



The higher-order lump, breather and hybrid solutions for the generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation in fluid mechanics

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Abstract Under investigation in this paper is the $(2 + 1)$ -dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation, which can be utilized to describe certain nonlinear phenomena in fluid mechanics. We obtain the higher-order lump, breather and hybrid solutions, and analyze the effects of the constant coefficients h_1 , h_2 , h_4 and h_5 in that equation on those solutions, since the higher-order lump solutions are generalized via the long-wave limit method, and since the higher-order breather solutions and hybrid solutions composed of the solitons, breathers and lumps are derived. With the help of the analytic and graphic analysis, we get the following: (1) amplitudes of the humps and valleys of the first-order lumps are related to h_1 , h_2 , h_4 and h_5 , proportional to h_4 while inversely proportional to h_2 . Velocities of the first-order lumps are proportional to h_4 . The second-order lumps describe the interaction between the two first-order lumps, which is elastic since those lumps keep their shapes, velocities and amplitudes unchanged after the interaction. Effects of h_2 and h_4 on the second-order lumps are graphically illustrated. (2) Amplitudes of the first-order breathers are proportional to h_2 . Interaction between the breather waves is graphically presented. Effects of h_2 and h_1 on the amplitudes and shapes of the

second-order breathers are graphically discussed. (3) Elastic interactions are graphically illustrated, between the first-order breathers and one solitons, the first-order lumps and one solitons, as well as the first-order breathers and first-order lumps. Also graphically illustrated, amplitudes of all those three kinds of hybrid solutions are inversely proportional to h_2 , and velocity of the one soliton is positively correlated to h_4 .

Keywords Fluid mechanics · $(2 + 1)$ -Dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation · Lump solutions · Breather solutions · Hybrid solutions

1 Introduction

Nonlinear evolution equations (NLEEs) have been proposed in such fields as fluid mechanics, plasma physics, fiber optics and ocean dynamics, and analytic solutions for the NLEEs such as the solitons, lumps and breathers have received people's attention [1–18]. Solitons have been seen to maintain their shapes and velocities unchanged during the propagation [19–26]. Lumps have been considered as the waves localized in all directions in the space [27–29]. Breathers, localized waves with oscillatory patterns, have been classified as three types, namely the Akhmediev breathers, Kuznetsov–Ma breathers and Peregrine solitons [30–36]. Interactions among those nonlinear waves have attracted the researchers' attention [37–39].

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Refs. [40,41] have considered the $(2 + 1)$ -dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation for certain nonlinear phenomena in fluid mechanics, which reads

$$u_t + h_1 u_{xxx} + h_2 uu_x + h_3 u_{xxx} + h_4 \partial_x^{-1} u_{yy} + h_5 u_{xxy} + h_6 (u_x \partial_x^{-1} u_y + uu_y) + h_7 (u_x u_{xx} + uu_{xx}) + h_8 u^2 u_x = 0, \tag{1}$$

where $u = u(x, y, t)$ is the amplitude of the relevant wave, x and y are the running coordinates, t is the time, the subscripts represent the partial derivatives, ∂_x^{-1} represents the integral with respect to x , and the coefficients h_ℓ 's ($\ell=1,2,\dots,8$) are the real constants. Some special cases of Eq. (1) in fluid mechanics and plasma physics have been seen as follows:

- When $h_1 = 0, h_2 = 0, h_3 = 1, h_4 = -5, h_5 = -5, h_6 = -15, h_7 = 15, h_8 = 45$, Eq. (1) has been reduced to the $(2 + 1)$ -dimensional B-type Kadomtsev–Petviashvili equation which describes the shallow water waves in a fluid or electrostatic wave potential in a plasma, where u is a wave amplitude function of the scaled space coordinates x, y and time coordinate t [42–47].
- When $h_2 = 6h_1, h_6 = 4h_5, h_3 = h_4 = h_7 = h_8 = 0$, Eq. (1) has been degenerated to the $(2 + 1)$ -dimensional generalized breaking soliton equation which describes the interaction of a Riemann wave propagating along the y axis and a long wave propagating along the x axis, where u represents the amplitude or elevation of the Riemann wave [48–50].
- When $h_3 = 1, h_7 = 15, h_8 = 45, h_1 = h_2 = h_4 = h_5 = h_6 = 0$ and u is independent of y , Eq. (1) has been reduced to the fifth-order Sawada-Kotera equation which describes the long waves in the shallow water under the gravity and in a one-dimensional nonlinear lattice [51–55].

N -soliton solutions of Eq. (1) have been obtained via the Hirota bilinear method, where N is a positive integer, and periodic-wave solutions of Eq. (1) have also been constructed via the Riemann theta function [40]. Lump, lumpoff and rogue wave solutions for Eq. (1) have been derived [41].

However, to our knowledge, Eq. (1) is still worthy of further study. Firstly, the higher-order lump solutions of Eq. (1) have not been reported, and interaction

between the lumps has not been investigated. Secondly, breather solutions of Eq. (1) have not been obtained. Thirdly, although Ref. [41] has obtained the lumpoff solutions of Eq. (1) which describe the inelastic interaction between the lumps and one solitons, elastic interactions there have not been investigated, between the first-order breathers and one solitons, the first-order lumps and one solitons, as well as the first-order breathers and first-order lumps. Motivated by those, we will present this paper.

Through the dependent variable transformation [40],

$$u = 12h_1 h_2^{-1} (\ln f)_{xx}, \tag{2}$$

Eq. (1) has been transformed into the following bilinear equation [40]:

$$(D_x D_t + h_1 D_x^4 + h_3 D_x^6 + h_4 D_y^2 + h_5 D_x^3 D_y) f \cdot f = 0, \tag{3}$$

where $2h_1 h_2^{-1} = 5h_3 h_7^{-1} = h_5 h_6^{-1} = h_7 h_8^{-1}$, $5h_3 h_4 = -h_5^2$, $f = f(x, y, t)$ is a real function, and D is the Hirota's bilinear differential operator defined as [56]

$$D_x^{m_1} D_y^{m_2} D_t^{m_3} f(x, y, t) \cdot g(x, y, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^{m_1} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^{m_2} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^{m_3} f(x, y, t) g(x', y', t')|_{x=x', y=y', t=t'}, \tag{4}$$

with $g(x', y', t')$ being a function of the formal variables x', y' and t' , while m_1, m_2 and m_3 being the non-negative integers. N -soliton solutions for Eq. (1) have been expressed as Expression (2) [40] with

$$f = \sum_{\mu=0,1} \exp \left(\sum_{i=1}^N \mu_i \eta_i + \sum_{i<j}^N \mu_i \mu_j A_{ij} \right),$$

$$\eta_i = p_i x + k_i y + \omega_i t + \phi_i, \quad (i = 1, 2, \dots, N)$$

$$\omega_i = -h_1 p_i^3 - h_3 p_i^5 - h_5 p_i^2 k_i - h_4 p_i^{-1} k_i^2,$$

$$e^{A_{ij}} = -\frac{R_{ij}}{S_{ij}}, \quad (i < j, j = 2, 3, \dots, N)$$

$$R_{ij} = h_3 (p_i - p_j)^6 + h_1 (p_i - p_j)^4 + h_4 (k_i - k_j)^2 + (p_i - p_j)(\omega_i - \omega_j) + h_5 (p_i - p_j)^3 (k_i - k_j),$$

$$S_{ij} = h_3 (p_i + p_j)^6 + h_1 (p_i + p_j)^4 + h_4 (k_i + k_j)^2 + (p_i + p_j)(\omega_i + \omega_j) + h_5 (p_i + p_j)^3 (k_i + k_j), \tag{5}$$

where p_i 's, k_i 's and ϕ_i 's are the complex constants, $\sum_{\mu=0,1}$ denotes a summation over all the possible

combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, \dots, \mu_N = 0, 1$, and the notation $\sum_{i < j}^N$ indicates a summation over all the possible pairs (i, j) chosen from the set $\{1, 2, \dots, N\}$, with the condition $i < j$.

To sum up, this paper will investigate the higher-order lump, breather and hybrid solutions for Eq. (1), and study the effects of h_1, h_2, h_4 and h_5 on those solutions. In Sect. 2, employing the long-wave limit method [57, 58], we will construct the L th-order lump solutions, where L is a positive integer, and analyze the effects of h_1, h_2, h_4 and h_5 on the velocities and amplitudes of the first-order lumps. We will graphically discuss the interaction between the two first-order lumps, and the effects of h_2 and h_4 on the first-order lumps. In Sect. 3, we will construct the M th-order breathers, where M is a positive integer. Effects of h_1 and h_2 on the second-order breathers will be discussed. Interaction between the two first-order breather waves will be graphically presented. In Sect. 4, hybrid solutions composed of the first-order breathers and one solitons, the first-order lumps and one solitons, as well as the first-order lumps and first-order breathers will be obtained. Elastic interactions of those three kinds of hybrid solutions will be graphically illustrated. We will also discuss the effects of the coefficients on those hybrid solutions. In Sect. 5, we will give our conclusions.

2 The higher-order lump solutions for Eq. (1)

In this part, we will construct the L th-order lumps via the long-wave limit of the N -soliton solutions for Eq. (1).

To construct the L th-order lumps for Eq. (1), setting $\phi_i = I\pi, p_i = \sigma P_i, k_i = \sigma K_i, N = 2L$, (6) and then taking $\sigma \rightarrow 0$, we can generalize the L th-order lumps from Solutions (2) and (5) as

$$u = 12h_1h_2^{-1}(\ln f)_{xx}, \tag{7}$$

where $I = \sqrt{-1}, \sigma$ is a real constant, P_i 's and K_i 's are the complex constants,

$$f = \prod_{i=1}^{2L} \theta_i + \frac{1}{2} \sum_{i,j}^{2L} B_{ij} \prod_{\Gamma \neq i,j}^{2L} \theta_\Gamma + \dots + \frac{1}{L!2^L} \times \sum_{i,j,\dots,\kappa,\nu}^{2L} \overbrace{B_{ij} B_{rs} \dots B_{\kappa\nu}}^L \prod_{\Upsilon \neq i,j,\dots,\kappa,\nu}^{2L} \theta_\Upsilon + \dots, \tag{8}$$

$$\theta_i = P_i x + K_i y - h_4 \frac{K_i^2}{P_i} t,$$

$$B_{ij} = \frac{6P_i^2 P_j^2 [2h_1 P_i P_j + h_5 (P_j K_i + P_i K_j)]}{h_4 (P_j K_i - P_i K_j)^2},$$

$\sum_{i,j,\dots,\kappa,\nu}^{2L}$ means the summation over all the possible combinations of i, j, \dots, κ, ν , which are taken from $1, 2, \dots, 2L$ and are all different from each other.

To obtain the nonsingular solutions, we set

$$P_r = P_{L+r}^* = P_{r1} + I P_{r2}, \tag{9}$$

$$K_r = K_{L+r}^* = K_{r1} + I K_{r2}, \quad (r = 1, 2, \dots, L)$$

and substitute Eqs. (9) into Eqs. (7) and (8), where the superscript $*$ represents the complex conjugation, P_{r1} 's, P_{r2} 's, K_{r1} 's and K_{r2} 's are the real constants. Thus, we obtain the nonsingular lump solutions for Eq. (1) under the condition $B_{r,L+r} > 0$.

When $L = 1$, the first-order lump solutions for Eq. (1) can be written as

$$u = 12h_1h_2^{-1}(\ln f)_{xx}, \tag{10}$$

where

$$f = \theta_1 \theta_2 + B_{12},$$

$$\theta_1 = P_1 x + K_1 y - h_4 \frac{K_1^2}{P_1} t,$$

$$\theta_2 = P_2 x + K_2 y - h_4 \frac{K_2^2}{P_2} t,$$

$$B_{12} = \frac{6P_1^2 P_2^2 [2h_1 P_1 P_2 + h_5 (P_2 K_1 + P_1 K_2)]}{h_4 (P_2 K_1 - P_1 K_2)^2}. \tag{11}$$

When one takes

$$\begin{aligned} P_1 &= P_2^* = P_{11} + IP_{12}, \\ K_1 &= K_2^* = K_{11} + IK_{12} \end{aligned} \tag{12}$$

in Solutions (10), the corresponding solutions are non-singular under the condition $B_{12} > 0$. Solutions (10) can also be expressed as

$$u = \frac{24h_1 [-(X' + \Lambda_1 Y')^2 + \Lambda_2^2 Y'^2 + \Lambda_3]}{h_2 [(X' + \Lambda_1 Y')^2 + \Lambda_2^2 Y'^2 + \Lambda_3]^2}, \tag{13}$$

with

$$\begin{aligned} X' &= x + \frac{h_4(K_{11}^2 + K_{12}^2)}{P_{11}^2 + P_{12}^2}t, \\ Y' &= y - \frac{2h_4(K_{11}P_{11} + K_{12}P_{12})}{P_{11}^2 + P_{12}^2}t, \\ \Lambda_1 &= \frac{K_{11}P_{11} + K_{12}P_{12}}{P_{11}^2 + P_{12}^2}, \quad \Lambda_2 = \frac{K_{12}P_{11} - K_{11}P_{12}}{P_{11}^2 + P_{12}^2}, \\ \Lambda_3 &= \frac{-3[h_5(K_{11}P_{11} + K_{12}P_{12}) + h_1(P_{11}^2 + P_{12}^2)]}{h_4(P_{11}^2 + P_{12}^2)^{-1}(K_{12}P_{11} - K_{11}P_{12})^2}, \end{aligned} \tag{14}$$

where $\Lambda_3 > 0$ must be satisfied. We find that $u \rightarrow 0$ when $x \rightarrow \infty$ and $y \rightarrow \infty$ from Expressions (13). Hence, Solutions (10) are the first-order lumps moving on the constant backgrounds along the line

$$(K_{11}^2 + K_{12}^2)y - 2(K_{11}P_{11} + K_{12}P_{12})x = 0. \tag{15}$$

Velocity components of the first-order lumps along the x and y directions, V_x and V_y , are obtained as

$$\begin{aligned} V_x &= -\frac{h_4(K_{11}^2 + K_{12}^2)}{P_{11}^2 + P_{12}^2}, \\ V_y &= \frac{2h_4(K_{11}P_{11} + K_{12}P_{12})}{P_{11}^2 + P_{12}^2}. \end{aligned} \tag{16}$$

For Solutions (10), when $t = 0$, there are three extreme points at $(0, 0)$, $(\sqrt{3\Lambda_3}, 0)$ and $(-\sqrt{3\Lambda_3}, 0)$ on the $x - y$ plane. Amplitudes of the humps and valleys of

the first-order lumps are derived as

$$\begin{aligned} \wp_1 &= \left| \frac{-8h_1h_4(K_{12}P_{11} - K_{11}P_{12})^2(P_{11}^2 + P_{12}^2)^{-1}}{h_2[h_5(K_{11}P_{11} + K_{12}P_{12}) + h_1(P_{11}^2 + P_{12}^2)]} \right|, \\ \wp_2 &= \left| \frac{h_1h_4(K_{12}P_{11} - K_{11}P_{12})^2(P_{11}^2 + P_{12}^2)^{-1}}{h_2[h_5(K_{11}P_{11} + K_{12}P_{12}) + h_1(P_{11}^2 + P_{12}^2)]} \right|. \end{aligned} \tag{17}$$

From Expressions (16) and (17), we find that the velocities of the first-order lumps are proportional to the coefficient h_4 . Amplitudes of the humps and valleys of the first-order lumps are related to the coefficients h_1, h_2, h_4 and h_5 : both the amplitudes of the humps and valleys of the first-order lumps are proportional to h_4 , while inversely proportional to h_2 . Amplitudes of the humps of the first-order lumps are eight times as large as those of the valleys of the first-order lumps.

We need to point out that The First-Order Lump Solutions (10) are constructed via the long-wave limit method, which is different from the method adopted in Ref. [41]. Compared with those of the lump solutions obtained in Ref. [41], the trajectories of The First-Order Lump Solutions (10) always pass through the point $(0, 0)$ on the $x - y$ plane. The First-Order Lump Solutions (10) depend on four parameters, namely P_{11}, P_{12}, K_{11} and K_{12} , while the first-order lump solutions obtained in Ref. [41] depend on more parameters.

When $L = 2$, the second-order lump solutions for Eq. (1) can be obtained via Solutions (7), as

$$u = 12h_1h_2^{-1}(\ln f)_{xx}, \tag{18}$$

where

$$\begin{aligned} f &= \theta_1\theta_2\theta_3\theta_4 + B_{12}\theta_3\theta_4 + B_{13}\theta_2\theta_4 + B_{14}\theta_2\theta_3 \\ &+ B_{23}\theta_1\theta_4 + B_{24}\theta_1\theta_3 + B_{34}\theta_1\theta_2 \\ &+ B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}. \end{aligned} \tag{19}$$

The second-order lumps describe the interaction between the two first-order lumps. As shown in Fig. 1, the two first-order lumps move along two lines

$$\begin{aligned} (K_{11}^2 + K_{12}^2)y - 2(K_{11}P_{11} + K_{12}P_{12})x &= 0, \\ (K_{21}^2 + K_{22}^2)y - 2(K_{21}P_{21} + K_{22}P_{22})x &= 0, \end{aligned} \tag{20}$$

on the $x - y$ plane. It can be seen that the two first-order lumps have different shapes, velocities and amplitudes.

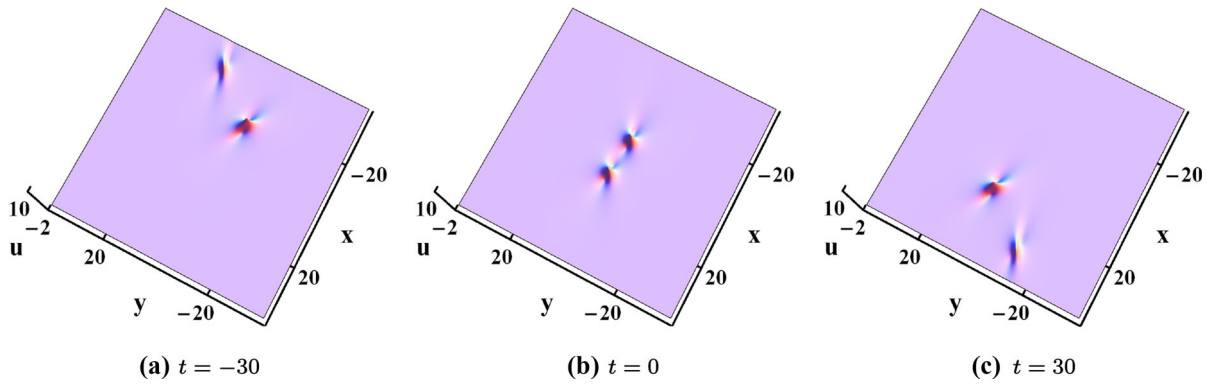


Fig. 1 The second-order lump via Solutions (18) with $P_1 = P_3^* = -1 - I$, $P_2 = P_4^* = -I$, $K_1 = K_3^* = 2 - I$, $K_2 = K_4^* = 2 - I$, $h_1 = 1$, $h_2 = 2$, $h_4 = -\frac{1}{5}$, $h_5 = 1$.

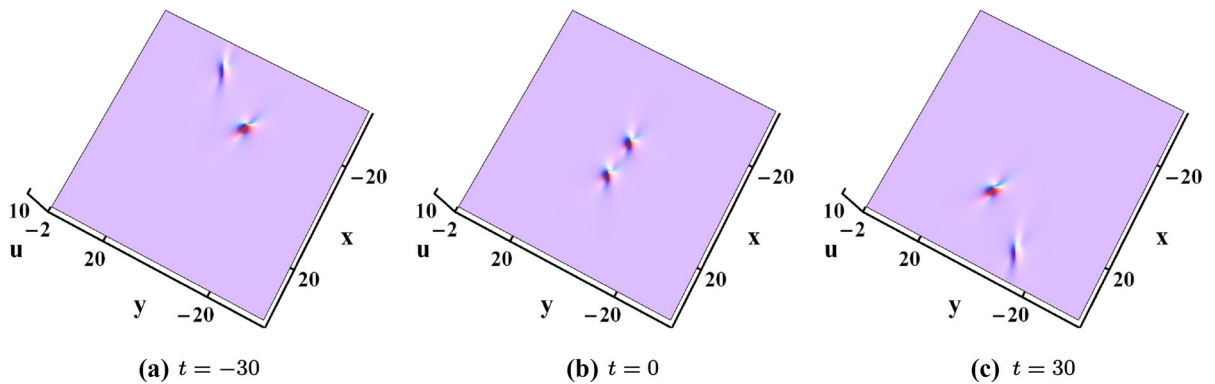


Fig. 2 The same as Fig. 1 except that $h_2 = 4$.

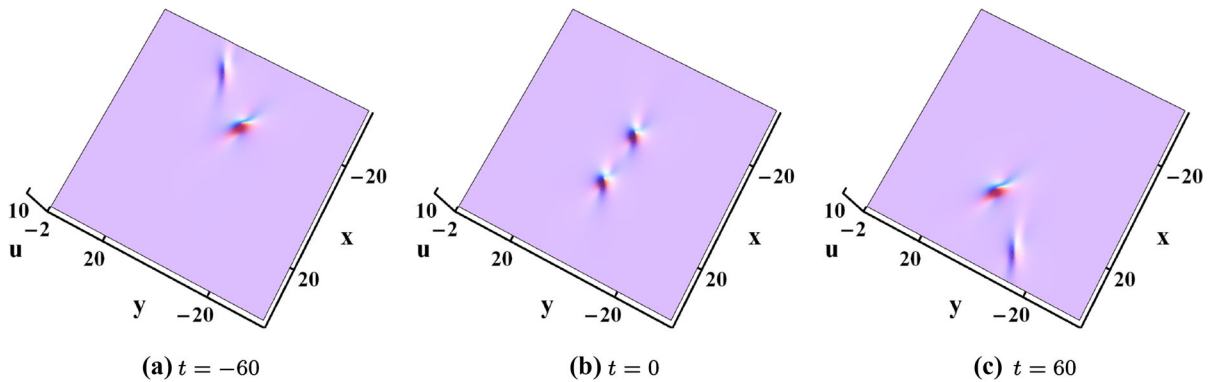


Fig. 3 The same as Fig. 1 except that $h_4 = -\frac{1}{10}$.

They move close, and then interact with each other. When $t = 0$, due to the interaction, it can be seen that there are two humps with the same amplitude. Finally, the two first-order lumps separate and move further and

further. We can find that the interaction between those two lumps is elastic since the shapes, velocities and amplitudes of those lumps have no changes the interaction. To investigate the effect of h_2 , based on the

parameters in Fig. 1, we set $h_2 = 4$ and fix the other parameters. Comparing Figs. 1 and 2, we find that the amplitudes of the two first-order lumps are inversely proportional to h_2 . When $h_4 = -\frac{1}{10}$ and the other parameters are the same as those in Fig. 1, a second-order lump is depicted in Fig. 3. Comparing Figs. 1 and 3, we find that both the velocities and amplitudes of those first-order lumps are proportional to h_4 .

3 The higher-order breather solutions for Eq. (1)

In this part, we will construct the M th-order breathers under certain parameter constraints in the N -soliton solutions for Eq. (1).

Motivated by the procedure in Refs. [29,37], we take

$$\begin{aligned} N &= 2M, \quad p_\tau = p_{M+\tau}^* = p_{\tau 1} + I p_{\tau 2}, \\ k_\tau &= k_{M+\tau}^* = k_{\tau 1} + I k_{\tau 2}, \\ \phi_\tau &= \phi_{M+\tau}^* = \phi_{\tau 1} + I \phi_{\tau 2}, \quad (\tau = 1, 2, \dots, M) \end{aligned} \tag{21}$$

in Solutions (2) and (5), where $p_{\tau 1}$'s, $p_{\tau 2}$'s, $k_{\tau 1}$'s, $k_{\tau 2}$'s, $\phi_{\tau 1}$'s and $\phi_{\tau 2}$'s are the real constants. The M th-order breather solutions of Eq. (1) are given as

$$u = 12h_1 h_2^{-1} (\ln f)_{xx}, \tag{22}$$

with

$$\begin{aligned} f &= \sum_{\mu=0,1} \exp \left(\sum_{i=1}^{2M} \mu_i \eta_i + \sum_{i<j}^{2M} \mu_i \mu_j A_{ij} \right), \\ \eta_\tau &= \eta_{M+\tau}^* = \eta_{\tau 1} + I \eta_{\tau 2}, \\ \eta_{\tau 1} &= p_{\tau 1} x + k_{\tau 1} y - (p_{\tau 1}^2 + p_{\tau 2}^2)^{-1} \\ &\quad \times \{ h_4 [(k_{\tau 1}^2 - k_{\tau 2}^2) p_{\tau 1} + 2k_{\tau 1} k_{\tau 2} p_{\tau 2}] \\ &\quad + (p_{\tau 1}^2 + p_{\tau 2}^2) [h_5 ((p_{\tau 1}^2 - p_{\tau 2}^2) k_{\tau 1} - 2k_{\tau 2} p_{\tau 1} p_{\tau 2}) \\ &\quad + h_1 p_{\tau 1} (p_{\tau 1}^2 - 3p_{\tau 2}^2) + h_3 p_{\tau 1} (p_{\tau 1}^4 - 10p_{\tau 1}^2 p_{\tau 2}^2 \\ &\quad + 5p_{\tau 2}^4)] \} t + \phi_{\tau 1}, \\ \eta_{\tau 2} &= p_{\tau 2} x + k_{\tau 2} y - (p_{\tau 1}^2 + p_{\tau 2}^2)^{-1} \\ &\quad \times \{ h_4 [(-k_{\tau 1}^2 + k_{\tau 2}^2) p_{\tau 2} + 2k_{\tau 1} k_{\tau 2} p_{\tau 1}] \\ &\quad + (p_{\tau 1}^2 + p_{\tau 2}^2) [h_5 ((p_{\tau 1}^2 - p_{\tau 2}^2) k_{\tau 2} + 2k_{\tau 1} p_{\tau 1} p_{\tau 2}) \\ &\quad + h_1 p_{\tau 2} (3p_{\tau 1}^2 - p_{\tau 2}^2) + h_3 p_{\tau 2} (5p_{\tau 1}^4 - 10p_{\tau 1}^2 p_{\tau 2}^2 \\ &\quad + p_{\tau 2}^4)] \} t + \phi_{\tau 2}, \\ e^{A_{ij}} &= -\frac{R_{ij}}{S_{ij}}, \end{aligned}$$

$$\begin{aligned} R_{ij} &= h_3(p_i - p_j)^6 + h_1(p_i - p_j)^4 + h_4(k_i - k_j)^2 \\ &\quad + (p_i - p_j)(\omega_i - \omega_j) + h_5(p_i - p_j)^3(k_i - k_j), \\ S_{ij} &= h_3(p_i + p_j)^6 + h_1(p_i + p_j)^4 + h_4(k_i + k_j)^2 \\ &\quad + (p_i + p_j)(\omega_i + \omega_j) + h_5(p_i + p_j)^3(k_i + k_j), \end{aligned} \tag{23}$$

and $e^{A_{\tau, M+\tau}} > 1$ to ensure that the solutions are non-singular.

When $M = 1$, Solutions (22) reduce to the first-order breather solutions of Eq. (1),

$$u = 12h_1 h_2^{-1} (\ln f)_{xx}, \tag{24}$$

with

$$\begin{aligned} f &= 1 + e^{\eta_1} + e^{\eta_1^*} + e^{\eta_1 + \eta_1^* + A_{12}}, \\ \eta_1 &= \eta_2^* = \eta_{11} + I \eta_{12}, \\ \eta_{11} &= p_{11} x + k_{11} y - (p_{11}^2 + p_{12}^2)^{-1} \\ &\quad \times \{ h_4 [(k_{11}^2 - k_{12}^2) p_{11} + 2k_{11} k_{12} p_{12}] \\ &\quad + (p_{11}^2 + p_{12}^2) [h_5 ((p_{11}^2 - p_{12}^2) k_{11} - 2k_{12} p_{11} p_{12}) \\ &\quad + h_1 p_{11} (p_{11}^2 - 3p_{12}^2) + h_3 p_{11} (p_{11}^4 - 10p_{11}^2 p_{12}^2 \\ &\quad + 5p_{12}^4)] \} t + \phi_{11}, \\ \eta_{12} &= p_{12} x + k_{12} y - (p_{11}^2 + p_{12}^2)^{-1} \\ &\quad \times \{ h_4 [(-k_{11}^2 + k_{12}^2) p_{12} + 2k_{11} k_{12} p_{11}] \\ &\quad + (p_{11}^2 + p_{12}^2) [h_5 ((p_{11}^2 - p_{12}^2) k_{12} + 2k_{11} p_{11} p_{12}) \\ &\quad + h_1 p_{12} (3p_{11}^2 - p_{12}^2) + h_3 p_{12} (5p_{11}^4 - 10p_{11}^2 p_{12}^2 \\ &\quad + p_{12}^4)] \} t + \phi_{12}, \\ e^{A_{12}} &= 1 - \frac{\mathfrak{R}}{\mathfrak{S}}, \\ \mathfrak{R} &= 3(p_{11}^2 + p_{12}^2)^2 \{ h_5 (k_{11} p_{11} + k_{12} p_{12}) \\ &\quad + (p_{11}^2 + p_{12}^2) [h_1 + 5h_3 (p_{11}^2 - p_{12}^2)] \}, \\ \mathfrak{S} &= h_4 (k_{12} p_{11} - k_{11} p_{12})^2 + p_{11}^2 (p_{11}^2 + p_{12}^2) \\ &\quad \times [3h_1 + 5h_3 (3p_{11}^2 - p_{12}^2)] + h_5 p_{11} (p_{11}^2 + p_{12}^2) \\ &\quad \times [2k_{12} p_{11} p_{12} + k_{11} (3p_{11}^2 + p_{12}^2)], \end{aligned} \tag{25}$$

and $e^{A_{12}} > 1$. The first-order breathers via Solutions (24) are localized along the direction of $\eta_{11} = 0$ and periodic along the direction of $\eta_{12} = 0$. Periods of those breathers are $\frac{2\pi}{p_{12}}$ along the x direction and $\frac{2\pi}{k_{12}}$ along the y direction. There are three kinds of the first-order breathers with different values of p_{11} and k_{11} in Solutions (24). When $p_{11} = 0$ and $k_{11} \neq 0$, the first-order breathers are parallel to the x axis on the $x - y$ plane; When $p_{11} \neq 0$ and $k_{11} = 0$, the first-order breathers are parallel to the y axis on the $x - y$ plane; More generally, when $p_{11} \neq 0$ and $k_{11} \neq 0$, the

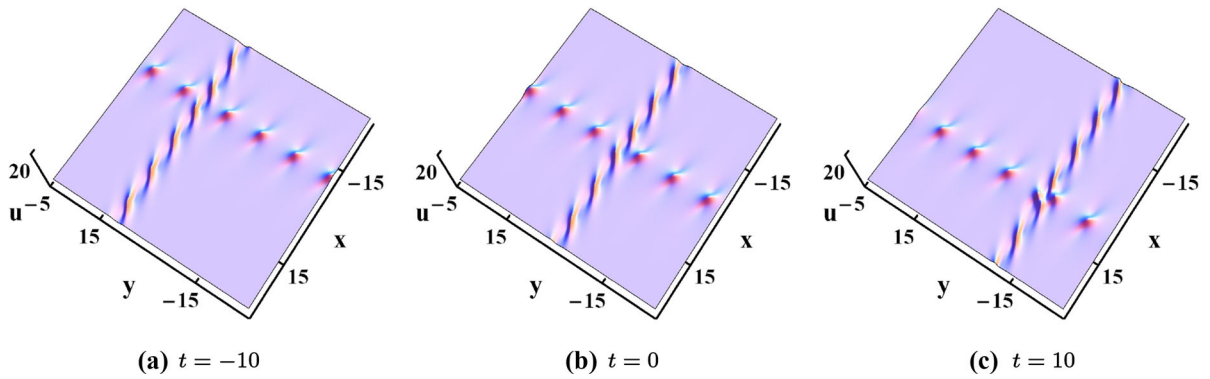


Fig. 4 The second-order breather via Solutions (22) with $M = 2$, $p_1 = p_3^* = \frac{3}{10} - \frac{1}{10}I$, $p_2 = p_4^* = \frac{3}{5}I$, $k_1 = k_3^* = \frac{3}{5}I$, $k_2 = k_4^* = 1 + I$, $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$, $h_1 = 1$, $h_2 = 2$, $h_4 = -\frac{1}{5}$, $h_5 = 1$.

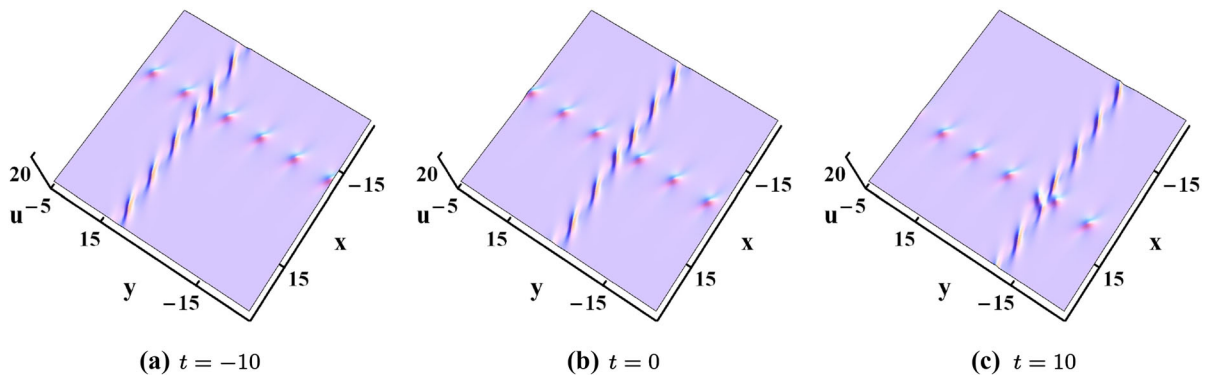


Fig. 5 The same as Fig. 4 except that $h_2 = 4$.

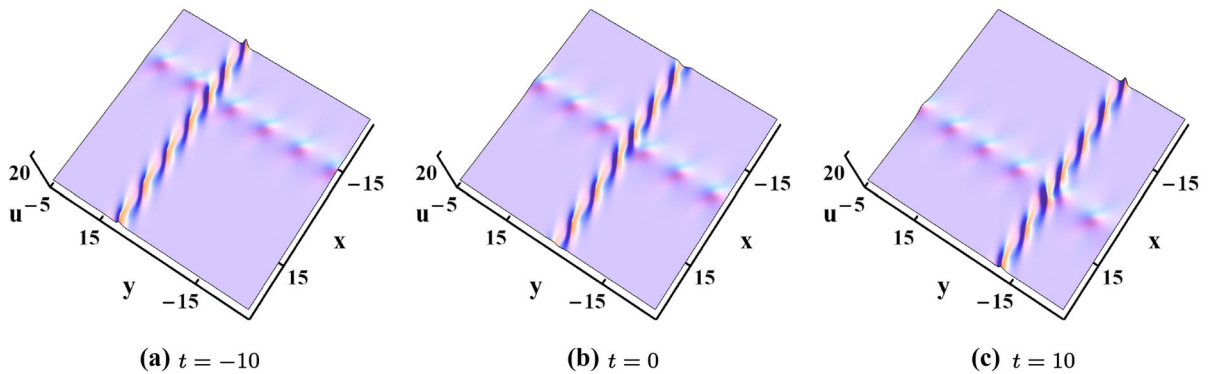


Fig. 6 The same as Fig. 4 except that $h_1 = 2$.

first-order breathers are parallel to the line $y = -\frac{p_{11}}{k_{11}}x$ on the $x - y$ plane. When $p_{12} = k_{12} = \phi_{12} = 0$, The First-Order Breather Solutions (24) reduce to the one-soliton solutions for Eq. (1). From Expressions (24)

and (25), we also notice that the amplitudes of the first-order breathers are inversely proportional to h_2 .

Similarly, we can obtain the second-order breather solutions of Eq. (1) with $M = 2$ in Solutions (22). The second-order breathers describe the interaction

between the two first-order breathers. As analyzed above, the second-order breathers can be composed of different kinds of the first-order breathers. For instance, we may construct a second-order breather which consists of two first-order breathers, one of which is parallel to the x axis, while the other is parallel to the y axis, as shown in Fig. 4. Those two first-order breathers keep interacting with each other on the $x - y$ plane. As t goes on, the interaction region keeps moving. To investigate the effect of h_2 , based on the parameters in Fig. 4, we set $h_2 = 4$ and fix the other parameters. Comparing Figs. 4 and 5, we find that the amplitudes of the two first-order breathers are inversely proportional to h_2 . When $h_1 = 2$ and the other parameters are the same as those in Fig. 4, interaction between the two first-order breather is illustrated in Fig. 6. Effects of h_1 on the amplitudes and shapes of the first-order breathers can be seen via comparing Figs. 4 with 6.

4 Hybrid solutions for Eq. (1)

In this section, we will focus on three types of hybrid solutions for Eq. (1). We will construct the hybrid solutions consisting of the first-order breathers and one solitons, of the first-order lumps and one solitons, and of the first-order lumps and first-order breathers.

4.1 Hybrid solutions consisting of the first-order breathers and one solitons for Eq. (1)

We take

$$\begin{aligned} N &= 3, \quad p_1 = p_2^* = p_{11} + Ip_{12}, \quad p_3 = p_{31}, \\ k_1 &= k_2^* = k_{11} + Ik_{12}, \quad k_3 = k_{31}, \\ \phi_1 &= \phi_2^* = \phi_{11} + I\phi_{12}, \quad \phi_3 = \phi_{31}, \end{aligned} \tag{26}$$

in Solutions (2) and (5), and get the following:

Hybrid solutions composed of the first-order breathers and one solitons describe the interaction between the first-order breathers and one solitons, as shown in Figs. 7 and 8. When $p_{11}k_{31} = p_{31}k_{11}$, the first-order breather is parallel to the one soliton on the $x - y$ plane, as shown in Fig. 7. The first-order breather and one soliton move in opposite directions. As t goes on, they move close, interact with each other at $t = 0$, and finally become further and further. When $t = 0$, due to the interaction, each peak of the first-order breather is divided into two peaks, and amplitude of any peak is lower than that of the first-order breather. After the interaction, both the breather and soliton keep their shapes, velocities and amplitudes unchanged, which means that the interaction between the first-order breather and one soliton is elastic. When $p_{11}k_{31} \neq p_{31}k_{11}$, the first-order breather and the one soliton are unparallel on the $x - y$ plane, and keep interacting with each other as t goes on. To investigate the effect of h_2 , we take $h_2 = 4$, and the other parameters are the same as those in Fig. 7. Comparing Figs. 7 and 8, we find that the amplitudes of both the first-order breathers and one solitons are inversely proportional to h_2 .

4.2 Hybrid solutions consisting of the first-order lumps and one solitons for Eq. (1)

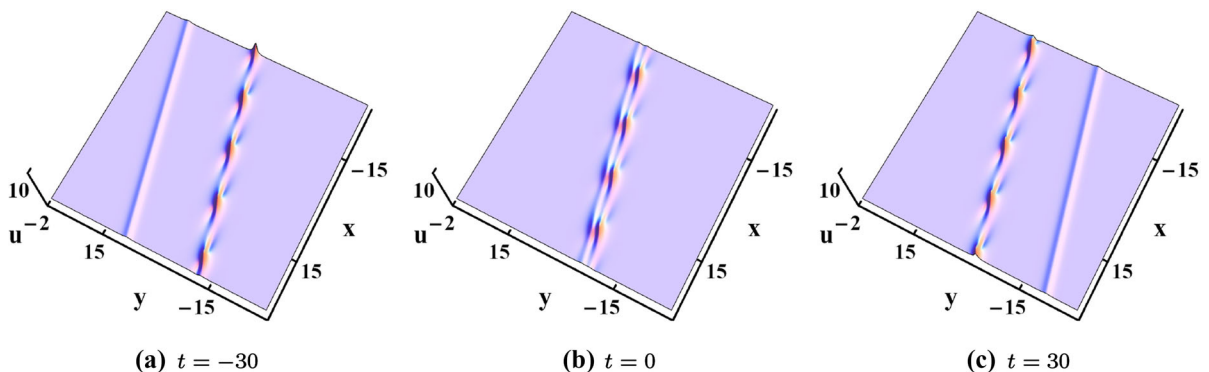


Fig. 7 Hybrid solution composed of a first-order breather and a single soliton via Solutions (2) and (5) with $N = 3, p_1 = p_2^* = \frac{3}{10} - \frac{3}{10}I, p_3 = \frac{1}{2}, k_1 = k_2^* = \frac{6}{5} + \frac{3}{5}I, k_3 = 2, \phi_1 = \phi_2 = \phi_3 = 0, h_1 = 1, h_2 = 2, h_4 = -\frac{1}{5}, h_5 = 1$.

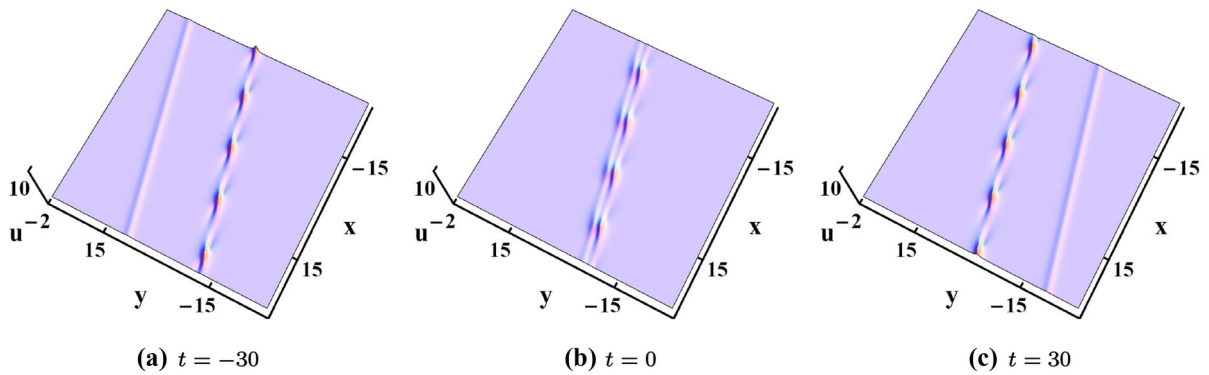


Fig. 8 The same as Fig. 7 except that $h_2 = 4$

Via the long wave limit method [57,58], hybrid solutions composed of the first-order lumps and one solitons for Eq. (1) can be derived via Solutions (2) and (5) with the parameters

$$\begin{aligned}
 N &= 3, \quad \phi_1 = \phi_2 = I\pi, \quad p_1 = \sigma P_1, \quad p_2 = \sigma P_2, \\
 k_1 &= \sigma K_1, \quad k_2 = \sigma K_2, \quad \sigma \rightarrow 0.
 \end{aligned}
 \tag{27}$$

Thus, we obtain the hybrid solutions composed of the first-order lumps and one solitons for Eq. (1) as

$$u = 12h_1h_2^{-1}(\ln f)_{xx},
 \tag{28}$$

where

$$\begin{aligned}
 f &= \theta_1\theta_2 + B_{12} + (B_{12} + \xi_{13}\xi_{23} + \xi_{13}\theta_2 \\
 &\quad + \xi_{23}\theta_1 + \theta_1\theta_2)\exp(\eta_3), \\
 \xi_{13} &= -\frac{\xi_{131}}{\xi_{132}}, \\
 \xi_{131} &= 6P_1^2p_3^2[h_5(P_1k_3 + K_1p_3) \\
 &\quad + 2h_1P_1p_3 + 5h_3P_1p_3^3], \\
 \xi_{132} &= h_5P_1p_3^3(2P_1k_3 + K_1p_3) + 3h_1P_1^2p_3^4 \\
 &\quad + 5h_3P_1^2p_3^6 - h_4(P_1k_3 - K_1p_3)^2, \\
 \xi_{23} &= -\frac{\xi_{231}}{\xi_{232}}, \\
 \xi_{231} &= 6P_2^2p_3^2[h_5(P_2k_3 + K_2p_3) \\
 &\quad + 2h_1P_2p_3 + 5h_3P_2p_3^3], \\
 \xi_{232} &= h_5P_2p_3^3(2P_2k_3 + K_2p_3) + 3h_1P_2^2p_3^4 \\
 &\quad + 5h_3P_2^2p_3^6 - h_4(P_2k_3 - K_2p_3)^2,
 \end{aligned}
 \tag{29}$$

and $\theta_1, \theta_2, B_{12}$ and η_3 are given by Expressions (5) and (11). In this case, we find that $\xi_{13} = \xi_{23}^*$, which ensures

that f be a real function. Velocities and amplitudes of the first-order lumps are given by Expressions (16) and (17). However, the velocity components of the one solitons along the x and y directions, $V_{x-soliton}$ and $V_{y-soliton}$, are obtained as

$$\begin{aligned}
 V_{x-soliton} &= \frac{h_1p_3^4 + h_3p_3^6 + h_5p_1^3k_3 + h_4k_3^2}{p_3^2}, \\
 V_{y-soliton} &= \frac{h_1p_3^4 + h_3p_3^6 + h_5p_1^3k_3 + h_4k_3^2}{p_3k_3},
 \end{aligned}
 \tag{30}$$

and the amplitudes of the one solitons are $|3p_3^2h_1h_2^{-1}|$.

Hybrid solutions composed of the first-order lumps and one solitons describe the interaction between the first-order lumps and one solitons, as shown in Figs. 9, 10 and 11. From Fig. 9, we can see that the velocities of the first-order lump and one soliton are different. At first, they move close, and then the first-order lump interacts with the one soliton at $t = 0$. When $t = 0$, due to the interaction, the maximum amplitude of u is lower than that of the first-order lump. Finally, they separate and the distance between them becomes further and further. After the interaction, no changes happen to the shapes, velocities, and amplitudes of the first-order lump and one soliton, which means that the interaction is elastic. Thus, properties of Hybrid Solutions (28) are different from those of the lumpoff solutions obtained in Ref. [41], which describe the inelastic interaction between the first-order lumps and one solitons.

To investigate the effect of h_2 , based on the parameters in Fig. 9, we set $h_2 = 4$ and fix the other parameters. Comparing Figs. 9 and 10, we find that the amplitudes of this kind of hybrid solutions are inversely propor-

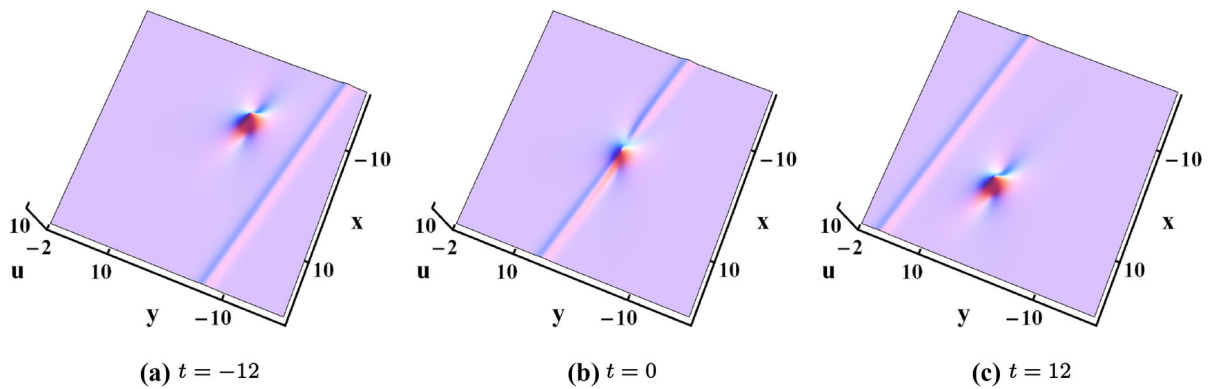


Fig. 9 Hybrid solution composed of a first-order lump and a single soliton via Solutions (28) with $P_1 = P_2^* = -1 - I$, $p_3 = -\frac{1}{2}$, $K_1 = K_2^* = 2 - I$, $k_3 = 2$, $\phi_3 = 0$, $h_1 = 1$, $h_2 = 2$, $h_4 = -\frac{1}{5}$, $h_5 = 1$.

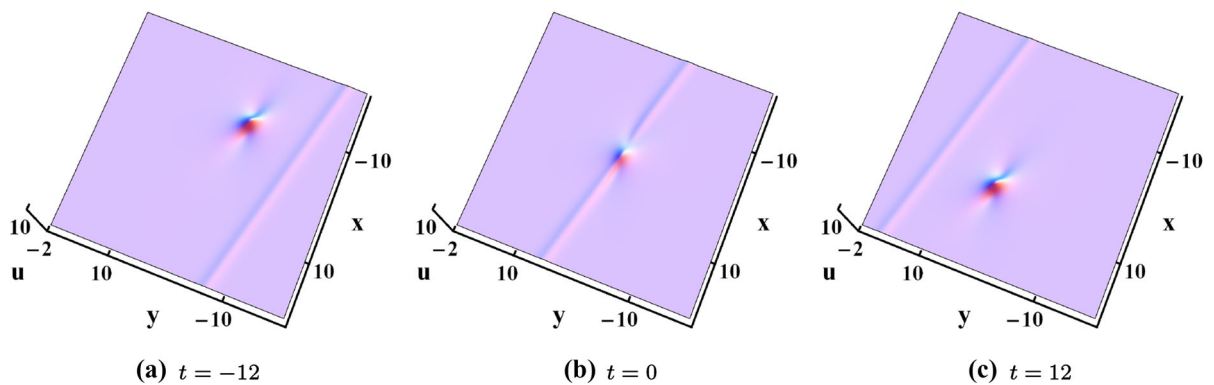


Fig. 10 The same as Fig. 9 except that $h_2 = 4$

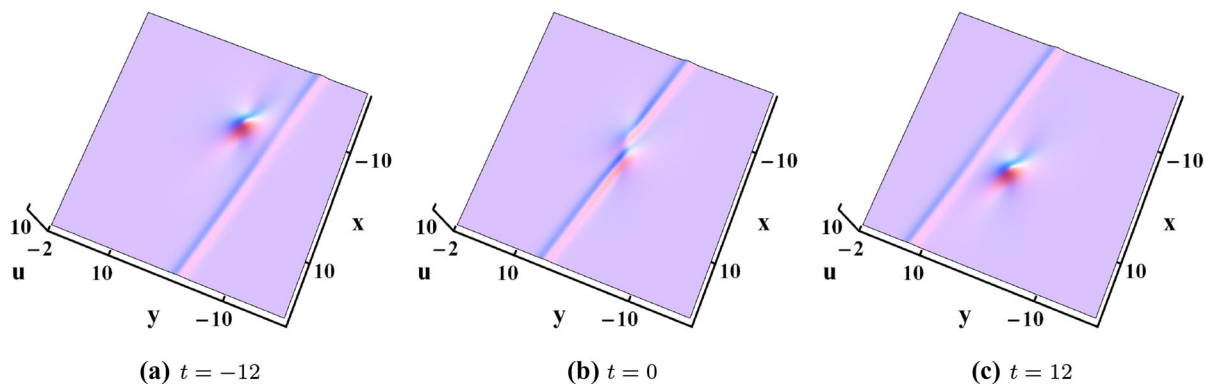


Fig. 11 The same as Fig. 9 except that $h_4 = -\frac{1}{10}$

tional to h_2 . When $h_4 = -\frac{1}{10}$ and the other parameters are the same as those in Fig. 9, interaction between a first-order lump and one soliton is illustrated in Fig. 11. Comparing Figs. 9 and 11, we find that the amplitude and velocity of the first-order lump are proportional

to h_4 . However, amplitude of the one soliton does not depend on h_4 , and velocity of the one soliton is positively correlated to h_4 . In this case, we also notice that the first-order lump crosses over the one soliton and has two peaks due to the interaction when $t = 0$.

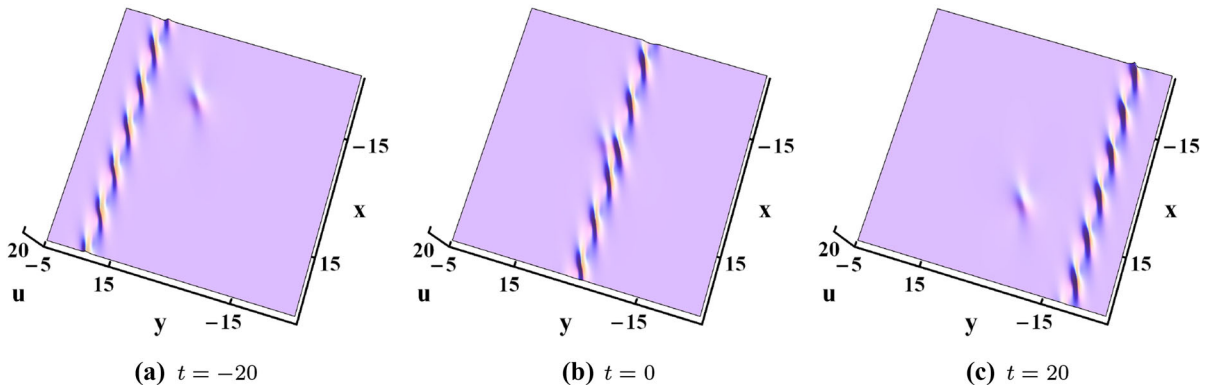


Fig. 12 Hybrid solution composed of a first-order breather and a first-order lump via Solutions (32) with $P_1 = P_2^* = 1 - \frac{3}{4}I$, $p_3 = p_4^* = \frac{3}{5}I$, $K_1 = K_2^* = 2 + I$, $k_3 = k_4^* = 1 + I$, $h_1 = 1$, $h_2 = 2$, $h_4 = -\frac{1}{5}$, $h_5 = 1$.

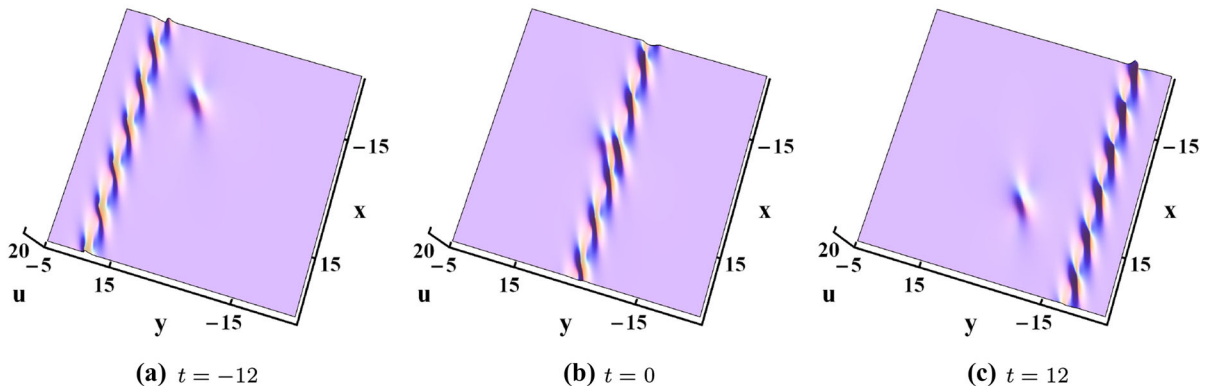


Fig. 13 The same as Fig. 12 except that $h_2 = 1$

4.3 Hybrid solutions consisting of the first-order breathers and first-order lumps for Eq. (1)

From Solutions (2) and (5) with

$$\begin{aligned}
 N &= 4, \phi_1 = \phi_2 = I\pi, p_1 = \sigma P_1, p_2 = \sigma P_2, \\
 k_1 &= \sigma K_1, k_2 = \sigma K_2, \sigma \rightarrow 0,
 \end{aligned}
 \tag{31}$$

hybrid solutions composed of the first-order breathers and first-order lumps for Eq. (1) are obtained as

$$u = 12h_1h_2^{-1}(\ln f)_{xx}, \tag{32}$$

with

$$\begin{aligned}
 f &= \theta_1\theta_2 + B_{12} + (B_{12} + \Delta_{13}\Delta_{23} + \Delta_{13}\theta_2 \\
 &\quad + \Delta_{23}\theta_1 + \theta_1\theta_2)\exp(\eta_3) + (B_{12} \\
 &\quad + \Delta_{14}\Delta_{24} + \Delta_{14}\theta_2 + \Delta_{24}\theta_1 + \theta_1\theta_2)\exp(\eta_4) \\
 &\quad + [\theta_1(\Delta_{23} + \Delta_{24}) + \theta_2(\Delta_{13} + \Delta_{14}) \\
 &\quad + B_{12} + \Delta_{13}\Delta_{23} + \Delta_{14}\Delta_{23} + \Delta_{13}\Delta_{24} \\
 &\quad + \Delta_{14}\Delta_{24} + \theta_1\theta_2]\exp(\eta_3 + \eta_4 + A_{34}), \\
 \Delta_{i,j} &= -\frac{\Delta_{i,j1}}{\Delta_{i,j2}}, \quad (i = 1, 2, j = 3, 4) \\
 \Delta_{i,j1} &= 6P_i^2 p_j^2 [h_5(P_i k_j + K_i p_j) \\
 &\quad + 2h_1 P_i p_j + 5h_3 P_i p_j^3], \\
 \Delta_{i,j2} &= h_5 P_i p_j^3 (2P_i k_j + K_i p_j) + 3h_1 P_i^2 p_j^4 \\
 &\quad + 5h_3 P_i^2 p_j^6 - h_4 (P_i k_j - K_i p_j)^2, \\
 p_3 &= p_4^* = p_{31} + I p_{32},
 \end{aligned}
 \tag{33}$$

under the condition $p_{31} = 0$ or $p_{32} = 0$, where $\eta_3, \eta_4, A_{34}, \theta_1, \theta_2$ and B_{12} are given by Expressions (5) and

(11), p_{31} and p_{32} are the real constants. In this case, we find that $\Delta_{13} = \Delta_{24}^*$ and $\Delta_{14} = \Delta_{23}^*$, which ensure that f be a real function.

Hybrid solutions composed of the first-order breathers and first-order lumps describe the interaction between the first-order breathers and first-order lumps, as shown in Figs. 12 and 13. As t goes on, the breather and lump move close, interact with each other, and finally move further and further. It is found that the interaction is elastic since the shapes, velocities and amplitudes of the first-order breather and lump do not change after the interaction. To investigate the effect of h_2 , based on the parameters in Fig. 12, we set $h_2 = 1$ and fix the other parameters. Comparing Figs. 12 and 13, we find that the amplitudes of this kind of hybrid solutions are inversely proportional to h_2 .

5 Conclusions

In this paper, we have studied the $(2 + 1)$ -dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation, i.e., Eq. (1), which can be utilized to describe certain nonlinear phenomena in fluid mechanics. The L th-Order Lump Solutions (7) have been constructed via the long-wave limit method. Based on Solutions (7), the first- and second-order lump solutions, i.e., Solutions (10) and (18), have been obtained. Via Expressions (16) and (17), it has been found that the velocities of the first-order lumps are proportional to the constant coefficient h_4 in Eq. (1). We have also found that both the amplitudes of the humps and valleys of the first-order lumps are related to the constant coefficients h_1, h_2, h_4 and h_5 in Eq. (1), proportional to h_4 while inversely proportional to h_2 . It has been found that the second-order lumps describe the interaction between two first-order lumps, which is elastic since those lumps keep their shapes, velocities and amplitudes unchanged after the interaction, as shown in Figs. 1, 2 and 3. Comparing Figs. 1 and 2, we have seen that the amplitudes of the second-order lumps are inversely proportional to h_2 . As shown in Figs. 1 and 3, we have seen that both the velocities and amplitudes of the first-order lumps are proportional to h_4 .

We have derived The M th-Order Breather Solutions (22). The First-Order Breather Solutions (24) have been obtained from Solutions (22) with $M = 1$. Solutions (24) have indicated that the amplitudes of the first-order breathers are inversely proportional to

h_2 . The second-order breather solutions have been obtained with $M = 2$ in Solutions (22). It has been found that the second-order breathers describe the interaction between two first-order breathers, as shown in Figs. 4, 5 and 6. Comparing Figs. 4 and 5, we have seen that the amplitudes of the second-order breathers are inversely proportional to h_2 . Effects of h_1 on the amplitudes and shapes of the second-order breathers have been shown in Figs. 4 and 6.

Hybrid solutions consisting of the first-order breathers and one solitons have been constructed via the substitution of Expressions (26) into Solutions (2) and (5). Interaction between a first-order breather and one soliton has been found to be elastic as both the breather and soliton keep their shapes, velocities and amplitudes unchanged after the interaction, as shown in Figs. 7 and 8. Comparing Figs. 7 and 8, we have seen that the amplitudes of both the first-order breathers and one solitons are inversely proportional to h_2 . Via the long-wave limit method, hybrid solutions composed of the first-order lumps and one solitons, i.e., Solutions (28), have been obtained from Solutions (2) and (5) with the parameters given in Expression (27), and graphically analyzed, as shown in Figs. 9, 10 and 11. It has been found that the interaction between a first-order lump and one soliton is elastic. Comparing Figs. 9 and 10, we have found that the amplitudes of this kind of hybrid solutions are inversely proportional to h_2 . Comparing Figs. 9 and 11, we have seen that the amplitude and velocity of the first-order lump are proportional to h_4 . However, we have found that the amplitude of the one soliton does not depend on h_4 , and the velocity of the one soliton is positive correlated to h_4 . Hybrid solutions composed of the first-order breathers and first-order lumps, i.e., Solutions (32), have been obtained from Solutions (2) and (5) with the parameters given in Expressions (31). As shown in Figs. 12 and 13, the interaction between a first-order breather and a first-order lump has been found to be elastic. Comparing Figs. 12 and 13, we have found that the amplitudes of this kind of hybrid solutions are inversely proportional to h_2 .

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Yuan, Y.Q., Tian, B., Qu, Q.X., Zhao, X.H., Du, X.X.: Periodic-wave and semirational solutions for the (2+1)-dimensional Davey-Stewartson equations on the surface water waves of finite depth. *Z. Angew. Math. Phys.* **71**, 46 (2020)
2. Chen, S.S., Tian, B., Sun Y., Zhang, C.R.: Generalized darboux transformations, rogue waves, and modulation instability for the coherently coupled nonlinear Schrödinger equations in nonlinear optics. *Ann. Phys. (Berlin)* **531**, 1900011 (2019)
3. Tanna, K., Vijayajayanthi, M., Lakshmanan, M.: Mixed solitons in a (2 + 1)-dimensional multicomponent long-wave-short-wave system. *Phys. Rev. E* **90**, 042901 (2014)
4. Sun, Y., Tian, B., Yuan, Y.Q.: Semi-rational solutions for a (2 + 1)-dimensional Davey-Stewartson system on the surface water waves of finite depth. *Nonlinear Dyn.* **94**, 3029–3040 (2018)
5. Du, Z., Tian, B., Qu, Q.X., Wu, X.Y., Zhao, X.H.: Vector rational and semi-rational rogue waves for the coupled cubic-quintic nonlinear Schrödinger system in a non-Kerr medium. *Appl. Numer. Math.* **153**, 179–187 (2020)
6. Peregrine, D.H.: Long waves on a beach. *J. Fluid Mech.* **27**, 815–827 (1967)
7. Gao, X.Y.: Mathematical view with observational/experimental consideration on certain (2 + 1)-dimensional waves in the cosmic/laboratory dusty plasmas. *Appl. Math. Lett.* **91**, 165–172 (2019)
8. Gao, X.Y., Guo, Y.J., Shan, W.R.: Water-wave symbolic computation for the Earth, Enceladus and Titan: the higher order Boussinesq-Burgers system, auto- and non-auto-Backlund transformations. *Appl. Math. Lett.* **104**, 106170 (2020)
9. Chen, Y.Q., Tian, B., Qu, Q.X., Li, H., Zhao, X.H., Tian, H.Y., Wang, M.: Reduction and analytic solutions of a variable-coefficient Korteweg-de Vries equation in a fluid, crystal or plasma. *Mod. Phys. Lett. B* **34**, 2050287 (2020)
10. Chen, Y.Q., Tian, B., Qu, Q.X., Li, H., Zhao, X.H., Tian, H.Y., Wang, M.: Ablowitz-Kaup-Newell-Segur system, conservation laws and Backlund transformation of a variable-coefficient Korteweg-de Vries equation in plasma physics, fluid dynamics or atmospheric science. *Int. J. Mod. Phys. B* **34**, 2050226 (2020)
11. Yang, D.Y., Tian, B., Qu, Q.X., Yuan, Y.Q., Zhang, C.R., Tian, H.Y.: Generalized Darboux transformation and the higher-order semi-rational solutions for a nonlinear Schrödinger system in a birefringent fiber. *Mod. Phys. Lett. B* (2020) (in press)
12. Du, X.X., Tian, B., Yuan, Y.Q., Du, Z.: Symmetry reductions, group-invariant solutions, and conservation laws of a (2+1)-dimensional nonlinear Schrödinger equation in a Heisenberg ferromagnetic spin chain. *Ann. Phys. (Berlin)* **531**, 1900198 (2019)
13. Gao, X.Y., Guo, Y.J., Shan, W.R.: Hetero-Backlund transformation and similarity reduction of an extended (2+1)-dimensional coupled Burgers system in fluid mechanics. *Phys. Lett. A* **384**, 126788 (2020)
14. Gao, X.Y., Guo, Y.J., Shan, W.R.: Viewing the Solar System via a variable-coefficient nonlinear dispersive-wave system. *Acta Mech.* **231**, 4415–4420 (2020)
15. Cristian, B., Michael, F., Stephane, B.: Deterministic optical rogue waves. *Phys. Rev. Lett.* **107**, 053901 (2011)
16. Hu, S.H., Tian, B., Du, X.X., Du, Z., Wu, X.Y.: Lie symmetry reductions and analytic solutions for the AB system in a nonlinear optical fiber. *J. Comput. Nonlinear Dyn.* **14**, 111001 (2019)
17. Yuan, Y.Q., Tian, B., Qu, Q.X., Zhang, C.R., Du, X.X.: Lax pair, binary Darboux transformation and dark solitons for the three-component Gross–Pitaevskii system in the spinor Bose–Einstein condensate. *Nonlinear Dyn.* **99**, 3001–3011 (2020)
18. Du, X.X., Tian, B., Qu, Q.X., Yuan, Y.Q., Zhao, X.H.: Lie group analysis, solitons, self-adjointness and conservation laws of the modified Zakharov–Kuznetsov equation in an electron-positron-ion magnetoplasma. *Chaos, Solitons Fract.* **134**, 109709 (2020)
19. Zhang, C.R., Tian, B., Qu, Q.X., Liu, L., Tian, H.Y.: Vector bright solitons and their interactions of the couple Fokas–Lenells system in a birefringent optical fiber. *Z. Angew. Math. Phys.* **71**, 18 (2020)
20. Chen, S.S., Tian, B., Liu, L., Yuan, Y.Q., Zhang, C.R.: Conservation laws, binary Darboux transformations and solitons for a higher-order nonlinear Schrödinger system. *Chaos Solitons Fract.* **118**, 337–346 (2019)
21. Zhang, C.R., Tian, B., Sun, Y., Yin, H.M.: Binary Darboux transformation and vector-soliton-pair interactions with the negatively coherent coupling in a weakly birefringent fiber. *EPL* **127**, 40003 (2019)
22. Zhao, X., Tian, B., Qu, Q.X., Yuan, Y.Q., Du X.X., Chu, M.X.: Dark-dark solitons for the coupled spatially modulated Gross-Pitaevskii system in the Bose-Einstein condensation. *Mod. Phys. Lett. B* **34**, 2050282 (2020)
23. Gao, X.Y., Guo, Y.J., Shan, W.R.: Bilinear forms through the binary Bell polynomials, N solitons and Backlund transformations of the Boussinesq-Burgers system for the shallow water. *Commun. Theor. Phys.* **72**, 095002 (2020)
24. Gao, X.Y., Guo, Y.J., Shan, W.R.: Shallow water in an open sea or a wide channel: Auto- and non-auto-Backlund transformations with solitons for a generalized (2+1)-dimensional dispersive long-wave system. *Chaos, Solitons Fract.* **138**, 109950 (2020)
25. Gao, X.Y., Guo, Y.J., Shan, W.R., Yuan, Y.Q., Zhang, C.R., Chen, S.S.: Magneto-optical/ferromagnetic-material computation: Backlund transformations, bilinear forms and N solitons for a generalized (3+1)-dimensional variable-coefficient modified Kadomtsev–Petviashvili system. *Appl. Math. Lett.* **111**, 106627 (2021)
26. Hu, S.H., Tian, B., Du, X.X., Liu, L., Zhang, C.R.: Lie symmetries, conservation laws and solitons for the AB system with time-dependent coefficients in nonlinear optics or fluid mechanics. *Pramana J. Phys.* **93**, 38 (2019)
27. Wang, M., Tian, B., Sun, Y., Zhang, Z.: Lump, mixed lump-stripe and rogue wave-stripe solutions of a (3+1)-

- dimensional nonlinear wave equation for a liquid with gas bubbles. *Comput. Math. Appl.* **79**, 576 (2020)
28. Wang, M., Tian, B., Qu, Q.X., Du, X.X., Zhang, C.R., Zhang, Z.: Lump, lumpoff and rogue waves for a $(2 + 1)$ -dimensional reduced Yu–Toda–Sasa–Fukuyama equation in a lattice or liquid. *Eur. Phys. J. Plus* **134**, 578 (2019)
 29. Li, W., Zhang, Y., Liu, Y.P.: Exact wave solutions for a $(3 + 1)$ -dimensional generalized B-type Kadomtsev–Petviashvili equation. *Comput. Math. Appl.* **77**, 3087–3101 (2019)
 30. Ma, Y.C.: The perturbed plane-wave solutions of the cubic Schrödinger equation. *Stud. Appl. Math.* **60**, 43–58 (1979)
 31. Peregrine, D.H.: Water waves, nonlinear Schrödinger equation and their solutions. *ANZIAM J.* **25**, 16–43 (1983)
 32. Akhmediev, N.N., Korneev, V.I.: Modulation instability and periodic solutions of the nonlinear Schrödinger equation. *Theor. Math. Phys.* **69**, 1089–1093 (1986)
 33. Yin, H.M., Tian, B., Zhao, X.C.: Chaotic breathers and breather fission/fusion for a vector nonlinear Schrödinger equation in a birefringent optical fiber or wavelength division multiplexed system. *Appl. Math. Comput.* **368**, 124768 (2020)
 34. Yin, H.M., Tian, B., Zhao X.C.: Magnetic breathers and chaotic wave fields for a higher-order $(2+1)$ -dimensional nonlinear Schrödinger-type equation in a Heisenberg ferromagnetic spin chain. *J. Magn. Magn. Mater.* **495**, 165871 (2020)
 35. Gao, X.Y., Guo, Y.J., Shan, W.R.: Comment on Bilinear form, solitons, breathers and lumps of a $(3+1)$ -dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation in ocean dynamics, fluid mechanics and plasma physics. *Eur. Phys. J. Plus* **135**, 631 (2020)
 36. Du, Z., Tian, B., Chai, H.P., Zhao, X.H.: Dark-bright semi-rational solitons and breathers for a higher-order coupled nonlinear Schrödinger system in an optical fiber. *Appl. Math. Lett.* **102**, 106110 (2020)
 37. An, H.L., Feng, D.L., Zhu, H.X.: General M -lump, high-order breather and localized interaction solutions to the $(2 + 1)$ -dimensional Sawada–Kotera equation. *Nonlinear Dyn.* **98**, 1275–1286 (2019)
 38. Hu, C.C., Tian, B., Yin, H.M., Zhang, C.R., Zhang, Z.: Dark breather waves, dark lump waves and lump wave-soliton interactions for a $(3 + 1)$ -dimensional generalized Kadomtsev–Petviashvili equation in a Fluid. *Comput. Math. Appl.* **78**, 166–177 (2019)
 39. Hu, C.C., Tian, B., Wu, X.Y., Yuan, Y.Q., Du, Z.: Mixed lump-kink and rogue wave-kink solutions for a $(3 + 1)$ -dimensional B-type Kadomtsev–Petviashvili equation in fluid mechanics. *Eur. Phys. J. Plus* **133**, 40 (2018)
 40. Feng, L.L., Tian, S.F., Yan, H., Wang, L., Zhang, T.T.: On periodic wave solutions and asymptotic behaviors to a generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation. *Eur. Phys. J. Plus* **131**, 241 (2016)
 41. Liu, W.H., Zhang, Y.F., Shi, D.D.: Analysis on lump, lumpoff and rogue waves with predictability to a generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation. *Commun. Theor. Phys.* **71**, 670–676 (2019)
 42. Liang, Y.Q., Wei, G.M., Li, X.N.: Painlevé integrability, similarity reductions, new soliton and soliton-like similarity solutions for the $(2 + 1)$ -dimensional BKP equation. *Nonlinear Dyn.* **62**, 195–202 (2010)
 43. Wazwaz, A.M.: Two B-type Kadomtsev–Petviashvili equations of $(2 + 1)$ and $(3 + 1)$ dimensions: multiple soliton solutions, rational solutions and periodic solutions. *Comput. Fluids* **86**, 357–362 (2013)
 44. Lan, Z.Z., Gao, Y.T., Yang, J.W., Su, C.Q., Wang, Q.M.: Solitons, Backlund transformation and Lax pair for a $(2 + 1)$ -dimensional B-type Kadomtsev–Petviashvili equation in the fluid/plasma mechanics. *Mod. Phys. Lett. B* **30**, 1650265 (2016)
 45. Meng, X.H.: The periodic solitary wave solutions for the $(2 + 1)$ -dimensional fifth-order KdV equation. *J. Appl. Math. Phys.* **2**, 639–643 (2014)
 46. Cao, C.W., Wu, Y.Y., Geng, X.G.: On quasi-periodic solutions of the $(2 + 1)$ dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada equation. *Phys. Lett. A* **256**, 59–65 (1999)
 47. Fang, T., Gao, C.N., Wang, H., Wang, Y.H.: Lump-type solution, rogue wave, fusion and fission phenomena for the $(2 + 1)$ -dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada equation. *Mod. Phys. Lett. B* **33**, 1950198 (2019)
 48. Lü, X., Li, J.: Integrability with symbolic computation on the Bogoyavlensky–Konoplechenko model: bell-polynomial manipulation, bilinear representation, and Wronskian solution. *Nonlinear Dyn.* **77**, 135–143 (2014)
 49. Qin, B., Tian, B., Liu, L.C., Meng, X.H., Liu, W.J.: Bäcklund transformation and multisoliton solutions in terms of wronskian determinant for $(2 + 1)$ -dimensional breaking soliton equations with symbolic computation. *Commun. Theor. Phys.* **54**, 1059–1066 (2010)
 50. Xin, X.P., Liu, X.Q., Zhang, L.L.: Explicit solutions of the Bogoyavlensky–Konoplechenko equation. *Appl. Math. Comput.* **215**, 3669–3673 (2010)
 51. Liu, C.F., Dai, Z.D.: Exact soliton solutions for the fifth-order Sawada–Kotera equation. *Appl. Math. Comput.* **206**, 272–275 (2008)
 52. Naher, H., Abdullah, F.A., Mohyud-Din, S.T.: Extended generalized Riccati equation mapping method for the fifth-order Sawada–Kotera equation. *AIP Adv.* **3**, 052104 (2013)
 53. Gupta, A.K., Ray, S.S.: Numerical treatment for the solution of fractional fifth-order Sawada–Kotera equation using second kind Chebyshev wavelet method. *Appl. Math. Model.* **39**, 5121–5130 (2015)
 54. Batwa, S., Ma, W.X.: Lump solutions to a $(2 + 1)$ -dimensional fifth-order KdV-like equation. *Adv. Math. Phys.* **2018**, 2062398 (2018)
 55. Wazwaz, A.M.: Abundant solitons solutions for several forms of the fifth-order KdV equation by using the tanh method. *Appl. Math. Comput.* **182**, 283–300 (2006)
 56. Hirota, R.: The direct method in soliton theory. Cambridge University Press, Cambridge (2004)
 57. Ablowitz, M.J., Satsuma, J.: Solitons and rational solutions of nonlinear evolution equations. *J. Math. Phys.* **19**, 2180–2186 (1978)
 58. Ablowitz, M.J., Satsuma, J.: Two-dimensional lumps in nonlinear dispersive systems. *J. Math. Phys.* **20**, 1496–1503 (1979)

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