



# Antidark solitons and soliton molecules in a $(3 + 1)$ -dimensional nonlinear evolution equation

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**Abstract** We investigate a  $(3 + 1)$ -dimensional nonlinear evolution equation which is a higher-dimensional generalization of the Korteweg–de Vries equation. On the basis of the decomposition approach, the  $N$ -antidark soliton solution on a finite background is constructed by using the Darboux transformation together with the limit technique. The asymptotic analysis for the  $N$ -antidark soliton solution is performed, and the collision between multiple antidark solitons is proved to be elastic. Under the velocity resonant mechanism, the antidark soliton molecules on the  $(x, t)$ ,  $(y, t)$ ,  $(y, z)$  and  $(t, z)$  planes are found instead of the  $(x, y)$  and  $(x, z)$  planes. Based on the three- and the four-antidark soliton solutions, the elastic collision between a soliton molecule and a common soliton and the elastic collision between two soliton molecules are analytically demonstrated, respectively. These results may be useful for the study of soliton molecules in hydrodynamics and nonlinear optics.

**Keywords** Antidark solitons · Soliton molecules · Darboux transformation;  $(3 + 1)$ -dimensional nonlinear evolution equation · Asymptotic analysis

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## 1 Introduction

Soliton molecules, bound states of solitons, have been drastically studied to a certain extent due to their significant applications in a variety of contexts including optics [1–4] and Bose–Einstein condensates [5, 6], to name a few. In 2017, Herink et al. track the formation of stable soliton molecules and reveal rapid internal motions for diverse bound states in a femtosecond laser oscillator of real-time access to multipulse interactions [3]. In 2018, Liu et al. observe the entire buildup process of soliton molecules to explore the complex soliton interaction dynamics in a mode-locked laser [4]. In 2019, Zakharov et al. demonstrate the experimental observation of shaped breathing soliton molecules in a standard single-mode fiber [7].

The formation of soliton molecules has always been an important task to exhibit the bound states of solitons analogous to molecules in numerous fields of physics from theoretical and experimental perspectives. More recently, Lou et al. have developed the velocity resonant mechanism to obtain soliton molecules in many integrable systems such as the defocusing Hirota equation [8], the fifth-order Korteweg–de Vries (KdV) equation [9] and the Sharma–Tasso–Olver–Burgers equation [10]. Intricate soliton molecules such as dark molecule, kink–kink molecule, kink–breather molecule and breather–breather molecule have been found. Furthermore, Li et al. have studied soliton molecules in the complex modified KdV equation

[11], the (2 + 1)-dimensional Sawada–Kotera equation [12] and the (2 + 1)-dimensional B-type Kadomtsev–Petviashvili equation [13].

Concerning the realistic physical environment, one should not confine the soliton molecule investigations to (1 + 1)-dimensional and (2 + 1)-dimensional models, although the findings of integrable models in higher dimensions are not an easy work. As a matter of fact, the oceanic rogue waves, solitons and lumps are (2 + 1)-dimensional phenomena [14–20] and in ultrafast optics and fluids the more complex multidimensional dynamics should be considered [21–31]. Therefore, the extension of soliton molecules in higher-dimensional descriptions such as the (3 + 1)-dimensional equations is essential. In this paper, we consider a (3 + 1)-dimensional nonlinear evolution equation ((3 + 1)D NEE):

$$3w_{xz} + (2w_t - w_{xxx} - 2ww_x)_y - 2(w_x \partial_x^{-1} w_y)_x = 0, \quad (1)$$

where  $w = w(x, y, t, z)$  is a real function, the subscripts denote the partial derivatives and  $\partial_x^{-1}$  is defined by  $(\partial_x^{-1} f)(x) = \int_{-\infty}^x f(s) ds$ . It is easy to find that in terms of a simple scale transformation  $w \rightarrow u$ ,  $t \rightarrow -6\sqrt{3}T$ ,  $x \rightarrow \sqrt{3}X$ , the center part of Eq. (1) can become the standard KdV equation  $u_T + u_{XXX} + 6uu_X = 0$ . Consequently, Eq. (1) can be viewed as a (3 + 1)-dimensional generalization of the KdV equation and has potential applications in hydrodynamics, nonlinear optics, and so on.

The (3 + 1)D NEE (1) was first proposed by Geng when studying algebraic–geometrical solutions for multidimensional nonlinear evolution equations [32]. As pointed out by Geng, Eq. (1) can be decomposed into three (1 + 1)-dimensional AKNS equations, i.e., the nonlinear Schrödinger (NLS) equation, the complex modified KdV (cmKdV) equation and the Lakshmanan–Porsezian–Daniel (LPD) equation, and hence justifies its physical application in turn. In the past few years, finding exact solutions for the (3 + 1)D NEE (1) has attracted considerable attention, and its soliton solutions, lump solutions and various types of interactional solutions on the zero background have been constructed via Hirota’s bilinear method [33–35]. Particularly, Wazwaz has derived the multiple-soliton solutions and multiple singular soliton solutions for Eq. (1) through the simplified Hirota’s bilinear method

[36]. Additionally, Wang et al. [37] have recently studied its rational solutions that have been shown to exhibit doubly localized lumps and line rogue waves on a finite background by utilizing the Darboux transformation (DT) method. Note that in our previous work [37], the construction of rational solutions for Eq. (1) is based on the fact that Eq. (1) is decomposed into three (1 + 1)-dimensional AKNS equations with the focusing case, while for the defocusing case, to our knowledge, has not been considered by any authors. Based on this point of view, our paper aims to investigate antidark solitons and soliton molecules in Eq. (1) on a finite background instead of zero background [33–36] through the DT method [38–42] and limit technique [43–52].

The paper can be arranged as follows. In Sect. 2, based on the decomposition approach [32, 53, 54], we firstly decompose Eq. (1) into the NLS equation, the cmKdV equation and the LPD equation with the defocusing case. Then, we introduce a quartet Lax pair and construct the  $N$ -fold DT for the linear eigenvalue problem. In Sect. 3, we derive the  $N$ -antidark soliton solution represented in a compact determinant form as well as Hirota’s bilinear  $N$ -soliton solution form [55]. The asymptotic behavior [56–58] for the  $N$ -antidark soliton solution is discussed, and the property of elastic collision between multiple antidark solitons is preserved. In Sect. 4, under the velocity resonant mechanism, we demonstrate that the soliton molecules can exist on the  $(x, t)$ ,  $(y, t)$ ,  $(y, z)$  and  $(t, z)$  planes rather than the  $(x, y)$  and  $(x, z)$  planes. The elastic collision between a soliton molecule and a common soliton and the elastic collision between two soliton molecules are analytically shown with the help of the three- and the four-antidark soliton solutions, respectively. Finally, we give our conclusion.

## 2 Decomposition and Darboux transformation

In this section, we shall decompose the (3 + 1)D NEE (1) into the NLS equation, the mKdV equation and the LPD equation with the defocusing case. To this end, we consider the first three members of the AKNS hierarchy:

$$iq_y + \frac{1}{2}q_{xx} - q^2 r = 0, \quad ir_y - \frac{1}{2}r_{xx} + r^2 q = 0, \quad (2)$$

$$q_t + q_{xxx} - 6qrq_x = 0, \quad r_t + r_{xxx} - 6qrr_x = 0, \quad (3)$$

and

$$\begin{aligned}
 &iq_z + \frac{1}{2}(q_{xxxx} - 8qrq_{xx} \\
 &+ 6q^3r^2 - 4q_xr_xq - 6r^2q_x^2 - 2q^2r_{xx}) = 0, \\
 &ir_z - \frac{1}{2}(r_{xxxx} - 8qrr_{xx} + 6r^3q^2 \\
 &- 4q_xr_xr - 6qr_x^2 - 2r^2q_{xx}) = 0.
 \end{aligned} \tag{4}$$

It is not difficult to verify that Eqs. (2)–(4) are compatible since the flows determined by them can commute. We thus have the following proposition.

**Proposition 1** *Let (q, r) be a compatible solution of Eqs. (2)–(4), then the constraint*

$$w = -3qr \tag{5}$$

solves the (3 + 1)D NEE (1).

*Proof* In terms of Eqs. (2)–(4), one obtains

$$\begin{aligned}
 w_y &= -\frac{3}{2}i(rq_{xx} - qr_{xx}), \partial_x^{-1}w_y \\
 &= -\frac{3}{2}i(rq_x - qr_x), \\
 w_{xxx} &= -\frac{3}{2}i(rq_{xxx} - qr_{xxx})_x \\
 &\quad - 3i(r_xq_{xxx} - q_xr_{xxx})_x, \\
 w_t &= 3(qr_{xxx} + r_{xxx}q) - 2w^2w_x, w_x\partial_x^{-1}w_y \\
 &= \frac{9}{2}i(r^2q_x^2 - q^2r_x^2), \\
 w_{ty} &= -\frac{3}{2}i(qr_{xxxx} - rq_{xxxx})_x \\
 &\quad - \frac{3}{2}i(q_{xxx}r_x - r_{xxx}q_x) - 2(w\partial_x^{-1}w_y)_{xx}, \\
 w_{zx} &= -\frac{3}{2}i(rq_{xxx} - qr_{xxx})_x \\
 &\quad + 2(ww_y + w_x\partial_x^{-1}w_y)_x,
 \end{aligned}$$

which lead to the (3 + 1)D NEE (1). □

Next, we show that Eqs. (2)–(4) are completely integrable and can be cast into a quartet Lax pair

$$\Phi_x = U\Phi, U = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}, \tag{6}$$

$$\Phi_y = V^{(1)}\Phi, V^{(1)} = \begin{pmatrix} -i\lambda^2 - \frac{i}{2}qr & \lambda q + \frac{i}{2}q_x \\ \lambda r - \frac{i}{2}r_x & i\lambda^2 + \frac{i}{2}qr \end{pmatrix}, \tag{7}$$

$$\Phi_t = V^{(2)}\Phi, V^{(2)} = \begin{pmatrix} V_{11}^{(2)} & V_{12}^{(2)} \\ V_{21}^{(2)} & -V_{11}^{(2)} \end{pmatrix}, \tag{8}$$

$$\Phi_z = V^{(3)}\Phi, V^{(3)} = \begin{pmatrix} V_{11}^{(3)} & V_{12}^{(3)} \\ V_{21}^{(3)} & -V_{11}^{(3)} \end{pmatrix}, \tag{9}$$

where  $\Phi = (\psi, \varphi)^T$  is the eigenfunction,  $\lambda$  is the spectral parameter and

$$\begin{aligned}
 V_{11}^{(2)} &= -4i\lambda^3 - 2i\lambda qr + rq_x - qr_x, \\
 V_{12}^{(2)} &= 4\lambda^2q + 2i\lambda q_x + 2q^2r - q_{xx}, \\
 V_{21}^{(2)} &= 4\lambda^2r - 2i\lambda r_x + 2r^2q - r_{xx}, \\
 V_{11}^{(3)} &= 4i\lambda^4 + 2i\lambda^2qr - \lambda(rq_x - qr_x) \\
 &\quad + \frac{i}{2}(3q^2r^2 - qr_{xx} - rq_{xx} + q_xr_x), \\
 V_{12}^{(3)} &= -4\lambda^3q - 2i\lambda^2q_x + \lambda(q_{xx} - 2q^2r) \\
 &\quad + \frac{i}{2}(q_{xxx} - 6qrq_x), \\
 V_{21}^{(3)} &= -4\lambda^3r + 2i\lambda^2r_x + \lambda(r_{xx} - 2r^2q) \\
 &\quad - \frac{i}{2}(r_{xxx} - 6qrr_x).
 \end{aligned}$$

The compatible conditions of this linear eigenvalue problem, i.e.,  $U_{t_m} - V_x^{(m)} + UV^{(m)} - V^{(m)}U = 0$  ( $t_m = y, t, z, m = 1, 2, 3$ ), can reproduce Eqs. (2)–(4), respectively. In view of Eq. (5), we know that there are two kinds of decomposition for Eq. (1), namely  $w = 3|q|^2$  the focusing reduction of the linear eigenvalue problem

$$r = -q^*, U^\dagger(\lambda) = -U(\lambda^*), V^{(j)\dagger}(\lambda) = -V^{(j)}(\lambda^*), \tag{10}$$

and  $w = -3|q|^2$  the defocusing reduction of the linear eigenvalue problem

$$\begin{aligned}
 r &= q^*, U^\dagger(\lambda) = -\Lambda U(\lambda^*)\Lambda, V^{(j)\dagger}(\lambda) \\
 &= -\Lambda V^{(j)}(\lambda^*)\Lambda,
 \end{aligned} \tag{11}$$

where  $\Lambda = \text{diag}(1, -1)$ ,  $j = 1, 2, 3$ . Noteworthy, the focusing case has been considered in our recent work [37] and the rational solutions for Eq. (1) have been investigated. In this paper, we discuss the defocusing case and concentrate on the antidark solitons and soliton molecules in Eq. (1).

At this point, assuming that  $\Phi_i$  ( $i = 1, 2, \dots, N$ ) are  $N$  solutions for the linear eigenvalue problem (6)–(9) with the symmetry reduction (11) at  $q = q[0]$  and

$\lambda = \lambda_i$ , then the  $N$ -fold DT takes the form

$$T[N] = \sum_{j=1}^N \frac{T_j}{\lambda - \lambda_j^*} = I - XM^{-1}(\lambda - D)^{-1}X^\dagger A, \tag{12}$$

where

$$X = (\Phi_1, \Phi_2, \dots, \Phi_N), \quad D = \text{diag}(\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*),$$

$$M = \begin{pmatrix} \frac{\Phi_1^\dagger \Phi_1}{\lambda_1 - \lambda_1^*} & \frac{\Phi_1^\dagger \Phi_2}{\lambda_2 - \lambda_1^*} & \dots & \frac{\Phi_1^\dagger \Phi_N}{\lambda_N - \lambda_1^*} \\ \frac{\Phi_2^\dagger \Phi_1}{\lambda_1 - \lambda_2^*} & \frac{\Phi_2^\dagger \Phi_2}{\lambda_2 - \lambda_2^*} & \dots & \frac{\Phi_2^\dagger \Phi_N}{\lambda_N - \lambda_2^*} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Phi_N^\dagger \Phi_1}{\lambda_1 - \lambda_N^*} & \frac{\Phi_N^\dagger \Phi_2}{\lambda_2 - \lambda_N^*} & \dots & \frac{\Phi_N^\dagger \Phi_N}{\lambda_N - \lambda_N^*} \end{pmatrix}.$$

Then, by applying

$$\begin{aligned} T[N]_x + T[N]U &= U[N]T[N], \quad U[N] = U(\lambda, q[N]), \\ T[N]_y + T[N]V^{(1)} &= V^{(1)}[N]T[N], \\ V^{(1)}[N] &= V^{(1)}(\lambda, q[N]), \\ T[N]_t + T[N]V^{(2)} &= V^{(2)}[N]T[N], \\ V^{(2)}[N] &= V^{(2)}(\lambda, q[N]), \\ T[N]_z + T[N]V^{(3)} &= V^{(3)}[N]T[N], \\ V^{(3)}[N] &= V^{(3)}(\lambda, q[N]), \end{aligned}$$

we have

$$q[N] = q[0] + 2i \sum_{j=1}^N T_j^{[12]},$$

and

$$|q[N]|^2 = |q[0]|^2 + i \frac{\partial}{\partial x} \sum_{j=1}^N (T_j^{[11]} - T_j^{[22]}).$$

Additionally, using

$$\left( \frac{\Phi_i^\dagger \Phi_j}{\lambda_j - \lambda_i^*} \right)_x = -i \Phi_i^\dagger \Phi_j,$$

one obtains the new potential expressed in a compact determinant form

$$q[N] = q[0] - 2i \frac{\det(M_1)}{\det(M)}, \tag{13}$$

where

$$M_1 = \begin{pmatrix} \frac{\Phi_1^\dagger \Phi_1}{\lambda_1 - \lambda_1^*} & \frac{\Phi_1^\dagger \Phi_2}{\lambda_2 - \lambda_1^*} & \dots & \frac{\Phi_1^\dagger \Phi_N}{\lambda_N - \lambda_1^*} & \varphi_1^* \\ \frac{\Phi_2^\dagger \Phi_1}{\lambda_1 - \lambda_2^*} & \frac{\Phi_2^\dagger \Phi_2}{\lambda_2 - \lambda_2^*} & \dots & \frac{\Phi_2^\dagger \Phi_N}{\lambda_N - \lambda_2^*} & \varphi_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\Phi_N^\dagger \Phi_1}{\lambda_1 - \lambda_N^*} & \frac{\Phi_N^\dagger \Phi_2}{\lambda_2 - \lambda_N^*} & \dots & \frac{\Phi_N^\dagger \Phi_N}{\lambda_N - \lambda_N^*} & \varphi_N^* \\ \psi_1 & \psi_2 & \dots & \psi_N & 0 \end{pmatrix},$$

and its intensity

$$|q[N]|^2 = |q[0]|^2 - \frac{\partial^2}{\partial x^2} \ln \det(M). \tag{14}$$

### 3 Antidark solitons and elastic collisions

For our studies, we start from the general plane-wave solution of Eqs. (2)–(4), that is,

$$q[0] = ce^{i\theta}, \quad \theta = ax + by + dt + ez, \tag{15}$$

where  $c$  and  $a$  are real constants and stand for the background and frequency of the complex field envelope  $q$ , respectively, and  $b, d, e$  are wave numbers such that

$$b = -\frac{1}{2}a^2 - c^2, \quad d = a^3 + 6ac^2, \quad e = \frac{1}{2}a^4 + 6a^2c^2 + 3c^4.$$

Substituting Eq. (15) and  $\lambda = \lambda_j$  ( $j = 1, 2, \dots, N$ ) into the quartet Lax pair (6)–(9), we can arrive at the fundamental solution

$$\Phi_j = \begin{pmatrix} 1 & 1 \\ 2ic & 2ic \end{pmatrix} \begin{pmatrix} e^{i\omega_j} \\ \alpha_j(\lambda_j^* - \lambda_j)e^{i\omega_j^*} \end{pmatrix},$$

where  $\alpha_j$  is an arbitrary complex constant,

$$\chi_j = 2\lambda_j + \sqrt{(2\lambda_j + a)^2 - 4c^2},$$

$$\chi_j^* = 2\lambda_j - \sqrt{(2\lambda_j + a)^2 - 4c^2},$$

and

$$\omega_j = \left( \lambda_j - \frac{1}{2}\chi_j \right) (x - v_{j1}y - v_{j2}t - v_{j3}z),$$

with

$$v_{j1} = -\left( \lambda_j - \frac{1}{2}a \right), \quad v_{j2} = -(4\lambda_j^2 - 2a\lambda_j + 2c^2 + a^2),$$

$$v_{j3} = 4\lambda_j^3 - 2a\lambda_j^2 + (a^2 + 2c^2)\lambda_j - \frac{1}{2}a(a^2 + 6c^2).$$

Here,

$$\text{Im}(\chi_j) = \sqrt{-(2\lambda_j + a)^2 + 4c^2} > 0,$$

such that

$$-c - \frac{1}{2}a < \lambda_j < c - \frac{1}{2}a, \lambda_j \in \mathbb{R}. \tag{16}$$

On account of the  $N$ -fold DT and the limit  $\lambda_j^* \rightarrow \lambda_j$ , we can derive the  $N$ -dark soliton solution of Eqs. (2)–(4), as

$$q[N] = ce^{i\theta} \left[ 1 - 2i \frac{\det(\widehat{M}_1)}{\det(\widehat{M})} \right], \tag{17}$$

where

$$\widehat{M} = \begin{pmatrix} \frac{1}{\chi_1 - \chi_1^*} [e^{-2\text{Im}(\omega_1)} + \beta_1] & \frac{1}{\chi_2 - \chi_1^*} e^{-i(\omega_1^* - \omega_2)} & \dots & \frac{1}{\chi_N - \chi_1^*} e^{-i(\omega_1^* - \omega_N)} \\ \frac{1}{\chi_1 - \chi_2^*} e^{-i(\omega_2^* - \omega_1)} & \frac{1}{\chi_2 - \chi_2^*} [e^{-2\text{Im}(\omega_2)} + \beta_2] & \dots & \frac{1}{\chi_N - \chi_2^*} e^{-i(\omega_2^* - \omega_N)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_1 - \chi_N^*} e^{-i(\omega_N^* - \omega_1)} & \frac{1}{\chi_2 - \chi_N^*} e^{-i(\omega_N^* - \omega_2)} & \dots & \frac{1}{\chi_N - \chi_N^*} [e^{-2\text{Im}(\omega_N)} + \beta_N] \end{pmatrix},$$

$$\widehat{M}_1 = \begin{pmatrix} \frac{1}{\chi_1 - \chi_1^*} [e^{-2\text{Im}(\omega_1)} + \beta_1] & \frac{1}{\chi_2 - \chi_1^*} e^{-i(\omega_1^* - \omega_2)} & \dots & \frac{1}{\chi_N - \chi_1^*} e^{-i(\omega_1^* - \omega_N)} & -\frac{i}{2(\chi_1^* + a)} e^{-i\omega_1^*} \\ \frac{1}{\chi_1 - \chi_2^*} e^{-i(\omega_2^* - \omega_1)} & \frac{1}{\chi_2 - \chi_2^*} [e^{-2\text{Im}(\omega_2)} + \beta_2] & \dots & \frac{1}{\chi_N - \chi_2^*} e^{-i(\omega_2^* - \omega_N)} & -\frac{i}{2(\chi_2^* + a)} e^{-i\omega_2^*} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\chi_1 - \chi_N^*} e^{-i(\omega_N^* - \omega_1)} & \frac{1}{\chi_2 - \chi_N^*} e^{-i(\omega_N^* - \omega_2)} & \dots & \frac{1}{\chi_N - \chi_N^*} [e^{-2\text{Im}(\omega_N)} + \beta_N] & -\frac{i}{2(\chi_N^* + a)} e^{-i\omega_N^*} \\ e^{i\omega_1} & e^{i\omega_2} & \dots & e^{i\omega_N} & 0 \end{pmatrix},$$

with

$$\beta_j = -\text{Im}(\chi_j) \text{Im} \left[ \alpha_j^* \left( 1 - \frac{4c^2}{(\chi_j + a)^2} \right) \right] > 0,$$

and

$$\text{Im}(\omega_j) = m_j(x - \nu_{j1}y - \nu_{j2}t - \nu_{j3}z),$$

$$m_j = -\frac{1}{2} \text{Im}(\chi_j) < 0.$$

At this time, returning to Eq. (5) with the symmetry reduction  $r = q^*$  and Eq. (14), we can present the  $N$ -antidark soliton solution for Eq. (1), namely

$$w[N] = -3 \left[ c^2 - \frac{\partial^2}{\partial x^2} \ln \det(\widehat{M}) \right]. \tag{18}$$

Explicitly, for  $N = 1$  in the above formula, we can get the one-antidark soliton solution

$$w[1] = -3 \left[ c^2 - m_1^2 \text{sech}^2(K_1) \right],$$

where

$$K_1 = m_1(x - \nu_{11}y - \nu_{12}t - \nu_{13}z) + \frac{1}{2} \ln \beta_1.$$

It can be computed that the maximum amplitude of  $w[1]$  is

$$w[1]_{\max} = -3(c^2 - m_1^2) = -\frac{3}{4}(2\lambda_1 + a)^2, \tag{19}$$

and it is localized at the line

$$x - \nu_{11}y - \nu_{12}t - \nu_{13}z = -\frac{\ln \beta_1}{2m_1}.$$

In addition, we find  $w[1] \rightarrow -3c^2$  for  $x \rightarrow \pm\infty$ , which implies that the antidark soliton propagates on a finite background instead of the zero background.

To proceed, the two-antidark soliton solution can be explicitly written as

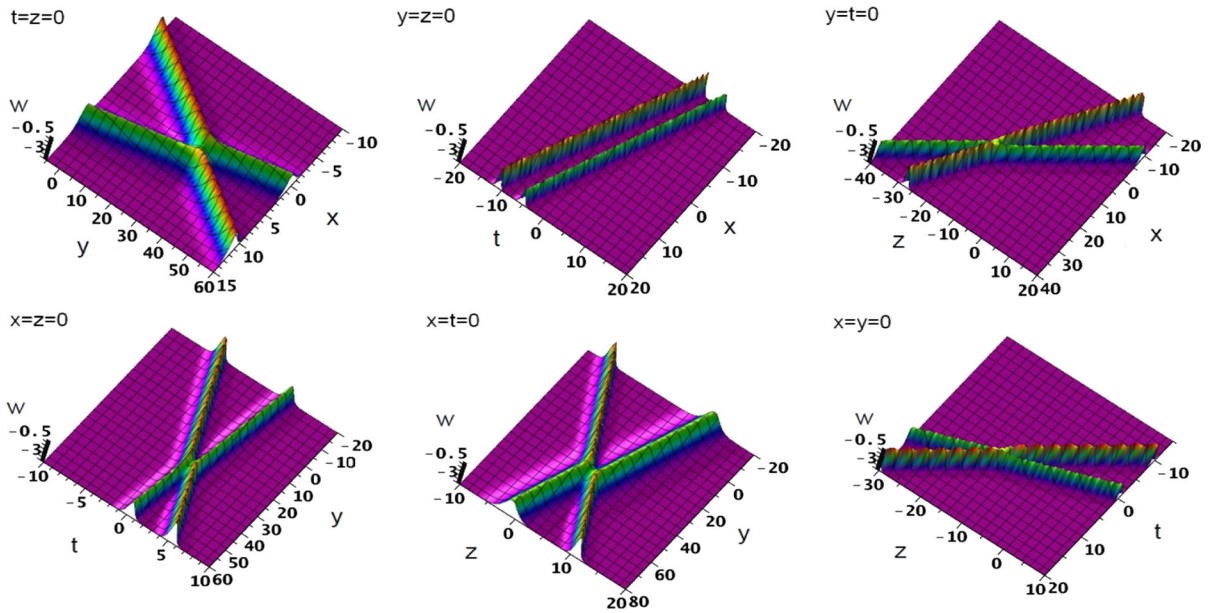
$$w[2] = -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_1} + e^{-2K_2} + e^{-2(K_1+K_2+A_{12})}] \right), \tag{20}$$

where

$$A_{12} = \ln \frac{|\chi_1 - \chi_2^*|}{|\chi_1 - \chi_2|},$$

$$K_l = m_l(x - \nu_{l1}y - \nu_{l2}t - \nu_{l3}z) + \frac{1}{2} \ln \beta_l, \quad l = 1, 2.$$

Following the standard asymptotic analysis process, we have:



**Fig. 1** Collision of two antidark solitons in Eq. (20). The parameters are  $c = 1, a = \frac{1}{2}, \lambda_1 = 0, \lambda_2 = \frac{1}{3}, \chi_1 = \frac{1}{2}\sqrt{15}, \chi_2 = \frac{2}{3} + \frac{1}{6}\sqrt{95}, \beta_1 = \frac{1}{10,000}, \beta_2 = 10,000$

(i) If  $K_1 = m_1(x - v_{11}y - v_{12}t - v_{13}z) + \frac{1}{2} \ln \beta_1 = \text{constant}$ ,

$$w[2] \rightarrow \begin{cases} -3 \left[ c^2 - m_1^2 \text{sech}^2(K_1) \right], & K_2 \rightarrow +\infty, \\ -3 \left[ c^2 - m_1^2 \text{sech}^2(K_1 + A_{12}) \right], & K_2 \rightarrow -\infty. \end{cases}$$

(ii) If  $K_2 = m_2(x - v_{21}y - v_{22}t - v_{23}z) + \frac{1}{2} \ln \beta_2 = \text{constant}$ ,

$$w[2] \rightarrow \begin{cases} -3 \left[ c^2 - m_2^2 \text{sech}^2(K_2 + A_{12}) \right], & K_1 \rightarrow -\infty, \\ -3 \left[ c^2 - m_2^2 \text{sech}^2(K_2) \right], & K_1 \rightarrow +\infty. \end{cases}$$

We exhibit in Fig. 1 the collision of two antidark solitons on the  $(x, y)$ ,  $(x, t)$ ,  $(x, z)$ ,  $(y, t)$ ,  $(y, z)$  and  $(t, z)$  planes. It is shown that the collision is elastic since the amplitude, velocity and shape of each soliton are unchanged after the collision except for a phase difference which is given by  $A_{12} = \ln \frac{|\chi_1 - \chi_2^*|}{|\chi_1 - \chi_2|}$ .

Furthermore, the explicit three-antidark soliton solution is found to be

$$w[3] = -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln \left[ 1 + e^{-2K_1} + e^{-2K_2} + e^{-2K_3} + e^{-2(K_1+K_2+A_{12})} + e^{-2(K_1+K_3+A_{13})} + e^{-2(K_2+K_3+A_{23})} + e^{-2(K_1+K_2+K_3+A_{12}+A_{13}+A_{23})} \right] \right), \tag{21}$$

where

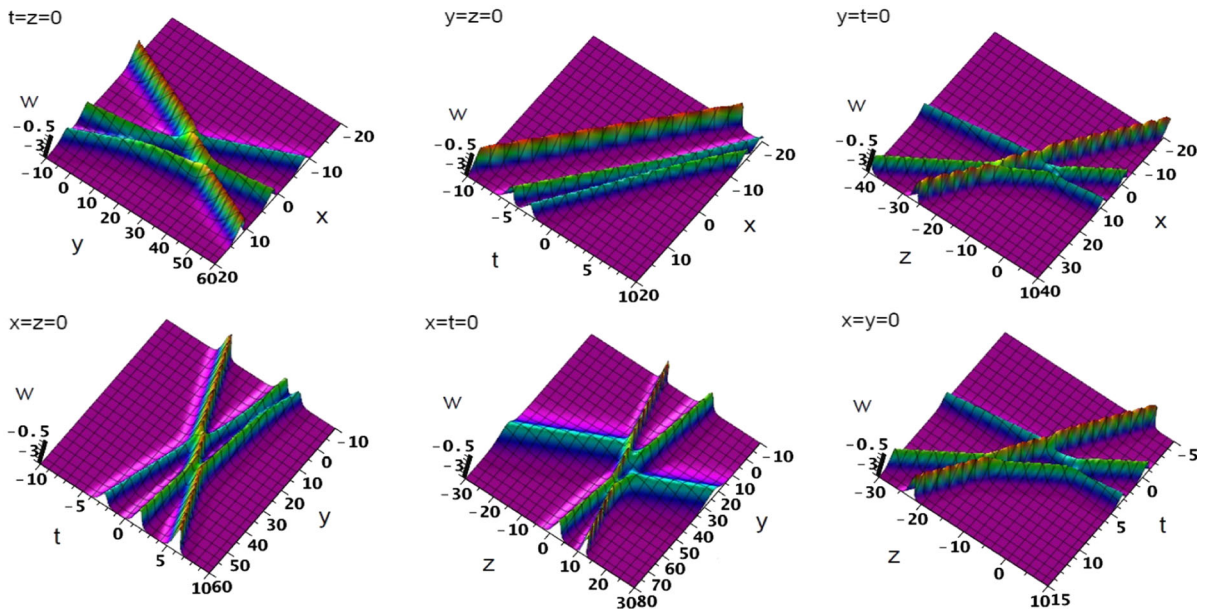
$$A_{jl} = \ln \frac{|\chi_j - \chi_l^*|}{|\chi_j - \chi_l|},$$

$$K_l = m_l(x - v_{l1}y - v_{l2}t - v_{l3}z) + \frac{1}{2} \ln \beta_l, \quad j < l = 1, 2, 3.$$

Similarly, we have the following asymptotic behaviors:

(i) If  $K_1 = m_1(x - v_{11}y - v_{12}t - v_{13}z) + \frac{1}{2} \ln \beta_1 = \text{constant}$ ,

$$w[3] \rightarrow \begin{cases} -3 \left[ c^2 - m_1^2 \text{sech}^2(K_1) \right], & K_2 \rightarrow +\infty, K_3 \rightarrow +\infty, \\ -3 \left[ c^2 - m_1^2 \text{sech}^2(K_1 + A_{12} + A_{13}) \right], & K_2 \rightarrow -\infty, K_3 \rightarrow -\infty. \end{cases}$$



**Fig. 2** Collision of three antidark solitons in Eq. (21). The parameters are  $c = 1, a = \frac{1}{2}, \lambda_1 = 0, \lambda_2 = \frac{1}{3}, \lambda_3 = \frac{1}{2}, \chi_1 = \frac{1}{2}\sqrt{15}, \chi_2 = \frac{2}{3} + \frac{1}{6}\sqrt{95}, \chi_3 = 1 + \frac{1}{2}\sqrt{7}, \beta_1 = \frac{1}{10,000}, \beta_2 = 10,000, \beta_3 = 20,000$

(ii) If  $K_2 = m_2(x - v_{21}y - v_{22}t - v_{23}z) + \frac{1}{2} \ln \beta_2 =$  constant,

$$w[3] \rightarrow \begin{cases} -3 \left[ c^2 - m_2^2 \operatorname{sech}^2(K_2 + A_{23}) \right], & K_1 \rightarrow +\infty, K_3 \rightarrow -\infty, \\ -3 \left[ c^2 - m_2^2 \operatorname{sech}^2(K_2 + A_{12}) \right], & K_1 \rightarrow -\infty, K_3 \rightarrow +\infty. \end{cases}$$

(iii) If  $K_3 = m_3(x - v_{31}y - v_{32}t - v_{33}z) + \frac{1}{2} \ln \beta_3 =$  constant,

$$w[3] \rightarrow \begin{cases} -3 \left[ c^2 - m_3^2 \operatorname{sech}^2(K_3 + A_{13} + A_{23}) \right], & K_1 \rightarrow -\infty, K_2 \rightarrow -\infty, \\ -3 \left[ c^2 - m_3^2 \operatorname{sech}^2(K_3) \right], & K_1 \rightarrow +\infty, K_2 \rightarrow +\infty. \end{cases}$$

The elastic collision property of three antidark solitons is kept, as seen in Fig. 2.

Next, by calculating the determinant of the  $N \times N$  Cauchy-type matrix in Eq. (18), we can put forward the  $N$ -antidark soliton solution for Eq. (1) in Hirota’s bilinear  $N$ -soliton solution form

$$w[N] = -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln \left[ \sum_{\mu=0,1} e^{-2(\sum_{j=1}^N \mu_j K_j + \sum_{1 \leq j < l \leq N} \mu_j \mu_l A_{jl})} \right] \right), \tag{22}$$

where

$$A_{jl} = \ln \frac{|\chi_j - \chi_l^*|}{|\chi_j - \chi_l|}, \quad K_l = m_l(x - v_{l1}y - v_{l2}t - v_{l3}z) + \frac{1}{2} \ln \beta_l, \quad j < l = 1, 2, \dots, N.$$

The detailed derivation for the above formula is given in Appendix. Further, we make the asymptotic analysis for the  $N$ -antidark soliton solution by assuming  $K_k = m_k(x - v_{k1}y - v_{k2}t - v_{k3}z) + \frac{1}{2} \ln \beta_k =$  constant ( $1 \leq k \leq N$ ), we conclude that

$$w[N] \rightarrow \begin{cases} -3 [c^2 - m_k^2 \operatorname{sech}^2(K_k + \Delta_k^-)], & K_i \rightarrow +\infty, K_j \rightarrow -\infty \\ -3 [c^2 - m_k^2 \operatorname{sech}^2(K_k) + \Delta_k^+], & K_i \rightarrow -\infty, K_j \rightarrow +\infty \end{cases}, \quad 1 \leq i < k < j \leq N,$$

where

$$\Delta_k^- = \begin{cases} \ln \prod_{l=k+1}^n \frac{|\chi_k - \chi_l^*|}{|\chi_k - \chi_l|}, & 1 \leq k \leq N - 1, \\ 0, & k = N, \end{cases}$$

and

$$\Delta_k^+ = \begin{cases} 0, & k = 1, \\ \ln \prod_{l=1}^{k-1} \frac{|\chi_k - \chi_l^*|}{|\chi_k - \chi_l|}, & 2 \leq k \leq N. \end{cases}$$

#### 4 Antidark soliton molecules

In this section, we utilize the velocity resonant method to discuss the possible formation of antidark soliton molecules on the  $(x, y)$ ,  $(x, t)$ ,  $(x, z)$ ,  $(y, t)$ ,  $(y, z)$  and  $(t, z)$  planes. We first consider the resonant condition on each plane.

(i) On the  $(x, y)$  plane, the resonant condition is  $v_{11} = v_{21}$  and we have

$$\lambda_2 = \lambda_1. \tag{23}$$

(ii) On the  $(x, t)$  plane, the resonant condition is  $v_{12} = v_{22}$  and we obtain

$$\lambda_2 = \frac{1}{2}(a - 2\lambda_1). \tag{24}$$

(iii) On the  $(x, z)$  plane, the resonant condition is  $v_{13} = v_{23}$  and it holds that

$$\lambda_2 = \frac{1}{4}(a - 2\lambda_1) \pm \frac{1}{2} \sqrt{-3 \left( \lambda_1 - \frac{1}{6}a \right)^2 - \frac{2}{3}a^2 - 2c^2}. \tag{25}$$

(iv) On the  $(y, t)$  plane, the resonant condition is  $\frac{v_{11}}{v_{21}} = \frac{v_{12}}{v_{22}}$  and it follows that

$$\lambda_2 = \frac{a\lambda_1 + c^2}{2\lambda_1 - a}. \tag{26}$$

(v) On the  $(y, z)$  plane, the resonant condition is  $\frac{v_{11}}{v_{21}} = \frac{v_{13}}{v_{23}}$  and it yields

$$\lambda_2 = \frac{1}{4}(a - 2\lambda_1) \pm \frac{1}{4} \sqrt{\frac{-8\lambda_1^3 - 4a\lambda_1^2 + 2a^2\lambda_1 + a^3 + 16ac^2}{a - 2\lambda_1}}. \tag{27}$$

(vi) On the  $(t, z)$  plane, the resonant condition is  $\frac{v_{12}}{v_{22}} = \frac{v_{13}}{v_{23}}$  and it implies

$$\lambda_2 = \frac{1}{2(4\lambda_1^2 - 2a\lambda_1 + 2c^2 + a^2)} (2a\lambda_1^2 - a^2\lambda_1 - 2ac^2 \pm \sqrt{v}), \tag{28}$$

where

$$v = -(12a^2 + 32c^2)\lambda_1^4 + 4a(a^2 - 4c^2)\lambda_1^3 - (3a^4 + 32c^4)\lambda_1^2 - 8ac^2(a^2 + c^2)\lambda_1 + 2c^2(a^4 + 2a^2c^2 - 4c^4).$$

From Eqs. (23) and (25), one can find that it is impossible to choose two different real spectral parameters to yield the resonant conditions, and hence the antidark soliton molecules on the  $(x, y)$  and  $(x, z)$  planes cannot be formed. Nevertheless, the antidark soliton molecules on the  $(x, t)$ ,  $(y, t)$ ,  $(y, z)$  and  $(t, z)$  planes can be obtained as long as Eqs. (24), (26), (27) and (28) are satisfied, respectively.

Meanwhile, we would like to say that the interesting collision between a soliton molecule (SM) and a common soliton (S) is also elastic, which can be proved by the other asymptotic analysis of  $w[3]$ :

(i) If  $K_1, K_2 = \text{constant}$ ,

$$w[3]_{\text{SM+S}} \rightarrow \begin{cases} -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_1} + e^{-2K_2} + e^{-2(K_1+K_2+A_{12})}] \right), & K_3 \rightarrow +\infty, \\ -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2\tilde{K}_1} + e^{-2\tilde{K}_2} + e^{-2(\tilde{K}_1+\tilde{K}_2+A_{12})}] \right), & K_3 \rightarrow -\infty, \end{cases}$$



where

$$\tilde{K}_1 = K_1 + A_{13}, \quad \tilde{K}_2 = K_2 + A_{23}.$$

Finally, we discuss the elastic collision of two antidark soliton molecules based on the four-antidark soliton solution. By use of the asymptotic analysis, we have:

(i) If  $K_1, K_2 = \text{constant}$ ,

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$$w[4]_{SM+SM} \rightarrow \begin{cases} -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_1} + e^{-2K_2} + e^{-2(K_1+K_2+A_{12})}] \right), & K_3, K_4 \rightarrow +\infty, \\ -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2\tilde{K}_1} + e^{-2\tilde{K}_2} + e^{-2(\tilde{K}_1+\tilde{K}_2+A_{12})}] \right), & K_3, K_4 \rightarrow -\infty, \end{cases}$$


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(ii) If  $K_1, K_3 = \text{constant}$ ,

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$$w[3]_{SM+S} \rightarrow \begin{cases} -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2\hat{K}_1} + e^{-2\hat{K}_3} + e^{-2(\hat{K}_1+\hat{K}_3+A_{13})}] \right), & K_2 \rightarrow -\infty, \\ -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_1} + e^{-2K_3} + e^{-2(K_1+K_3+A_{13})}] \right), & K_2 \rightarrow +\infty, \end{cases}$$


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where

$$\hat{K}_1 = K_1 + A_{12}, \quad \hat{K}_3 = K_3 + A_{23}.$$

(iii) If  $K_2, K_3 = \text{constant}$ ,

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$$w[3]_{SM+S} \rightarrow \begin{cases} -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_2} + e^{-2K_3} + e^{-2(K_2+K_3+A_{23})}] \right), & K_1 \rightarrow +\infty, \\ -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2\bar{K}_2} + e^{-2\bar{K}_3} + e^{-2(\bar{K}_2+\bar{K}_3+A_{23})}] \right), & K_1 \rightarrow -\infty, \end{cases}$$


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where

$$\bar{K}_2 = K_2 + A_{12}, \quad \bar{K}_3 = K_3 + A_{13}.$$

The antidark soliton molecule and the elastic collision between a soliton molecule and a common soliton on the  $(x, t)$ ,  $(y, t)$ ,  $(y, z)$  and  $(t, z)$  planes can be presented by choosing adequate parameters, as seen in Figs. 3, 4, 5 and 6.

where

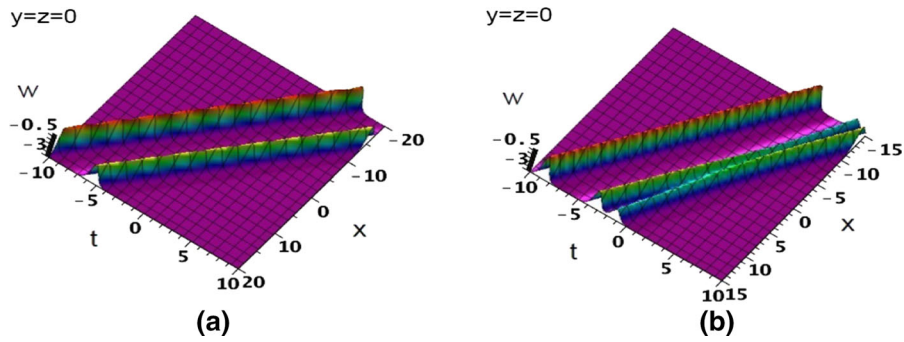
$$\tilde{\tilde{K}}_1 = K_1 + A_{13} + A_{14}, \quad \tilde{\tilde{K}}_2 = K_2 + A_{23} + A_{24}.$$

(ii) If  $K_1, K_3 = \text{constant}$ ,

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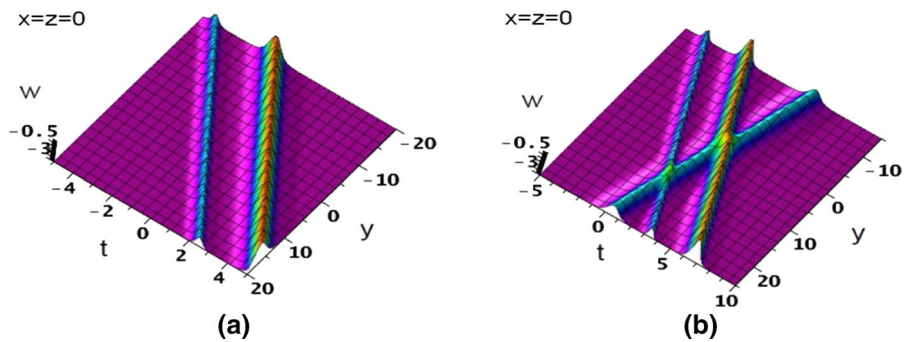

$$w[4]_{SM+SM} \rightarrow \begin{cases} -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2\tilde{\tilde{K}}_1} + e^{-2\tilde{\tilde{K}}_3} + e^{-2(\tilde{\tilde{K}}_1+\tilde{\tilde{K}}_3+A_{13})}] \right), & K_2, K_4 \rightarrow -\infty, \\ -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_1} + e^{-2K_3} + e^{-2(K_1+K_3+A_{13})}] \right), & K_2, K_4 \rightarrow +\infty, \end{cases}$$


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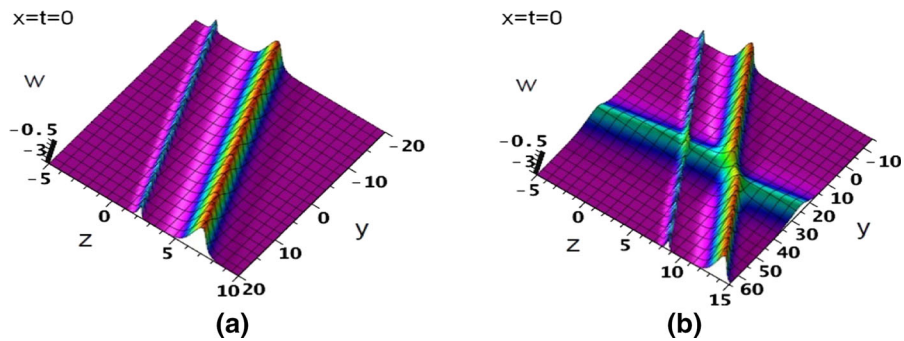
**Fig. 3** **a** Antidark soliton molecule in Eq. (20) and **b** collision between an soliton molecule and a common soliton in Eq. (21) on the  $(x, t)$  plane. The parameters are  $c = 1, a = \frac{1}{2}, \lambda_1 = 0, \lambda_2 =$

$$\frac{1}{4}, \lambda_3 = \frac{1}{2}, \chi_1 = \frac{i}{2}\sqrt{15}, \chi_2 = \frac{1}{2} + i\sqrt{3}, \chi_3 = 1 + \frac{i}{2}\sqrt{7}, \beta_1 = \frac{1}{10,000}, \beta_2 = 10,000, \beta_3 = 20,000$$



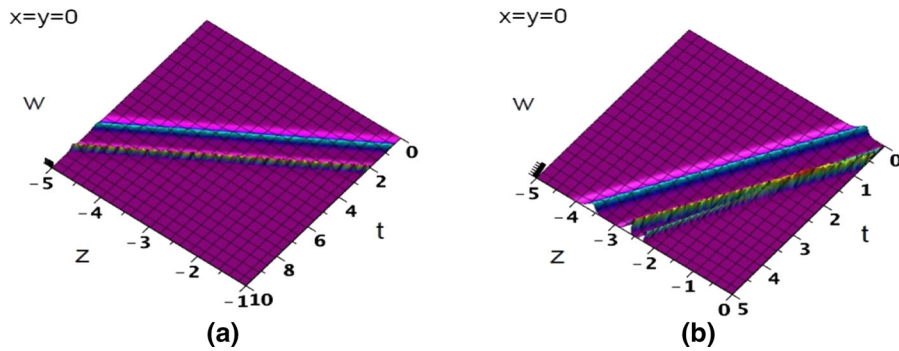
**Fig. 4** **a** Antidark soliton molecule in Eq. (20) and **b** collision between an soliton molecule and a single soliton in Eq. (21) on the  $(y, t)$  plane. The parameters are  $c = 1, a = \frac{1}{2}, \lambda_1 = -1, \lambda_2 =$

$$-\frac{1}{5}, \lambda_3 = \frac{1}{2}, \chi_1 = -2 + \frac{i}{2}\sqrt{7}, \chi_2 = -\frac{2}{5} + \frac{i}{10}\sqrt{399}, \chi_3 = 1 + \frac{i}{2}\sqrt{7}, \beta_1 = \frac{1}{10,000}, \beta_2 = 10,000, \beta_3 = 20,000$$



**Fig. 5** **a** Antidark soliton molecule in Eq. (20) and **b** collision between an soliton molecule and a single soliton in Eq. (21) on the  $(y, z)$  plane. The parameters are  $c = 1, a =$

$$\frac{1}{2}, \lambda_1 = -1, \lambda_2 = \frac{5}{8} - \frac{\sqrt{545}}{40}, \lambda_3 = \frac{1}{2}, \chi_1 = -2 + \frac{i}{2}\sqrt{7}, \chi_2 = -\frac{\sqrt{545}}{20} + \frac{5}{4} + \frac{i}{20}\sqrt{-170 + 70\sqrt{545}}, \chi_3 = 1 + \frac{i}{2}\sqrt{7}, \beta_1 = \frac{1}{10,000}, \beta_2 = 10,000, \beta_3 = 20,000$$



**Fig. 6** **a** Antidark soliton molecule in Eq. (20) and **b** collision between an soliton molecule and a single soliton in Eq. (21) on the  $(t, z)$  plane. The parameters are  $c = 1, a = 2, \lambda_1 =$

$$-\frac{1}{4}, \lambda_2 = -\frac{11}{58} - \frac{5}{58}\sqrt{35}, \lambda_3 = -\frac{3}{2}, \chi_1 = -\frac{1}{2} + \frac{i}{2}\sqrt{7}, \chi_2 = -\frac{5}{29}\sqrt{35} - \frac{11}{29} + \frac{i}{29}\sqrt{280 + 470\sqrt{35}}, \chi_3 = -3 + i\sqrt{3}, \beta_1 = \frac{1}{10,000}, \beta_2 = 10,000, \beta_3 = 20,000$$

where

$$\tilde{K}_1 = K_1 + A_{12} + A_{14}, \tilde{K}_3 = K_3 + A_{23} + A_{34}.$$

(iii) If  $K_1, K_4 = \text{constant}$ ,

$$w[4]_{\text{SM+SM}} \rightarrow \begin{cases} -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_1} + e^{-2K_4} + e^{-2(K_1+K_4+A_{14})}] \right), & K_2, K_3 \rightarrow +\infty, \\ -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2\tilde{K}_1} + e^{-2\tilde{K}_4} + e^{-2(\tilde{K}_1+\tilde{K}_4+A_{14})}] \right), & K_2, K_3 \rightarrow -\infty, \end{cases}$$

where

$$\hat{K}_1 = K_1 + A_{12} + A_{13}, \hat{K}_4 = K_4 + A_{24} + A_{34}.$$

(iv) If  $K_2, K_3 = \text{constant}$ ,

$$w[4]_{\text{SM+SM}} \rightarrow \begin{cases} -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2\hat{K}_2} + e^{-2\hat{K}_3} + e^{-2(\hat{K}_2+\hat{K}_3+A_{23})}] \right), & K_1, K_4 \rightarrow -\infty, \\ -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_2} + e^{-2K_3} + e^{-2(K_2+K_3+A_{23})}] \right), & K_1, K_4 \rightarrow +\infty, \end{cases}$$

where

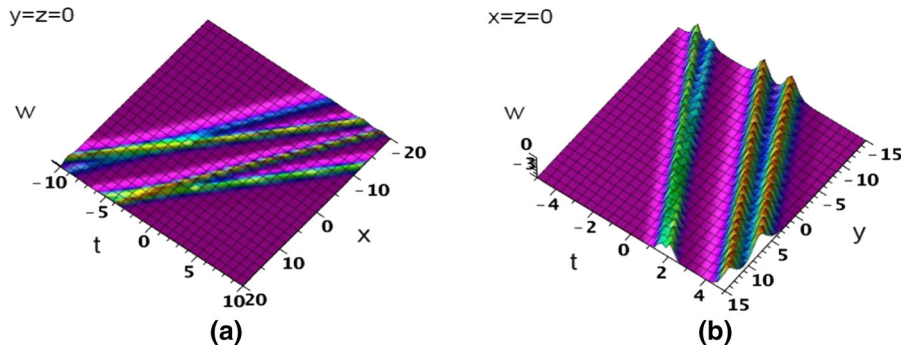
$$\hat{K}_2 = K_2 + A_{12} + A_{24}, \hat{K}_3 = K_3 + A_{13} + A_{34}.$$

(v) If  $K_2, K_4 = \text{constant}$ ,

$$w[4]_{\text{SM+SM}} \rightarrow \begin{cases} -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_2} + e^{-2K_4} + e^{-2(K_2+K_4+A_{24})}] \right), & K_1, K_3 \rightarrow +\infty, \\ -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2\tilde{K}_2} + e^{-2\tilde{K}_3} + e^{-2(\tilde{K}_2+\tilde{K}_3+A_{24})}] \right), & K_1, K_3 \rightarrow -\infty, \end{cases}$$

where

$$\tilde{K}_2 = K_2 + A_{12} + A_{23}, \tilde{K}_4 = K_4 + A_{14} + A_{34}.$$



**Fig. 7 a, b** Collision of two antidark soliton molecules in Eq. (22) for  $N = 4$  on the  $(x, t)$  and  $(y, t)$  planes. The parameters are **a**  $c = 1, a = \frac{1}{2}, \lambda_1 = 0, \lambda_2 = \frac{1}{4}, \lambda_3 = \frac{1}{3}, \lambda_4 = \frac{7}{12}, \chi_1 = \frac{i}{2}\sqrt{15}, \chi_2 = \frac{1}{2} + i\sqrt{3}, \chi_3 = -\frac{2}{3} + \frac{1}{6}\sqrt{143}, \chi_4 = \frac{7}{6} + \frac{1}{3}\sqrt{11}$ ; **b**

$c = 1, a = \frac{1}{2}, \lambda_1 = -1, \lambda_2 = -\frac{1}{5}, \lambda_3 = -\frac{1}{3}, \lambda_4 = -\frac{5}{7}, \chi_1 = -2 + \frac{i}{2}\sqrt{7}, \chi_2 = -\frac{2}{5} + \frac{i}{10}\sqrt{399}, \chi_3 = -\frac{2}{3} + \frac{i}{6}\sqrt{143}, \chi_4 = -\frac{10}{7} + \frac{i}{14}\sqrt{615}$ . The other parameters are  $\beta_1 = \frac{1}{10,000}, \beta_2 = 10,000, \beta_3 = 20,000, \beta_4 = \frac{1}{20,000}$

(vi) If  $K_3, K_4 = \text{constant}$ ,

$$w[4]_{SM+SM} \rightarrow \begin{cases} -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2\tilde{K}_3} + e^{-2\tilde{K}_4} + e^{-2(\tilde{K}_3 + \tilde{K}_4 + A_{34})}] \right), & K_1, K_2 \rightarrow -\infty, \\ -3 \left( c^2 - \frac{\partial^2}{\partial x^2} \ln [1 + e^{-2K_3} + e^{-2K_4} + e^{-2(K_3 + K_4 + A_{34})}] \right), & K_1, K_2 \rightarrow +\infty, \end{cases}$$

where

$$\tilde{K}_3 = K_3 + A_{13} + A_{23}, \tilde{K}_4 = K_4 + A_{14} + A_{24}.$$

For illustration, we display in Fig. 7(a) and 7(b) the collisions of two antidark soliton molecules on the  $(x, t)$  and  $(y, t)$  planes, respectively. The collisions on the  $(x, t)$  and  $(y, t)$  planes can be similarly presented, and here we omit exhibiting them.

### 5 Conclusion

In summary, based on a quartet lax pair, we have constructed the  $N$ -antidark soliton solution represented in a compact determinant form as well as the equivalent Hirota’s bilinear  $N$ -soliton solution form for the (3+1)D NEE (1) by the  $N$ -fold DT along with the limit technique. The (3+1)D model, as a higher-dimensional generalization of the KdV equation, is decomposed to three integrable (1+1)D equations, i.e., the NLS equation, the cmKdV equation and the LPD equation. The two- and the three-antidark soliton solutions on the

$(x, y), (x, t), (x, z), (y, t), (y, z)$  and  $(t, z)$  planes have been graphically exhibited. The asymptotic analysis has been rigorously performed for the  $N$ -antidark soliton solution. Moreover, by virtue of the velocity resonant method, we have found that the soliton molecule that has two antidark solitons propagating with the same velocities can be formed on the  $(x, t), (y, t), (y, z)$  and  $t, z$  planes, while on the  $(x, y)$  and  $(x, z)$  planes it cannot be obtained. In addition, the elastic collision between a soliton molecule and a common soliton has been demonstrated by the asymptotic analysis method. Lastly, we have graphically and analytically discussed the elastic collision of two antidark soliton molecules on the basis of the four-antidark soliton solution. We hope these results may help understand the soliton molecule dynamics in fields ranging from hydrodynamics to nonlinear optics, and so on.

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**Compliance with ethical standards**

**Conflict of interest** The authors declare that there is no conflict of interests regarding the publication of this paper.

**Appendix: The derivation of Eq. (22)**

*Proof* Considering the following  $N \times N$  determinant with respect to  $\zeta$

$$g(\zeta) = \det \left( \frac{\beta_j}{\chi_l - \chi_j^*} \delta_{jl} \zeta + \frac{1}{\chi_l - \chi_j^*} e^{i(\theta_l - \theta_j^*)} \right)_{1 \leq j, l \leq N},$$

we have the expansion

$$g(\zeta) = \left( \prod_{i=1}^N \frac{\beta_i}{\chi_i - \chi_i^*} \right) (\zeta^N + a_1 \zeta^{N-1} + \dots + a_N).$$

Then, one can compute that

$$\begin{aligned} a_N &= g(0) / \left( \prod_{i=1}^N \frac{\beta_i}{\chi_i - \chi_i^*} \right) \\ &= \det \left( \frac{1}{\chi_l - \chi_j^*} e^{i(\theta_l - \theta_j^*)} \right) / \left( \prod_{i=1}^N \frac{\beta_i}{\chi_i - \chi_i^*} \right). \end{aligned}$$

In terms of the determinant of the following Cauchy matrix

$$\det \left( \frac{1}{\chi_l - \chi_j^*} \right) = \prod_{i=1}^N \frac{1}{\chi_i - \chi_i^*} \prod_{1 \leq j < l} \frac{|\chi_j - \chi_l|^2}{|\chi_j - \chi_l^*|^2},$$

one obtains

$$\begin{aligned} a_N &= e^{i(\theta_1 - \theta_1^*) + i(\theta_2 - \theta_2^*) + \dots + i(\theta_N - \theta_N^*)} \frac{1}{\beta_1 \beta_2 \dots \beta_N} \\ &\quad \prod_{1 \leq j < l} \frac{|\chi_j - \chi_l|^2}{|\chi_j - \chi_l^*|^2} \\ &= e^{-2(\sum_{j=1}^N K_j + \sum_{1 \leq j < l} \mu_j \mu_l A_{jl})}. \end{aligned}$$

Furthermore, we can calculate that

$$\begin{aligned} a_{N-1} &= g'(0) / \left( \prod_{i=1}^N \frac{\beta_i}{\chi_i - \chi_i^*} \right) \\ &= e^{i(\theta_2 - \theta_2^*) + i(\theta_3 - \theta_3^*) + \dots + i(\theta_N - \theta_N^*)} \frac{1}{\beta_2 \beta_3 \dots \beta_N} \\ &\quad \prod_{2 \leq j < l} \frac{|\chi_j - \chi_l|^2}{|\chi_j - \chi_l^*|^2} \\ &\quad + e^{i(\theta_1 - \theta_1^*) + i(\theta_3 - \theta_3^*) + \dots + i(\theta_N - \theta_N^*)} \frac{1}{\beta_1 \beta_3 \dots \beta_N} \\ &\quad \prod_{1 \leq j < l, \substack{j \neq 2 \\ j \neq 2}} \frac{|\chi_j - \chi_l|^2}{|\chi_j - \chi_l^*|^2} + \dots \\ &\quad + e^{i(\theta_1 - \theta_1^*) + i(\theta_2 - \theta_2^*) + \dots + i(\theta_{N-1} - \theta_{N-1}^*)} \\ &\quad \frac{1}{\beta_1 \beta_2 \dots \beta_{N-1}} \prod_{1 \leq j < l} \frac{|\chi_j - \chi_l|^2}{|\chi_j - \chi_l^*|^2}. \end{aligned}$$

Continuing the above process by following

$$\begin{aligned} a_{N-2} &= \frac{1}{2!} g''(0) / \left( \prod_{i=1}^N \frac{\beta_i}{\chi_i - \chi_i^*} \right), \dots, \\ a_1 &= \frac{1}{(N-1)!} g^{N-1}(0) / \left( \prod_{i=1}^N \frac{\beta_i}{\chi_i - \chi_i^*} \right), \end{aligned}$$

one can infer that

$$\begin{aligned} \widehat{M} &= g(1) = \det \left( \frac{\beta_j}{\chi_l - \chi_j^*} \delta_{jl} + \frac{1}{\chi_l - \chi_j^*} e^{i(\theta_l - \theta_j^*)} \right) \\ &= \sum_{\mu=0,1} e^{-2(\sum_{j=1}^n \mu_j K_j + \sum_{1 \leq j < l} \mu_j \mu_l A_{jl})}. \end{aligned}$$

□

**References**

1. Stratmann, M., Pagel, T., Mitschke, F.: Experimental observation of temporal soliton molecules. *Phys. Rev. Lett.* **95**, 143902 (2005)
2. Hause, A., Mitschke, F.: Higher-order equilibria of temporal soliton molecules in dispersion-managed fibers. *Phys. Rev. A* **88**, 063843 (2013)
3. Herink, G., Kurtz, F., Jalali, B., Solli, D.R., Ropers, C.: Real-time spectral interferometry probes the internal dynamics of femtosecond soliton molecules. *Science* **356**, 50 (2017)
4. Liu, X.M., Yao, X.K., Cui, Y.D.: Real-time observation of the buildup of soliton molecules. *Phys. Rev. Lett.* **121**, 023905 (2018)
5. Lakomy, K., Nath, R., Santos, L.: Spontaneous crystallization and filamentation of solitons in dipolar condensates. *Phys. Rev. A* **85**, 033618 (2012)
6. Lakomy, K., Nath, R., Santos, L.: Soliton molecules in dipolar Bose-Einstein condensates. *Phys. Rev. A* **86**, 013610 (2012)

7. Xu, G., Gelash, A., Chabchoub, A., Zakharov, V., Kibler, B.: Breather Wave Molecules. *Phys. Rev. Lett.* **122**, 084101 (2019)
8. Xu, D.H., Lou, S.Y.: Dark soliton molecules in nonlinear optics. *Acta Phys. Sin.* **69**, 014208 (2020)
9. Lou, S.Y.: Soliton molecules and asymmetric solitons in fluid systems via velocity resonance, [arXiv:1909.03399](https://arxiv.org/abs/1909.03399) (2019)
10. Yan, Z.W., Lou, S.Y.: Soliton molecules in Sharma-Tasso-Olver-Burgers equation. *Appl. Math. Lett.* **104**, 106271 (2020)
11. Zhang, Z., Yang, X.Y., Li, B.: Soliton molecules and novel smooth positons for the complex modified KdV equation. *Appl. Math. Lett.* **103**, 106168 (2020)
12. Dong, J.J., Li, B., Yuen, M.W.: Soliton molecules and mixed solutions of the (2+1)-dimensional bidirectional Sawada-Kotera equation. *Commun. Theor. Phys.* **72**, 025002 (2020)
13. Yang, X.Y., Fan, R., Li, B.: Soliton molecules and some novel interaction solutions to the (2+1)-dimensional B-type Kadomtsev-Petviashvili equation. *Phys. Scr.* **95**, 045213 (2020)
14. Kundu, A., Naskar, T.: Arbitrary bending of optical solitonic beam regulated by boundary excitations in a doped resonant medium. *Physica D* **276**, 21 (2014)
15. Chen, S.H., Soto-Crespo, J.M., Baronio, F., Grelu, P., Mihalache, D.: Rogue-wave bullets in a composite (2+1)D nonlinear medium. *Opt. Express* **24**, 15251 (2016)
16. Qiu, D.Q., Zhang, Y.S., He, J.S.: The rogue wave solutions of a new (2+1)-dimensional equation. *Commun. Nonlinear Sci. Numer. Simulat.* **30**, 307 (2016)
17. Kundu, A., Mukherjee, A., Naskar, T.: Modelling rogue waves through exact dynamical lump soliton controlled by ocean currents. *Proc. R. Soc. A* **470**, 20130576 (2017)
18. Hu, W.C., Huang, W.H., Shen, J., Lu, Z.M.: The generation of symmetric and asymmetric lumps by a bottom topography. *Wave Motion* **75**, 62 (2017)
19. Hu, W.C., Huang, W.H., Lu, Z.M., Stepanyants, Y.: Interaction of multi-lumps within the Kadomtsev-Petviashvili equation. *Wave Motion* **77**, 243 (2018)
20. Wang, X., Wang, L.: Darboux transformation and nonautonomous solitons for a modified Kadomtsev-Petviashvili equation with variable coefficients. *Comput. Math. Appl.* **75**, 4201 (2018)
21. Chen, S.H., Dudley, J.M.: Spatiotemporal nonlinear optical self-similarity in three dimensions. *Phys. Rev. Lett.* **102**, 233903 (2009)
22. Mihalache, D.: Multidimensional localized structures in optics and Bose-Einstein condensates: A selection of recent studies. *Rom. J. Phys.* **59**, 295 (2014)
23. Leonetti, M., Conti, C.: Observation of three dimensional optical rogue waves through obstacles. *Appl. Phys. Lett.* **106**, 254103 (2015)
24. Wazwaz, A.M.: Two B-type Kadomtsev-Petviashvili equations of (2+1) and (3+1) dimensions: multiple soliton solutions, rational solutions and periodic solutions. *Comput. Fluids* **86**, 357 (2013)
25. Wazwaz, A.M., Xu, G.Q.: Modified Kadomtsev-Petviashvili equation in (3+1) dimensions: multiple front-wave solutions. *Commun. Theor. Phys.* **63**, 727 (2015)
26. Wazwaz, A.M.: New (3+1)-dimensional nonlinear evolution equations with mKdV equation constituting its main part: multiple soliton solutions. *Chaos Solitons Fractals* **76**, 93 (2015)
27. Qian, C., Rao, J.G., Liu, Y.B., He, J.S.: Rogue waves in the three-dimensional Kadomtsev-Petviashvili equation. *Chin. Phys. Lett.* **33**, 110201 (2016)
28. Wazwaz, A.M., El-Tantawy, S.A.: A new integrable (3+1)-dimensional KdV-like model with its multiple-soliton solutions. *Nonlinear Dyn.* **83**, 1529 (2016)
29. Wazwaz, A.M., El-Tantawy, S.A.: A new (3+1)-dimensional generalized Kadomtsev-Petviashvili equation. *Nonlinear Dyn.* **84**, 1107 (2016)
30. Wazwaz, A.M.: Multiple-soliton solutions for extended (3+1)-dimensional Jimbo-Miwa equations. *Appl. Math. Lett.* **64**, 21 (2017)
31. Sergyeyev, A.: New integrable (3+1)-dimensional systems and contact geometry. *Lett. Math. Phys.* **108**, 359 (2018)
32. Geng, X.G.: Algebraic-geometrical solutions of some multidimensional nonlinear evolution equations. *J. Phys. A Math. Gen.* **36**, 2289 (2003)
33. Zhaqilao: Rogue waves and rational solutions of a (3+1)-dimensional nonlinear evolution equation. *Phys. Lett. A* **377**, 3021 (2013)
34. Shi, Y.B., Zhang, Y.: Rogue waves of a (3+1)-dimensional nonlinear evolution equation. *Commun. Nonlinear. Sci. Numer. Simulat.* **44**, 120 (2017)
35. Chen, M.D., Li, X., Wang, Y., Li, B.: A pair of resonance stripe solitons and lump solutions to a reduced (3+1)-dimensional nonlinear evolution equation. *Commun. Theor. Phys.* **67**, 595 (2017)
36. Wazwaz, A.M.: A (3+1)-dimensional nonlinear evolution equation with multiple soliton solutions and multiple singular soliton solutions. *Appl. Math. Comput.* **215**, 1548 (2009)
37. Wang, X., Wei, J., Geng, X.G.: Rational solutions for a (3+1)-dimensional nonlinear evolution equation. *Commun. Nonlinear. Sci. Numer. Simulat.* **83**, 105116 (2020)
38. Matveev, V.B., Salle, M.A.: *Darboux Transformations and Solitons*. Springer, Berlin (1991)
39. Gu, C.H., Hu, H.S., Zhou, Z.X.: *Darboux Transformations in Integrable Systems: Theory and their Applications to Geometry*. Springer, New York (2005)
40. Li, R.M., Geng, X.G.: On a vector long wave-short wave-type model. *Stud. Appl. Math.* **144**, 164 (2020)
41. Li, R.M., Geng, X.G.: Rogue periodic waves of the sine-Gordon equation. *Appl. Math. Lett.* **102**, 106147 (2020)
42. Geng, X.G., Li, R.M., Xue, B.: A vector general nonlinear Schrödinger equation with (m+n) components. *J. Nonlinear Sci.* **30**, 991 (2020)
43. Guo, B.L., Ling, L.M., Liu, Q.P.: Nonlinear Schrödinger equation: generalized Darboux transformation and rogue wave solutions. *Phys. Rev. E* **85**, 026607 (2012)
44. Tao, Y.S., He, J.S.: Multisolitons, breathers, and rogue waves for the Hirota equation generated by the Darboux transformation. *Phys. Rev. E* **85**, 026601 (2012)
45. Ling, L.M., Zhao, L.C., Guo, B.L.: Darboux transformation and multi-dark soliton for N-component nonlinear Schrödinger equations. *Nonlinearity* **28**, 3243 (2015)
46. Wang, X., Liu, C., Wang, L.: Rogue waves and W-shaped solitons in the multiple self-induced transparency system. *Chaos* **27**, 093106 (2017)

47. Zhang, G.Q., Yan, Z.Y., Wen, X.Y.: Multi-dark-dark solitons of the integrable repulsive AB system via the determinants. *Chaos* **27**, 083110 (2017)
48. Wang, L., Liu, C., Wu, X., Wang, X., Sun, W.R.: Dynamics of superregular breathers in the quintic nonlinear Schrödinger equation. *Nonlinear Dyn.* **94**, 977 (2018)
49. Wang, X., Zhang, J.L., Wang, L.: Conservation laws, periodic and rational solutions for an extended modified Korteweg-de Vries equation. *Nonlinear Dyn.* **92**, 1507 (2018)
50. Wang, L., Wu, X., Zhang, H.Y.: Superregular breathers and state transitions in a resonant erbium-doped fiber system with higher-order effects. *Phys. Lett. A* **382**, 2650 (2018)
51. Wang, X., Wei, J., Wang, L., Zhang, J.L.: Baseband modulation instability, rogue waves and state transitions in a deformed Fokas-Lenells equation. *Nonlinear Dyn.* **97**, 343 (2019)
52. Sun, W.R., Wang, L.: Solitons, breathers and rogue waves of the coupled Hirota system with  $4 \times 4$  Lax pair. *Commun. Nonlinear Sci. Numer. Simulat.* **82**, 105055 (2020)
53. Geng, X.G., Wei, J., Zeng, X.: Algebro-geometric integration of the modified Belov-Chaltikian lattice hierarchy. *Theor. Math. Phys.* **199**, 675 (2019)
54. Wei, J., Geng, X.G., Zeng, X.: The Riemann theta function solutions for the hierarchy of Bogoyavlensky lattices. *Trans. Am. Math. Soc.* **371**, 1483 (2019)
55. Hirota, R.: *The Direct Method in Soliton Theory*. Cambridge University Press, Cambridge (2004)
56. Chen, D.Y.: *Introduction on Solitons*. Science Press, Beijing (2006)
57. Geng, X.G., Liu, H.: The nonlinear steepest descent method to long-time asymptotics of the coupled nonlinear Schrödinger equation. *J. Nonlinear Sci.* **28**, 739 (2018)
58. Liu, H., Geng, X.G., Xue, B.: The Deift-Zhou steepest descent method to long-time asymptotics for the Sasatsuma equation. *J. Differ. Equ.* **265**, 5984 (2018)

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