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ORIGINAL PAPER

Stability analysis and synthesis of stabilizing controls for a class of nonlinear mechanical systems

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Abstract This paper is concerned with the problems of stability and stabilization for a class of nonlinear mechanical systems. It is assumed that considered systems are under the action of linear gyroscopic forces, nonlinear homogeneous positional forces and nonlinear homogeneous dissipative forces of positional–viscous friction. An approach to strict Lyapunov functions construction for such systems is proposed. With the aid of these functions, sufficient conditions of the asymptotic stability and estimates of the convergence rate of solutions are found. Moreover, systems with delay in the positional forces are studied, and new delay-independent stability conditions are derived. The obtained results are used for developing new approaches to the synthesis of stabilizing controls with delay in feedback law.

Keywords Nonlinear mechanical system · Asymptotic stability · Lyapunov function · Decomposition · Delay · Stabilization

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1 Introduction

Stability analysis of nonlinear mechanical systems is fundamental and challenging research problem due to its broad applications. The general approach to the problem is the Lyapunov direct method [1–3].

However, it is worth mentioning that in numerous real-world applications motions of mechanical systems are described by essentially nonlinear multivariate systems of differential equations of the second order [4–8]. For such systems, the explicit construction of Lyapunov functions taking the nonlinear dynamics into account remains a difficult problem [1, 3, 9, 10].

An efficient tool to overcome this difficulty is the decomposition method [3, 11]. The method is successfully used in various forms for the investigation of stability of wide classes of mechanical systems, see, for example, [1, 12–16] and references therein.

One of the forms of decomposition of mechanical systems is based on the reducing stability problem for an original second-order system to that for two independent first-order auxiliary subsystems. With the aid of such an approach, in [12, 13], asymptotic stability conditions for linear time-invariant gyroscopic systems were found. In [15, 17, 18], results of [12, 13] were extended to some classes on nonlinear nonstationary mechanical systems.

In the present contribution, this approach is used for the stability analysis of mechanical systems with linear gyroscopic forces, nonlinear homogeneous positional forces and nonlinear homogeneous dissipative forces of positional–viscous friction.

It is known that experimental investigation of elastic properties for a significant number of materials applied in contemporary mechanical and civil engineering gives the nonlinear strain–stress relation [4, 6, 7, 19]. For instance, in [19], nonlinear homogeneous positional forces were used for the construction of seismic mitigation devices. Furthermore, in models of some mechanical systems it is necessary to take into account the dependence of damping coefficients on generalized coordinates (see [6]). In particular, nonlinear homogeneous forces of positional–viscous friction were applied in [20, 21] for modeling dynamics of a gimbal gyro.

It should be noted that sufficient conditions of the asymptotic stability for the considered class of systems were derived in [17]. However, results of [17] are based on constructing weak Lyapunov functions and using Barbashin–Krasovskii theorem [1]. Derivatives of weak Lyapunov functions with respect to investigated systems are only nonnegative. It is well known [9] that such Lyapunov functions are insufficient to analyze general nonlinear systems. These functions are not well suited to robustness analysis, since their negative semi-definite derivatives along trajectories could become positive under arbitrarily small perturbations of the dynamics. This has motivated a great deal of significant research on methods to explicitly construct strict Lyapunov functions, i.e., functions with negative definite derivatives (see, for instance, [9, 10, 22–24]).

In this paper, new constructions of strict Lyapunov functions for considered nonlinear mechanical systems are proposed. With the aid of these functions, not only asymptotic stability conditions but also estimates of the convergence rate of solutions are derived. Moreover, systems with delay in positional forces are studied, and new delay-independent stability conditions are found. In addition, the obtained results permit us to propose new approaches to the synthesis of stabilizing controls with delay in feedback law.

2 Preliminaries

In the sequel, $\mathbb{R}$ denotes the field of real numbers, and $\mathbb{R}^n$ the n-dimensional Euclidean space. The Euclidean norm will be used for vectors.

For a given number $\tau_0 > 0$, let $C^1([-\tau_0, 0], \mathbb{R}^n)$ be the space of continuously differentiable functions $\varphi(\theta) : [-\tau_0, 0] \to \mathbb{R}^n$ with the uniform norm

$$\|\varphi\|_{\tau_0} = \max_{\theta\in[-\tau_0,0]} (\|\varphi(\theta)\| + \|\dot{\varphi}(\theta)\|).$$

Definition 1 (see [2]) A function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of the order $\mu > 0$ (with respect to the standard dilation) if $f(c\mathbf{x}) = c^\mu f(\mathbf{x})$ for any $c > 0$ and $\mathbf{x} \in \mathbb{R}^n$.

Remark 1 It is known [2] that if $f(\mathbf{x})$ is a continuous homogeneous of the order μ function, then

$$a_1\|\mathbf{x}\|^\mu \le f(\mathbf{x}) \le a_2\|\mathbf{x}\|^\mu$$

for $\mathbf{x} \in \mathbb{R}^n$, where

$$a_1 = \min_{\|\mathbf{x}\|=1} f(\mathbf{x}), \quad a_2 = \max_{\|\mathbf{x}\|=1} f(\mathbf{x}),$$

and in the case where $f(\mathbf{x})$ is positive definite, the constant a_1 is positive.

We will use the following lemmas, see [25].

Lemma 1 *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$W(\mathbf{x},\mathbf{y}) = \|\mathbf{x}\|^\alpha + \|\mathbf{y}\|^\beta - c\|\mathbf{x}\|^\gamma\|\mathbf{y}\|^\delta.$$

Here $c, \alpha, \beta, \gamma, \delta$ are positive constants. Then function $W(\mathbf{x},\mathbf{y})$ is positive definite for any $c > 0$ if and only if $\gamma/\alpha + \delta/\beta > 1$. In the case where $\gamma/\alpha + \delta/\beta = 1$, function $W(\mathbf{x},\mathbf{y})$ is positive definite for sufficiently small positive values of c.

Lemma 2 *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$W(\mathbf{x},\mathbf{y}) = \|\mathbf{x}\|^\alpha + \|\mathbf{y}\|^\beta + c_1\|\mathbf{x}\|^\eta\|\mathbf{y}\|^\zeta - c_2\|\mathbf{x}\|^\gamma\|\mathbf{y}\|^\delta.$$

Here $c_1, c_2, \alpha, \beta, \gamma, \delta, \eta, \zeta$ are positive constants. If

$$\frac{\eta}{\alpha} + \frac{\zeta}{\beta} < 1,$$

then function $W(\mathbf{x},\mathbf{y})$ is positive definite for any $c_1 > 0$ and $c_2 > 0$ if and only if

$$\gamma + \delta \frac{\alpha - \eta}{\zeta} > \alpha, \quad \gamma \frac{\beta - \zeta}{\eta} + \delta > \beta.$$

3 Statement of the problem

In [13], stability of the linear gyroscopic system

$$\ddot{\mathbf{x}}(t) + (\mathbf{B} + h\mathbf{G})\dot{\mathbf{x}}(t) + \mathbf{C}\mathbf{x}(t) = \mathbf{0} \quad (1)$$

was studied. Here $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{B}, \mathbf{G}, \mathbf{C}$ are constant matrices, and h is a positive parameter. It was assumed that $\mathbf{B}$ is a symmetric positive definite matrix of dissipative forces, and $\mathbf{G}$ is a skew-symmetric and nonsingular matrix of gyroscopic forces.

To derive stability conditions, an expansion of the roots of the characteristic equation for (1) in the series with respect to the negative powers of h was used. It was proved that, if the auxiliary subsystem

$$\dot{\mathbf{x}}(t) = -\mathbf{G}^{-1}\mathbf{C}\mathbf{x}(t) \quad (2)$$

is asymptotically stable, then, for sufficiently large values of h, system (1) is also asymptotically stable. Thus, the stability problem for original second-order system (1) can be reduced to that one for first-order subsystem (2).

The objective of the present paper is an extension of the Merkin's result to mechanical systems with nonlinear force fields.

Let motions of a mechanical system be defined by the equations

$$\ddot{\mathbf{x}}(t) + (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\dot{\mathbf{x}}(t) + \mathbf{Q}(\mathbf{x}(t)) = \mathbf{0}. \quad (3)$$

Here $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{G}$ is a constant matrix, entries of the matrix $\mathbf{F}(\mathbf{x})$ are continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$ homogeneous functions of the order $\sigma > 1$; components of the vector $\mathbf{Q}(\mathbf{x})$ are continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$ homogeneous functions of the order $\lambda > 1$.

System (3) admits the equilibrium position

$$\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}. \quad (4)$$

We will look for asymptotic stability conditions of the equilibrium position.

Assumption 1 The matrix $\mathbf{G}$ is skew-symmetric and nonsingular.

Remark 2 If Assumption 1 is fulfilled, then n is an even number [13].

Assumption 2 For every $\mathbf{x} \neq \mathbf{0}$, the matrix $\mathbf{F}(\mathbf{x}) + \mathbf{F}^\top(\mathbf{x})$ is positive definite.

Remark 3 Under Assumption 2, there exists a positive constant c such that $\dot{\mathbf{x}}^\top \mathbf{F}(\mathbf{x})\dot{\mathbf{x}} \geq c\|\mathbf{x}\|^\sigma \|\dot{\mathbf{x}}\|^2$ for all $\mathbf{x}, \dot{\mathbf{x}} \in \mathbb{R}^n$, i.e., the forces $-\mathbf{F}(\mathbf{x})\dot{\mathbf{x}}$ are dissipative ones.

Thus, the considered system is under the action of linear gyroscopic forces $-\mathbf{G}\dot{\mathbf{x}}$, nonlinear homogeneous positional forces $-\mathbf{Q}(\mathbf{x})$ and nonlinear homogeneous dissipative forces of positional–viscous friction $-\mathbf{F}(\mathbf{x})\dot{\mathbf{x}}$. Such systems are widely applied in nonlinear mechanics (see, for instance, [4, 6, 19]). For example, they can be used for modeling dynamics of a gimbal gyro [20, 21] or for modeling magnetic suspension control system of a gyro rotor (see [26, 27]).

Moreover, system (3) may be treated as a vector Lienard equation [28]. Such an equation is widely used for modeling mechanical and electromechanical systems [28–32].

Remark 4 It is worth noting that, in the present paper, mechanical systems with unity mass matrices are considered. However, with the aid of the standard technique (see [1]), the obtained results can be extended to holonomic mechanical systems in the Lagrangian form.

In addition, let the following assumptions be fulfilled:

Assumption 3 The inequality

$$\lambda > \sigma + 1 \quad (5)$$

holds.

Assumption 4 The zero solution of the auxiliary subsystem

$$\dot{\mathbf{x}}(t) = -\mathbf{G}^{-1}\mathbf{Q}(\mathbf{x}(t)) \quad (6)$$

is asymptotically stable.

Remark 5 It is worth noting that system (3) is nonlinear and nonhomogeneous, whereas (6) is a homogeneous system. Therefore, known approaches for the stability analysis of homogeneous systems (see [2, 33]) can be applied to (6).

Let us determine conditions under which the asymptotic stability of the zero solution of (6) implies that equilibrium position (4) of system (3) is also asymptotically stable.

The main contributions of this paper are described below:

(i) An original approach to the construction of a strict Lyapunov function for (3) is proposed.
(ii) Conditions are derived under which the stability problem for second-order system (3) can be reduced to that one for auxiliary first-order subsystem (6). It should be noted that, compared with the linear case [13], an important feature of the obtained result is that, to guarantee the asymptotic stability, there is no need to use a large parameter at the vector of gyroscopic forces.
(iii) Estimates of the convergence rate of solutions are derived.
(iv) New delay-independent stability conditions for mechanical systems with delay in positional forces are found.
(v) On the basis of the obtained results, an original approach to the stabilization of nonlinear mechanical systems is proposed.

4 Stability conditions and estimates of solutions

To determine stability conditions for (3), with the aid of a special substitution, we will represent the original system as a complex system describing interaction of two subsystems. Next, we will construct a strict Lyapunov function for the complex system. Such approach will permits us not only to obtain conditions of the asymptotic stability, but also to estimate convergence rate of solutions.

Theorem 1 *Under Assumptions 1–4, equilibrium position (4) of system (3) is asymptotically stable.*

Proof From Assumptions 1 and 2 it follows that the matrix $\mathbf{F}(\mathbf{x})+\mathbf{G}$ is nonsingular for all $\mathbf{x}\in\mathbb{R}^n$. Let

$$\mathbf{z}(t)=\dot{\mathbf{x}}(t)+(\mathbf{F}(\mathbf{x}(t))+\mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t)). \tag{7}$$

Then

$$\begin{aligned}\dot{\mathbf{z}}(t)=&-(\mathbf{F}(\mathbf{x}(t))+\mathbf{G})\dot{\mathbf{x}}(t)-\mathbf{Q}(\mathbf{x}(t))\\&+\frac{\partial\Big((\mathbf{F}(\mathbf{x}(t))+\mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)}{\partial\mathbf{x}}\dot{\mathbf{x}}(t)\\=&-(\mathbf{F}(\mathbf{x}(t))+\mathbf{G})\mathbf{z}(t)\\&+\frac{\partial\Big((\mathbf{F}(\mathbf{x}(t))+\mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)}{\partial\mathbf{x}}\\&\Big(\mathbf{z}(t)-(\mathbf{F}(\mathbf{x}(t))+\mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big).\end{aligned}$$

Thus, substitution (7) transforms system (3) to the following one:

$$\begin{aligned}\dot{\mathbf{x}}(t)=&\mathbf{z}(t)-(\mathbf{F}(\mathbf{x}(t))+\mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t)),\\\dot{\mathbf{z}}(t)=&-(\mathbf{F}(\mathbf{x}(t))+\mathbf{G})\mathbf{z}(t)\\&+\frac{\partial\Big((\mathbf{F}(\mathbf{x}(t))+\mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)}{\partial\mathbf{x}}\\&\Big(\mathbf{z}(t)-(\mathbf{F}(\mathbf{x}(t))+\mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big).\end{aligned}\tag{8}$$

It is known (see [2, 33]) that if the zero solution of (6) is asymptotically stable, then, for any $\nu_1>1$, there exists a Lyapunov function $V_1(\mathbf{x})$ such that

(a) it is homogeneous of the order ν_1;
(b) it is continuously differentiable for $\mathbf{x}\in\mathbb{R}^n$;
(c) it is positive definite, while its derivative with respect to (6) is negative definite.

Construct a Lyapunov function for complex system (4) in the form

$$V(\mathbf{x},\mathbf{z})=V_1(\mathbf{x})+\eta\|\mathbf{z}\|^{\nu_2}-\varepsilon\|\mathbf{z}\|^{\beta-1}\mathbf{z}^\top\mathbf{x}, \tag{9}$$

where $\eta>0$, $\varepsilon>0$, $\nu_2>1$, $\beta\ge1$.

Then

$$\begin{aligned}\alpha_1\|\mathbf{x}\|^{\nu_1}+\eta\|\mathbf{z}\|^{\nu_2}-\varepsilon\|\mathbf{z}\|^{\beta}\|\mathbf{x}\|\le V(\mathbf{x},\mathbf{z})\\\le\alpha_2\|\mathbf{x}\|^{\nu_1}+\eta\|\mathbf{z}\|^{\nu_2}+\varepsilon\|\mathbf{z}\|^{\beta}\|\mathbf{x}\|\end{aligned}$$

for $\mathbf{x},\mathbf{z}\in\mathbb{R}^n$. Here α_1,α_2 are positive constants.

Differentiating function (9) with respect to system (4), we obtain

$$
\begin{aligned}
\dot{V} =& \left(\frac{\partial V_1(\mathbf{x}(t))}{\partial \mathbf{x}}\right)^\top \Big(\mathbf{z}(t) - (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big) \\
&+ \eta\nu_2\|\mathbf{z}(t)\|^{\nu_2-2}\mathbf{z}^\top(t)\Big(-(\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\mathbf{z}(t) \\
&+ \frac{\partial\Big((\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)}{\partial \mathbf{x}} \\
&\Big(\mathbf{z}(t) - (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)\Big) - \varepsilon\|\mathbf{z}(t)\|^{\beta+1} \\
&+ \varepsilon\|\mathbf{z}(t)\|^{\beta-1}\mathbf{z}^\top(t)(\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t)) \\
&- \varepsilon\mathbf{x}^\top(t)\frac{\partial\Big(\|\mathbf{z}(t)\|^{\beta-1}\mathbf{z}(t)\Big)}{\partial \mathbf{z}}\Big(-(\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\mathbf{z}(t) \\
&+ \frac{\partial\Big((\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)}{\partial \mathbf{x}} \\
&\Big(\mathbf{z}(t) - (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)\Big) \\
\le& -\left(\frac{\partial V_1(\mathbf{x}(t))}{\partial \mathbf{x}}\right)^\top \mathbf{G}^{-1}\mathbf{Q}(\mathbf{x}(t)) + \left\|\frac{\partial V_1(\mathbf{x}(t))}{\partial \mathbf{x}}\right\|\|\mathbf{z}(t)\| \\
&+ \left\|\frac{\partial V_1(\mathbf{x}(t))}{\partial \mathbf{x}}\right\|\|\mathbf{G}^{-1} - (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\|\|\mathbf{Q}(\mathbf{x}(t))\| \\
&- \eta\nu_2\|\mathbf{z}(t)\|^{\nu_2-2}\mathbf{z}^\top(t)(\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\mathbf{z}(t) \\
&+ \eta\nu_2\|\mathbf{z}(t)\|^{\nu_2-1}\left\|\frac{\partial\Big((\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)}{\partial \mathbf{x}}\right\|\Big\|\mathbf{z}(t) \\
&- (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big\| - \varepsilon\|\mathbf{z}(t)\|^{\beta+1} \\
&+ \varepsilon\|\mathbf{z}(t)\|^{\beta}\|(\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\|\|\mathbf{Q}(\mathbf{x}(t))\| \\
&+ \varepsilon\|\mathbf{x}(t)\|\left\|\frac{\partial\Big(\|\mathbf{z}(t)\|^{\beta-1}\mathbf{z}(t)\Big)}{\partial \mathbf{z}}\right\|\Big\|-(\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\mathbf{z}(t) \\
&+ \frac{\partial\Big((\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)}{\partial \mathbf{x}} \\
&\Big(\mathbf{z}(t) - (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\mathbf{Q}(\mathbf{x}(t))\Big)\Big\|.
\end{aligned}
$$

From Assumptions 1, 2, 4 it follows that

$$
\begin{aligned}
&-\left(\frac{\partial V_1(\mathbf{x}(t))}{\partial \mathbf{x}}\right)^\top \mathbf{G}^{-1}\mathbf{Q}(\mathbf{x}(t)) \le -\alpha_3\|\mathbf{x}(t)\|^{\nu_1+\lambda-1}, \\
&\quad -\eta\nu_2\|\mathbf{z}(t)\|^{\nu_2-2}\mathbf{z}^\top(t)(\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\mathbf{z}(t) \\
&\quad = -\eta\nu_2\|\mathbf{z}(t)\|^{\nu_2-2}\mathbf{z}^\top(t)\mathbf{F}(\mathbf{x}(t))\mathbf{z}(t) \\
&\quad \le -\eta\alpha_4\|\mathbf{x}(t)\|^{\sigma}\|\mathbf{z}(t)\|^{\nu_2}
\end{aligned}
$$

for $\mathbf{x}(t), \mathbf{z}(t) \in \mathbb{R}^n$, where $\alpha_3 > 0$, $\alpha_4 > 0$.

Taking into account Remark 1, it is easy to show that that there exist positive numbers $\Delta_1, \alpha_5, \alpha_6, \alpha_7$ such that the estimate

$$
\begin{aligned}
\dot{V} \le& -\alpha_3\|\mathbf{x}(t)\|^{\nu_1+\lambda-1} - \eta\alpha_4\|\mathbf{x}(t)\|^{\sigma}\|\mathbf{z}(t)\|^{\nu_2} \\
&- \varepsilon\|\mathbf{z}(t)\|^{\beta+1} + \varepsilon\alpha_5\|\mathbf{x}(t)\|\|\mathbf{z}(t)\|^{\beta-1}(\|\mathbf{z}(t)\| + \|\mathbf{x}(t)\|^{2\lambda-1}) \\
&+ \eta\alpha_6\|\mathbf{x}(t)\|^{\lambda-1}\|\mathbf{z}(t)\|^{\nu_2-1}(\|\mathbf{z}(t)\| + \|\mathbf{x}(t)\|^{\lambda}) \\
&+ \alpha_7\|\mathbf{x}(t)\|^{\nu_1-1}(\|\mathbf{z}(t)\| + \|\mathbf{x}(t)\|^{\sigma+\lambda})
\end{aligned}
$$

is valid for $\|\mathbf{x}(t)\| < \Delta_1$, $\mathbf{z}(t) \in \mathbb{R}^n$.

Applying Lemmas 1 and 2, it can be verified that if

$$\nu_1 + \lambda - \sigma - 1 = \lambda\nu_2, \tag{10}$$

$$\beta = \nu_2 + \sigma - 1, \tag{11}$$

then, for sufficiently small values of ε and sufficiently large values of η, one can choose $\Delta_2 > 0$ such that

$$
\begin{aligned}
&\frac{1}{2}(\alpha_1\|\mathbf{x}(t)\|^{\nu_1} + \eta\|\mathbf{z}(t)\|^{\nu_2}) \le V(\mathbf{x}(t), \mathbf{z}(t)) \\
&\quad \le 2(\alpha_2\|\mathbf{x}(t)\|^{\nu_1} + \eta\|\mathbf{z}(t)\|^{\nu_2}),
\end{aligned}
\tag{12}
$$

$$\dot{V} \le -\frac{1}{2}\Big(\alpha_3\|\mathbf{x}(t)\|^{\nu_1+\lambda-1} + \varepsilon\|\mathbf{z}(t)\|^{\beta+1}\Big) \tag{13}$$

for $\|\mathbf{x}(t)\| + \|\mathbf{z}(t)\| < \Delta_2$.

Thus, (9) will be a strict Lyapunov function for complex system (4). Hence, the zero solution of (4) is asymptotically stable.

From the properties of substitution (7), it follows that equilibrium position (4) of system (3) is also asymptotically stable. □

Next, let us show that, with the aid of Lyapunov function (9), estimates of the convergence rate for solutions of system (3) can be obtained.

Theorem 2 *Under Assumptions 1–4, there exist positive numbers $\tilde{\Delta}, c_1, c_2$ such that if for a solution $\mathbf{x}(t)$ of (3) the inequalities $t_0 \ge 0$, $\|\mathbf{x}(t_0)\| + \|\dot{\mathbf{x}}(t_0)\| < \tilde{\Delta}$ hold, then*

$$
\begin{aligned}
&\|\mathbf{x}(t)\| \le c_1(t - t_0 + 1)^{-\mu}, \\
&\quad \|\dot{\mathbf{x}}(t)\| \le c_2(t - t_0 + 1)^{-1/\sigma}
\end{aligned}
\tag{14}
$$

for $t \ge t_0$, where $\mu = 1/(\lambda - 1)$ for $\lambda > 1 + \sigma(\sigma + 1)$, and $\mu = \zeta/(\sigma(\sigma + 1))$ for $\lambda \le 1 + \sigma(\sigma + 1)$. Here ζ is an arbitrary chosen number from the interval (0,1).

Proof Consider Lyapunov function (9). We will assume that, for chosen values of parameters $\nu_1, \nu_2, \eta, \varepsilon, \beta$ of this function, equalities (10), (11) hold and estimates (12), (13) are valid for $\|\mathbf{x}(t)\| + \|\mathbf{z}(t)\| < \tilde{\Delta}_1$, where $\tilde{\Delta}_1 = \text{const} > 0$.

Using inequalities (12), (13) and properties of homogeneous functions (see [2, 33]), we obtain

$$\begin{aligned}\dot{V} &\le -\tilde{c}_1\left(\|\mathbf{x}(t)\|^{\nu_1(\omega+1)} + \|\mathbf{z}(t)\|^{\nu_2(\omega+1)}\right)\\ &\le -\tilde{c}_2(2\alpha_2\|\mathbf{x}(t)\|^{\nu_1} + 2\eta\|\mathbf{z}(t)\|^{\nu_2})^{\omega+1}\\ &\le -\tilde{c}_2 V^{\omega+1}(\mathbf{x}(t), \mathbf{z}(t))\end{aligned}$$

for $\|\mathbf{x}(t)\| + \|\mathbf{z}(t)\| < \tilde{\Delta}_1$, where $\tilde{c}_1, \tilde{c}_2$ are positive constants, $\omega = \max\{(\lambda - 1)/\nu_1;\ \sigma/\nu_2\}$.

The zero solution of (4) is asymptotically stable. Hence, there exist a number $\tilde{\Delta}_2 > 0$ such that if $t_0 \ge 0$, $0 < \|\mathbf{x}_0\| + \|\mathbf{z}_0\| < \tilde{\Delta}_2$, then

$$\dot{\widetilde{V}}(t) \le -\tilde{c}_2 \widetilde{V}^{\omega+1}(t) \tag{15}$$

for $t \ge t_0$. Here $(\mathbf{x}^\top(t), \mathbf{z}^\top(t))^\top$ is the solution of (4) satisfying the conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{z}(t_0) = \mathbf{z}_0$, and $\widetilde{V}(t) = V(\mathbf{x}(t), \mathbf{z}(t))$.

Integrating differential inequality (15) and taking into account estimates (12), we obtain that

$$\begin{aligned}\frac{1}{2}(\alpha_1\|\mathbf{x}(t)\|^{\nu_1} + \eta\|\mathbf{z}(t)\|^{\nu_2}) &\le \widetilde{V}(t)\\ &\le \left(\widetilde{V}(t_0) + \omega\tilde{c}_2(t - t_0)\right)^{-\frac{1}{\omega}}\\ &\le \tilde{c}_3(1 + t - t_0)^{-\frac{1}{\omega}}\end{aligned}$$

for $t \ge t_0$, where $\tilde{c}_3 = \text{const} > 0$.

Hence,

$$\|\mathbf{x}(t)\| \le d_1(t - t_0 + 1)^{-\frac{1}{\omega\nu_1}}, \quad \|\mathbf{z}(t)\| \le d_2(t - t_0 + 1)^{-\frac{1}{\omega\nu_2}}$$

for $t \ge t_0$, where d_1 and d_2 are positive constants.

From substitution (7) it follows that if $\tilde{\Delta}_2$ is sufficiently small, then

$$\|\dot{\mathbf{x}}(t)\| \le d_3\|\mathbf{x}(t)\|^{\lambda} + \|\mathbf{z}(t)\|, \quad d_3 = \text{const} > 0,$$

for $t \ge t_0$.

It is easy to verify that $1/\nu_2 < \lambda/\nu_1$. Therefore, one can choose $\tilde{\Delta}_3 > 0$ and $d_4, d_5 > 0$ such that

$$\|\mathbf{x}(t)\| \le d_4(t - t_0 + 1)^{-\frac{1}{\omega\nu_1}}, \tag{16}$$

$$\|\dot{\mathbf{x}}(t)\| \le d_5(t - t_0 + 1)^{-\frac{1}{\omega\nu_2}} \tag{17}$$

for $t_0 \ge 0$, $\|\mathbf{x}(t_0)\| + \|\dot{\mathbf{x}}(t_0)\| < \tilde{\Delta}_3$, $t \ge t_0$.

Finally, let us note that, to derive more precise estimate (16) (in the sense of minimization of the exponent), one should pass to the limit in the exponent as $\nu_1 \to +\infty$, whereas, to derive more precise estimate (17), one should pass to the limit in the corresponding exponent as $\nu_2 \to 1$. As a result, we arrive at inequalities (14). □

Remark 6 In the case where $\lambda \le \sigma(\sigma + 1)$, values of $\tilde{\Delta}, c_1, c_2$ in Theorem 2 depend on chosen number ζ. The more close is the parameter ζ to 1, the more precise is the estimate for $\|\mathbf{x}(t)\|$ in the sense of minimization of the exponent.

5 Delay-independent stability conditions

Consider a nonlinear mechanical system with delay in positional forces. Let equations of motion be of the form

$$\ddot{\mathbf{x}}(t) + (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\dot{\mathbf{x}}(t) + \mathbf{Q}(\mathbf{x}(t)) + \mathbf{L}(\mathbf{x}(t - \tau(t))) = \mathbf{0}. \tag{18}$$

Here components of the vector $\mathbf{L}(\mathbf{x})$ are continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$ homogeneous functions of the order $\lambda > 1$, $\tau(t)$ is a continuous delay that is nonnegative and bounded for $t \ge 0$, and the rest notation is the same as for (3).

Denote $\tau_0 = \sup_{t \ge 0} \tau(t)$. We will assume that initial functions for solutions of (18) belong to the space $C^1([-\tau_0, 0], \mathbb{R}^n)$. Let $\mathbf{x}_t$ denote the restriction of a solution $\mathbf{x}(t)$ of (18) to the segment $[t - \tau_0, t]$, i.e., $\mathbf{x}_t:\ \theta \to \mathbf{x}(t + \theta)$, $\theta \in [-\tau_0, 0]$.

It is well known (see, for example, [34]) that delay may seriously affect on the stability and others dynamical properties of a system. Moreover, in numerous practical problems, values of delays could be unknown. Therefore, delay-independent stability conditions are very important in applications [34, 35].

We will show that the approach to a strict Lyapunov function construction proposed in the previous section and the original technique of application of the Razumikhin condition for nonlinear systems developed in [36, 37] permit us to obtain delay-independent conditions of asymptotic stability for equilibrium position (4) of system (18).

Construct the auxiliary delay-free subsystem

$$\dot{\mathbf{x}}(t) = -\mathbf{G}^{-1}\widetilde{\mathbf{Q}}(\mathbf{x}(t)), \tag{19}$$

where $\widetilde{\mathbf{Q}}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) + \mathbf{L}(\mathbf{x})$.

Assumption 5 The zero solution of (19) is asymptotically stable.

Theorem 3 *Let Assumptions 1, 2, 3, 5 be fulfilled. Then equilibrium position (4) of system (18) is asymptotically stable for an arbitrary continuous delay that is nonnegative and bounded for $t \geq 0$.*

Proof The substitution

$$(\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\dot{\mathbf{x}}(t) + \widetilde{\mathbf{Q}}(\mathbf{x}(t)) = (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\mathbf{z}(t)$$

transforms (18) to the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{z}(t) - (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\widetilde{\mathbf{Q}}(\mathbf{x}(t)),\\ \dot{\mathbf{z}}(t) &= -(\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\mathbf{z}(t)\\ &\quad + \frac{\partial\Big((\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\widetilde{\mathbf{Q}}(\mathbf{x}(t))\Big)}{\partial \mathbf{x}}\\ &\quad \Big(\mathbf{z}(t) - (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})^{-1}\widetilde{\mathbf{Q}}(\mathbf{x}(t))\Big)\\ &\quad + \mathbf{L}(\mathbf{x}(t)) - \mathbf{L}(\mathbf{x}(t-\tau(t)))).\end{aligned} \tag{20}$$

Choose a Lyapunov function for (20) in form (9), where parameters v_1, v_2, β satisfy conditions (10) and (11). Then, for sufficiently small values of ε and sufficiently large values of η, there exist positive numbers $\Delta, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ such that, for the function $V(\mathbf{x}, \mathbf{z})$ and its derivative with respect to system (20), the estimates

$$\begin{aligned}&\alpha_1\|\mathbf{x}(t)\|^{v_1} + \frac{\eta}{2}\|\mathbf{z}(t)\|^{v_2} \leq V(\mathbf{x}(t), \mathbf{z}(t))\\ &\quad \leq \alpha_2\|\mathbf{x}(t)\|^{v_1} + 2\eta\|\mathbf{z}(t)\|^{v_2},\\ &\dot{V} \leq -\alpha_3\|\mathbf{x}(t)\|^{v_1+\lambda-1} - \eta\alpha_4\|\mathbf{x}(t)\|^{\sigma}\|\mathbf{z}(t)\|^{v_2}\\ &\quad - \frac{\varepsilon}{2}\|\mathbf{z}(t)\|^{\beta+1} + \alpha_5\Big(\|\mathbf{x}(t)\|\|\mathbf{z}(t)\|^{\beta-1}\\ &\quad + \|\mathbf{z}(t)\|^{v_2-1}\Big)\|\mathbf{L}(\mathbf{x}(t)) - \mathbf{L}(\mathbf{x}(t-\tau(t)))\|\end{aligned} \tag{21}$$

hold for $\|\mathbf{x}(t)\| + \|\mathbf{z}(t)\| < \Delta$.

Consider a solution $(\mathbf{x}^\top(t), \mathbf{z}^\top(t))^\top$ of (20). Let the inequality $\|\mathbf{x}(\xi)\| + \|\mathbf{z}(\xi)\| < \Delta$ and the Razumikhin condition $V(\mathbf{x}(\xi), \mathbf{z}(\xi)) \leq 2V(\mathbf{x}(t), \mathbf{z}(t))$ be fulfilled for $\xi \in [t-\tau_0, t]$. Then from (21) it follows that

$$\|\mathbf{x}(\xi)\| \leq d_1(\|\mathbf{x}(t)\| + \|\mathbf{z}(t)\|^{v_2/v_1}), \tag{22}$$

$$\|\mathbf{z}(\xi)\| \leq d_2(\|\mathbf{x}(t)\|^{v_1/v_2} + \|\mathbf{z}(t)\|) \tag{23}$$

for $\xi \in [t-\tau_0, t]$. Here d_1 and d_2 are positive constants.

Let $L_1(\mathbf{x}), \ldots, L_n(\mathbf{x})$ be components of the vector $\mathbf{L}(\mathbf{x})$. With the aid of the mean value theorem, we obtain

$$\begin{aligned}&|L_i(\mathbf{x}(t)) - L_i(\mathbf{x}(t-\tau(t)))|\\ &\quad = \tau(t)\left|\dot{\mathbf{x}}^\top(t-\chi_i\tau(t))\frac{\partial L_i(\mathbf{x}(t-\chi_i\tau(t)))}{\partial \mathbf{x}}\right|\\ &\quad \leq d_3\|\dot{\mathbf{x}}(t-\chi_i\tau(t))\|\|\mathbf{x}(t-\chi_i\tau(t))\|^{\lambda-1}\\ &\quad \leq d_4\|\dot{\mathbf{x}}(t-\chi_i\tau(t))\|\big(\|\mathbf{x}(t)\|^{\lambda-1}\\ &\qquad + \|\mathbf{x}(t) - \mathbf{x}(t-\chi_i\tau(t))\|^{\lambda-1}\big), \quad i = 1, \ldots, n.\end{aligned}$$

Here d_3, d_4 are positive constants, $\chi_i \in (0, 1)$.

Applying the mean value theorem once again to estimate the terms $\|\mathbf{x}(t) - \mathbf{x}(t-\chi_i\tau(t))\|^{\lambda-1}$ and using inequalities (22), (23), we have

$$\begin{aligned}&\|\mathbf{L}(\mathbf{x}(t)) - \mathbf{L}(\mathbf{x}(t-\tau(t)))\| \leq d_5\Big(\|\mathbf{x}(t)\|^{\lambda-1+v_1/v_2}\\ &\quad + \|\mathbf{x}(t)\|^{\lambda v_1/v_2} + \|\mathbf{z}(t)\|^{\lambda} + \|\mathbf{x}(t)\|^{\lambda-1}\|\mathbf{z}(t)\|\Big),\end{aligned}$$

where $d_5 = \text{const} > 0$.

Using this inequality and applying Lemma 2, it can be shown that if Δ is sufficiently small, then

$$\dot{V} \leq -\frac{1}{2}\Big(\alpha_3\|\mathbf{x}(t)\|^{v_1+\lambda-1} + \frac{\varepsilon}{2}\|\mathbf{z}(t)\|^{\beta+1}\Big).$$

Hence (see [34]), the zero solution of (20) is asymptotically stable. □

The constructed strict Lyapunov function permits us to derive estimates of the convergence rate of solutions for time-delay system (18), as well.

Theorem 4 *Let Assumptions 1, 2, 3, 5 be fulfilled. Then there exist positive numbers $\tilde{\Delta}, c_1, c_2$ such that if for a solution $\mathbf{x}(t)$ of (18) the inequalities $t_0 \geq 0$, $\|\mathbf{x}_{t_0}\|_{\tau_0} < \tilde{\Delta}$ hold, then*

$$\|\mathbf{x}(t)\| \leq c_1(t-t_0+1)^{-\mu}, \quad \|\dot{\mathbf{x}}(t)\| \leq c_2(t-t_0+1)^{-1/\sigma}$$

for $t \geq t_0$, where $\mu = 1/(\lambda-1)$ for $\lambda > 1 + \sigma(\sigma+1)$, and $\mu = \zeta/(\sigma(\sigma+1))$ for $\lambda \leq 1 + \sigma(\sigma+1)$. Here ζ is an arbitrary chosen number from the interval (0,1).

The proof of the theorem is a similar to that of Theorem 2.

6 Synthesis of stabilizing controls

Consider the system

$$\ddot{\mathbf{x}}(t) + (\mathbf{F}(\mathbf{x}(t)) + \mathbf{G})\dot{\mathbf{x}}(t) + \mathbf{Q}(\mathbf{x}(t)) = \mathbf{U}. \quad (24)$$

Here $\mathbf{U}$ is a control vector and the rest notation is the same as for (3).

Let equilibrium position (4) of the corresponding uncontrolled ($\mathbf{U} \equiv \mathbf{0}$) system is unstable.

We are going to design a feedback control law to stabilize the equilibrium position in the case where there exists a delay in the control scheme. Assume that the delay $\tau(t)$ is a continuous function that is nonnegative and bounded for $t \geq 0$.

Remark 7 It is worth noting that, for a linear control law, we cannot guarantee the stabilization for an arbitrary continuous nonnegative and bounded delay [34].

Define a control vector by the formula

$$\mathbf{U} = -h\mathbf{G}\frac{\partial W(\mathbf{x}(t-\tau(t)))}{\partial \mathbf{x}}, \quad (25)$$

where $W(\mathbf{x})$ is a twice continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$ positive definite homogeneous of the order $\lambda + 1$ function and h is a positive parameter.

Theorem 5 *Under Assumptions 1–3, one can choose a number $h_0 > 0$ such that equilibrium position (4) of system (24) closed by control (25) is asymptotically stable for any $h \geq h_0$ and any continuous delay that is nonnegative and bounded for $t \geq 0$.*

Proof Let us show that, for sufficiently large values of h, all the conditions of Theorem 3 are satisfied for the closed-loop system. To do this, it is sufficient to verify the fulfilment of Assumption 5.

In this case, subsystem (19) is of the form

$$\dot{\mathbf{x}}(t) = -\mathbf{G}^{-1}\mathbf{Q}(\mathbf{x}(t)) - h\frac{\partial W(\mathbf{x}(t))}{\partial \mathbf{x}}. \quad (26)$$

Choose a Lyapunov function for (26) as follows: $V(\mathbf{x}) = \|\mathbf{x}\|^2$. Then

$$\dot{V} \leq (a_1 - ha_2)\|\mathbf{x}(t)\|^{\lambda+1}.$$

Here a_1 and a_2 are positive constants independent of h. Hence, for sufficiently large values of h, the zero solution of (26) is asymptotically stable. □

Next, consider the case where positional forces in (24) are potential, i.e.,

$$\mathbf{Q}(\mathbf{x}) = \frac{\partial \Pi(\mathbf{x})}{\partial \mathbf{x}}, \quad (27)$$

where the potential energy $\Pi(\mathbf{x})$ is a twice continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$ negative definite homogeneous of the order $\lambda + 1$ function.

Using the Lyapunov function

$$V(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}\|\dot{\mathbf{x}}\|^2 + \Pi(\mathbf{x})$$

and applying the Krasovskii instability theorem (see [1]), it is easy to verify that the equilibrium position of uncontrolled ($\mathbf{U} \equiv \mathbf{0}$) system (24) with positional forces (27) is unstable.

Construct a control vector by the formula

$$\mathbf{U} = -h\|\mathbf{x}(t-\tau(t))\|^{\lambda-1}\mathbf{G}\mathbf{x}(t-\tau(t)), \quad (28)$$

where h is a positive parameter.

Theorem 6 *Let Assumptions 1–3 be fulfilled. Then equilibrium position (4) of system (24) with potential positional forces (27) and control (28) is asymptotically stable for any $h > 0$ and any continuous delay that is nonnegative and bounded for $t \geq 0$.*

Proof Let us show that, for any $h > 0$, all the conditions of Theorem 3 are satisfied for the closed-loop system. To do this, it is sufficient to verify the fulfilment of Assumption 5.

Consider subsystem (19) corresponding to the closed-loop system. We obtain

$$\dot{\mathbf{x}}(t) = -\mathbf{G}^{-1}\frac{\partial \Pi(\mathbf{x}(t))}{\partial \mathbf{x}} - h\|\mathbf{x}(t)\|^{\lambda-1}\mathbf{x}(t). \quad (29)$$

Let

$$V(\mathbf{x}) = -\Pi(\mathbf{x}). \quad (30)$$

Function (30) is positive definite. Calculate the derivative of (30) with respect to system (29). Taking into account that $\mathbf{G}^{-1}$ is skew-symmetric matrix and using the Euler formula for homogeneous functions (see [2]), we obtain

$$\dot{V} = h\|\mathbf{x}(t)\|^{\lambda-1}\mathbf{x}^{\top}(t)\frac{\partial\Pi(\mathbf{x}(t))}{\partial\mathbf{x}} = h(\lambda+1)\|\mathbf{x}(t)\|^{\lambda-1}\Pi(\mathbf{x}(t)).$$

Hence,

$$\dot{V} \leq -ah\|\mathbf{x}(t)\|^{2\lambda}, \quad a = \text{const} > 0.$$

Thus, for any $h > 0$, the zero solution of (29) is asymptotically stable.

Application of Theorem 3 to the closed-loop system completes the proof. □

Remark 8 If $\tau(t) \equiv 0$, then control forces (28) are circular or nonconservative (see [1, 13]). It is well known [13, 20], that the influence of linear circular forces on the stability of mechanical systems is ambiguous: on the one hand, they can provide the asymptotic stability of a system; on the other hand, they can destabilize it. In this section, nonlinear homogeneous circular forces are used to stabilize a nonlinear mechanical system with linear gyroscopic forces, nonlinear homogeneous potential forces and nonlinear homogeneous dissipative forces of positional–viscous friction. It is important that we can guarantee the stabilization even in the case where there is a delay in the feedback law and control circular forces are small compared with destabilizing potential forces (for arbitrary small values of parameter h).

7 Examples

Consider some examples to demonstrate the effectiveness of the obtained results.

7.1 Example 1

Let system (3) be of the form

$$\ddot{\mathbf{x}}(t) + \|\mathbf{x}(t)\|^{\sigma}\dot{\mathbf{x}}(t) + \mathbf{G}\dot{\mathbf{x}}(t) + \|\mathbf{x}(t)\|^{\lambda-1}\mathbf{G}\mathbf{x}(t) = \mathbf{0}. \tag{31}$$

Here $n = 2$, $\mathbf{x}(t) = (x_1(t), x_2(t))^{\top}$, $\sigma > 1$, $\lambda > 1$,

$$\mathbf{G} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{32}$$

It is worth noting that such a system can be used for the modeling magnetic suspension control system of a gyro rotor (see [26, 27]).

Construct subsystem (6) corresponding to (31). We obtain

$$\dot{\mathbf{x}}(t) = -\|\mathbf{x}(t)\|^{\lambda-1}\mathbf{x}(t). \tag{33}$$

The zero solution of (33) is asymptotically stable. Hence (see Theorem 1), under condition (5), we can guarantee the asymptotic stability of the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of (31).

Next, assume that $\lambda = \sigma + 1$. Then system (31) admits the following family of solutions: $x_1(t) = \gamma\cos t$, $x_2(t) = \gamma\sin t$, where γ is an arbitrary constant. Therefore, in this case the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ is not asymptotically stable.

Thus, this example demonstrates that condition (5) cannot be relaxed.

7.2 Example 2

Consider the control system

$$\ddot{\mathbf{x}}(t) + b\|\mathbf{x}(t)\|^{\frac{4}{3}}\dot{\mathbf{x}}(t) + g\mathbf{G}\dot{\mathbf{x}}(t) - \|\mathbf{x}(t)\|^{2}\mathbf{x}(t) = \mathbf{U}, \tag{34}$$

where $n = 2$, $\mathbf{x}(t) = (x_1(t), x_2(t))^{\top}$, g and b are positive coefficients, the matrix $\mathbf{G}$ is defined by formula (32), $\mathbf{U} = (u_1, u_2)^{\top}$ is a control vector.

Our goal is to design a feedback control law stabilizing the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (34).

Assume that there is a delay in the control scheme, and the delay might be unknown and time-varying.

Let

$$\mathbf{U} = -h\|\mathbf{x}(t-\tau(t))\|^{2}\mathbf{G}\mathbf{x}(t-\tau(t)), \tag{35}$$

where $h = \text{const} > 0$. Verifying the conditions of Theorem 6, we obtain that the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (34) closed by control (35) is asymptotically stable for an arbitrary positive coefficient h and for any continuous delay that is nonnegative and bounded for $t \geq 0$.

For simulation, we take $b = 1/2$, $g = 1$, $h = 0.35$, $\tau = 1$ and $\mathbf{x}(t) = (0.22, 0.21)$ for $t \in [-1, 0]$. In Fig. 1, the dependence of $\|\mathbf{x}\|$ on t is presented. The results obtained confirm the theoretical conclusions.

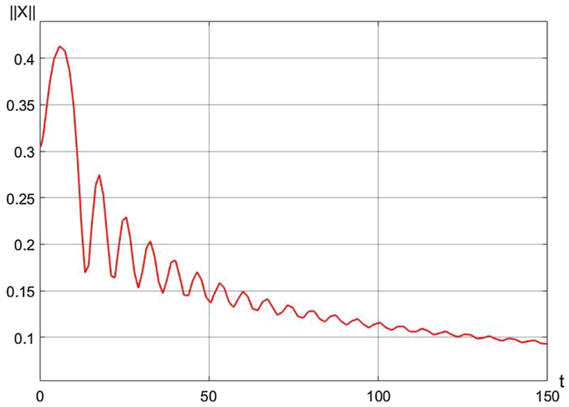

Fig. 1 Simulation results

8 Conclusion

In the present contribution, an original construction of a strict Lyapunov functions is proposed for a mechanical system with linear gyroscopic forces, nonlinear homogeneous positional forces and nonlinear homogeneous dissipative forces of positional–viscous friction. Using this function, new conditions of the asymptotic stability of a trivial equilibrium position and estimates of the convergence rate of solutions are obtained. Furthermore, delay-independent stability conditions are found for systems with time-varying delay in positional forces, and new approaches to the design of nonlinear stabilizing controls are proposed for the case where there is a delay in the in feedback law.

An interesting direction for further research is application of the developed approaches for stability analysis of nonlinear mechanical systems with switched force fields.

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Compliance with ethical standards

Conflicts of interest The author declares that he has no conflict of interest.

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