



Dynamics of solitons and breathers on a periodic waves background in the nonlocal Mel'nikov equation

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Abstract Dynamics of general line solitons and breathers on a periodic line waves (PLWs) background in the nonlocal Mel'nikov (MK) equation are investigated via the KP hierarchy reduction method. By constraining different parametric conditions for a general type of tau functions of the KP hierarchy, two families of mixed solutions to the nonlocal MK equation are derived. The first family of mixed solutions illustrates general line solitons on a PLWs background. The simplest case of such mixed solutions shows the two-line solitons on a PLWs background, and the two-line solitons possess five different patterns: a mixture of one-dark-soliton and one-antidark-soliton, two-antidark-soliton, two-dark-soliton, degenerated two-dark-soliton, and degenerated two-anti-dark-soliton. The high-order mixed solutions display superposition of several individual simplest solutions. The second family of mixed solutions demonstrates general breathers on a PLWs background or on a nonzero constant background. The breathers are periodic in time and do not move in the (x, y) -plane as time propagates.

Keywords Nonlocal Mel'nikov equation · Soliton solution · Breather solution · Periodic line waves background · KP hierarchy reduction method

1 Introduction

The existence of various soliton solutions, such as breathers, solitons and rogue waves, is one of the most important characters in most of the integrable systems [1–5]. These soliton solutions can be well used to illustrate elastic or inelastic interaction between various types of nonlinear waves. Very recently, a new kind of integrable systems termed as nonlocal systems was proposed due to their potential application in nonlinear optics [6,7]. The first one example of such integrable systems is the parity-time (PT)-symmetric nonlocal nonlinear schödinger (NLS) equation [8]:

$$iq_t(x, t) = q_{xx}(x, t) + 2\delta q(x, t)q^*(-x, t)q(x, t). \quad (1)$$

This newly proposed nonlocal NLS equation was derived from the coupled NSL equations

$$\begin{aligned} iq_t(x, t) &= q_{xx}(x, t) + 2q(x, t)^2 r(x, t), \\ -ir_t(x, t) &= r_{xx}(x, t) + 2r(x, t)^2 q(x, t), \end{aligned} \quad (2)$$

under the following reductions

$$r(x, t) = \delta q^*(-x, t). \quad (3)$$

Similar to the classical NLS equation, the nonlocal NLS equation (1) also admits solitons, breathers and rogue waves, and some other mixed solutions [9–32]. Considering other types of reductions for the coupled NSL

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equations (2), some other nonlocal NLS equations possessing different PT-symmetries have also been introduced and discussed [10]. Taking the following reduction

$$r(x, t) = \delta q(x, -t) \tag{4}$$

the coupled system (2) leads to the reverse time nonlocal NLS:

$$iq_t(x, t) = q_{xx}(x, t) + 2\delta q(x, t)^2 q(x, -t). \tag{5}$$

Considering another reduction, namely

$$r(x, t) = \delta q(-x, -t) \tag{6}$$

the coupled system (2) leads to the following reverse space-time nonlocal NLS:

$$iq_t(x, t) = q_{xx}(x, t) + 2\delta q(x, t)^2 q(-x, -t). \tag{7}$$

In addition to these nonlocal NLS equations given by (1), (5) and (7), families of other integrable systems possessing different kinds of PT-symmetries have been proposed [33–52]; examples of these newly proposed evolution equations include the nonlocal modified Kortweg-de Vries equation, the nonlocal sine-Gordon equation, the nonlocal derivative NLS equation. Usually, compared to the (1+1)-dimensional nonlocal systems, the (2+1)-dimensional integrable nonlocal systems correspond to more PT-symmetric versions. For example, the local Davey–Stewartson (DS) equation:

$$\begin{aligned} iA_t &= A_{xx} + \delta A_{yy} + (\epsilon AA^* - 2Q)A, \\ Q_{xx} - \delta Q_{yy} &= (\epsilon AA^*)_{xx}, \quad \epsilon = \pm 1, \delta = \pm 1, \end{aligned} \tag{8}$$

corresponds to the following nonlocal DS equations:

- The fully PT-symmetric nonlocal DS equations

$$\begin{aligned} iA_t &= A_{xx} + \delta A_{yy} + (\epsilon AA^*(-x, -y, t) - 2Q)A, \\ Q_{xx} - \delta Q_{yy} &= (\epsilon AA^*(-x, -y, t))_{xx}, \end{aligned} \tag{9}$$

- The partially PT-symmetric nonlocal DS equations

$$\begin{aligned} iA_t &= A_{xx} + \delta A_{yy} + (\epsilon AA^*(-x, -y, t) - 2Q)A, \\ Q_{xx} - \delta Q_{yy} &= (\epsilon AA^*(-x, -y, t))_{xx}, \end{aligned} \tag{10}$$

- The reverse space-time nonlocal DS:

$$\begin{aligned} iA_t &= A_{xx} + \delta A_{yy} + (\epsilon AA(-x, -y, -t) - 2Q)A, \\ Q_{xx} - \delta Q_{yy} &= (\epsilon AA(-x, -y, -t))_{xx}, \end{aligned} \tag{11}$$

- The partial reverse space-time nonlocal DS:

$$\begin{aligned} iA_t &= A_{xx} + \delta A_{yy} + (\epsilon AA(-x, y, -t) - 2Q)A, \\ Q_{xx} - \delta Q_{yy} &= (\epsilon AA(-x, y, -t))_{xx}, \end{aligned} \tag{12}$$

- The reverse time nonlocal DS:

$$\begin{aligned} iA_t &= A_{xx} + \delta A_{yy} + (\epsilon AA(x, y, -t) - 2Q)A, \\ Q_{xx} - \delta Q_{yy} &= (\epsilon AA(x, y, -t))_{xx}. \end{aligned} \tag{13}$$

Families of soliton solutions, including line solitons, breathers, lumps and rogue waves on a periodic line waves (PLWs) or a nonzero constant background, have been reported for these nonlocal DS equation [30,39,40,53,54].

In this paper, we mainly investigate another (2+1)-dimensional nonlocal Mel’nikov (MK) equation [51,52]

$$\begin{aligned} u_{xt} + u_{xxxx} + 3(u^2)_{xx} - 3u_{yy} \\ + \kappa(\Phi\Phi^*(-x, y, -t))_{xx} &= 0, \\ i\Phi_y &= u\Phi + \Phi_{xx}. \end{aligned} \tag{14}$$

The associated local equation of this nonlocal equation is the MK equation [55–57]

$$\begin{aligned} u_{xt} + u_{xxxx} + 3(u^2)_{xx} - 3u_{yy} \\ + \kappa(\Phi\Phi^*(x, y, t))_{xx} &= 0, \\ i\Phi_y &= u\Phi + \Phi_{xx}, \end{aligned} \tag{15}$$

where κ is a real constant and u and Φ are a real long-wave (LW) field and a complex short-wave (SW) one, respectively. The local MK equation (15) was first proposed to describe the interaction of LW and SW packets by Mel’nikov [55–57]. This equation is integrable and admits solitons solutions [59–64] as other (2+1)-dimensional integrable systems, such as bright and dark soliton, breather and rational localized solutions. Some soliton solutions in the nonlocal MK equation (14) have been reported, including bright solitons under zero boundary condition, dark solitons under nonzero boundary condition [52], breathers on a nonzero constant or PLWs background, lumps on a nonzero constant background or a PLWs background, and solutions showing the superposition of lumps and breathers on a nonzero constant or PLWs background [51]. However, there are some open issues about the nonlocal MK equation (14) that have not been well investigated. A typical example is that the solitons on a PLWs background have not been reported. Besides, Liu et.al. [52] derived breathers and semi-rational solutions comprised of lumps and breathers on a constant or PLWs background to the nonlocal MK equation (14); these solutions are not expressed by determinants. In the present work, we would investigate the following arguments to the nonlocal MK system defined in (14):

- The general line solitons on a PLWs background in the nonlocal MK equation (14).
- The mixed solutions describing breathers on a nonzero constant or PLWs background in the nonlocal MK equation defined in (14) expressed by determinants.

The structure of the paper is organized as follows. In Sect. 2, dynamics of line solitons on a PLWs background in the nonlocal MK equation (14) are analyzed in detail. In Sect. 3, dynamical features of general periodic solutions expressed via Gram-type determinants, which contain PLWs background, breathers on a nonzero constant or PLWs background, are investigated. The conclusions for this paper are discussed in Sect. 4.

2 General line solitons on a PLWs background

In this section, we would comprehensively analyze the dynamics of general line solitons on a PLWs background for the nonlocal MK equation (14). To this end, we combine the general line solitons on a nonzero constant background with the PLWs, and the corresponding solutions are expressed by $(2N + 1) \times (2N + 1)$ determinants. We note that Liu et al. constructed general line solitons on a constant background [52], and the corresponding soliton solutions are given by $2N \times 2N$ determinants. Below, we present the general line solitons on a background of PLWs to the nonlocal MK equation (14) by the following theorem, whose proof procedure would be given in the Appendix in detail.

Theorem 1 *The nonlocal MK equation (14) has the following solitons sitting on a PLWs background*

$$\Phi = \sqrt{2} \frac{g}{f}, u = (2 \log f)_{xx} \tag{16}$$

with functions f and g given by the following $(2N + 1) \times (2N + 1)$ determinants

$$\begin{aligned}
 f(x, t) &= \begin{vmatrix} b_s \delta_{sj} e^{-\zeta_s} + \frac{1}{p_s + p_j^*} & \frac{1}{p_s - p_j^*} & \dots & \dots & \frac{1}{p_s - p_{2N+1}^*} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{p_s - p_j^*} & -b_s^* \delta_{sj} e^{-\zeta_s^*(-x, y, -t)} - \frac{1}{p_s + p_j^*} & \dots & \dots & \dots \\ \frac{1}{p_{2N+1} + p_j^*} & \frac{1}{p_{2N+1} - p_j^*} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}, \\
 g(x, t) &= \begin{vmatrix} b_s \delta_{sj} e^{-\zeta_s} - \frac{p_s}{p_j^*} \frac{1}{p_s + p_j^*} & \frac{p_s}{p_j^*} \frac{1}{p_s - p_j^*} & \dots & \dots & \frac{p_s}{p_j^*} \frac{1}{p_s - p_{2N+1}^*} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{p_s}{p_j^*} \frac{1}{p_s - p_j^*} & -b_s^* \delta_{sj} e^{-\zeta_s^*(-x, y, -t)} + \frac{p_s}{p_j^*} \frac{1}{p_s + p_j^*} & \dots & \dots & \dots \\ \frac{p_{2N+1}}{p_j^*} \frac{1}{p_{2N+1} + p_j^*} & \frac{p_{2N+1}}{p_j^*} \frac{1}{p_{2N+1} - p_j^*} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},
 \end{aligned}
 \tag{17}$$

and

$$\begin{aligned}
 \zeta_s &= 2p_{sR}x + 4p_{sR}p_{sI}y - 2p_{sR} \left[4(p_{sR}^2 - 3p_{sI}^2) \right. \\
 &\quad \left. + \frac{\kappa}{p_{sR}^2 + p_{sI}^2} \right] t + \zeta_{0,s}, \\
 \zeta_{2N+1} &= 2ip_{2N+1,I}x + 2p_{2N+1,R}p_{2N+1,I}y \\
 &\quad + 2ip_{2N+1,I} \left[4(3p_{2N+1,R}^2 - p_{2N+1,I}^2) \right. \\
 &\quad \left. + \frac{\kappa}{p_{2N+1,R}^2 + p_{2N+1,I}^2} \right] t + \zeta_{0,2N+1},
 \end{aligned}
 \tag{18}$$

where b_s, p_s are complex and $\zeta_{0,s}$ are real for $s = 0, 1, \dots, N$, the real part of p_{2N+1} are zero (i.e., $p_{2N+1,R} = 0$), and b_{2N+1} is real.

To examine dynamics of the two line solitons on a PLWs background in the nonlocal MK equation (14), we have to consider a periodic solution, which can illustrate a PLWs background. By taking $N = 0$ in formula (17), a PLW solution could be yielded, and functions f and g read as

$$\begin{aligned}
 f &= ib_1 e^{-2ip_{1I}x - 2p_{1R}p_{1I}y - 2ip_{1I}[4(3p_{1R}^2 - p_{1I}^2) + \frac{\kappa}{p_{1R}^2 + p_{1I}^2}]t - \zeta_{0,1}} \\
 &\quad + \frac{1}{2ip_{1I}} \\
 &= -b_1 \sin \chi e^{-2p_{1R}p_{1I}y - \zeta_{0,1}} - \frac{1}{2p_{1I}} \\
 &\quad + ib_1 \cos \chi e^{-2p_{1R}p_{1I}y - \zeta_{0,1}}, \\
 g &= ib_1 e^{-2ip_{1I}x - 2p_{1R}p_{1I}y - 2ip_{1I}[4(3p_{1R}^2 - p_{1I}^2) + \frac{\kappa}{p_{1R}^2 + p_{1I}^2}]t - \zeta_{0,1}} \\
 &\quad - \frac{1}{2ip_{1I}} \frac{p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}},
 \end{aligned}
 \tag{19}$$

where $\chi = -2p_{1I}x - 2p_{1I}[4(3p_{1R}^2 - p_{1I}^2) + \frac{\kappa}{p_{1R}^2 + p_{1I}^2}]t$, and $b_1, p_{1R}, p_{1I}, \zeta_{0,1}$ are arbitrary real parameters.

As discussed in earlier works of the literature [30], this type of periodic solutions also exists in the nonlocal DS equation, which provides a PLWs background. It can be directly obtained that this solution always possesses a series of singular points when $p_{1R} \neq 0$. Hence, we take $p_{1R} = 0$ to avoid the singular solutions,

Theorem 1. The solutions Φ and u to the nonlocal MK equation (14) are generated from formula (16):

$$\Phi = \sqrt{2} \frac{g_1}{f_1}, u = (2 \log f_1)_{xx} \tag{21}$$

where f_1 and g_1 are given by the following 3×3 determinants

$$f_1(x, t) = \begin{vmatrix} b_1 e^{-\zeta_1} + \frac{1}{p_1 + p_1^*} & \frac{1}{p_1 - p_1^*} & \frac{1}{p_1 - p_3^*} \\ -\frac{1}{p_1 - p_1^*} & -b_1^* e^{-\zeta_1^*(-x, y, -t)} - \frac{1}{p_1 + p_1^*} & -\frac{1}{p_1 + p_3^*} \\ \frac{1}{p_3 + p_3^*} & \frac{1}{p_3 - p_3^*} & ib_3 e^{-\zeta_3} + \frac{1}{p_3 - p_3^*} \end{vmatrix},$$

$$g_1(x, t) = \begin{vmatrix} b_1 e^{-\zeta_1} - \frac{p_1}{p_1^*} \frac{1}{p_1 + p_1^*} & \frac{p_1}{p_1^*} \frac{1}{p_1 - p_1^*} & \frac{p_1}{p_3^*} \frac{1}{p_1 - p_3^*} \\ -\frac{p_1}{p_1^*} \frac{1}{p_1 - p_1^*} & -b_1^* e^{-\zeta_1^*(-x, y, -t)} + \frac{p_1}{p_1^*} \frac{1}{p_1 + p_1^*} & \frac{p_1}{p_3^*} \frac{1}{p_1 + p_3^*} \\ -\frac{p_3}{p_1^*} \frac{1}{p_3 + p_3^*} & \frac{p_3}{p_1^*} \frac{1}{p_3 - p_3^*} & ib_3 e^{-\zeta_3} + \frac{p_3}{p_3^*} \frac{1}{p_3 - p_3^*} \end{vmatrix},$$

and these corresponding solutions are independent of y . Besides, the real part of the function f denoted as f_r satisfies $f_r \neq 0$ when $|\frac{e^{\zeta_{01}}}{2b_1 p_{1I}}| > 1$. In this case, these periodic solutions are regular, and this character also holds for the mixed solutions demonstrating line solitons on a PLWs background, which will be discussed in the following context. The maximum amplitudes of the PLWs solutions $|\Phi|$ and $|u|$ are

$$|\Phi|_{\max} = \sqrt{2} \left| \frac{2b_1 p_{1I} e^{-\zeta_{0,1}} + 1}{2b_1 p_{1I} e^{-\zeta_{0,1}} - 1} \right|, \tag{20}$$

$$|u|_{\max} = \frac{16b_1 p_{1I}^3 e^{-\zeta_{0,1}}}{(2b_1 p_{1I} e^{-\zeta_{0,1}} - 1)^2}.$$

It is noteworthy that the shift parameter $\zeta_{0,1}$ can also determine the maximum amplitudes of the PLWs, which is different from the local MK equation. Figure 1 shows the PLWs background.

In what follows, we show the line solitons on a PLWs background, which are demonstrated by the solutions (16) in Theorem 1 with $N \geq 1$. We first examine $N = 1$ in Theorem 1, i.e., the simplest solutions in

and

$$\zeta_1 = 2p_{1R}x + 4p_{1R}p_{1I}y - 2p_{1R} \left[4(p_{1R}^2 - 3p_{1I}^2) + \frac{\kappa}{p_{1R}^2 + p_{1I}^2} \right] t + \zeta_{0,1}, \tag{23}$$

$$\zeta_3 = 2ip_{3I}x + 2ip_{3I} \left[-4p_{3I}^2 + \frac{\kappa}{p_{3I}^2} \right] t + \zeta_{0,3},$$

where $p_1 = p_{1R} + ip_{1I}$, $p_3 = ip_{3I}$, b_1 is a freely complex parameter, and $\zeta_{0,1}$, $\zeta_{0,3}$, b_3 are real.

This solution demonstrates two line solitons moving on a PLWs background; the line orientations of the two line solitons are in the $(p_{1I}, 1)$ direction and $(p_{1I}, -1)$ direction. Hence, only the parameter p_{1I} controls the direction of the two line solitons. As discussed in Ref. [30], the patterns of the two line solitons are the same in PLWs background and constant background under same parametric condition. Below, we analyze the asymptotic behaviors of the two line soliton solutions on a constant background, which are obtained by deleting the third column and the third row of determinant expression of functions f and g (22):

$$f = -|b_1|^2 e^{-(\zeta_1 + \zeta_1^*(-x, y, -t))} - \frac{b_1 e^{-\zeta_1} + b_1^* e^{-\zeta_1^*(-x, y, -t)}}{2p_{1R}} - \frac{p_{1R}^2 + p_{1I}^2}{4p_{1R}^2 p_{1I}^2},$$

$$g = -|b_1|^2 e^{-(\zeta_1 + \zeta_1^*(-x, y, -t))} - \frac{b_1 e^{-\zeta_1} + b_1^* e^{-\zeta_1^*(-x, y, -t)}}{2p_{1R}} \left(\frac{p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}} \right) - \frac{p_{1R}^2 + p_{1I}^2}{4p_{1R}^2 p_{1I}^2} \left(\frac{p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}} \right)^2. \tag{24}$$

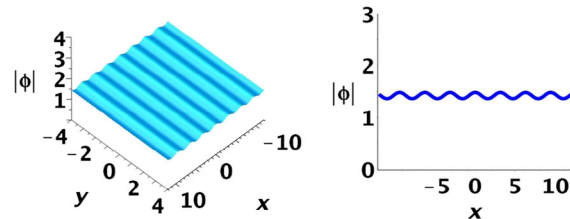


Fig. 1 The first-order periodic line wave solution (16) with functions f and g given by (19) with parameters $\kappa = 1, b_1 = \frac{1}{4}, p_{1R} = 0, p_{1I} = 1, \zeta_1^0 = \pi$. The right panel is a section diagram along $y = 0$ of the left panel (Colour online)

We assume $p_{1R} > 0$ and define the solitons along lines $2p_{1R}x + 4p_{1R}p_{1I}y - 2p_{1R}[4(p_{1R}^2 - 3p_{1I}^2) + \frac{\kappa}{p_{1R}^2+p_{1I}^2}]t + \zeta_{0,1}$ and $-2p_{1R}x + 4p_{1R}p_{1I}y + 2p_{1R}[4(p_{1R}^2 - 3p_{1I}^2) + \frac{\kappa}{p_{1R}^2+p_{1I}^2}]t + \zeta_{0,1}$ as soliton 1 and soliton 2, respectively, which lead to the following asymptotic forms:

(i) Before collision ($x \rightarrow -\infty$)

Soliton 1 ($\zeta_1 \approx 0, \zeta_1^*(-x, y, -t) \rightarrow +\infty$):

$$\Phi_1^- \simeq \sqrt{2} \frac{b_1 e^{-\zeta_1} + \frac{1}{2p_{1R}} \left(\frac{-p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}} \right)}{b_1 e^{-\zeta_1} + \frac{1}{2p_{1R}}}, \tag{25}$$

Soliton 2 ($\zeta_1^*(-x, y, -t) \approx 0, \zeta_1 \rightarrow -\infty$):

$$\Phi_2^- \simeq \sqrt{2} \left(\frac{-p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}} \right) \frac{b_1^* e^{-(\zeta_1^*(-x, y, -t) + \Delta)} + \frac{1}{2p_{1R}} \left(\frac{-p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}} \right)}{b_1^* e^{-(\zeta_1^*(-x, y, -t) + \Delta)} + \frac{1}{2p_{1R}}}, \tag{26}$$

where $e^{-\Delta} = \frac{p_{1I}^2}{p_{1R}^2 + p_{1I}^2}$.

(ii) After collision ($x \rightarrow +\infty$) Soliton 1 ($\zeta_1 \approx 0, \zeta_1^*(-x, y, -t) \rightarrow -\infty$):

$$\Phi_1^- \simeq \sqrt{2} \left(\frac{-p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}} \right) \frac{b_1 e^{-(\zeta_1 + \Delta)} + \frac{1}{2p_{1R}} \left(\frac{-p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}} \right)}{b_1 e^{-(\zeta_1 + \Delta)} + \frac{1}{2p_{1R}}}, \tag{27}$$

Soliton 2 ($\zeta_1^*(-x, y, -t) \approx 0, \zeta_1 \rightarrow +\infty$):

$$\Phi_2^+ \simeq \sqrt{2} \frac{b_1^* e^{-\zeta_1^*(-x, y, -t)} + \frac{1}{2p_{1R}} \left(\frac{-p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}} \right)}{b_1^* e^{-\zeta_1^*(-x, y, -t)} + \frac{1}{2p_{1R}}}. \tag{28}$$

Here Φ_j^+ and Φ_j^- represent the asymptotic form of the j th soliton before and after interaction for $j = 1, 2$. Then, we obtain $\Phi_1^+(\zeta_1) = \left(\frac{-p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}} \right) \Phi_1^-(\zeta_1 + \Delta)$, $\Phi_2^+(\zeta_1^*(-x, y, -t)) = \left(\frac{-p_{1R} - ip_{1I}}{p_{1R} + ip_{1I}} \right) \Phi_2^+(\zeta_1^*(-x, y, -t) - \Delta)$. Since $|\frac{-p_{1R} + ip_{1I}}{p_{1R} - ip_{1I}}| = 1, |\Phi_j^-| = |\Phi_j^+|$. These identities indicate the amplitude, shape and velocities of the j th are consistent before and after collision. Below, we classify the patterns of the two line soliton. Along the two lines $2p_{1R}x + 4p_{1R}p_{1I}y - 2p_{1R}[4(p_{1R}^2 - 3p_{1I}^2) + \frac{\kappa}{p_{1R}^2+p_{1I}^2}]t + \zeta_{0,1} + \frac{1}{2} \ln(4(b_{1R}^2 + b_{1I}^2)^2 p_{1R}^2)$ and $-2p_{1R}x + 4p_{1R}p_{1I}y + 2p_{1R}[4(p_{1R}^2 - 3p_{1I}^2) + \frac{\kappa}{p_{1R}^2+p_{1I}^2}]t + \zeta_{0,1} + \frac{1}{2} \ln(4(b_{1R}^2 + b_{1I}^2)^2 p_{1R}^2)$,

the asymptotic line solitons attain the max amplitudes denoted as $|\Phi_1|_{\max}$ and $|\Phi_2|_{\max}$:

$$|\Phi_1|_{\max} = \sqrt{2} \sqrt{1 - \frac{4(b_{1R}p_{1R} - b_{1I}p_{1I}) p_{1R}^2 \sqrt{(b_{1R}^2 + b_{1I}^2)} p_{1R}^2}{\Gamma}}, \tag{29}$$

$$|\Phi_1|_{\max} = \sqrt{2} \sqrt{1 - \frac{4(b_{1R}p_{1R} + b_{1I}p_{1I}) p_{1R}^2 \sqrt{(b_{1R}^2 + b_{1I}^2)} p_{1R}^2}{\Gamma}},$$

where

$$\Gamma = \left(b_{1R}p_{1R}p_{1I} + b_{1I}p_{1R}^2 + \sqrt{(b_{1R}^2 + b_{1I}^2)p_{1R}^2 p_{1I}} \right)^2 + p_{1R}^2 \left(b_{1R}p_{1R} - b_{1I}p_{1I} - \sqrt{(b_{1R}^2 + b_{1I}^2)p_{1R}^2} \right)^2,$$

According to the above analysis, the patterns of the two-line soliton solution Φ are listed as follows:

- Two-dark soliton if $b_{1R}p_{1R} - b_{1I}p_{1I} > 0, b_{1R}p_{1R} + b_{1I}p_{1I} > 0$;
- Two-antidark soliton if $b_{1R}p_{1R} - b_{1I}p_{1I} < 0, b_{1R}p_{1R} + b_{1I}p_{1I} < 0$;
- One-antidark soliton and one-dark soliton if $(b_{1R}p_{1R} - b_{1I}p_{1I})(b_{1R}p_{1R} + b_{1I}p_{1I}) < 0$;
- A degenerate two-line soliton if $(b_{1R}p_{1R} - b_{1I}p_{1I})(b_{1R}p_{1R} + b_{1I}p_{1I}) = 0$;

It should be note that here we analysis the asymptotic behaviors of the two-line soliton as $x \rightarrow \pm\infty$, which were analyzed as $t \rightarrow \pm\infty$ in [52]. Besides, we classify the patterns of the two-line soliton in the (x, y) -plane under different parametric conditions according to the asymptotic analysis that were not given in Ref. [52].

By doing similar analysis of the solution of the long wave component u , solution u only possesses two-antidark-soliton. Five different types of two-line-soliton solution Φ on a background of PLWs are shown in Fig. 2: two-antidark-soliton (Fig. 2a), one-antidark-soliton and one-dark-soliton (Fig. 2b), two-dark-soliton (Fig. 2c), a degenerate two-dark-soliton (Fig. 2d) and a degenerate two-antidark-soliton (Fig. 2e). Figure 2f illustrates the two-antidark-soliton solution of the long wave component u . We note that the classification of the patterns of two-line-soliton on a constant background was not given in [52].

The higher-order line solitons on a PLWs background can be yielded from Theorem 1 with larger N . In addition to the degenerated $2N$ -line soliton, the

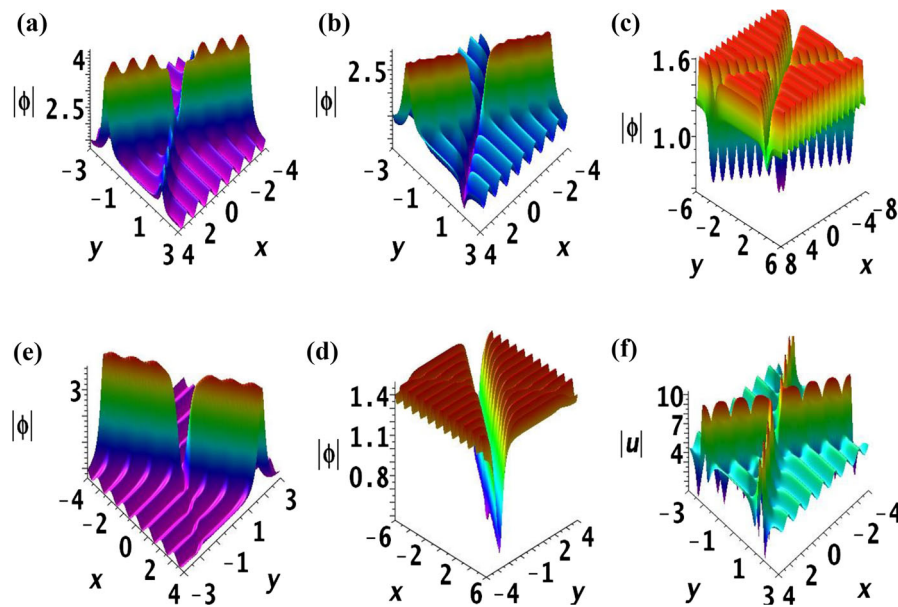


Fig. 2 Several different types of two line solitons on a PLWs background given by (22) in the nonlocal MK equation (14) at the time $t = 0$: **a** Two-antidark-soliton on a PLWs background in the SW component $|\Phi|$ with parameters $\kappa = 1, p_1 = -1 - i, p_3 = -3i, b_1 = 1 + i, b_3 = -\frac{i}{4}, \zeta_{0,1} = 0, \zeta_{0,3} = \pi$; **b** One-antidark-soliton and one-dark-soliton on a PLWs background in the SW component $|\Phi|$ with parameters $\kappa = 1, p_1 = -1 - i, p_3 = -3i, b_1 = 1 + \frac{i}{2}, b_3 = -\frac{i}{4}, \zeta_{0,1} = 0, \zeta_{0,3} = \pi$; **c** Two-dark-soliton on a PLWs background in the SW component $|\Phi|$ with parameters $\kappa = 1, p_1 = -1 - i, p_3 = -3i, b_1 = -1 - 2i, b_3 =$

$-\frac{i}{4}, \zeta_{0,1} = 0, \zeta_{0,3} = \pi$; **d** A degenerated two-dark-soliton on a PLWs background in the SW component $|\Phi|$ with parameters $\kappa = 1, p_1 = -1 + i, p_3 = -3i, b_1 = 1 - i, b_3 = -\frac{i}{4}, \zeta_{0,1} = 0, \zeta_{0,3} = \pi$; **e** A degenerated two-antidark-soliton on a PLWs background in the SW component $|\Phi|$ with parameters $\kappa = 1, p_1 = 1 + i, p_3 = -3i, b_1 = 1 - i, b_3 = -\frac{i}{4}, \zeta_{0,1} = 0, \zeta_{0,3} = \pi$; and **f** Two-antidark-soliton in the LW component $|u|$ with parameters $\kappa = 1, p_1 = -1 - i, p_3 = -3i, b_1 = 1 + \frac{i}{2}, b_3 = -\frac{i}{4}, \zeta_{0,1} = 0, \zeta_{0,3} = \pi$ (Colour online)

SW component $|\Phi|$ possesses $2N + 1$ different patterns: N' -antidark-soliton and $2N - N'$ -dark-soliton ($0 \leq N' \leq 2N$) on a PLWs background. The LW component $|u|$ only has one patterns: $2N$ -antidark-soliton on the PLWs background. For instance, with $N = 2$ and different parameters in Theorem 1, the corresponding solutions illustrate four-line solitons on a PLWs background. Figure 3a–e shows five different patterns of four-line-soliton solution on a background of PLWs in the SW component $|\Phi|$, namely four-antidark-soliton (see Fig. 3a), three-antidark-soliton and one-dark-soliton (see Fig. 3b), two-antidark-soliton and two-dark-soliton (see Fig. 3c), one-antidark-soliton and three-dark-soliton (see Fig. 3d) and four-dark-soliton (see Fig. 3e). Figure 3f demonstrates the four-antidark-soliton in the LW component $|u|$. For larger N , these higher-order mixed solutions can be directly obtained. However, these high-order mixed solutions do not have qualitatively different behaviors, but more line solitons

interacting with each other on a PLWs background, and the interaction may give rise to more line wave patterns.

3 General breathers on a background of constant and PLWs

In this section, we would comprehensively focus on the features of the breathers on two types of background in the nonlocal MK equation (14), namely the constant background and the PLWs background. These solutions are presented via determinant forms in the present paper, which are different from the results in Ref. [51], in which the solutions are given by complicated algebraic formula. As discussed in earlier works in the literature [51], breather solution is a type of periodic solutions. Furthermore, the PLWs background is also described by a periodic solution; thus, breathers on a constant or PLWs background are a kind of periodic solutions. Below, we would first give a general type of

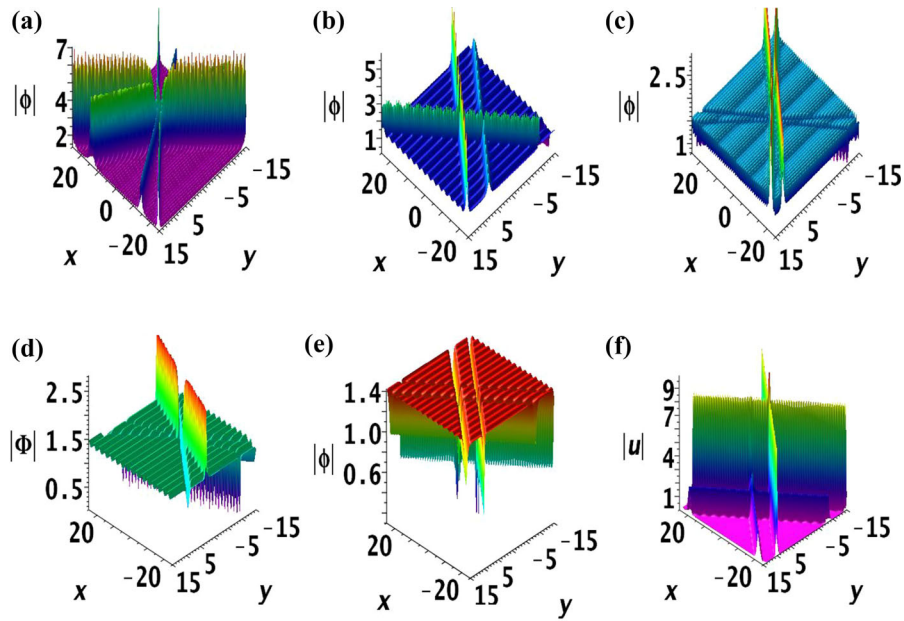


Fig. 3 Five different types of four-line-soliton solution on a PLWs background in the SW component $|\Phi|$, and one type of four-line-soliton solution on a background of PLWs in the LW component $|u|$, given by (22) to the nonlocal MK equation (14) at the time $t = 0$: **a** Four-antidark-soliton in the SW component $|\Phi|$ with parameters $\kappa = 1, p_1 = -\frac{1}{2} - \frac{1}{2}i, p_2 = -1 - \frac{4}{5}i, p_5 = -3i, b_1 = 1 + \frac{1}{2}i, b_2 = 1 + \frac{2}{5}i, b_5 = -\frac{1}{4}i, \zeta_{0,1} = 0, \zeta_{0,2} = 0, \zeta_{0,5} = -\pi$; **b** Three-antidark-soliton and one-dark-soliton in the SW component $|\Phi|$ with parameters $\kappa = 1, p_1 = -\frac{1}{2} - i, p_2 = -1 - i, p_5 = -i, b_1 = 1 + 2i, b_2 = 1 + \frac{1}{2}i, b_5 = -i, \zeta_{0,1} = 3\pi, \zeta_{0,2} = -3\pi, \zeta_{0,5} = -\frac{1}{2}\pi$; **c** Two-antidark-soliton and two-dark-soliton in the SW com-

ponent $|\Phi|$ with parameters $\kappa = 1, p_1 = -1 - i, p_2 = -1 - \frac{4}{5}i, p_5 = -3i, b_1 = 1 + \frac{1}{2}i, b_2 = 1 + \frac{2}{5}i, b_5 = -\frac{4}{5}i, \zeta_{0,1} = 0, \zeta_{0,2} = 0, \zeta_{0,5} = \pi$; **d** One-antidark-soliton and three-dark-soliton in the SW component $|\Phi|$ with parameters $\kappa = 1, p_1 = 1 - i, p_2 = -\frac{1}{2} - 2i, p_5 = -i, b_1 = 1, b_2 = 1 - \frac{1}{2}i, b_5 = -i, \zeta_{0,1} = 0, \zeta_{0,1} = -4\pi, \zeta_{0,2} = 4\pi, \zeta_{0,5} = 0$; **e** Four-dark-soliton in the SW component $|\Phi|$ with parameters $\kappa = 1, p_1 = 1 - i, p_2 = -2 - i, p_5 = -i, b_1 = 1, b_2 = 1, b_5 = -i, \zeta_{0,1} = 5\pi, \zeta_{0,2} = -5\pi, \zeta_{0,5} = -\frac{3}{2}\pi$; and **f** Four-antidark-soliton in the LW component $|u|$ with parameters $\kappa = 1, p_1 = 1 - i, p_2 = -2 - i, p_5 = -i, b_1 = 1, b_2 = 1, b_5 = -i, \zeta_{0,1} = 5\pi, \zeta_{0,2} = -5\pi, \zeta_{0,5} = -\frac{3}{2}\pi$ (Colour online)

periodic solutions to the nonlocal MK equation (14) in the following theorem, whose proof procedure for the theorem would be provided in Appendix in detail.

Theorem 2 *The nonlocal MK equation (14) has periodic solutions*

$$\Phi = \sqrt{2} \frac{g}{f}, u = (2 \log f)_{xx} \tag{30}$$

with functions f and g given by the following $N \times N$ determinants

$$\begin{aligned} f(x, t) &= \Gamma_0 \left| \frac{\delta_{sj}}{\omega_s e^{\zeta_s t}} + \frac{1}{\frac{1}{2}(\omega_s + \omega_j) + i(\lambda_s - \lambda_j)} \right|_{1 \leq s, j \leq N}, \\ g(x, t) &= \Gamma_0 \left| \frac{\delta_{sj}}{\omega_s e^{\zeta_s t}} - \frac{\omega_s + 2i\lambda_s}{\omega_j - 2i\lambda_j} \frac{1}{\frac{1}{2}(\omega_s + \omega_j) + i(\lambda_s - \lambda_j)} \right|_{1 \leq s, j \leq N}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} \zeta_s &= i\omega_s x - 2\lambda_s \omega_s y + i\omega_s \left[(12\lambda_s^2 - \omega_s^2) \right. \\ &\quad \left. + \frac{4\kappa}{4\lambda_s^2 + \omega_s^2} \right] t + \zeta_{0,s}, \end{aligned} \tag{32}$$

where $\Gamma_0 = \prod_{s=1}^N \omega_s e^{\zeta_s}$, and $\omega_s, \lambda_s, \zeta_{0,s}$ are real parameters.

Remark 1 With different parametric conditions, these periodic solutions have the following three different behaviors:

- Breathers on a nonzero constant background when the parameters fulfill the following parametric constraints:

$$\begin{aligned} N &= 2M, \omega_{M+j} = -\omega_j, \lambda_{M+j} \\ &= -\lambda_j, \zeta_{0,M+s} = \zeta_{0,s}, j = 1, 2, \dots \end{aligned} \tag{33}$$

- Breathers on a PLWs background when the parameters fulfill the following parametric constraints:

$$\begin{aligned}
 N &= 2M + 1, \omega_{M+j} = -\omega_j, \lambda_{M+j} \\
 &= -\lambda_j, \zeta_{0,M+s} = \zeta_{0,s}, \lambda_{2M+1} = 0, j = 1, 2, \dots
 \end{aligned}
 \tag{34}$$

- Periodic line waves when the parameters do not satisfy the parametric conditions given by (33) and (34).

In what follows, we consider the dynamics of the breathers on a background of constant and PLWs, respectively.

3.1 Breathers on a constant background

The first-order breather on a constant background is generated by taking the following parameters in Theorem 2

$$N = 2, \omega_2 = -\omega_1, \lambda_2 = -\lambda_1, \zeta_{0,2} = \zeta_{0,1},
 \tag{35}$$

and the functions f and g of solutions (30) read as

$$\begin{aligned}
 f &= \left| \begin{array}{cc} \frac{1}{\omega_1 e^{\zeta_1}} + \frac{1}{\omega_1} & \frac{1}{2i\lambda_1} \\ \frac{1}{-2i\lambda_1} & \frac{1}{-\omega_1 e^{\zeta_1^*}} - \frac{1}{\omega_1} \end{array} \right|, \\
 g &= \left| \begin{array}{cc} \frac{1}{\omega_1 e^{\zeta_1}} - \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{\omega_1} & \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{2i\lambda_1} \\ -\frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{2i\lambda_1} & -\frac{1}{\omega_1 e^{\zeta_1^*}} + \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{\omega_1} \end{array} \right|,
 \end{aligned}
 \tag{36}$$

After simple algebra, the final expression for the first-order breather solution reads

$$\begin{aligned}
 \Phi &= \sqrt{2} e^{\delta_1 + i\delta_2} \frac{\cos \tilde{\eta} + \sqrt{\delta_0} \cosh(\tilde{\xi} + \tilde{\Delta} - \delta_1 - i\delta_2)}{\cos \tilde{\eta} + \sqrt{\delta_0} \cosh(\tilde{\xi} + \tilde{\Delta})} \\
 u &= \frac{2\omega_1^2 \sqrt{\delta_0} (\sqrt{\delta_0} + \cos \tilde{\eta} \cosh(\tilde{\xi} + \tilde{\Delta}))}{(\cos \tilde{\eta} + \sqrt{\delta_0} \cosh(\tilde{\xi} + \tilde{\Delta}))^2}
 \end{aligned}
 \tag{37}$$

with

$$\begin{aligned}
 \tilde{\eta} &= \omega_1 x + \omega_1 [(12\lambda_1^2 - \omega_1^2) + \frac{4\kappa}{4\lambda_1^2 + \omega_1^2}] t, \\
 \tilde{\xi} &= -2\lambda_1 \omega_1 y + \zeta_{0,1},
 \end{aligned}
 \tag{38}$$

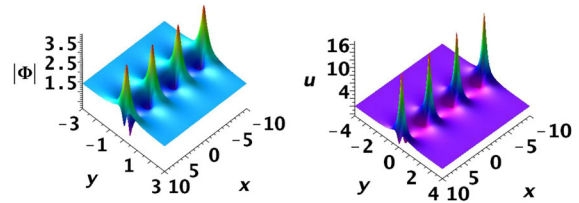


Fig. 4 The one-breather solution for the nonlocal Mel’nikov equation (14) at the time $t = 0$ with parameters $\lambda_1 = 1, \omega_1 = 1, \kappa = 1, \zeta_{0,1} = 0$ (Colour online)

where $\delta_0 = 1 + \frac{\omega_1^2}{4\lambda_1^2}, \tilde{\Delta} = \ln(\frac{1}{\sqrt{\delta_0}}), \delta_1 = |\frac{2i\lambda_1 + \omega_1}{2i\lambda_1 - \omega_1}|, \delta_2 = \arg \frac{2i\lambda_1 + \omega_1}{2i\lambda_1 - \omega_1}$. From the above expression, one can clearly get that $(|\Phi|, u) \rightarrow (\sqrt{2}, 0)$ when $\tilde{\xi} \rightarrow \pm\infty$. The periodic feature of the one-breather is controlled by $\tilde{\eta}$, and the localized feature of the one-breather is controlled by $\tilde{\eta}$. Hence, it is clear that, in the (x, y) -plane, this one-breather solution is only periodic along x and t , whose corresponding periods are $\frac{2\pi}{\omega_1}$ and $\frac{2\pi}{\omega[(12\lambda_1^2 - \omega_1^2) + \frac{4\kappa}{4\lambda_1^2 + \omega_1^2}]}$, respectively. It should be emphasized that the one breather does not move along y direction as time t propagates. Figure 4 shows an example of that one-breather solution at $t = 0$.

The high-order breathers on the constant background could be yielded from Theorem 2 by taking larger $N = 2M$, and other parameters meeting the parametric condition defined in (33). The corresponding solutions consist of M individual breathers, and the M individual breathers interact with each other and would generate interesting curly wave patterns. For example, with $M = 2$ (i.e., $N = 4$), the two-breather solution could be derived from Theorem 2:

$$\Phi = \sqrt{2} \frac{\bar{g}_2}{f_2}, u = (2 \log \bar{f}_2)_{xx},
 \tag{39}$$

where functions f and g are

$$\begin{aligned} \overline{f_2} &= \begin{vmatrix} \frac{1}{\omega_1 e^{\zeta_1}} + \frac{1}{\frac{1}{2}(\omega_1 + \omega_1) + i(\lambda_1 - \lambda_1)} & \frac{1}{\frac{1}{2}(\omega_1 + \omega_2) + i(\lambda_1 - \lambda_2)} & \frac{1}{\frac{1}{2}(\omega_1 + \omega_3) + i(\lambda_1 - \lambda_3)} & \frac{1}{\frac{1}{2}(\omega_1 + \omega_4) + i(\lambda_1 - \lambda_4)} \\ \frac{1}{\frac{1}{2}(\omega_2 + \omega_1) + i(\lambda_2 - \lambda_1)} & \frac{1}{\omega_2 e^{\zeta_2}} + \frac{1}{\omega_2} & \frac{1}{\frac{1}{2}(\omega_2 + \omega_3) + i(\lambda_2 - \lambda_3)} & \frac{1}{\frac{1}{2}(\omega_2 + \omega_4) + i(\lambda_2 - \lambda_4)} \\ \frac{1}{\frac{1}{2}(\omega_3 + \omega_1) + i(\lambda_3 - \lambda_1)} & \frac{1}{\frac{1}{2}(\omega_3 + \omega_2) + i(\lambda_3 - \lambda_2)} & \frac{1}{\omega_3 e^{\zeta_3}} + \frac{1}{2\omega_3} & \frac{1}{\frac{1}{2}(\omega_3 + \omega_4) + i(\lambda_3 - \lambda_4)} \\ \frac{1}{\frac{1}{2}(\omega_4 + \omega_1) + i(\lambda_4 - \lambda_1)} & \frac{1}{\frac{1}{2}(\omega_4 + \omega_2) + i(\lambda_4 - \lambda_2)} & \frac{1}{\frac{1}{2}(\omega_4 + \omega_3) + i(\lambda_4 - \lambda_3)} & \frac{1}{\omega_4 e^{\zeta_4}} + \frac{1}{\omega_4} \end{vmatrix}, \\ \overline{g_2} &= \begin{vmatrix} \frac{1}{\omega_1 e^{\zeta_1}} + \overline{\omega}_{11} \frac{1}{\frac{1}{2}(\omega_1 + \omega_1) + i(\lambda_1 - \lambda_1)} & \overline{\omega}_{12} \frac{1}{\frac{1}{2}(\omega_1 + \omega_2) + i(\lambda_1 - \lambda_2)} & \overline{\omega}_{13} \frac{1}{\frac{1}{2}(\omega_1 + \omega_3) + i(\lambda_1 - \lambda_3)} & \overline{\omega}_{14} \frac{1}{\frac{1}{2}(\omega_1 + \omega_4) + i(\lambda_1 - \lambda_4)} \\ \overline{\omega}_{21} \frac{1}{\frac{1}{2}(\omega_2 + \omega_1) + i(\lambda_2 - \lambda_1)} & \frac{1}{\omega_2 e^{\zeta_2}} + \overline{\omega}_{22} \frac{1}{\omega_2} & \overline{\omega}_{23} \frac{1}{\frac{1}{2}(\omega_2 + \omega_3) + i(\lambda_2 - \lambda_3)} & \overline{\omega}_{24} \frac{1}{\frac{1}{2}(\omega_2 + \omega_4) + i(\lambda_2 - \lambda_4)} \\ \overline{\omega}_{31} \frac{1}{\frac{1}{2}(\omega_3 + \omega_1) + i(\lambda_3 - \lambda_1)} & \overline{\omega}_{32} \frac{1}{\frac{1}{2}(\omega_3 + \omega_2) + i(\lambda_3 - \lambda_2)} & \frac{1}{\omega_3 e^{\zeta_3}} + \overline{\omega}_{33} \frac{1}{2\omega_3} & \overline{\omega}_{34} \frac{1}{\frac{1}{2}(\omega_3 + \omega_4) + i(\lambda_3 - \lambda_4)} \\ \overline{\omega}_{41} \frac{1}{\frac{1}{2}(\omega_4 + \omega_1) + i(\lambda_4 - \lambda_1)} & \overline{\omega}_{42} \frac{1}{\frac{1}{2}(\omega_4 + \omega_2) + i(\lambda_4 - \lambda_2)} & \overline{\omega}_{43} \frac{1}{\frac{1}{2}(\omega_4 + \omega_3) + i(\lambda_4 - \lambda_3)} & \frac{1}{\omega_4 e^{\zeta_4}} + \overline{\omega}_{44} \frac{1}{\omega_4} \end{vmatrix}, \end{aligned} \tag{40}$$

with

$$\begin{aligned} \zeta_s &= i\omega_s x - 2\lambda_s \omega_s y + i\omega_s \left[\left(12\lambda_s^2 - \omega_s^2 \right) \right. \\ &\quad \left. + \frac{4\kappa}{4\lambda_s^2 + \omega_s^2} \right] t + \zeta_{0,s}, \\ \overline{\omega}_{sj} &= -\frac{\omega_s + 2i\lambda_s}{\omega_j - 2i\lambda_j}, \end{aligned} \tag{41}$$

for $s, j = 1, 2, 3, 4$, and parameters satisfy

$$\begin{aligned} \lambda_3 &= -\lambda_1, \lambda_4 = -\lambda_2, \omega_3 = -\omega_1, \\ \omega_4 &= -\omega_2, \zeta_{0,3} = \zeta_{0,1}, \zeta_{0,4} = \zeta_{0,2}. \end{aligned} \tag{42}$$

According to the analysis for the one-breather solution given by (47), these two breathers do not move along y on the (x, y) -plane as time t alters. To observe the different wave patterns generated from the interaction of the two breathers, we take a fixed value of the parameter $\zeta_{0,2}$ and different values of $\zeta_{0,1}$, which give rise to the location of one breather is fixed, and the other breather alters. Hence, one breather moves to the fixed breather and then interacts with each other. In this evolution, different wave patterns could be observed. To clearly observe the interaction wave patterns, we take the following parameter choices

$$\begin{aligned} \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = -1, \omega_1 = 1, \omega_2 = \frac{1}{2}, \\ \omega_3 = -1, \omega_4 = -\frac{1}{2}, \kappa = 1, \zeta_{0,2} = 0. \end{aligned} \tag{43}$$

We denote the breather determined by ζ_1 as breather-1 and the other one determined by ζ_2 as breather-2. Under parameter choices given by (43), the period of breather-2 is two times of the breather-1. Evolution of the interaction between of the two breathers is illustrated in Fig. 5 with different $\zeta_{0,1}$. When $\zeta_{0,1} \gg 0$, the two breathers are far from each other and out of interaction. When $\zeta_{0,1} \rightarrow 0$, the two breathers are close to

each other and begin to interact with each other. Due to the interaction, the period of the breather-2 changes, and the wave structures become periodic triangle patterns (see Fig. 5b). In the intermediate state of the evolution, the two breathers merge into one breather and form periodic fundamental patterns (see Fig. 5c). At the final state of the evolution, the two breathers separate with each other completely, and the two breathers are out of interaction again (see Fig. 5d).

3.2 Breathers on a PLWs background

To consider the dynamical behaviors of breathers on a PLWs background, we have to give the first-order PLWs solution, which is used to provide the PLWs background. By taking $N = 1$ in formula (31), the first-

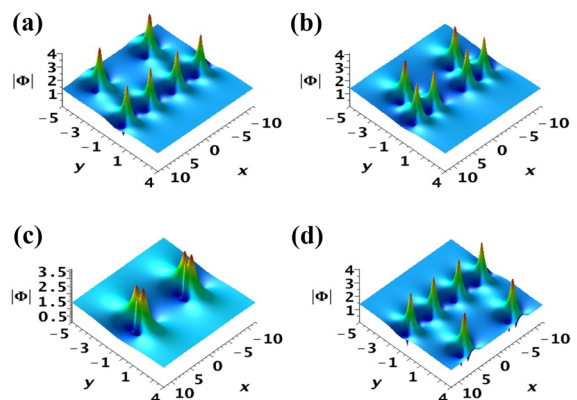


Fig. 5 The two-breather on a constant background for the non-local Me’nikov equation (14) with parameters given by (43) and different choices of parameter $\zeta_{0,2}$: **a** $\zeta_{0,1} = -\frac{\pi}{2}$; **b** $\zeta_{0,1} = 0$; **c** $\zeta_{0,1} = 0.0174\pi$ and **d** $\zeta_{0,1} = \pi$ (Colour online)

order periodic line wave solution could be yielded, and the functions f and g read as

$$\begin{aligned}
 f &= 1 + e^{i\omega_1 x - 2\lambda_1 \omega_1 y + i\omega_1 \left[(12\lambda_1^2 - \omega_1^2) + \frac{4\kappa}{4\lambda_1^2 + \omega_1^2} \right] t + \zeta_{0,1}} \\
 &= 1 + e^{\zeta_{0,1} - 2\lambda_1 \omega_1 y} \cos \chi - i \sin \chi, \tag{44} \\
 g &= 1 - \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} e^{i\omega_1 x - 2\lambda_1 \omega_1 y + i\omega_1 \left[(12\lambda_1^2 - \omega_1^2) + \frac{4\kappa}{4\lambda_1^2 + \omega_1^2} \right] t + \zeta_{0,1}},
 \end{aligned}$$

where $\chi = \omega_1 x + \omega_1 \left[(12\lambda_1^2 - \omega_1^2) + \frac{4\kappa}{4\lambda_1^2 + \omega_1^2} \right] t$, and $c_1, \omega_1, \lambda_1, \zeta_{0,1}$ are arbitrary real parameters. The periodic solution given by (44) is equivalent to the periodic solution defined in (19), and it is singular when $\lambda_1 = 0, 1 > e^{\zeta_{0,1}}$ or $\lambda_1 = 0, e^{\zeta_{0,1}} < -1$.

By taking $N = 3$ and the following parameter choices in Theorem 2

$$\omega_2 = -\omega_1, \lambda_2 = -\lambda_1, \lambda_3 = 0, \zeta_{0,2} = \zeta_{0,1}, \tag{45}$$

the one-breather on a PLWs background is obtained

$$\Phi = \sqrt{2} \frac{g_2}{f_2}, u = (2 \log f_2)_{x,x}, \tag{46}$$

where functions f_2 and g_2 are

$$\begin{aligned}
 f_2 &= \begin{vmatrix} \frac{1}{\omega_1 e^{\zeta_1}} + \frac{1}{\omega_1} & & \frac{1}{2i\lambda_1} & \frac{1}{\frac{1}{2}(\omega_1 + \omega_3) + i\lambda_1} \\ \frac{1}{-2i\lambda_1} & \frac{1}{-\omega_1 e^{\zeta_1^*}} - \frac{1}{\omega_1} & & \frac{1}{\frac{1}{2}(\omega_2 + \omega_3) + i\lambda_2} \\ \frac{1}{\frac{1}{2}(\omega_3 + \omega_1) - i\lambda_1} & \frac{1}{\frac{1}{2}(\omega_3 + \omega_2) - i\lambda_2} & & \frac{1}{\omega_3 e^{\zeta_3}} + \frac{1}{\omega_3} \end{vmatrix}, \\
 g_2 &= \begin{vmatrix} \frac{1}{\omega_1 e^{\zeta_1}} - \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{\omega_1} & \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{2i\lambda_1} & -\frac{\omega_1 + 2i\lambda_1}{\omega_3} & \frac{1}{\frac{1}{2}(\omega_1 + \omega_3) + i\lambda_1} \\ -\frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{2i\lambda_1} & -\frac{1}{\omega_1 e^{\zeta_1^*}} + \frac{\omega_1 + 2i\lambda_1}{\omega_1 - 2i\lambda_1} \frac{1}{\omega_1} & -\frac{\omega_1 + 2i\lambda_1}{\omega_3} & \frac{1}{\frac{1}{2}(\omega_2 + \omega_3) + i\lambda_2} \\ \frac{\omega_3}{\omega_1 - 2i\lambda_1} \frac{1}{\frac{1}{2}(\omega_3 + \omega_1) - i\lambda_1} & -\frac{\omega_3}{\omega_1 - 2i\lambda_1} \frac{1}{\frac{1}{2}(\omega_3 + \omega_2) - i\lambda_2} & \frac{1}{\omega_3 e^{\zeta_3}} - \frac{1}{\omega_3} & \end{vmatrix}, \tag{47}
 \end{aligned}$$

and

$$\begin{aligned}
 \zeta_1 &= i\omega_1 x - 2\lambda_1 \omega_1 y + i\omega_1 \left[(12\lambda_1^2 - \omega_1^2) \right. \\
 &\quad \left. + \frac{4\kappa}{4\lambda_1^2 + \omega_1^2} \right] t + \zeta_{0,1}, \tag{48} \\
 \zeta_3 &= i\omega_3 x + i\omega_3 \left(-\omega_3^2 + \frac{4\kappa}{\omega_3^2} \right) t + \zeta_{0,3}.
 \end{aligned}$$

This solution is a combination of the one-breather solution in (47) and the first-order PLWs solution in (44). According to the analysis for these two solutions, the solution given by (46) still keeps periodic along x

and t , and localized along y . Besides, since these two periodic solutions are also periodic along time t , the breather would be fixed on the PLWs background as time changes in the (x, y) -plane. However, the location of the breather can be controlled by the parameter $\zeta_{0,3}$. Figure 6 shows the dynamics of the one-breather on the PLWs background with different values of $\zeta_{0,3}$. It is directly to obtain that the shape, velocity and amplitude of the one breather do not alter when its location changes; hence, the collision between the one-breather and the PLWs is elastic collision.

In what follows, we consider the high-order breathers on a PLWs background. The M -breather on a PLWs background can be generated from Theorem 2 with $N = 2M + 1$ and other parameters satisfying the parameters condition (34). For instance, with $N = 5$ and

$$\begin{aligned}
 \lambda_3 &= -\lambda_1, \lambda_4 = -\lambda_2, \lambda_5 = 0, \omega_3 = -\omega_1, \omega_4 \\
 &= -\omega_2, \zeta_{0,3} = \zeta_{0,1}, \zeta_{0,4} = \zeta_{0,2}, \tag{49}
 \end{aligned}$$

the two breathers on a PLWs background would be generated, and Fig. 7 shows the dynamical features of that

mixed solutions. In order to compare the two-breather on the PLWs background with the two-breather on the constant background shown in Fig. 5, we take the same parameter choices in Fig. 7 as the parameter choices in Fig. 5 except the value of $\zeta_{0,1}$. Here, the parameter $\zeta_{0,1}$ is used to control the distance between the two breathers. As can be seen from Fig. 7, when the breather moves toward to the fixed breather, the wave patterns of the two breathers range from two usual breathers (see Fig. 7a) to triangular patterns (see Fig. 7b, c), then to fundamental patterns (see Fig. 7d), and to triangular

Fig. 6 The one-breather on a PLWs background in the nonlocal MK equation (14) at the time $t = 0$ with $\lambda_1 = 1, \lambda_3 = 0, \omega_1 = 1, \omega_3 = 2, \kappa = 1, \zeta_{0,3} = -\frac{\pi}{2}$: **a** $\zeta_{0,1} = -\pi$; **b** $\zeta_{0,1} = 0$; and **c** $\zeta_{0,1} = \pi$ (Colour online)

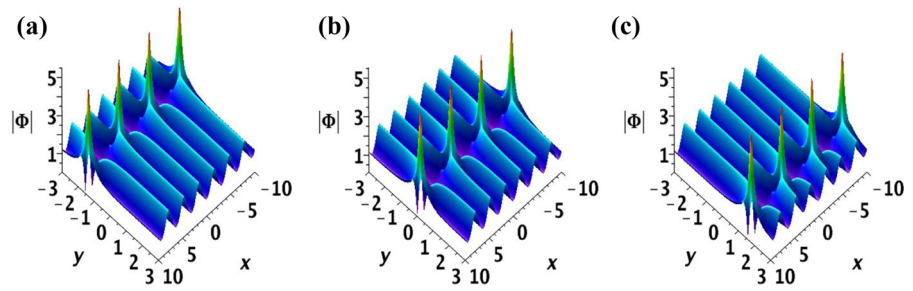
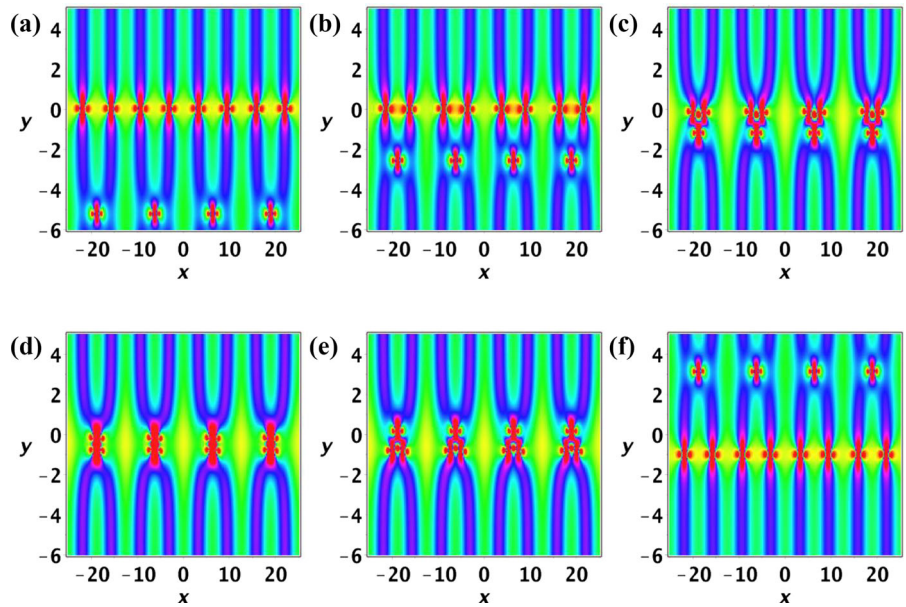


Fig. 7 The two-breather on a PLWs background for the nonlocal Me'nikov equation (14) at the time $t = 0$ with parameters

$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = 0, \omega_1 = 1, \omega_2 = \frac{1}{2}, \omega_3 = -1, \omega_4 = -\frac{1}{2}, \omega_5 = 1, \kappa = 1, \zeta_{0,2} = \zeta_{0,1}, \zeta_{0,3} = 0, \zeta_{0,4} = 0, \zeta_{0,5} = -\frac{4}{3}\pi$, and different values of parameter $\zeta_{0,1}$: **a** $\zeta_{0,1} = -5\pi$; **b** $\zeta_{0,1} = -3\pi$; **c** $\zeta_{0,1} = -\pi$; **d** $\zeta_{0,1} = \frac{\pi}{2}$; **e** $\zeta_{0,1} = \pi$ and **f** $\zeta_{0,1} = 3\pi$ (Colour online)



patterns again (see Fig. 7e), and finally to two usual breathers (see Fig. 7f).

4 Summary and discussion

In this paper, we mainly consider dynamics of line solitons and breathers on a PLWs background in the nonlocal MK equation (14). These dynamical behaviors are described by two families of mixed solutions: one consists of line solitons and PLWs, and the other one comprises breathers and PLWs. These two families of mixed solutions are derived by constraining two different parametric conditions of tau functions of the KP hierarchy. For the general line solitons on a PLWs background, the SW component $|\Phi|$ possesses five different wave patterns in the simplest (first-order) solutions: two-antidark-soliton on a PLWs background, one-antidark-soliton and one-dark-soliton on a PLWs background, two-dark-soliton on the PLWs background,

a degenerated two-dark-soliton, a degenerated two-antidark-soliton (see Fig. 2a–e), while the LW component $|u|$ only displays the two-antidark-soliton on a PLWs background (see Fig. 2f). The high-order mixed solutions demonstrate more line solitons on the PLWs background, which shows the features of superposition of several individual first-order mixed solutions (see Fig. 3). For the breathers on a PLWs background, both of SW component $|\Phi|$ and LW component $|u|$ only display one type of breather on a PLWs background, namely bright breather on a PLWs background. Besides, the breathers on a constant background can also be reduced from the breathers on a PLWs background under particular parametric conditions.

By comparing with the solutions containing line solitons and breathers to the nonlocal MK equation (14) discussed in Refs. [51, 52], the solutions investigated in the present paper can be summarized as follows:

- The line solitons on PLWs background to the nonlocal MK equation (14) and their dynamical features are investigated in detail in this paper, while under investigation in Ref. [52] is the line solitons on a constant background. Besides, the patterns of the two-line-soliton are classified according to the asymptotic analysis.
- The breathers on a constant or PLWs background were also reported in Ref. [51]. However, we construct these solutions by employing the KP hierarchy reduction method in the present paper, and these solutions are presented by Gram-type determinants. In Ref. [51], these solutions were constructed by employing the Hirota’s direct method with perturbation expansion, and these solutions are expressed by complicated algebraic formulae.
- We have constructed more periodic solutions to the nonlocal MK equation, which have not been reported before.

Since rational and semi-rational solutions can be derived from periodic solutions by using the long wave limit technique, a natural extension of the present work could be constructed rational and semi-rational solutions for the nonlocal MK equation (14) by taking a long wave limit of the periodic solutions given in Theorem 2. The rational solutions would demonstrate lumps on a background of PLWs or constant, and the semi-rational solutions show a mixture of lumps, breathers and line solitons on a PLWs or constant background. We would continue to explore this interesting topic for the MK equation (14).

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Compliance with ethical standards

Conflict of interest We declare that we have no conflict of interests.

Appendix A

In this Appendix, we will present the proof procedure for Theorems 1 and 2. Since the solutions given in Theorems 1 and 2 are under nonzero boundary conditions, we employ the employing the following depen-

dent variable transformations

$$\Psi = \sqrt{2} \frac{g(x, y, t)}{f(x, y, t)}, u = 2(\log f(x, y, t))_{xx}, \tag{50}$$

to translate the nonlocal MK equation (14) into the following bilinear equation

$$\begin{aligned} (D_x^2 - iD_y)g \cdot f &= 0, \\ (D_x^4 + D_x D_t - 3D_y^2)f \cdot f &= 2\kappa[f^2 - gg^*(-x, y, -t)]. \end{aligned} \tag{51}$$

According to the Sato theory [65–70], the following bilinear equations in the single component KP hierarchy

$$\begin{aligned} (D_{x_1}^2 - D_{x_2})\tau_{n+1} \cdot \tau_n &= 0, \\ D_{x_1} D_{x_{-1}} - 2)\tau_n \cdot \tau_n + 2\tau_{n+1}\tau_{n-1} &= 0, \\ (D_{x_1}^4 - 3D_{x_1} D_{x_3} + 4D_{x_2}^2)\tau_n \cdot \tau_n &= 0, \end{aligned} \tag{52}$$

possess the following tau functions expressed via Gramian determinant

$$\tau_n = \det_{1 \leq i, j \leq N} (m_{ij}^{(n)}), \tag{53}$$

where the matrix elements are defined as

$$\begin{aligned} m_{sj}^{(n)} &= \tilde{b}_s \delta_{sj} + \frac{p_s + r_s}{p_s + q_j} \left(-\frac{p_s}{q_j}\right)^n e^{\xi_s + \eta_j}, \\ \xi_s &= \frac{1}{p_s} x_{-1} + p_s x_1 + p_s^2 x_2 + p_s^3 x_3 + \xi_{0,s}, \\ \eta_j &= \frac{1}{q_j} x_{-1} + q_j x_1 - q_j^2 x_2 + q_j^3 x_3 + \eta_{0,j}, \end{aligned} \tag{54}$$

and $p_s, r_s, q_j, b_s, \xi_{i0}$ and $\eta_{0,j}$ are arbitrary complex constants.

To construct periodic solutions given in Theorem 1, we first take the variable transformations

$$x_{-1} = \kappa t, x_1 = x, x_2 = -iy, x_3 = -4t, \tag{55}$$

and then the tau function defined in (53) can be rewritten as

$$\tau_n(x, y, t) = \prod_{s=1}^N (p_s + r_s) e^{\zeta_s} \det_{1 \leq i, j \leq N} (\hat{m}_{ij}^{(n)}), \tag{56}$$

where

$$\hat{m}_{sj}^{(n)} = \tilde{b}_s \delta_{sj} e^{-\zeta_s} \frac{1}{p_s + r_s} + \frac{1}{p_s + q_j} \left(-\frac{p_s}{q_j}\right)^n, \tag{57}$$

with

$$\begin{aligned} \zeta_s &= \xi_s + \eta_s \\ &= (p_s + q_s)x - i(p_s^2 - q_s^2)y \\ &\quad + \left(\frac{\kappa}{p_s} + \frac{\kappa}{q_s} - 4p_s^3 - 4q_s^3\right)t + \zeta_{0,s}, \end{aligned} \tag{58}$$

In what follows, we consider $(2M + 1) \times (2M + 1)$ matrix for τ_n (i.e., $N = 2M + 1$ in (53)) and take the parameters satisfying the following constraint conditions

$$\begin{aligned} r_j &= -p_j + 1, p_{M+s} = -p_s, q_{M+s} = -q_s, q_s \\ &= p_s^*, q_{2M+1} = -p_{2M+1}^*, r_j = -p_j + 1, \\ \tilde{b}_j &= b_j, b_{M+s} = -b_s^*, \end{aligned} \tag{59}$$

$\tilde{b}_{2M+1} = ib_{2M+1}, \zeta_{0,M+s} = \zeta_{0,s}$, for $j = 1, 2, \dots, 2M$ and $s = 1, 2, \dots, M$. Since

$$\begin{aligned} \zeta_s &= \xi_s + \eta_s \\ &= (p_s + p_s^*)x - i(p_s^2 - p_s^{*2})y \\ &\quad + \left(\frac{\kappa}{p_s} + \frac{\kappa}{p_s^*} - 4p_s^3 - 4p_s^{*3} \right) t + \zeta_{0,s}, \end{aligned}$$

$$\begin{aligned} \zeta_{2M+1} &= \xi_{2M+1} + \eta_{2M+1} \\ &= (p_{2M+1} - p_{2M+1}^*)x \\ &\quad - i(p_{2M+1}^2 - p_{2M+1}^{*2})y \\ &\quad + \left(\frac{\kappa}{p_{2M+1}} - \frac{\kappa}{p_{2M+1}^*} \right. \\ &\quad \left. - 4p_{2M+1}^3 + 4p_{2M+1}^{*3} \right) t + \zeta_{0,2M+1}, \end{aligned} \tag{60}$$

thus

$$\begin{aligned} \zeta_{M+s}^*(-x, y, -t) &= \zeta_s(x, y, t), \zeta_{2M+1}^*(-x, y, -t) \\ &= \zeta_{2M+1}(x, y, t), \end{aligned} \tag{61}$$

and further obtain

$$\begin{aligned} \widehat{m}_{M+s,j}^{*(n)}(-x, y, -t) &= -b_s^* \delta_{M+s,j} e^{-\zeta_s} \\ &\quad - \frac{1}{p_s^* + p_{M+j}} \left(-\frac{p_{M+j}}{p_s^*} \right)^{-n} \\ &= -\widehat{m}_{s,M+j}^{(-n)}(x, y, t), \end{aligned} \tag{62}$$

similarly, the following conditions can also be derived

$$\begin{aligned} \widehat{m}_{s,M+j}^{*(n)}(-x, y, -t) &= -\widehat{m}_{M+s,j}^{(-n)}(x, y, t), \\ \widehat{m}_{M+s,M+j}^{*(n)}(-x, y, -t) &= -\widehat{m}_{j,s}^{(-n)}(x, y, t), \\ \widehat{m}_{2M+1,2M+1}^{*(n)}(-x, y, -t) &= -\widehat{m}_{2M+1,2M+1}^{(-n)}(x, y, t), \\ \widehat{m}_{2M+1,j}^{*(n)}(-x, y, -t) &= -\widehat{m}_{j,2M+1}^{(-n)}(x, y, t), \tag{63} \\ \widehat{m}_{2M+1,M+j}^{*(n)}(-x, y, -t) &= -\widehat{m}_{j,2M+1}^{(-n)}(x, y, t), \\ \widehat{m}_{s,2M+1}^{*(n)}(-x, y, -t) &= -\widehat{m}_{2M+1,M+s}^{(-n)}(x, y, t), \\ \widehat{m}_{M+s,2M+1}^{*(n)}(-x, y, -t) &= -\widehat{m}_{2M+1,s}^{(-n)}(x, y, t), \end{aligned}$$

which results in the following nonlocal symmetry condition:

$$\tau_n^*(-x, y, -t) = (-1)^{3N} \tau_{-n}(x, y, t). \tag{64}$$

By defining

$$f(x, y, t) = \tau_0(x, y, t), g(x, y, t) = \tau_1(x, y, t), \tag{65}$$

then solutions to the nonlocal MK equation given in Theorem 1 would be obtained that completes the proof for Theorem 1.

Finally, we give the proof for Theorem 2. To derive more general periodic solutions to the nonlocal Mel'nikov equation (14), we choose different variable transformations

$$x_{-1} = -ikt, x_1 = ix, x_2 = iy, x_3 = 4it. \tag{66}$$

The tau function τ_n is rewritten as

$$\tau_n = \prod_{s=1}^N (p_s + r_s) e^{\xi_s + \eta_s} \det_{1 \leq s, j \leq N} (\overline{m}_{s,j}^{(n)}), \tag{67}$$

where the matrix elements $\overline{m}_{s,j}^{(n)}$ given by (67) become the following formula

$$\begin{aligned} \overline{m}_{s,j}^{(n)} &= \frac{\tilde{b}_s \delta_{sj}}{(p_s + r_s) e^{\xi_s + \eta_j}} + \frac{1}{p_s + q_j} \left(-\frac{p_s}{q_j} \right)^n, \\ \xi_s &= ip_s x + ip_s^2 y + \left(4ip_s^3 - \frac{i\kappa}{p_s} \right) t + \xi_{0,s}, \\ \eta_j &= iq_j x - iq_j^2 y + \left(4iq_j^3 - \frac{i\kappa}{q_j} \right) t + \eta_{0,j}. \end{aligned} \tag{68}$$

Under the parameters satisfying the following constraint conditions

$$\tilde{b}_{sj} = 1, r_j = q_j, q_j = p_j^*, \tag{69}$$

and $\zeta_{0,s}$ are real for $j = 1, 2, \dots, N$, then

$$\begin{aligned} \zeta_s &= \xi_s + \eta_s \\ &= i(p_s + p_s^*)x + i(p_s^2 - p_s^{*2})y \\ &\quad + i \left(4p_s^3 + 4p_s^{*3} - \frac{\kappa}{p_s} - \frac{\kappa}{p_s^*} \right) t + \zeta_{0,s}, \end{aligned} \tag{70}$$

and one can derive the following condition

$$\zeta_s^*(-x, y, -t) = \zeta_s(x, y, t), \tag{71}$$

which can further yield

$$\begin{aligned} m_{s,j}^{*(n)}(-x, y, -t) &= m_{j,s}^{(-n)}(x, y, t), \tau_n^*(-x, y, -t) \\ &= \tau_{-n}(x, y, t). \end{aligned} \tag{72}$$

Again, defining $f = \tau_0, g = \tau_1$ and taking

$$p_s = \frac{\omega_s}{2} + i\lambda_s, q_s = \frac{\omega_s}{2} - i\lambda_s, \tag{73}$$

the periodic solutions for the nonlocal MK equation given by Theorem 2 are obtained that completes Theorem 2.

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