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Opinion dynamics with the increasing peer pressure and prejudice on the signed graph

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Abstract In this paper, we propose the opinion dynamics model with the increasing peer pressure and the stubborn agents. Cooperation and competition between individuals are considered simultaneously in a social network. Similar to the signed DeGroot model, we adopt a weighted average update rule in our model. We derive conditions under which opinions converge to a fixed opinion distribution. In particular, we find conditions under which opinions reach consensus or polarization (bipartite consensus). Two examples are provided to illustrate the effectiveness of the obtained results.

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1 Introduction

Social networks are constituted by social agents (individuals and communities) and social relations (friendship or competition) among them [1]. With the rapid development of computer technology, recent decades have witnessed the explosive development of online social media. Nowadays, social networks, no matter offline or online, are important media for information diffusion [2]. Hence, social networks have attracted extensive attention from natural, engineering and social sciences [3,4]. Studies of social networks are mainly about information propagation [5], social learning [6], opinion formation [7], opinion dynamics [8] and so on. Among them, opinion dynamics focuses on the basic problem of how individuals are influenced by the presence of others in a social group, or how individuals in a social network interact and exchange their opinions about a topic or many topics. As one of the foundational problem in sociology, opinion dynamics has drawn considerable attention among the investigations of social networks [9–12].

Here, the term "opinion" is broadly referred to individuals' displayed cognitive orientations to objects (e.g., topics or issues); it includes the displayed attitudes (signed orientations) and beliefs (subjective certainties) [13]. Unlike many multi-agent systems in which agents collaborate to achieve a common goal and it is easier to reach consensus, for opinion dynamics systems, opinions of social actors often disagree, and clusters [14,15] and polarization (or bipartite consensus) [16–19] usually occur due to the differences in backgrounds such as politics, economy, culture and beliefs among individuals. Here, clusters mean the limiting value to which the agent opinions converge. With some abuse of terminology, it is also referred to the set of agents whose opinions converge to the same value as a cluster. Polarization is a special kind of clusters. It means that the social network separates into two clusters of opposing opinions. Obviously, it is significant and challenging to develop opinion dynamics models which admit mathematically rigorous analysis and can capture these main properties of the real social networks [20–22].

In order to explain clusters and polarization phenomena, the idea of "homophily" or "biased assimilation" [23] often is adopted to mathematically model the evolution of opinions. The term "homophily" means that agents readily adopt opinions of like-minded individuals but accept the more deviant opinions with discretion. The mainstream model represented by this idea is the bounded confidence (BC) model [23], where agents do not completely consider opinions outside their confidence intervals. To a certain extent, BC models can capture the characteristic of opinion evolution, so BC models have attracted significant attention, for example, homogeneous BC models [24–26], heterogeneous BC models [27,28], multidimensional BC models [29]. These BC models all display the clustering characteristic of opinions. However, their rigorous mathematical analysis is an open problem. For instance, it is very difficult to predict the structure of opinion clusters for a given initial condition [13].

Another possible explanation of opinion disagreement is agent's "innate beliefs" or "prejudices" [5,30], which means that agents with prejudices have the will to maintain their initial opinions (prejudices). Recently, empirical work [31,32] has shown that the existence of innate beliefs is often hidden but it can continuously influence the evolution of opinions. Such a class social actor is called the stubborn agents and the model associated with it is generally called the Friedkin–Johnsen (FJ) model [5,30]. Unlike the DeGroot model [33], where each actor is completely open to interpersonal influence and updates its opinion based on the convex hull spanned by opinions of himself and neighbors; in the FJ model, the stubborn agents always consider their prejudices for every iteration of opinions. It implies that it is difficult to affect or alter the stubborn agents' opinion, even completely impossible. So, in the FJ model, it is difficult for social group to reach opinion agreement, i.e., agents often form multiple clusters. In special cases, the FJ model also can be understood by the game-theoretic model [5,34] and Leontier economic model [35]. In [36], the stability problem for the FJ model was investigated and a sufficient condition of stability was derived. Furthermore, authors studied the convergence speed of the FJ model in [34]. In [13], the convergence problem for multidimensional FJ model was investigated. Due to the existence of the stubborn agents, the FJ model cannot achieve consensus in general. So an interesting question is under what circumstances the FJ model can derive less clusters even a cluster, i.e., consensus. The empirical evidence [37] showed that the FJ model may reach an agreement over a sequence of issues, and authors of the literature [38] explained this point theoretically. Authors argued in [39] that when considering the peer pressure between individuals, the modified FJ model also can obtain consensus if the peer pressure is increasing and unbounded.

Apart from homophily and prejudices, competition or confrontation between social actors is also a source of clusters and polarization. In the real world, antagonism, competition, indifference or distrust between individuals and their groups are ubiquitous; for example, the British scientists' Charles Darwin's "natural selection" emphasizes the competition between individuals [40,41]. Competition, confrontation and distrust are usually modeled by repulsive couplings or negative ties [42] among the agents, i.e., the signed graph, where the positive edges represent friendly and cooperative interactions and the negative edges correspond to the antagonistic counterpart. Recently, opinion dynamics on the signed graphs has attracted significant attention [42–46]. Due to the antagonistic interaction, the evolution of opinions on the signed graphs is more complicated. In fact, as shown in [17], opinion dynamics with cooperative and competitive interactions may result in clusters, polarity, consensus or neutrality under different opinion protocols. Furthermore, in [18], a necessary and sufficient condition of bipartite consensus was obtained, which depends on a gauge transformation. It implies that the polarization of opinions and the structural balance graph [18] are closely related. In [47], the convergence of the DeGroot model was studied on the signed graph. In [48], the bipartite consensus problem for the high-order opinion dynamic system was investigated. In [41,45], opinion dynamics with switching topology was researched on the signed graphs. Considering some special contexts where people may be more concerned with the signs of opinions and ignore their size, the sign-consensus problem for the signed networks was investigated in [49–51]. In particular, in [49], the influence of stubbornness and competition on opinion signs was investigated for continuous-time opinion dynamics model and sing-consensus conditions were obtained, which depend on the eventually positive matrix [52] and are independent of the structural balance.

Although "homophily," "stubbornness" and "competition" may lead to clusters and polarization, most of the aforementioned models of opinion dynamics focus on one factor of them. In fact, the empirical work showed that these factors may be coexisting in real social networks [5,8]. Besides, the peer pressure is also ubiquitous in the real social networks. For example, purchasing behaviors [53], health behaviors [54] and beliefs and cultural norms [55] are all linked to the peer pressure. Furthermore, authors of the literature [39] found that the peer pressure has a positive effect on agreement, and enormous pressure is more likely to cause opinion consensus. Therefore, in this paper, we will consider various factors ("stubbornness," "competition" and "peer pressure") to model opinion dynamics so that our model can capture the clustering property of the real social network and reveal a richer mechanism of opinion evolution. Furthermore, we also want to observe whether huge peer pressure still results in consensus under the case, which stubbornness and competition are considered in social networks.

The main contributions of this paper are as follow: Firstly, a novel opinion dynamics model, where "stubbornness," "competition" and "peer pressure" are considered simultaneously, is proposed. Then, by using the contraction maps, we obtain conditions under which opinions converge to a fixed opinion distribution. These fixed opinions in general correspond to multiple clusters. In particular, we find that in special case, the peer pressure still results in consensus or bipartite consensus in our model. In other words, we obtain conditions under which our model will form fewer clusters, i.e., polarization or consensus.

The rest of the paper is organized as follows: basic definitions and properties of graphs and models are recalled in Sect. 2; the convergence of our model is discussed in Sect. 3; in Sect. 4, we give two examples to illustrate the effectiveness of the obtained results. Finally, in Sect. 5, we give our conclusion.

Notations Throughout this paper, $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n$ denote, respectively, the $m \times n$ real matrix and the n-dimensional real space. Suppose $A \in \mathbb{R}^{m\times n}$ or $A \in \mathbb{R}^n$, $A \geq 0$ means all elements of $A \geq 0$ are not less than 0, and A^{T} denotes its transpose. $\mathbf{1}_n$ denotes the n-dimensional vector $[1, 1, \ldots, 1]^{\mathrm{T}}$. We use diag$[m_1, m_2, \ldots, m_n]$ to denote the diagonal matrix whose diagonal elements are $m_1, m_2, \ldots, m_n$. The notation $[-1, 1]^n$ denotes the set $\{x|x_i \in [-1, 1], x \in \mathbb{R}^n\}$.

2 Model and preliminaries

Consider a set of n nodes denoted by $V = \{1, 2, \ldots, n\}$ and the subset $E \subset V \times V$, $G = (V, E)$ is called a digraph with the set of nodes (or vertices) V and the set of edges E. A path from a vertex i to another vertex j is a sequence of distinct vertices starting with i and ending with j, in which each vertex is adjacent to its next vertex. We say that the digraph $G = (V, E)$ contains a spanning tree if there is a vertex i such that there exists a path from i to every other vertex in $G = (V, E)$ where the node i is called a root node. Furthermore, if each node is a root node, then a digraph $G = (V, E)$ is said to be strongly connected. The neighbor set of the vertex i is defined by $\mathbb{N}_i = \{j \in V|(j, i) \in E\}$. We say that $G = (V, E)$ is an undirected graph if $\forall j \in \mathbb{N}_i$ means $i \in \mathbb{N}_j$. For an undirected graph, the strong connectivity means connectivity. Suppose a matrix $A \in \mathbb{R}^{n\times n}$ satisfies: $a_{ij} \neq 0 \Longleftrightarrow (j, i) \in E$, then matrix A is called the weighted adjacency matrix. In this paper, we assume $a_{ii} > 0$ for all $i \in V$, i.e., each vertex has a self-loops. If the adjacency matrix A is assumed both positive and negative values, then it is called the signed adjacency matrix and its associate graph is called the signed graph $G(A)$. For the signed adjacency matrix A, its Laplacian matrix $L = D - A$ where $D =$diag$[d_1, d_2, \ldots, d_n]$ and $d_i = \sum_{j=1}^{n} |a_{ij}|$ for all $i \in V$ [56].

The state of agent $i \in V$ at k is a continuous value $x_i(k) \in [-1, 1]$ that represents the disclosed opinion or position on a topic. The constant prejudice $x_i^+ \in [-1, 1]$ represents agent i inherent bias which may differ from the opinion disclosed to the public.

$s_i \geq 0$ models the tendency of agent to maintain its prejudice x_i^+ in public. If $s_i = 0$, i.e., agent i has no willingness to maintain prejudice which is called a non-stubborn agent; otherwise, the agent is called a stubborn agent and s_i is called stubbornness. Let $V_1 = \{i|s_i > 0, i \in V\}$ denote the set of all stubborn agents and $V_2 = \{i|s_i = 0, i \in V\}$ denote the set of all non-stubborn agents. As pointed out in [18], competition, antagonism or distrust between stubborn agents and non-stubborn agents are ubiquitous, which are usually modeled by repulsive couplings or negative ties among the agents. Meanwhile, the relationship between stubborn agents (or non-stubborn agents) is cooperative and friendly, which can be usually modeled by positive ties. It means that $a_{ij} \geq 0$ for $\forall i, j \in V_l (l \in \{1, 2\})$ and $a_{ij} \leq 0$ for $\forall i \in V_p, j \in V_q, p \neq q, (p, q \in \{1, 2\})$. In other words, in this paper, the signed graph $G(A)$ is structural balance [18]. We also say that the weighted adjacency matrix A is structural balance. The vector $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^{\mathrm{T}}$ denotes the set of all disclosed opinions, while the set of constant private prejudice is $x^+ = [x_1^+, x_2^+, \ldots, x_n^+]^{\mathrm{T}}$. For convenience, we refer to publicly disclosed opinions simply as opinions in the remainder of this paper.

Inspired by the literature [5,39], in this paper, we adopt the following social stress function.

$$\Theta_i(x_i(k), x(k-1), k) = s_i(x_i(k) - x_i^+)^2 + \rho(k) \sum_{i=1}^{n} |a_{ij}|(x_i(k) - \mathrm{sgn}(a_{ij})x_j(k-1))^2, \quad (1)$$

where $\rho(k) > 0$ is the peer pressure coefficient which is also call the peer pressure. $\mathrm{sgn}(\cdot)$ is a sign function defined as follows:

$$\mathrm{sgn}(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ -1 & \text{if } z < 0. \end{cases} \quad (2)$$

The state of agent i is updated by minimizing its social stress. In this paper, we assume that $\rho(k)$ is an increasing function of k. Intuitively, we should assume that each agent experiences a distinct peer pressure effect. But, for simplicity, we can assume agents are exposed to the same degree of pressure. As noted in [39], the reason is as follows: their adoption rate depends on their stubbornness (s_i) and the relative weights of influence by their neighbors (a_{ij}). It implies that in fact each agent has a distinct peer pressure. According to [5], under these assumptions, the necessary conditions are sufficient for minimizing $\Theta_i(x_i(k), x(k-1), k)$. So it is easy to obtain the optimal state $x_i(k)$ for agent i at k:

$$x_i(k) = \frac{s_i x_i^+ + \rho(k) \sum_{j=1}^{n} a_{ij} x_j(k-1)}{s_i + \rho(k) d_i}, \quad (3)$$

where $k \in \mathbf{Z} = \{1, 2, \ldots\}$. Let $S = \mathrm{diag}[s_1, s_2, \ldots, s_n]$ and $X^+ = \mathrm{diag}[x_1^+, s_2^+, \ldots, s_n^+]$; then, the model (3) can be rewritten in the following compact form

$$x(k) = (S + \rho(k)D)^{-1}(Sx^+ + \rho(k)Ax(k-1)). \quad (4)$$

Remark 1 Empirical and experimental science shows that individuals always consider their own and neighbors' opinions when making decisions. So, in this paper, we suppose $a_{ii} > 0$ for all $i \in V$. It implies $s_i + \rho(k)d_i > 0$, i.e., (3) can be well-defined. In fact, when some $a_{ii} = 0$, (3) can still be established; we only need to assume $G(A)$ is connected.

Remark 2 Recently, opinion dynamics with stubborn agents has been investigated in [13,38,57,58]. It should be pointed out that in [13,38,57,58], only the cooperation between individuals was considered, and the competition and distrust between individuals were neglected. In other words, all elements of the weighted adjacency A are assumed to be nonnegative. Meanwhile, opinion dynamics on signed graphs has been studied in [17,18,42–45,50,51]. However, in these papers, it was implicitly assumed that each actor was completely open to interpersonal influence, i.e., they did not consider the stubborn behavior of social actors. Besides, peer pressure between individuals was not considered in the literature mentioned above. In [39], authors investigated how opinions evolve under the existence of stubborn agents and peer pressure. But they did not consider the influence of competition and distrust between individuals for opinion evolution. However, in our model, in the presence of stubborn agents, we simultaneously consider the possible cooperation and competition between individuals and the agreement pressure between individuals. Hence, our model may be more general than most of opinion dynamics models mentioned above and can reveal a richer mechanism of opinion evolution.

In order to derive our main results, we also need the following definitions, assumptions and lemmas

Assumption 1 The signed digraph $G(A)$ contains a spanning tree and there is at least one root node which is a stubborn agent.

Definition 1 If $\lim_{k\to\infty}|x_i(k)| = \alpha > 0$ for all $i \in V$, and there exist i and j satisfying that $\lim_{k\to\infty} x_i(k) = -\lim_{k\to\infty} x_j(k)$, then we claim that system (3) can achieve bipartite consensus or polarization. In particular, if $\lim_{k\to\infty} x_i(k) = \alpha$ for all $i \in V$, we say that system (3) reaches consensus.

Lemma 1 ([59]) *Assume* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{A} \geq 0$. *If* $G(\mathbf{A})$ *contains a spanning tree, then the Laplacian matrix* $\mathbf{L}$ *of matrix* $\mathbf{A}$ *has a simple eigenvalue at 0.*

Lemma 2 ([60]) *Assume* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{A} \geq 0$ *is a stochastic matrix and its all diagonal elements* $a_{ii} > 0$. *If* $G(\mathbf{A})$ *contains a spanning tree, then* $\mathbf{A}$ *has an eigenvalue* $\lambda = 1$ *with algebraic multiplicity equal to one and all the other eigenvalues satisfy* $|\lambda| < 1$.

3 Main results

In this section, we will consider the convergence of system (4). Firstly, we will consider opinions $x(k)$ as a sequence of contraction maps. We will show that every map has the unique fixed point. Then, we will show that all these unique fixed points converge to the constant c. Finally, we will obtain that $x(k)$ also converges to this constant c. For this purpose, we also need the following lemmas.

Lemma 3 *Let* $\Gamma = diag[\sigma_1, \sigma_2, \ldots \sigma_n]$ *where* σ_i *is defined as follows:*

$$\sigma_i = \begin{cases} 1 & \text{if the } i\text{th agent is a stubborn agent} \\ -1 & \text{otherwise,} \end{cases} \tag{5}$$

then, for system (4), *we have* $\Gamma A\Gamma \geq 0$ *and* $\Gamma S = S$.

Proof Let $\mathbf{E} = (e_{ij}) = \Gamma A\Gamma$. Through simple calculations, one can obtain that $e_{ij} = \sigma_i\sigma_j a_{ij}$. If $i \in V_1$ ($j \in V_1$) and $j \in V_2$ ($i \in V_2$), then $a_{ij} \leq 0$. According to the definition of σ_i, it implies $e_{ij} \geq 0$. Similarly, one can easily obtain that $e_{ij} \geq 0$ for $\forall i, j \in V_1(V_2)$. It shows that $\Gamma A\Gamma \geq 0$. Note that $s_i = 0$ and $\sigma_i = -1$ when the agent i is a non-stubborn agent, when the agent i is a stubborn agent, $s_i > 0$ and $\sigma_i = 1$. Hence, we have $\Gamma S = S$. In this paper, the diagonal matrix Γ will play a key role in deriving our main results. □

Lemma 4 *Suppose Assumption 1 holds, then the matrix* $(S + \rho(k)L)$ *is invertible for system* (4) *where* $L = D - A$.

Proof According to Lemma 3, the gauge transformation Γ satisfies $\Gamma A\Gamma \geq 0$. Let

$$\begin{aligned} B &= \Gamma(S + \rho(k)L)\Gamma \\ &= S + \rho(k)(D - \Gamma A\Gamma). \end{aligned} \tag{6}$$

Noting that $\Gamma^{-1} = \Gamma$, we only need to prove that B is invertible. Let

$$C = \begin{bmatrix} 0 & 0 \\ S\mathbf{1}_n & \rho(k)\Gamma A\Gamma \end{bmatrix}. \tag{7}$$

Noting that $S\mathbf{1}_n \geq 0$ and $\Gamma A\Gamma \geq 0$, so $C \geq 0$. According to Assumption 1, there is at least one stubborn root node in $G(A)$ which is assumed to be agent i. So $s_i > 0$. It is obvious that the graphs $G(A)$ and $G(\rho(k)\Gamma A\Gamma)$ are exactly the same; hence, the graph $G(C)$ contains a spanning tree. According to the matrix C, one can easily obtain the Laplacian matrix L_c of it.

$$L_c = \begin{bmatrix} 0 & 0 \\ -S\mathbf{1}_n & B \end{bmatrix}. \tag{8}$$

According to Lemma 1, L_c has a simple eigenvalue at 0. So all eigenvalues of the matrix B are not 0, i.e., the matrix B is invertible. This completes the proof of Lemma 4. □

Next, we will establish a relationship between opinions $x(k)$ and contraction maps. Suppose $f_k(x) = (S + \rho(k)D)^{-1}(Sx^+ + \rho(k)Ax)$, $F_k = f_k \circ f_{k-1} \circ \ldots \circ f_1$. Obviously, we have $x(k) = f_k(x(k-1))$ and $x(k) = F_k(x(0))$. Furthermore, we have the following lemma.

Lemma 5 [39] *If* f_k *is an analytic contraction function in a domain with* $f_k(\mathfrak{D}) \subseteq \mathfrak{D}$ *for all* $k \in \mathbf{Z}$, *then* $F = \lim_{k\to\infty} F_k$ *is a constant function c, i.e.,* $F(x) = c$ *for all* $x \in \mathfrak{D}$ *and the fixed point* $\overline{x}(k)$ *of* f_k *also converges to the constant c. It implies that* $\lim_{k\to\infty} x(k) = \lim_{k\to\infty} \overline{x}(k)$.

Next, we will explain that f_k is a contraction map on the domain $[-1, 1]^n$.

Lemma 6 *Suppose Assumption 1 holds, then, for all* $k \in \mathbf{Z}$, f_k *is a contraction map and its unique fixed point* $\overline{x}(k) = (S + \rho(k)L)^{-1}Sx^+$.

Proof By adding a row and a column to $\rho(k)(S+\rho(k)D)^{-1}A$, we construct the following auxiliary matrix

$$\mathbf{B}=\begin{bmatrix}\rho(k)(S+\rho(k)D)^{-1}A & (S+\rho(k)D)^{-1}S\mathbf{1}_n\\ 0 & 1\end{bmatrix}. \tag{9}$$

Let

$$\mathbf{C}=\begin{bmatrix}\Gamma & 0\\ 0 & 1\end{bmatrix}, \tag{10}$$

then we have

$$\mathbf{CBC}=\begin{bmatrix}\rho(k)(S+\rho(k)D)^{-1}\Gamma A\Gamma & (S+\rho(k)D)^{-1}\Gamma S\mathbf{1}_n\\ 0 & 1\end{bmatrix}. \tag{11}$$

According to Lemma 3, $\Gamma A\Gamma \geq 0$ and $\Gamma S = S \geq 0$, so we have $\mathbf{CBC} \geq 0$. Furthermore,

$$\begin{aligned}&[\rho(k)(S+\rho(k)D)^{-1}\Gamma A\Gamma,\\ &\quad (S+\rho(k)D)^{-1}\Gamma S\mathbf{1}_n]\mathbf{1}_{n+1}\\ &=\rho(k)(S+\rho(k)D)^{-1}\Gamma A\Gamma\mathbf{1}_n\\ &\quad+(S+\rho(k)D)^{-1}\Gamma S\mathbf{1}_n\\ &=\rho(k)(S+\rho(k)D)^{-1}D\mathbf{1}_n\\ &\quad+(S+\rho(k)D)^{-1}S\mathbf{1}_n\\ &=\mathbf{1}_n.\end{aligned} \tag{12}$$

Hence, **CBC** is a stochastic matrix. According to Assumption 1, $G(A)$ contains a spanning tree, and there is at least one root node which is a stubborn agent. Similar to Lemma 4, one can know that $G(\mathbf{CBC})$ has a spanning tree. Note that all diagonal elements of matrix **CBC** are greater than 0; according to Lemma 2, **CBC** has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda| < 1$. Hence, all eigenvalues λ_i of matrix $\rho(k)(S+\rho(k)D)^{-1}\Gamma A\Gamma$ satisfy $|\lambda_i| < 1$. It implies that all eigenvalues λ_i of matrix $\rho(k)(S+\rho(k)D)^{-1}A$ satisfy $|\lambda_i| < 1$. Therefore, $\lim_{i\to\infty}(\rho(k)(S+\rho(k)D)^{-1}A)^i = 0$. Equivalently, if $\|\cdot\|$ denotes the matrix operator norm, then $\|\rho(k)(S+\rho(k)D)^{-1}A\| < 1$. For any $x, y \in [-1,1]^n$, one has that

$$\begin{aligned}&\|f_k(x)-f_k(y)\|\\ &=\|\rho(k)(S+\rho(k)D)^{-1}A(x-y)\|\\ &\leq\|\rho(k)(S+\rho(k)D)^{-1}A\|\|(x-y)\|\\ &\leq\|(x-y)\|.\end{aligned} \tag{13}$$

It implies that f_k is a contraction map on a compact set. According to Banach fixed-point theorem, f_k has an unique fixed point. Suppose $\overline{x}(k)$ is the unique fixed point of f_k, then $\overline{x}(k) = f_k(\overline{x}(k))$. So

$$\begin{aligned}&\overline{x}(k)=(S+\rho(k)D)^{-1}(Sx^++\rho(k)A\overline{x}(k)) \\ &\Rightarrow (S+\rho(k)D)\overline{x}(k)=Sx^++\rho(k)A\overline{x}(k)\\ &\Rightarrow (S+\rho(k)L)\overline{x}(k)=Sx^+\\ &\Rightarrow \overline{x}(k)=(S+\rho(k)L)^{-1}Sx^+.\end{aligned} \tag{14}$$

This completes the proof. □

Combining Lemma 5 with Lemma 6, one can easily obtain the following result.

Theorem 1 *For system (4), if Assumption 1 is true and $\rho(k)$ is increasing and bounded, then*

$$\lim_{k\to\infty} x(k)=(S+\rho^*L)^{-1}Sx^+, \tag{15}$$

where $\rho^* = \lim_{k\to\infty}\rho(k)$.

Proof Since $\rho(k)$ is increasing and bounded, according to the monotone convergence principle, it converges to a finite constant ρ^*. From Lemma 4, $(S+\rho^*L)$ is well-defined and invertible. Furthermore, we know that matrix inversion is continuous. According to Lemmas 5 and 6, we can obtain

$$\begin{aligned}\lim_{k\to\infty} x(k)&=\lim_{k\to\infty}\overline{x}(k)\\ &=\lim_{k\to\infty}(S+\rho(k)L)^{-1}Sx^+\\ &=(S+\lim_{k\to\infty}\rho(k)L)^{-1}Sx^+\\ &=(S+\rho^*L)^{-1}Sx^+.\end{aligned} \tag{16}$$

This completes the proof of Theorem 1. □

In Theorem 1, we assume that $G(A)$ is a directed signed graph. If $G(A)$ is an undirected graph, we can obtain the following corollary.

Corollary 1 *For system (4), suppose $G(A)$ is an undirected signed graph and contains a spanning tree. If $\rho(k)$ is increasing and bounded, then*

$$\lim_{k\to\infty} x(k) = (S + \rho^* L)^{-1} S x^+, \tag{17}$$

where $\rho^* = \lim_{k\to\infty} \rho(k)$.

Proof Since $G(A)$ is an undirected graph and has a spanning tree, $G(A)$ must be connected. It implies that each node in $G(A)$ is the root node. So, Lemmas 4–6 hold. According to Theorem 1, the conclusion is obviously established. □

According to (15) and (17), if $\rho(k)$ is bounded, then system (4) converges to multiple clusters in general. The following theorem shows that if $\rho(k)$ is unbounded, then system (4) forms two clusters at most. It implies that increasing the peer pressure can weaken the opinion differences and reduce the number of clusters.

Theorem 2 *For system* (4)*, assume* $A = A^{\mathrm{T}}$ *and the signed graph* $G(A)$ *is connected. If* $\rho(k)$ *is increasing and unbounded, then*

$$\lim_{k\to\infty} x(k) = \frac{\sum_{i\in V_1} s_i x_i^+}{\sum_{i\in V_1} s_i}\Gamma. \tag{18}$$

It implies that if $\sum_{i\in V_1} s_i x_i^+ \neq 0$*, system* (4) *will achieve bipartite consensus; otherwise, system* (4) *will reach consensus.*

Remark 3 Theorem 2 shows that in the case of increasing and unbounded peer pressure, opinion absolute value of all the agents always converges to the constant $|\frac{\sum_{i\in V_1} s_i x_i^+}{\sum_{i\in V_1} s_i}|$ which depends on the average of their the prejudices weighted by stubbornness of the stubborn agents. It is irrespective of the non-stubborn agents and the weighting of the edges in the network, so long as $A = A^{\mathrm{T}}$ and $G(A)$ contains a spanning tree.

Remark 4 The literature [13,38,57] presented that when considering the stubborn agents opinions usually form multiple clusters. However, Theorem 2 reveals that as the peer pressure is increasing and unbounded, the considered social groups will form two clusters at most. Corollary 2 also illustrates this point. This shows that the peer pressure may make opinions to reach an agreement in the presence of stubbornness and competition.

Remark 5 It is well-known that for BC models, we can only explain its convergence, and it is difficult to predict the value of the final opinion [24–29]. But, for our model, we give the value of the final opinion in Theorems 1 and 2. It implies that it is easier to mathematically analyze our model than BC models. Furthermore, our model also considers more factors to explain clustering phenomenon. These show that our model not only captures the main features of opinion evolution but also allows mathematically rigorous analysis, which is in line with the requirement of mathematical modeling.

Proof Firstly, in order to proof Theorem 2, we first give a lemma.

Lemma 7 *Assume* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{A} = \mathbf{A}^{\mathrm{T}} \geq 0$*. The graph* $G(\mathbf{A})$ *is connected and the diagonal matrix* $\mathbf{S} = diag[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n] \geq 0$ *and* $\mathbf{S} \neq 0$*. If* $\rho(k)$ *is increasing and unbounded, then*

$$\lim_{k\to\infty} (\mathbf{S} + \rho(k)\mathbf{L})^{-1} = \frac{1}{\sum_{i=1}^n \mathbf{s}_i}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}, \tag{19}$$

where $\mathbf{L}$ *is the Laplacian matrix of the nonnegative matrix* $\mathbf{A}$.

Lemma 7 is a direct extension of Theorem 1 of the literature [39]. The specific proof is omitted here.

According to the proof of Lemmas 4–6, one can easily obtain that Lemmas 4–6 still hold when $A \geq 0$ is a symmetric matrix, and the graph $G(A)$ is connected. It implies that

$$\begin{aligned}\lim_{k\to\infty} x(k) &= \lim_{k\to\infty} \overline{x}(k)\\ &= \lim_{k\to\infty} (S + \rho(k)L)^{-1} S x^+. \end{aligned}\tag{20}$$

Furthermore, according to the definition of Γ, we have

$$\begin{aligned}&\lim_{k\to\infty} (S + \rho(k)L)^{-1}\\ &= \lim_{k\to\infty} \Gamma(\Gamma S\Gamma + \rho(k)\Gamma L\Gamma)^{-1}\Gamma\\ &= \Gamma(\lim_{k\to\infty} (\Gamma S\Gamma + \rho(k)\Gamma L\Gamma)^{-1})\Gamma\\ &= \Gamma(\lim_{k\to\infty} (S + \rho(k)\Gamma L\Gamma)^{-1})\Gamma. \end{aligned}\tag{21}$$

Combining Lemma 7 with $\Gamma S = S$ and noting that $s_i = 0$ when $i \in V_2$, one can obtain

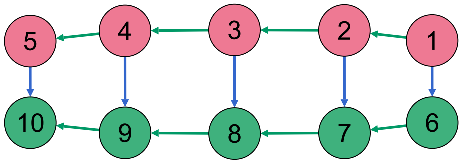

Fig. 1 Network topology in Example 1

$$\begin{aligned}\lim_{k\to\infty} x(k) &= \Gamma(\frac{1}{\sum_{i=1}^{n} s_i}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}})\Gamma S x^+ \\ &= \frac{1}{\sum_{i=1}^{n} s_i}\Gamma(\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}})S x^+ \\ &= \frac{\sum_{i=1}^{n} s_i x_i^+}{\sum_{i=1}^{n} s_i}\Gamma \\ &= \frac{\sum_{i\in V_1} s_i x_i^+}{\sum_{i\in V_1} s_i}\Gamma. \end{aligned} \tag{22}$$

So, according to (22) and the definition of Γ, when $\sum_{i\in V_1} s_i x_i^+ \neq 0$, system (4) achieves bipartite consensus and all agents are divided into two clusters, i.e., the stubborn agent set and the non-stubborn agent set. When $\sum_{i\in V_1} s_i x_i^+ = 0$, $\lim_{k\to\infty} x(k) = 0$, i.e., opinions reach consensus. □

Remark 6 By Theorems 1 and 2, one can find that for system (4), the final opinions depend on the prejudices of stubborn agents. The non-stubborn agents do not contribute to the formation of the final opinions, which means that the stubborn agents will grasp the progress of the situation. This is also in line with our intuition, because compared with the non-stubborn agents, it is more difficult to influence opinions of the stubborn agents. This is also consistent with the results of the literature [13,38,39,57,58].

In Theorems 1 and 2, graph $G(A)$ is the signed graph, i.e., some elements of A are negative . In fact, according to the proof of lemmas and theorems, our results still hold when $A \geq 0$. Hence, we have the following corollary.

Corollary 2 *For system (4), we assume $A \geq 0$ and $G(A)$ satisfies Assumption 1. If $\rho(k)$ is increasing and bounded, then*

$$\lim_{k\to\infty} x(k) = (S + \rho^* L)^{-1} S x^+, \tag{23}$$

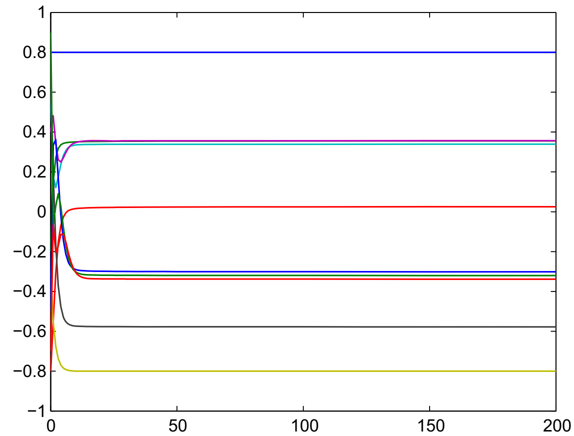

Fig. 2 Opinion evolution in Example 1 when $\rho(k) = 5 - 1/k$

where $\rho^* = \lim_{k\to\infty} \rho(k)$. In particular, if $\rho(k)$ is increasing and unbounded and $A = A^{\mathrm{T}}$, then

$$\lim_{k\to\infty} x(k) = \frac{\sum_{i=1}^{n} s_i x_i^+}{\sum_{i=1}^{n} s_i}\mathbf{1}_n. \tag{24}$$

In other words, system (4) will reach consensus.

Remark 7 Recently, opinion dynamics with the initial prejudices and peer pressure has been investigated in [39]. It is worth mentioning that it is assumed that $A = A^{\mathrm{T}} \geq 0$ in [39]. It means that $G(A)$ is an undirected graph. Under this assumption, authors have pointed out that if $\rho(k)$ is increasing and $G(A)$ is connected, then opinions will converge to the constant opinions. Note that for (23), we do not require $A = A^{\mathrm{T}}$; so Corollary 2 shows that our results further extend the findings of [39].

4 Numerical example

In the section, we will give two examples to observe the state evolution of systems in order to verify our obtained results.

Example 1 Consider a network with ten agents. The network topology is shown in Fig. 1. Note we assume that each node has a self-loop in this paper. So, for the sake of convenience, we omit these self-loops in Figs. 1, 6 and 7. The red node represents the stubborn agent, and the green node is the non-stubborn agent. According to Fig. 1, one can find that the

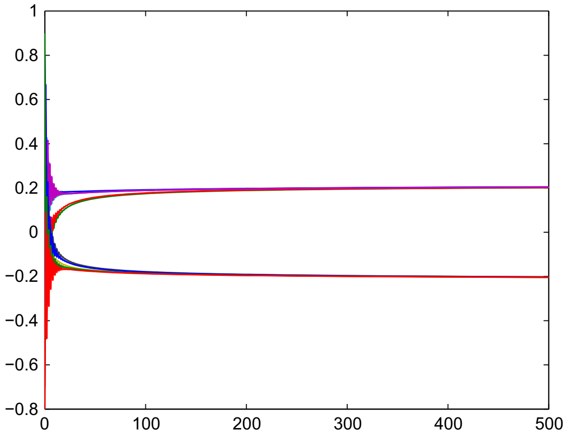

Fig. 3 Opinion evolution when $G(A)$ is assumed to be a undirected graph and $\rho(k) = k/2$ in Example 1

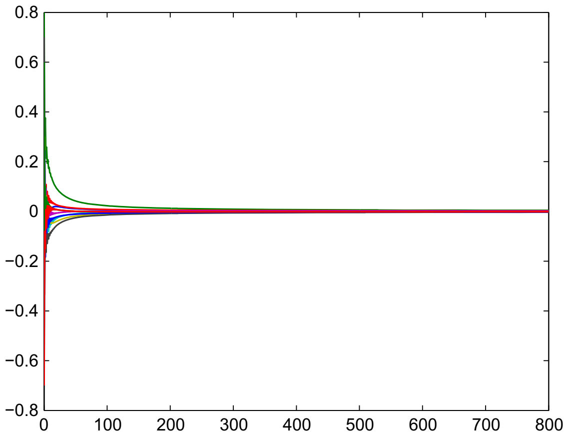

Fig. 4 System (4) reach consensus in Example 1

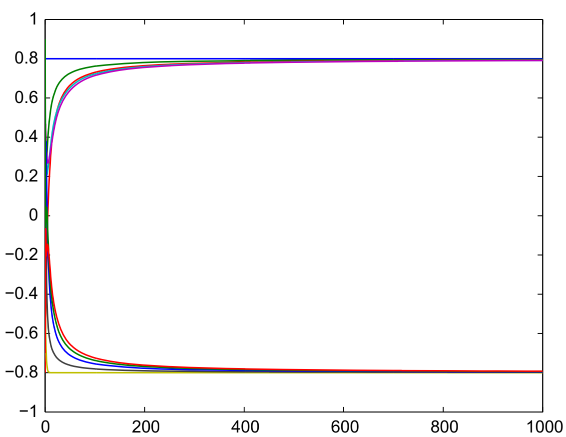

Fig. 5 Opinion evolution when $G(A)$ is a directed graph and $\rho(k) = k/2$ in Example 1

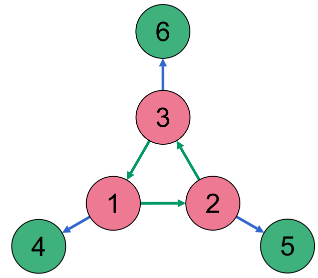

Fig. 6 Network topology of $G_1(A)$ in Example 2

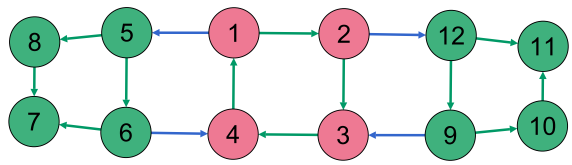

Fig. 7 Network topology of $G_2(A)$ in Example 2

graph $G(A)$ satisfies the conditions of Theorem 1. Let S =diag[2, 4, 2, 6, 2, 0, 0, 0, 0, 0], the initial preference $x^+ = [0.8, -0.2, -0.8, 0.6, 0.4, -0.3, 0.7, -0.5, 0.9, -0.7]$ and the peer pressure $\rho(k) = 5 - 1/k$. Figure 2 gives the opinion evolution of system (4). According to Fig. 2, opinions finally converge to multiple clusters. Furthermore, if we assume that $G(A)$ is an undirected graph and $\rho(k) = k/2$, then the conditions of Theorem 2 is satisfied. Note $\sum_{i=1}^{10} s_i x_i^+ = 3.6 \neq 0$, it implies that opinions will finally form two clusters, i.e., bipartite consensus. By Fig. 3, one can find this point. If we let $x^+ = [-0.5, 0.8, 0.2, -0.6, 0.5, -0.3, 0.7, -0.4, 0.6, -0.7]$, then $\sum_{i=1}^{10} s_i x_i^+ = 0$, then system (4) will reach consensus. You can see Fig. 4.

Remark 8 In fact, in Example 1, for the digraph $G(A)$ showed in Fig. 1, if we let $\rho(k) = k/2$, we have found that opinions also derive bipartite consensus. Figure 5 shows this point. Is this accidental? So we have done a lot of simulation experiments, we found that if Assumption 1 holds and $\rho(k)$ is increasing and unbounded, then system (4) will reach bipartite consensus or consensus. This is in line with the conclusion of Theorem 2. The following Example 2 further shows this point. However, unfortunately, we cannot explain it theoretically. This will be the focus of our future work. In other words, in the future work, we will try to delete the condition $A = A^{\mathrm{T}}$ in Theorem 2.

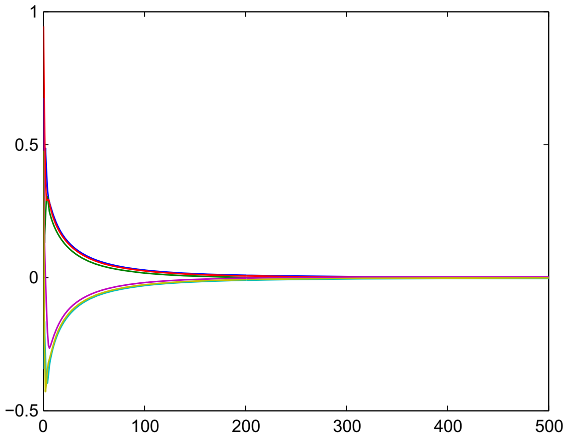

Fig. 8 Opinion evolution when $G_1(A)$ is a directed graph and $\rho(k) = \sqrt{k}$ in Example 2

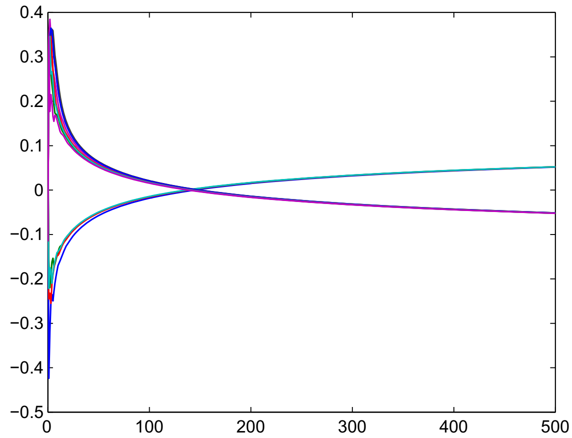

Fig. 11 Opinion evolution when $G_2(A)$ is a directed graph and $\rho(k) = \sqrt{k}$ in Example 2

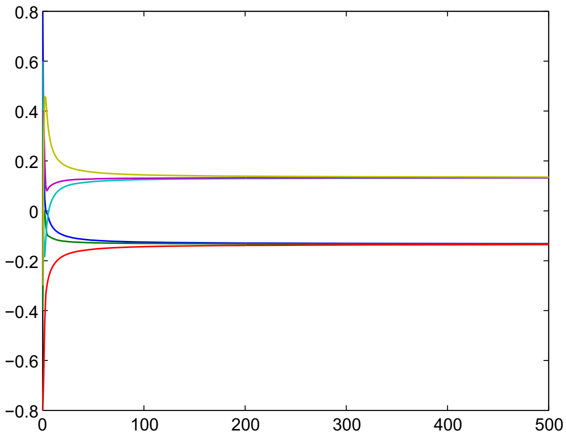

Fig. 9 Opinion evolution when $G_1(A)$ is a directed graph and $\rho(k) = k/2$ in Example 2

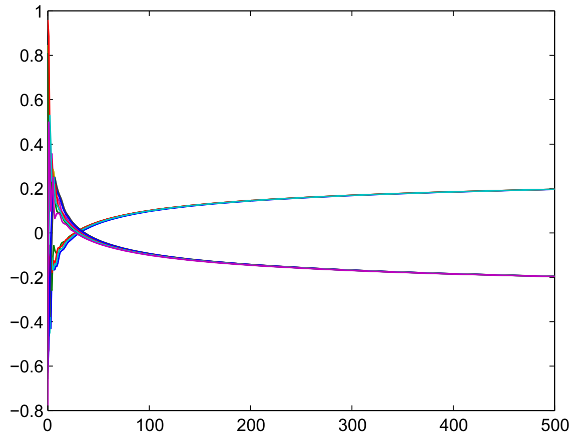

Fig. 12 Opinion evolution when $G_2(A)$ is a directed graph and $\rho(k) = k/2$ in Example 2

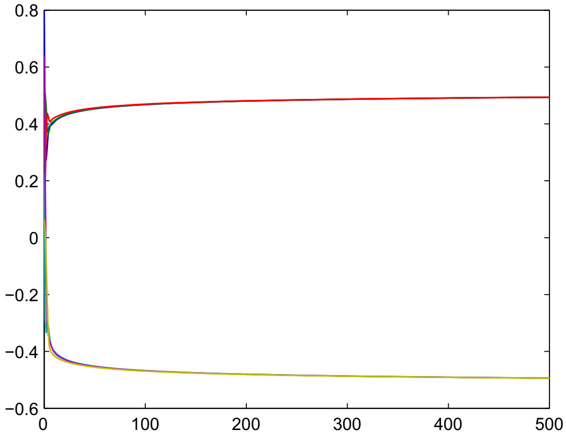

Fig. 10 Opinion evolution when $G_1(A)$ is a directed graph and $\rho(k) = e^k$ in Example 2

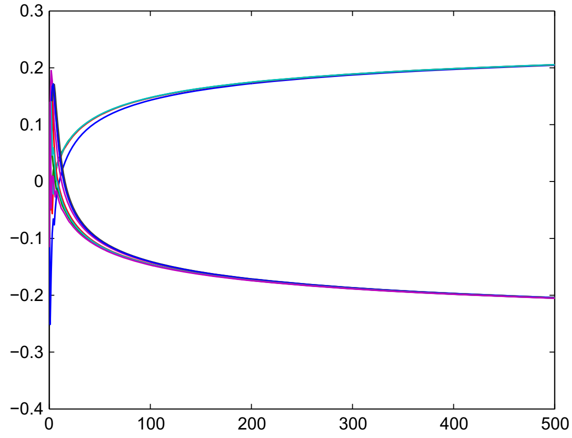

Fig. 13 Opinion evolution when $G_2(A)$ is a directed graph and $\rho(k) = e^k$ in Example 2

Example 2 In this example, we consider two the signed networks $G_1(A)$ and $G_2(A)$ which are showed in Figs. 6 and 7, respectively. Similar to Example 1, the red node represents the stubborn agent, and the green node is the non-stubborn agent. According to Figs. 6 and 7, one can find that the graphs $G_1(A)$ and $G_2(A)$ all satisfy Assumption 1. Let $\rho(k) = \sqrt{k}, k/2, e^k$, respectively. The stubbornness and prejudice are set randomly. The opinion trajectories for this example are shown in Figs. 8, 9, 10, 11, 12, 13. According to Figs. 8, 9, 10, 11, 12, 13, for the digraph $G_1(A)$ and $G_2(A)$, the system (4) finally reaches bipartite consensus or consensus.

5 Conclusion

In this paper, firstly, a novel opinion dynamics model has been proposed in which "stubbornness," "competition" and "peer pressure" are simultaneously considered. Then, the convergence of model has been investigated. We have found that if the peer pressure is increasing and bounded, opinions usually form multiple clusters in our model. Meanwhile, if the peer pressure is increasing and unbounded, our model will reach bipartite consensus even consensus. This shows that as the peer pressure increases indefinitely, the differences of opinion may decrease. Finally, two numerical examples have been given to illustrate the effectiveness of our results.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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