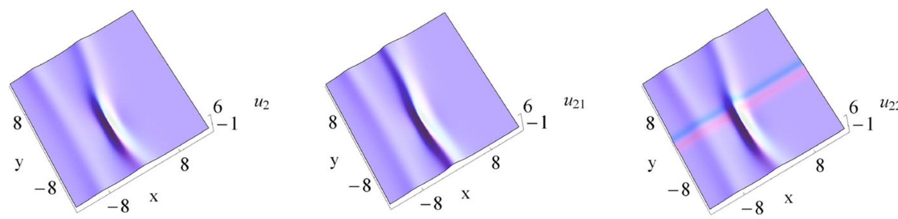


Fig. 2**a** **(a)** u_{21} **(b)** u_{22}

Fig. 7 The same as Fig. 2a except that $\lambda = 1$

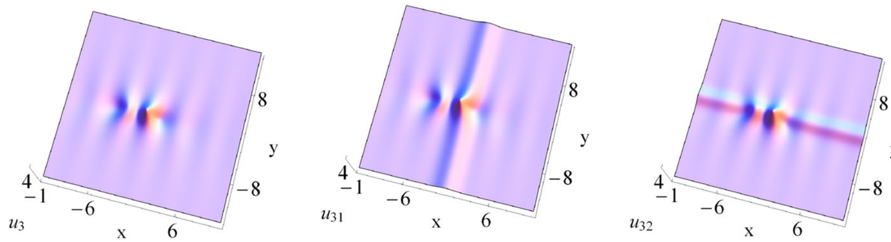


Fig. 4**a** **(a)** u_{31} **(b)** u_{32}

Fig. 8 The same as Fig. 4a except that $\lambda = 1$

Figure 6 depicts the mixed rogue–solitary waves via Solutions (19) under the noise perturbations. Figure 7 depicts the mixed rogue–solitary waves via Solutions (20) under the noise perturbations. Figure 8 depicts the mixed lump-periodic waves via Solutions (27) under the noise perturbations. Under the same noise perturbations, Solutions (20) are the most stable while Solutions (19) are the least stable among Solutions (19), (20) and (27).

6 Conclusions

In this paper, a generalized (2 + 1)-dimensional nonlinear wave equation in nonlinear optics, fluid mechanics and plasma physics, i.e., Eq. (3), has been studied. Via the Hirota bilinear method, Bilinear BT (10) has been worked out, based on which Lax Pair (14) has been constructed. Via an existing bilinear form, i.e., Bilinear Form (5), and symbolic computation, Mixed Rogue–Solitary Wave Solutions (19)–(21), Rogue–Periodic Wave Solutions (26) and Lump-Periodic Wave Solutions (27) have been derived.

Figure 1 has shown the interaction among the one rogue wave and two solitary waves with their shapes invariant via Solutions (19). Figure 2 has depicted the interaction among the one rogue wave and two solitary waves: Fig. 2a shows that the two solitary waves move along the same direction; One rogue wave appears between the two solitary waves, as shown in Fig. 2b; One rogue wave merges with the two solitary waves, as shown in Fig. 2c. Figure 3 has shown the interaction among the one rogue wave and two solitary waves with their shapes invariant via Solutions (21).

Interaction between the one rogue wave and one periodic wave has been depicted in Fig. 4: The one rogue wave possesses two peaks with different amplitudes; It is worth concluding that the interaction between the one rogue wave and one periodic wave is periodic, as shown in Fig. 4e, a. Figure 5 has depicted the periodic lump waves with the same amplitude.

Effects of the noise perturbations on Solutions (19), (20) and (27) have been shown in Figs. 6, 7 and 8, respectively: Under the same noise perturbations, Solutions (20) are the most stable while Solutions (19) are the least stable among Solutions (19), (20) and (27).

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Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

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