Nonlinear Dyn (2020) 99:1039–1052
https://doi.org/10.1007/s11071-019-05328-4

ORIGINAL PAPER

Solitons and periodic waves for the (2 + 1)-dimensional generalized Caudrey–Dodd–Gibbon–Kotera–Sawada equation in fluid mechanics

Gao-Fu Deng · **Yi-Tian Gao** · **Jing-Jing Su** ·
Cui-Cui Ding · **Ting-Ting Jia**

Received: 10 August 2019 / Accepted: 22 October 2019 / Published online: 21 November 2019

Abstract Fluid mechanics has the applications in a wide range of disciplines, such as oceanography, astrophysics, meteorology, and biomedical engineering. Under investigation in this paper is the (2 + 1)-dimensional generalized Caudrey–Dodd–Gibbon–Kotera–Sawada equation in fluid mechanics. Via the Pfaffian technique and certain constraint on the real constant α, the Nth-order Pfaffian solutions are derived. One- and two-soliton solutions are obtained via the Nth-order Pfaffian solutions. Based on the Hirota–Riemann method, one- and two-periodic wave solutions are constructed. With the help of the analytic and graphic analysis, we notice that: (1) of the one soliton, amplitude is irrelevant to γ, a real constant coefficient in the equation, velocity along the x direction is independent of γ, while velocity along the y direction is proportional to γ; (2) one soliton keeps its amplitude and velocity invariant during the propagation and total amplitude of the two solitons in the interaction region is lower than that of any soliton; (3) one-periodic wave can be viewed as a superposition of the overlapping solitary waves, placed one period apart; (4) periodic behaviors for the two-periodic wave exist along the x and y directions, respectively; (5) under certain limiting conditions, one-periodic wave solutions approach to the one-soliton solutions and two-periodic wave solutions approach to the two-soliton solutions.

Keywords Fluid mechanics · (2 + 1)-Dimensional generalized Caudrey–Dodd–Gibbon–Kotera–Sawada equation · Solitons · Periodic waves · Pfaffian technique · Hirota–Riemann method

1 Introduction

Fluid mechanics deals with the underlying mechanisms of liquids, gases or plasmas, and the forces on them [1–8]. It has the applications in a wide range of disciplines, such as oceanography, astrophysics, meteorology, and biomedical engineering [9–17]. For the insight into the fluid mechanics problems, people have focused their attention on the analytic solutions of the nonlinear evolution equations (NLEEs) to describe the nonlinear waves [18–27]. For example, soliton solutions have been derived for the (2 + 1)-dimensional Korteweg–de Vries (KdV) equation [28, 29], lump solutions have been obtained for the extended Kadomtsev–Petviashvili (KP) equation [32,33], rogue wave solutions have been constructed for the B-type KP equation [34–37], and periodic wave solutions have been studied for the (2+1)-dimensional extended shallow water wave equation [38]. Methods for deriving the analytic solutions of the NLEEs including the inverse scattering transform, Pfaffian technique, Lie symmetry method and Hirota–Riemann method have been pro-

G.-F. Deng · Y.-T. Gao (✉) · J.-J. Su · C.-C. Ding · T.-T. Jia
Ministry-of-Education Key Laboratory of Fluid Mechanics and National Laboratory for Computational Fluid Dynamics, Beijing University of Aeronautics and Astronautics, Beijing 100191, China
e-mail: gaoyt163@163.com

posed [39–46]. Among them, the Pfaffian technique has been used to construct the soliton solutions and the Hirota–Riemann method has been utilized to derive the periodic wave solutions of the NLEEs [47–52].

Ref. [53] has considered the (2 + 1)-dimensional generalized Caudrey–Dodd–Gibbon–Kotera–Sawada (gCDGKS) equation,

$$36u_t + \left(u_{xxxx} + 15uu_{xx} + 15u^3\right)_x - \alpha\partial_x^{-1}u_{yy} - \gamma\left(u_{xxy} + 3uu_y + 3u_x\partial_x^{-1}u_y\right) = 0, \quad (1)$$

where $u = u(x, y, t)$ is the differentiable function with respect to the variables x, y and t, α and γ are the real constants, the subscripts represent the partial derivatives, and ∂_x^{-1} represents the integral with respect to x. Soliton solutions for Eq. (1) have been constructed via the Hirota bilinear method, and lump solutions for Eq. (1) have been derived via the symbolic computation [53]. In fluid mechanics, special cases for Eq. (1) are given as follows:

- When $\alpha = \gamma = 5$, Eq. (1) has been reduced to the (2 + 1)-dimensional fifth-order KdV equation in fluid mechanics [54–56],

  $$36u_t + \left(u_{xxxx} + 15uu_{xx} + 15u^3\right)_x - 5\partial_x^{-1}u_{yy} - 5\left(u_{xxy} + 3uu_y + 3u_x\partial_x^{-1}u_y\right) = 0. \quad (2)$$

  Periodic solitary wave solutions for Eq. (2) have been constructed via the Hirota bilinear method [54]. Quasi-periodic solutions for Eq. (2) have been derived in terms of the Riemann theta functions [55]. Lump-type and rogue wave solutions for Eq. (2) have been obtained via the symbolic computation [56].
- When $\alpha = \gamma = 5$, $t = 36T'$ and $u_y = 0$, Eq. (1) has been reduced to the Sawada–Kotera equation for the long waves in shallow water under the gravity [57–61],

  $$u_{T'} + u_{xxxxx} + 15u_xu_{xx} + 15uu_{xxx} + 45u^2u_x = 0. \quad (3)$$

  Eq. (3) has also been seen in lattice dynamics, quantum mechanics and nonlinear optics [58]. Soliton solutions for Eq. (3) have been constructed via the Hirota bilinear method [59]. Periodic and rational solutions for Eq. (3) have been constructed via the (G'/G)-expansion method [60]. Traveling waves with different frequencies and velocities for Eq. (3) have been constructed via the three wave method [61].

Through the dependent transformation [53],

$$u = 2\,(\ln f)_{xx}\,, \quad (4)$$

where f is a real function of x, y and t, Eq. (1) has been written as the bilinear form [53],

$$\left(36D_xD_t + D_x^6 - \alpha D_y^2 - \gamma D_x^3D_y\right) f\cdot f = 0, \quad (5)$$

where the bilinear operators D_x, D_y and D_t are defined by [62]

$$D_x^lD_y^mD_t^n\theta(x,y,t)\cdot\vartheta(x',y',t') \equiv \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^l\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n \theta(x,y,t)\vartheta(x',y',t')|_{x'=x,\ y'=y,\ t'=t}, \quad (6)$$

with $\theta(x, y, t)$ being a differentiable function of x, y and t, $\vartheta(x', y', t')$ being a differentiable function of the independent variables x', y' and t', and l, m and n being the non-negative integers.

On the other hand, the Nth-order Pfaffian, i.e., $(1, 2, \ldots, 2N)$, has the following expansion [62]:

$$(1,2,\ldots,2N) = \sum_{j=2}^{2N}(-1)^j(1,j)\quad(2,3,\ldots,\hat{j},\ldots,2N), \quad (7)$$

where $\hat{j}$ means that the element j is omitted, $(2, 3, \ldots, \hat{j}, \ldots, 2N)$ is the $(N-1)$th-order Pfaffian, (r, j) is the antisymmetric element of the Pfaffian and defined as

$$(r,j) = c_{rj} + \int^x D_x\phi_r\cdot\phi_j\mathrm{d}x, \quad (8)$$

r, j and N are the positive integers, ϕ_r's and ϕ_j's are the real functions of x, y and t, and c_{rj} is a constant satisfying the condition $c_{rj} = -c_{jr}$. Pfaffian has been said to possess the following properties [62]:

$$(\alpha_1,\alpha_2,\ldots,\alpha_{2N},1,2,\ldots,2N)(1,2,\ldots,2N) = \sum_{j=2}^{2N}(-1)^j(\alpha_1,\alpha_j,1,2,\ldots,2N) (\alpha_2,\alpha_3,\ldots,\hat{\alpha}_j,\ldots,\alpha_{2N},1,2,\ldots,2N), \quad (9)$$

where α_j's are the real numbers, and $\hat{\alpha}_j$ means that the element α_j is omitted.

$$\begin{aligned}
&(d_0,d_1,d_2,d_3,\bullet)(\bullet)-(d_0,d_1,\bullet)(d_2,d_3,\bullet)\\
&+(d_0,d_2,\bullet)(d_1,d_3,\bullet)-(d_0,d_3,\bullet)(d_1,d_2,\bullet)=0,\\
&(d_n,r)=\frac{\partial^n\phi_r}{\partial x^n},\ (d_m,d_n)=0,\\
&(m,n=0,1,2,\ldots,2N-1),
\end{aligned}\tag{10}$$

where $(\bullet)=(1,2,\ldots,2N)$.

However, to our knowledge, soliton solutions via the Pfaffian technique and periodic wave solutions via the Hirota–Riemann method for Eq. (1) have not been investigated. In Sect. 2, the Nth-order Pfaffian solutions for Eq. (1) will be constructed via the Pfaffian technique, and soliton solutions for Eq. (1) will be derived via the Nth-order Pfaffian solutions. In Sect. 3, periodic wave solutions for Eq. (1) will be obtained via the Hirota–Riemann method, and asymptotic behaviors of the periodic wave solutions will be given. In Sect. 4, our conclusions will be presented.

2 Pfaffian solutions for Eq. (1)

In this section, we would like to construct the Pfaffian solutions for Eq. (1) via the Pfaffian technique. To derive the Nth-order Pfaffian $(1,2,\ldots,2N)$ satisfying Bilinear Form (5), we can set the differentiable functions ϕ_r's and ϕ_j's in Eq. (8) satisfying the following conditions:

$$\begin{aligned}
&\frac{\partial\phi_r}{\partial y}=\frac{5}{\gamma}\frac{\partial^3\phi_r}{\partial x^3},\quad \frac{\partial\phi_r}{\partial t}=\frac{1}{4}\frac{\partial^5\phi_r}{\partial x^5},\quad \frac{\partial\phi_j}{\partial y}=\frac{5}{\gamma}\frac{\partial^3\phi_j}{\partial x^3},\\
&\frac{\partial\phi_j}{\partial t}=\frac{1}{4}\frac{\partial^5\phi_j}{\partial x^5},\quad \alpha=\frac{\gamma^2}{5},
\end{aligned}\tag{11}$$

then we have

$$\begin{aligned}
\frac{\partial(r,j)}{\partial x}&=\frac{\partial\phi_r}{\partial x}\phi_j-\frac{\partial\phi_j}{\partial x}\phi_r\\
&=(d_1,r)(d_0,j)-(d_0,r)(d_1,j)\\
&=(d_0,d_1,r,j),\\
\frac{\partial(r,j)}{\partial y}&=\int\left(\frac{\partial^2\phi_r}{\partial x\partial y}\phi_j+\frac{\partial\phi_r}{\partial x}\frac{\partial\phi_j}{\partial y}\right.\\
&\quad\left.-\frac{\partial^2\phi_j}{\partial x\partial y}\phi_r-\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_r}{\partial y}\right)\mathrm{d}x\\
&=\frac{5}{\gamma}[(d_0,d_3,r,j)-2(d_1,d_2,r,j)],\\
\frac{\partial(r,j)}{\partial t}&=\int\left(\frac{\partial^2\phi_r}{\partial x\partial t}\phi_j+\frac{\partial\phi_r}{\partial x}\frac{\partial\phi_j}{\partial t}\right.\\
&\quad\left.-\frac{\partial^2\phi_j}{\partial x\partial t}\phi_r-\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_r}{\partial t}\right)\mathrm{d}x\\
&=\frac{1}{4}\Big[2(d_2,d_3,r,j)-2(d_1,d_4,r,j)\\
&\quad+(d_0,d_5,r,j)\Big].
\end{aligned}\tag{12}$$

According to Eqs. (12), the following differential conditions can be derived:

$$\begin{aligned}
\tau_N&=(\bullet),\\
\tau_{N,x}&=(d_0,d_1,\bullet),\\
\tau_{N,xx}&=(d_0,d_2,\bullet),\\
\tau_{N,xxx}&=(d_1,d_2,\bullet)+(d_0,d_3,\bullet),\\
\tau_{N,xxxx}&=2(d_1,d_3,\bullet)+(d_0,d_4,\bullet),\\
\tau_{N,xxxxx}&=2(d_2,d_3,\bullet)+3(d_1,d_4,\bullet)+(d_0,d_5,\bullet),\\
\tau_{N,xxxxxx}&=2(d_0,d_1,d_2,d_3,\bullet)+5(d_2,d_4,\bullet)\\
&\quad+4(d_1,d_5,\bullet)+(d_0,d_6,\bullet),\\
\tau_{N,y}&=\frac{5}{\gamma}[(d_0,d_3,\bullet)-2(d_1,d_2,\bullet)],
\end{aligned}\tag{13}$$

$$\begin{aligned}
\tau_{N,yy}&=\frac{-25}{\gamma^2}[-(d_0,d_6,\bullet)-2(d_2,d_4,\bullet)\\
&\quad+2(d_1,d_5,\bullet)+4(d_0,d_1,d_2,d_3,\bullet)],\\
\tau_{N,xy}&=\frac{5}{\gamma}[(d_0,d_4,\bullet)-(d_1,d_3,\bullet)],\\
\tau_{N,xxy}&=\frac{5}{\gamma}[(d_0,d_5,\bullet)-(d_2,d_3,\bullet)],\\
\tau_{N,xxxy}&=\frac{5}{\gamma}[(d_1,d_5,\bullet)+(d_0,d_6,\bullet)-(d_2,d_4,\bullet)\\
&\quad-(d_0,d_1,d_2,d_3,\bullet)],\\
\tau_{N,t}&=\frac{1}{4}[-2(d_1,d_4,\bullet)+2(d_2,d_3,\bullet)+(d_0,d_5,\bullet)],\\
\tau_{N,xt}&=\frac{1}{4}[-(d_1,d_5,\bullet)+(d_0,d_6,\bullet)\\
&\quad+2(d_0,d_1,d_2,d_3,\bullet)].
\end{aligned}\tag{14}$$

Combining Eqs. (9) and (10) with Eqs. (13) and (14), we obtain

$$\begin{aligned}
&\left(36D_xD_t+D_x^6-\alpha D_y^2-\gamma D_x^3D_y\right)\tau_N\cdot\tau_N\\
&\quad=\frac{2}{5}(\gamma^2\tau_{N,y}^2-\gamma^2\tau_N\tau_{N,yy}-15\gamma\tau_{N,xy}\tau_{N,xx}\\
&\qquad+15\gamma\tau_{N,x}\tau_{N,xxy}+5\gamma\tau_{N,y}\tau_{N,xxx}\\
&\qquad-5\gamma\tau_N\tau_{N,xxxy}-50\tau_{N,xxx}^2-180\tau_{N,t}\tau_{N,x}\\
&\qquad+180\tau_N\tau_{N,xt}+75\tau_{N,xx}\tau_{N,xxxx}\\
&\qquad-30\tau_{N,x}\tau_{N,xxxxx}+5\tau_N\tau_{N,xxxxxx})\\
&\quad=90\Big[(d_0,d_1,d_2,d_3,\bullet)(\bullet)-(d_0,d_1,\bullet)(d_2,d_3,\bullet)
\end{aligned}$$

$$+(d_0,d_2,\bullet)(d_1,d_3,\bullet)-(d_0,d_3,\bullet)(d_1,d_2,\bullet)\Big]$$
$$=0. \tag{15}$$

Thus, we find that $f=\tau_N$ satisfies Bilinear Form (5) and the Nth-order Pfaffian solutions for Eq. (1) can be derived as

$$u=2(\ln\tau_N)_{xx}. \tag{16}$$

To construct the soliton solutions for Eq. (1) via the Nth-Order Pfaffian Solutions (16), we can set ϕ_r's and ϕ_j's in Conditions (11) as

$$\phi_r=e^{k_r x+\frac{5k_r^3}{\gamma}y+\frac{k_r^5}{4}t},$$
$$\phi_j=e^{k_j x+\frac{5k_j^3}{\gamma}y+\frac{k_j^5}{4}t}, \tag{17}$$

where k_r's and k_j's are real constants. Motivated by Ref. [62], we set $c_{12}=c_{34}=1$, $c_{13}=c_{14}=c_{23}=c_{24}=0$, and obtain

$$(r,j)=c_{rj}+\frac{k_r-k_j}{k_r+k_j}\phi_r\phi_j. \tag{18}$$

Hereby, when $N=1$ and 2 in the Nth-Order Pfaffian Solutions (16), the one- and two-soliton solutions for Eq. (1) can be expressed as

$$u=2(\ln\tau_1)_{xx}, \tag{19}$$
$$u=2(\ln\tau_2)_{xx}, \tag{20}$$

with

$$\begin{aligned}
\tau_1&=(1,2)=1+A_1e^{\xi_1+\xi_2},\\
\tau_2&=(1,2,3,4)\\
&=(1,2)(3,4)-(1,3)(2,4)+(1,4)(2,3)\\
&=1+A_1e^{\xi_1+\xi_2}+A_2e^{\xi_3+\xi_4}+A_{12}e^{\xi_1+\xi_2+\xi_3+\xi_4},\\
A_1&=\frac{H_1-H_2}{H_1+H_2},\quad A_2=\frac{H_3-H_4}{H_3+H_4},\\
\xi_\varrho&=H_\varrho x+S_\varrho y+J_\varrho t,\\
A_{12}&=\frac{(H_1-H_4)(H_2-H_3)}{(H_1+H_4)(H_2+H_3)}+\frac{(H_1-H_2)(H_3-H_4)}{(H_1+H_2)(H_3+H_4)}\\
&\quad-\frac{(H_1-H_3)(H_2-H_4)}{(H_1+H_3)(H_2+H_4)},\\
H_\varrho&=k_\varrho,\ S_\varrho=\frac{5k_\varrho^3}{\gamma},\ J_\varrho=\frac{k_\varrho^5}{4},\quad(\varrho=1,2,3,4).
\end{aligned} \tag{21}$$

Equation (19) indicates that the amplitude of the one soliton is irrelevant to γ, the velocity along the x direction of the one soliton is independent of γ, while the velocity along the y direction is proportional to γ. Figure 1 shows the propagation of the one soliton, and we notice that the one soliton keeps its amplitude and velocity invariant. Figure 2 shows the interaction between the two solitons, and we find that the total amplitude of the interaction region is lower than that of any soliton.

3 Periodic wave solutions for Eq. (1)

In this section, we will utilize the Hirota–Riemann method [63] to construct the periodic wave solutions for Eq. (1).

3.1 Hirota–Riemann method for the NLEEs

Ref. [63] has considered a generalized (N+1)-dimensional NLEE:

$$\mathscr{F}\left(u,u_t,u_{x_1},u_{x_2},u_{x_N},\ldots\right)=0, \tag{22}$$

where $\mathscr{F}$ is a polynomial function and $x_1,x_2,\ldots,x_N$ are the space variables. Using the Hirota bilinear method and the dependent variable transformation,

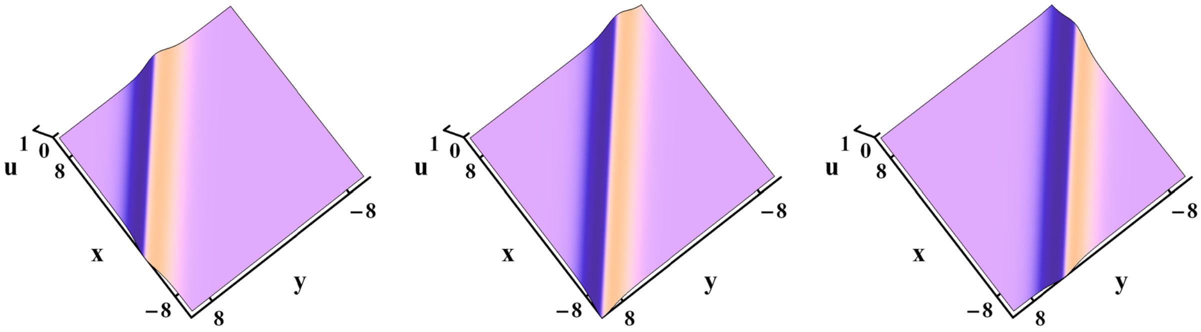
$$u=u_0+p\partial_{x_N}^q\ln\vartheta\left(\zeta,\lambda\right), \tag{23}$$

where $\partial_{x_N}^q$ represents the $q-th$ order partial derivatives with respect to x_N, $\vartheta\left(\zeta,\lambda\right)$ is the Riemann theta function, $\zeta=(\zeta_1,\zeta_2,\ldots,\zeta_N)^T$ (the superscript T signifies the vector transpose), $i\lambda=(i\lambda_{\mu\iota})$ is a positive definite and real-valued symmetric $N\times N$ matrix. $\zeta_\mu=Q_\mu x+B_\mu y+R_\mu t+\epsilon_\mu$, $(\mu,\iota=1,2,\ldots,N)$, p,q,N are the positive integers, and Q's, B's, R's, ϵ's and u_0 are all the real constants; Ref. [63] obtains the bilinear form for Eq. (22) as

$$\mathscr{F}\left(D_{x_1},D_{x_2},\ldots,D_{x_N},D_t,c\right)\vartheta\left(\zeta,\lambda\right)\cdot\vartheta\left(\zeta,\lambda\right)=0, \tag{24}$$

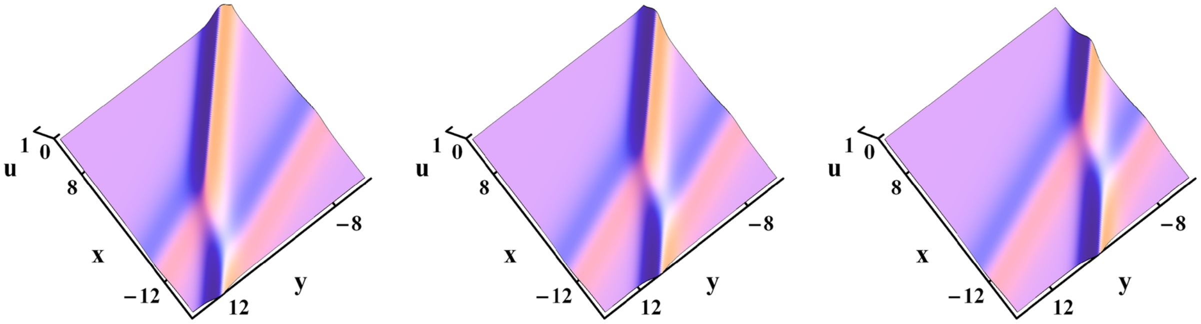
where c is an integration constant and must not be dropped in our present periodic case because the elliptic functions generally do not satisfy the equations with the zero integration constants. Then, the multi-periodic wave solutions for Eq. (22) can be constructed via the Riemann theta function,

$$\vartheta\left(\zeta,\lambda\right)=\sum_{\eta\in\mathbb{Z}^2}e^{\pi i\langle\eta\lambda,\eta\rangle+2\pi i\langle\zeta,\eta\rangle}, \tag{25}$$



(a) t=-300 **(b)** t=0 **(c)** t=300

Fig. 1 One soliton via Solutions (19) with $k_1 = 0.6,\ k_2 = 0.4$ and $\gamma = 1.2$



(a) t=-300 **(b)** t=0 **(c)** t=300

Fig. 2 Interaction between the two solitons via Solutions (20) with $k_1 = -0.52,\ k_2 = -0.5,\ k_3 = -0.35,\ k_4 = -0.24$ and $\gamma = 1.2$

where $i = \sqrt{-1}$, the integer value vector $\eta = (\eta_1, \eta_2, \ldots, \eta_N)^T \in \mathbb{Z}^N$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N)^T \in \mathbb{C}^N$, $\mathbb{Z}$ denotes the integer number, where $\mathbb{C}$ denotes the complex number. In this paper, taking the matrix λ to be pure imaginary matrix yields Riemann Theta Function (25) real-valued. For two vectors $f = (f_1, f_2, \ldots, f_N)^T$ and $g = (g_1, g_2, \ldots, g_N)^T$, their inner product is defined by

$$\langle f, g \rangle = f_1 g_1 + f_2 g_2 + \cdots + f_N g_N. \tag{26}$$

3.2 One-periodic wave solutions for Eq. (1)

In order to construct the periodic wave solutions for Eq. (1), we should consider a more generalized bilinear form than Bilinear Form (5) for Eq. (1) by introducing one more widely dependent transformation:

$$u = u_0 + 2\,[\ln \vartheta(\zeta, \lambda)]_{xx}\,. \tag{27}$$

Substituting Transformation (27) into Eq. (1), we can derive a generalized bilinear form as:

$$\begin{aligned} &\mathscr{L}(D_x, D_y, D_t)\vartheta(\zeta,\lambda)\cdot\vartheta(\zeta,\lambda) \\ &\quad = \Big(36 D_x D_t + D_x^6 + u_0 D_x^6 \\ &\qquad - \alpha D_y^2 - \gamma D_x^3 D_y + c\Big)\,\vartheta(\zeta,\lambda)\cdot\vartheta(\zeta,\lambda) \\ &\quad = 0. \end{aligned} \tag{28}$$

From Riemann Theta Function (25), we derive the one-Riemann theta function as

$$\vartheta(\zeta_1, \lambda_1) = \sum_{\eta=-\infty}^{+\infty} e^{\pi i \eta^2 \lambda_1 + 2\pi i \eta \zeta_1}, \tag{29}$$

where $\zeta_1 = Q_1 x + B_1 y + R_1 t + \epsilon$, λ_1 is a pure imaginary number and meets the condition $\mathrm{Im}(\lambda_1) > 0$, and ϵ is a real constant. Substituting Eq. (29) into (28), we have

$$
\begin{aligned}
&\mathscr{L}(D_x, D_y, D_t)\vartheta(\zeta_1, \lambda_1)\cdot\vartheta(\zeta_1, \lambda_1)\\
&\quad=\sum_{\varpi=-\infty}^{+\infty}\sum_{\eta=-\infty}^{+\infty}\mathscr{L}(D_x, D_y, D_t)\\
&\qquad e^{\pi i\eta^2\lambda_1+2\pi i\eta\zeta_1}\cdot e^{\pi i\varpi^2\lambda_1+2\pi i\varpi\zeta_1}\\
&\quad=\sum_{\varpi=-\infty}^{+\infty}\sum_{\eta=-\infty}^{+\infty}\mathscr{L}\Big[2i\pi(\eta-\varpi)Q_1, 2i\pi(\eta-\varpi)B_1,\\
&\qquad 2i\pi(\eta-\varpi)R_1\Big]e^{\pi i(\varpi^2+\eta^2)\lambda_1+2\pi i(\varpi+\eta)\zeta_1}\\
&\qquad\overset{\varpi'=\varpi+\eta}{=}\sum_{\varpi'=-\infty}^{+\infty}\tilde{\mathscr{L}}(\varpi')e^{2\pi i\varpi'\zeta_1},
\end{aligned}
\tag{30}
$$

with

$$
\begin{aligned}
&\tilde{\mathscr{L}}(\varpi')\\
&\quad=\sum_{\eta=-\infty}^{+\infty}\mathscr{L}\Big[2i\pi(2\eta-\varpi')Q_1, 2i\pi(2\eta-\varpi')B_1,\\
&\qquad 2i\pi(2\eta-\varpi')R_1\Big]e^{\pi i[\eta^2+(\eta-\varpi')^2]\lambda_1}\\
&\quad\overset{\eta=\eta'+1}{=}\sum_{\eta'=-\infty}^{+\infty}\mathscr{L}\Big\{2i\pi[2\eta'-(\varpi'-2)]Q_1,\\
&\qquad 2i\pi[2\eta'-(\varpi'-2)]B_1, 2i\pi[2\eta'-(\varpi'-2)]R_1\Big\}\\
&\qquad e^{\pi i[\eta'^2+(\eta'-(\varpi'-2))^2]\lambda_1}\cdot e^{2\pi i(\varpi'-1)\lambda_1}\\
&\quad=\tilde{\mathscr{L}}(\varpi'-2)e^{2\pi i(\varpi'-1)\lambda_1}\\
&\quad=\cdots=\begin{cases}\tilde{\mathscr{L}}(0)e^{\pi i\varpi'\lambda_1}, & \varpi' \text{ is even},\\ \tilde{\mathscr{L}}(1)e^{\pi i(\varpi'+1)\lambda_1}, & \varpi' \text{ is odd},\end{cases}\quad \varpi', \eta'\in\mathbb{Z},
\end{aligned}
\tag{31}
$$

Equation (31) implies that $\tilde{\mathscr{L}}(\varpi')$ for $\varpi'\in\mathbb{Z}$ are completely dominated by $\tilde{\mathscr{L}}(0)$ and $\tilde{\mathscr{L}}(1)$. If $\tilde{\mathscr{L}}(0)=\tilde{\mathscr{L}}(1)=0$, then $\mathscr{L}(D_x, D_y, D_t)\vartheta(\zeta_1, \lambda_1)\cdot\vartheta(\zeta_1, \lambda_1)=0$.

Based on Bilinear Form (28), the one-periodic wave[1] solutions can be derived by

$$
\begin{aligned}
\tilde{\mathscr{L}}(0)&=\sum_{\eta=-\infty}^{+\infty}\mathscr{L}\left(4\eta\pi iQ_1, 4\eta\pi iB_1, 4\eta\pi iR_1\right)e^{2\eta^2\pi i\lambda_1}\\
&=\sum_{\eta=-\infty}^{+\infty}\Big(-576\eta^2\pi^2Q_1R_1-4096\eta^6\pi^6Q_1^6\\
&\qquad-4096u_0\eta^6\pi^6Q_1^6+16\alpha\eta^2\pi^2B_1^2\\
&\qquad-256\gamma\eta^4\pi^4Q_1^3B_1+c\Big)e^{2i\pi\eta^2\lambda_1}=0,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathscr{L}}(1)&=\sum_{\eta=-\infty}^{+\infty}\mathscr{L}\Big[2i\pi(2\eta-1)Q_1, 2i\pi(2\eta-1)B_1,\\
&\qquad 2i\pi(2\eta-1)R_1\Big]e^{\pi i(2\eta^2-2\eta+1)\lambda_1}\\
&=\sum_{\eta=-\infty}^{+\infty}\Big[-144(2\eta-1)^2\pi^2Q_1R_1\\
&\qquad-64(2\eta-1)^6\pi^6Q_1^6-64u_0(2\eta-1)^6\pi^6Q_1^6\\
&\qquad+4\alpha(2\eta-1)^2\pi^2B_1^2-16\gamma(2\eta-1)^4\pi^4Q_1^3B_1\\
&\qquad+c\Big]e^{\pi i(2\eta^2-2\eta+1)\lambda_1}=0.
\end{aligned}
\tag{32}
$$

Through the notations

$$\Delta=e^{\pi i\lambda_1},\tag{33}$$

$$
\begin{aligned}
a_{11}&=-\sum_{\eta=-\infty}^{+\infty}576\eta^2\pi^2Q_1\Delta^{2\eta^2},\quad a_{12}=\sum_{\eta=-\infty}^{+\infty}\Delta^{2\eta^2},\\
a_{21}&=-\sum_{\eta=-\infty}^{+\infty}144(2\eta-1)^2\pi^2Q_1\Delta^{2\eta^2-2\eta+1},\\
a_{22}&=\sum_{\eta=-\infty}^{+\infty}\Delta^{2\eta^2-2\eta+1},\\
b_1&=\sum_{\eta=-\infty}^{+\infty}\Big(4096\eta^6\pi^6Q_1^6+4096u_0\eta^6\pi^6Q_1^6\\
&\qquad-16\alpha\eta^2\pi^2B_1^2+256\gamma\eta^4\pi^4Q_1^3B_1\Big)\Delta^{2\eta^2},\\
b_2&=\sum_{\eta=-\infty}^{+\infty}\Big[64(2\eta-1)^6\pi^6Q_1^6+64u_0(2\eta-1)^6\pi^6Q_1^6\\
&\qquad-4\alpha(2\eta-1)^2\pi^2B_1^2+16\gamma(2\eta-1)^4\pi^4Q_1^3B_1\Big]\\
&\qquad\Delta^{2\eta^2-2\eta+1},
\end{aligned}
\tag{34}
$$

Equation (32) can be rewritten as a linear system about R_1 and c, i.e.,

$$
\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}R_1\\ c\end{pmatrix}=\begin{pmatrix}b_1\\ b_2\end{pmatrix}.
\tag{35}
$$

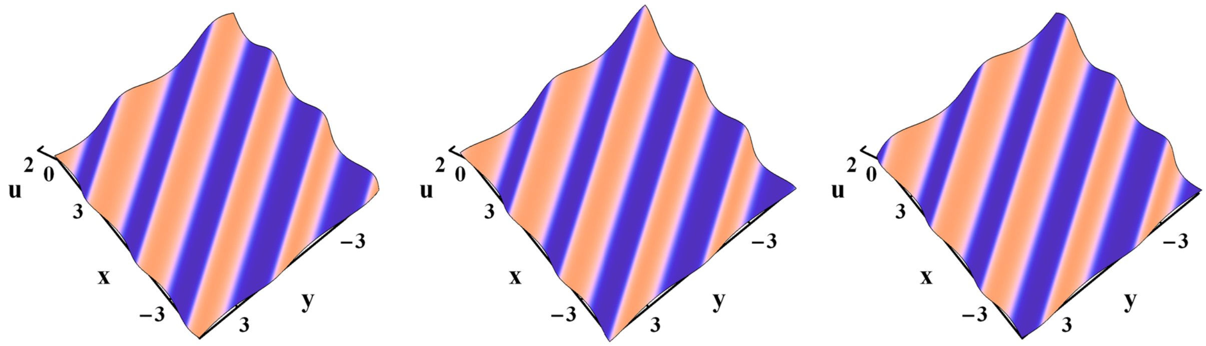
Solving System (35), we can derive the one-periodic wave solutions for Eq. (1) as

$$u=u_0+2\left[\ln\vartheta(\zeta_1, \lambda_1)\right]_{xx}.\tag{36}$$

Figure 3 shows that the one-periodic wave can be viewed as a superposition of the overlapping solitary waves, placed one period apart. In the following section, the asymptotic behaviors of One-Periodic Wave Solutions (36) will be studied. Equation (34) can be expanded as

---

[1] One-periodic wave implies the wave propagating with the constant period in the x, y and t directions [63].



(a) t=-10 **(b)** t=0 **(c)** t=10

Fig. 3 One-periodic wave via Solutions (36) with $\lambda_1 = i,\ Q_1 = 0.3,\ B_1 = 0.2$ and $\alpha = \gamma = u_0 = 1$

$$
\begin{aligned}
a_{11} &= -1152\pi^2 Q_1 \left(\varDelta^2 + 4\varDelta^8 + \cdots + \eta^2\varDelta^{2\eta^2} + \cdots\right),\\
a_{12} &= 1 + 2(\varDelta^2 + \varDelta^8 + \cdots + \varDelta^{2\eta^2} + \cdots),\\
a_{21} &= -288\pi^2 Q_1[\varDelta + 9\varDelta^5 + \cdots\\
&\quad + (2\eta-1)^2\varDelta^{2\eta^2-2\eta+1} + \cdots],\\
a_{22} &= 2(\varDelta + \varDelta^5 + \cdots + \varDelta^{2\eta^2-2\eta+1} + \cdots),\\
b_1 &= 2\Big(4096\pi^6 Q_1^6 + 4096u_0\pi^6 Q_1^6 - 16\alpha\pi^2 B_1^2\\
&\quad + 256\gamma\pi^4 Q_1^3 B_1\Big)\varDelta^2\\
&\quad + 2\Big(262144\pi^6 Q_1^6 + 262144u_0\pi^6 Q_1^6\\
&\quad - 64\alpha\pi^2 B_1^2 + 4096\gamma\pi^4 Q_1^3 B_1\Big)\varDelta^8 + \cdots\\
&\quad + \Big(4096\eta^6\pi^6 Q_1^6 + 4096u_0\eta^6\pi^6 Q_1^6\\
&\quad - 16\alpha\eta^2\pi^2 B_1^2\\
&\quad + 256\gamma\eta^4\pi^4 Q_1^3 B_1\Big)\varDelta^{2\eta^2} + \cdots,\\
b_2 &= 2\left(64\pi^6 Q_1^6 + 64u_0\pi^6 Q_1^6\right.\\
&\quad \left.- 4\alpha\pi^2 B_1^2 + 16\gamma\pi^4 Q_1^3 B_1\right)\varDelta\\
&\quad + 2\Big(46656\pi^6 Q_1^6 + 46656u_0\pi^6 Q_1^6 - 36\alpha\pi^2 B_1^2\\
&\quad + 1296\gamma\pi^4 Q_1^3 B_1\Big)\varDelta^5 + \cdots + [64(2\eta-1)^6\pi^6 Q_1^6\\
&\quad + 64u_0(2\eta-1)^6\pi^6 Q_1^6 - 4\alpha(2\eta-1)^2\pi^2 B_1^2\\
&\quad + 16\gamma(2\eta-1)^4\pi^4 Q_1^3 B_1]\varDelta^{2\eta^2-2\eta+1} + \cdots,
\end{aligned}
\tag{37}
$$

and substituting Eq. (37) into System (35), we have

$$
\begin{aligned}
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \varLambda_0 + \varLambda_1\varDelta + \varLambda_2\varDelta^2 + \cdots,\\
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \varTheta_0 + \varTheta_1\varDelta + \varTheta_2\varDelta^2 + \cdots,
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
&\varLambda_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \varLambda_1 = \begin{pmatrix} 0 & 0 \\ -288\pi^2 Q_1 & 2 \end{pmatrix},\\
&\varLambda_2 = \begin{pmatrix} -1152\pi^2 Q_1 & 2 \\ 0 & 0 \end{pmatrix},\ \varLambda_3 = \varLambda_4 = \mathbf{0},\\
&\varLambda_5 = \begin{pmatrix} 0 & 0 \\ -2592\pi^2 Q_1 & 2 \end{pmatrix}, \ldots,\ \varTheta_0 = \varTheta_3 = \varTheta_4 = \mathbf{0},\\
&\nu_1 = 128\pi^6 Q_1^6 + 128u_0\pi^6 Q_1^6 - 8\alpha\pi^2 B_1^2\\
&\quad + 32\gamma\pi^4 Q_1^3 B_1,\\
&\nu_2 = 8192\pi^6 Q_1^6 + 8192u_0\pi^6 Q_1^6 - 32\alpha\pi^2 B_1^2\\
&\quad + 512\gamma\pi^4 Q_1^3 B_1,\\
&\nu_5 = 93312\pi^6 Q_1^6 + 93312u_0\pi^6 Q_1^6 - 72\alpha\pi^2 B_1^2\\
&\quad + 2592\gamma\pi^4 Q_1^3 B_1,\\
&\varTheta_1 = \begin{pmatrix} 0 \\ \nu_1 \end{pmatrix},\ \varTheta_2 = \begin{pmatrix} \nu_2 \\ 0 \end{pmatrix},\ \varTheta_5 = \begin{pmatrix} 0 \\ \nu_5 \end{pmatrix},\\
&\quad \cdots.
\end{aligned}
\tag{39}
$$

Then, R_1 and c in System (35) can be rewritten as

$$
\begin{aligned}
&\begin{pmatrix} R_1 \\ c \end{pmatrix} = \varGamma_0 + \varGamma_1\varDelta + \varGamma_2\varDelta^2 + \cdots,\\
&\varGamma_0 = \begin{pmatrix} \frac{2\varTheta_0^{[1]} - \varTheta_1^{[2]}}{288\pi^2 Q_1} \\ \varTheta_0^{[1]} \end{pmatrix},\ \varGamma_1 = \begin{pmatrix} \frac{2\varTheta_1^{[1]} - (\varTheta_2 - \varLambda_2\varGamma_0)^{[2]}}{288\pi^2 Q_1} \\ \varTheta_1^{[1]} \end{pmatrix},\\
&\varGamma_n = \begin{pmatrix} \frac{2[\varTheta_{n+1} - \Sigma_{j=2}^{n}\varLambda_j\varGamma_{n-j}]^{[1]} - [\varTheta_{n+1} - \Sigma_{j=2}^{n+1}\varLambda_j\varGamma_{n-j+1}]^{[2]}}{288\pi^2 Q_1} \\ [\varTheta_{n+1} - \Sigma_{j=2}^{n}\varLambda_j\varGamma_{n-j}]^{[1]} \end{pmatrix},\\
&\quad n \geq 2,
\end{aligned}
\tag{40}
$$

where n is the positive integer, and $\varTheta^{[\kappa]}$ $(\kappa = 1, 2)$ denotes the κ-th elements of the two-dimensional vector $\varTheta$.

From Eq. (40), we have

$$\mathit{\Gamma}_0 = \begin{pmatrix} \frac{16\pi^4 Q_1^6+16u_0\pi^4 Q_1^6-\alpha B_1^2+4\gamma\pi^2 Q_1^3 B_1}{-36Q_1} \\ 0 \end{pmatrix}, \ \mathit{\Gamma}_1 = \mathbf{0},$$
$$\mathit{\Gamma}_2 = \begin{pmatrix} \frac{-32\pi^4 Q_1^6-32u_0\pi^4 Q_1^6+2\alpha B_1^2-8\gamma\pi^2 Q_1^3 B_1}{9Q_1} \\ -512\pi^6 Q_1^6 - 512u_0\pi^6 Q_1^6 + 32\alpha\pi^2 B_1^2 - 128\gamma\pi^4 Q_1^3 B_1 \end{pmatrix},$$
$$\cdots. \tag{41}$$

Substituting Eqs. (41) into (38) and setting $\Delta \to 0$, we can obtain

$$c \to 0,$$
$$R_1 \to \frac{16\pi^4 Q_1^6 + 16u_0\pi^4 Q_1^6 - \alpha B_1^2 + 4\gamma\pi^2 Q_1^3 B_1}{-36Q_1}. \tag{42}$$

If we assume

$$u_0 = 0, \quad Q_1 = \frac{k_1+k_2}{2i\pi}, \quad B_1 = \frac{5k_1^3+5k_2^3}{2\gamma i\pi},$$
$$\epsilon = \frac{-i\pi\lambda + \ln\frac{k_1-k_2}{k_1+k_2}}{2i\pi}, \quad \alpha = \frac{\gamma^2}{5}, \tag{43}$$

where k_1, k_2, α and γ are determined by Eq. (19), we have

$$\begin{aligned} 2i\pi\zeta_1 &= 2i\pi(Q_1x + B_1y + R_1t + \epsilon) \\ &= (k_1+k_2)x + \frac{5k_1^3+5k_2^3}{\gamma}y + \frac{k_1^5+k_2^5}{4}t \\ &\quad + \ln\frac{k_1-k_2}{k_1+k_2} - i\pi\lambda_1 \\ &= \xi_1 + \xi_2 + \ln\frac{k_1-k_2}{k_1+k_2} - i\pi\lambda_1. \end{aligned} \tag{44}$$

Combining Eqs. (29) and (44), we further obtain

$$\begin{aligned} \vartheta(\zeta_1, \lambda_1) &= \sum_{\eta=-\infty}^{+\infty} e^{\pi i\eta^2\lambda_1+2\pi i\eta\zeta_1} \\ &= 1 + (e^{2\pi i\zeta_1} + e^{-2\pi i\zeta_1})\Delta + \cdots \\ &= 1 + e^{\xi_1+\xi_2+\ln\frac{k_1-k_2}{k_1+k_2}} + e^{-\left(\xi_1+\xi_2+\ln\frac{k_1-k_2}{k_1+k_2}\right)}\Delta^2 + \cdots \\ &\overset{\Delta\to 0}{=} 1 + \frac{k_1-k_2}{k_1+k_2}e^{\xi_1+\xi_2}. \end{aligned} \tag{45}$$

From the above analysis, we find that One-Periodic Wave Solutions (36) approach to One-Soliton Solutions (19) under the limiting condition $\Delta \to 0$ [Δ is defined in (33)].

3.3 Two-periodic wave solutions for Eq. (1)

From Riemann Theta Function (25), we derive the two-Riemann theta function as:

$$\vartheta(\zeta, \lambda_2) = \sum_{\eta\in\mathbb{Z}^2} e^{\pi i\langle\lambda_2\eta,\eta\rangle+2\pi i\langle\zeta,\eta\rangle}, \tag{46}$$

where $\eta = (\eta_1, \eta_2)^T \in \mathbb{Z}^2$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$, $\mathbb{C}$ denotes the complex number, $\zeta_r = Q_rx + B_ry + R_rt + \epsilon_r$, $r = 1, 2$, Q_r's, B_r's, R_r's are all the constants, $-i\lambda_2$ is a real-valued 2×2 matrix:

$$\lambda_2 = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{pmatrix}, \ \mathrm{Im}(\lambda_{11}) > 0, \ \mathrm{Im}(\lambda_{22}) > 0,$$
$$\lambda_{12}^2 - \lambda_{11}\lambda_{22} > 0. \tag{47}$$

Substituting Eq. (46) into (28), we can derive

$$\begin{aligned} &\mathscr{L}(D_x, D_y, D_t)\vartheta(\zeta_1, \zeta_2, \lambda_2)\cdot\vartheta(\zeta_1, \zeta_2, \lambda_2) \\ &= \sum_{\varpi,\eta\in\mathbb{Z}^2} \mathscr{L}\Big(2i\pi\langle\eta-\varpi, Q\rangle, 2i\pi\langle\eta-\varpi, B\rangle, \\ &\quad 2i\pi\langle\eta-\varpi, R\rangle\Big)e^{2\pi i\langle\zeta,\eta+\varpi\rangle+\pi i(\langle\lambda_2\eta,\eta\rangle+\langle\lambda_2\varpi,\varpi\rangle)} \\ &\overset{\varpi'=\varpi+\eta}{=} \sum_{\varpi'\in\mathbb{Z}^2}\Big\{\sum_{\eta\in\mathbb{Z}^2}\mathscr{L}\Big(2i\pi\langle 2\eta-\varpi', Q\rangle, \\ &\quad 2i\pi\langle 2\eta-\varpi', B\rangle, 2i\pi\langle 2\eta-\varpi', R\rangle\Big) \\ &\quad e^{\pi i\left[\langle\lambda_2(\eta-\varpi'),\eta-\varpi'\rangle+\langle\lambda_2\eta,\eta\rangle\right]}\Big\}e^{2\pi i\langle\zeta,\varpi'\rangle} \\ &= \sum_{\varpi'\in Z^2}\tilde{\mathscr{L}}\left(\varpi'\right)e^{2\pi i\langle\zeta,\varpi'\rangle}, \end{aligned} \tag{48}$$

where $Q = (Q_1, Q_2)^T$, $B = (B_1, B_2)^T$, $R = (R_1, R_2)^T$ and $\varpi' = (\varpi_1', \varpi_2')^T$. From Eq. (48), and setting $\eta' = \eta - \delta_{\sigma,j}$, $(j = 1, 2)$, we can obtain

$$\begin{aligned} &\tilde{\mathscr{L}}\left(\varpi'\right) \\ &= \sum_{\eta\in\mathbb{Z}^2}\mathscr{L}\Big(2i\pi\langle 2\eta-\varpi', Q\rangle, 2i\pi\langle 2\eta-\varpi', B\rangle, \\ &\quad 2i\pi\langle 2\eta-\varpi', R\rangle\Big)e^{\pi i\left[\langle\lambda_2(\eta-\varpi'),\eta-\varpi'\rangle+\langle\lambda_2\eta,\eta\rangle\right]} \\ &= \sum_{\eta\in\mathbb{Z}^2}\mathscr{L}\Big\{2i\pi\sum_{\sigma=1}^{2}[2\eta_\sigma' - (\varpi_\sigma' - 2\delta_{\sigma,j})]Q_\sigma, \\ &\quad 2i\pi\sum_{\sigma=1}^{2}[2\eta_\sigma' - (\varpi_\sigma' - 2\delta_{\sigma,j})]B_\sigma, \\ &\quad 2i\pi\sum_{\sigma=1}^{2}[2\eta_\sigma' - (\varpi_\sigma' - 2\delta_{\sigma,j})]R_\sigma\Big\} \end{aligned}$$

$$
\begin{aligned}
&e^{\pi i \sum_{\sigma,\varsigma=1}^{2}\left[(\eta_\sigma^{'}+\delta_{\sigma,j})(\eta_\varsigma^{'}+\delta_{\varsigma,j})+(\varpi_\sigma^{'}-\eta_\sigma^{'}-\delta_{\sigma,j})(\varpi_\varsigma^{'}-\eta_\varsigma^{'}-\delta_{\varsigma,j})\right]\lambda_{\sigma,\varsigma}}\\
&=\begin{cases}\tilde{\mathscr{L}}(\varpi_1^{'}-2,\varpi_2^{'})e^{2\pi i(\varpi_1^{'}-1)\lambda_{11}+2\pi i\varpi_2^{'}\lambda_{12}}, & j=1,\\ \tilde{\mathscr{L}}(\varpi_1^{'},\varpi_2^{'}-2)e^{2\pi i(\varpi_2^{'}-1)\lambda_{22}+2\pi i\varpi_1^{'}\lambda_{12}}, & j=2,\end{cases}\\
&\quad \varpi^{'},\eta^{'}\in\mathbb{Z}^2,
\end{aligned}
\tag{49}
$$

where $\delta_{\sigma,j}$'s represent the Kronecker's delta [64]. Equation (49) implies that if $\tilde{\mathscr{L}}(0,0)=\tilde{\mathscr{L}}(1,0)=\tilde{\mathscr{L}}(0,1)=\tilde{\mathscr{L}}(1,1)=0$, then $\tilde{\mathscr{L}}(\varpi_1^{'},\varpi_2^{'})=0$ for all $\varpi_1^{'},\varpi_2^{'}\in\mathbb{Z}^2$, Eq. (46) is the solution for Eq. (28). Setting $\Psi_r=(\Psi_r^{[1]},\Psi_r^{[2]})^T, r=1,2,3,4, \Psi_1=(0,0)^T, \Psi_2=(1,0)^T, \Psi_3=(0,1)^T, \Psi_4=(1,1)^T$, we have

$$
\begin{aligned}
&\tilde{\mathscr{L}}(0,0)\\
&\quad=\sum_{\eta\in\mathbb{Z}^2}\mathscr{L}\Big(2i\pi\langle 2\eta-\Psi_1,Q\rangle,2i\pi\langle 2\eta-\Psi_1,B\rangle,\\
&\quad 2i\pi\langle 2\eta-\Psi_1,R\rangle\Big)e^{\pi i[\langle\lambda(\eta-\Psi_1),\eta-\Psi_1\rangle+\langle\lambda\eta,\eta\rangle]}=0,\\
&\tilde{\mathscr{L}}(1,0)\\
&\quad=\sum_{\eta\in\mathbb{Z}^2}\mathscr{L}\Big(2i\pi\langle 2\eta-\Psi_2,Q\rangle,2i\pi\langle 2\eta-\Psi_2,B\rangle,\\
&\quad 2i\pi\langle 2\eta-\Psi_2,R\rangle\Big)e^{\pi i[\langle\lambda(\eta-\Psi_2),\eta-\Psi_2\rangle+\langle\lambda\eta,\eta\rangle]}=0,\\
&\tilde{\mathscr{L}}(0,1)\\
&\quad=\sum_{\eta\in\mathbb{Z}^2}\mathscr{L}\Big(2i\pi\langle 2\eta-\Psi_3,Q\rangle,2i\pi\langle 2\eta-\Psi_3,B\rangle,\\
&\quad 2i\pi\langle 2\eta-\Psi_3,R\rangle\Big)e^{\pi i[\langle\lambda(\eta-\Psi_3),\eta-\Psi_3\rangle+\langle\lambda\eta,\eta\rangle]}=0,\\
&\tilde{\mathscr{L}}(1,1)\\
&\quad=\sum_{\eta\in\mathbb{Z}^2}\mathscr{L}\Big(2i\pi\langle 2\eta-\Psi_4,Q\rangle,2i\pi\langle 2\eta-\Psi_4,B\rangle,\\
&\quad 2i\pi\langle 2\eta-\Psi_4,R\rangle\Big)e^{\pi i[\langle\lambda(\eta-\Psi_4),\eta-\Psi_4\rangle+\langle\lambda\eta,\eta\rangle]}=0.
\end{aligned}
\tag{50}
$$

Combining Eqs. (28) and (50), we derive

$$
\begin{aligned}
&\sum_{\eta\in\mathbb{Z}^2}\Big[-144\pi^2\langle 2\eta-\Psi_r,Q\rangle\langle 2\eta-\Psi_r,R\rangle\\
&-64\pi^6\langle 2\eta-\Psi_r,Q\rangle^6-64u_0\pi^6\langle 2\eta-\Psi_r,Q\rangle^6\\
&+4\alpha\pi^2\langle 2\eta-\Psi_r,B\rangle^2\\
&-16\gamma\pi^4\langle 2\eta-\Psi_r,Q\rangle^3\langle 2\eta-\Psi_r,B\rangle+c\Big]\\
&e^{\pi i[\langle\lambda(\eta-\Psi_r),\eta-\Psi_r\rangle+\langle\lambda\eta,\eta\rangle]}=0.
\end{aligned}
\tag{51}
$$

Accordingly, Eq. (51) can be rewritten as a linear system,

$$
\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14}\\ g_{21} & g_{22} & g_{23} & g_{24}\\ g_{31} & g_{32} & g_{33} & g_{34}\\ g_{41} & g_{42} & g_{43} & g_{44}\end{pmatrix}\begin{pmatrix}R_1\\ R_2\\ u_0\\ c\end{pmatrix}=\begin{pmatrix}q_1\\ q_2\\ q_3\\ q_4\end{pmatrix},
\tag{52}
$$

with

$$
\begin{aligned}
&\mathscr{I}_1=e^{\pi i\lambda_{11}},\ \mathscr{I}_2=e^{\pi i\lambda_{22}},\ \mathscr{I}_3=e^{2\pi i\lambda_{12}},\\
&G=(g_{rj})_{4\times 4},\ q=(q_1,q_2,q_3,q_4)^T,\\
&\mathscr{A}_r(\eta)=\mathscr{I}_1^{\left\{\eta_1^2+\left(\eta_1-\Psi_r^{[1]}\right)^2\right\}}\mathscr{I}_2^{\left\{\eta_2^2+\left(\eta_2-\Psi_r^{[2]}\right)^2\right\}}\\
&\qquad\mathscr{I}_3^{\left\{\eta_1\eta_2+\left(\eta_1-\Psi_r^{[1]}\right)\left(\eta_2-\Psi_r^{[2]}\right)\right\}},\\
&g_{r1}=-144\pi^2\sum_{\eta\in\mathbb{Z}^2}\langle 2\eta-\Psi_r,Q\rangle\\
&\qquad\left(2\eta_1-\Psi_r^{[1]}\right)\mathscr{A}_r(\eta),\\
&g_{r2}=-144\pi^2\sum_{\eta\in\mathbb{Z}^2}\langle 2\eta-\Psi_r,Q\rangle\\
&\qquad\left(2\eta_2-\Psi_r^{[2]}\right)\mathscr{A}_r(\eta),\\
&g_{r3}=-64\pi^6\sum_{\eta\in\mathbb{Z}^2}\langle 2\eta-\Psi_r,Q\rangle^6\mathscr{A}_r(\eta),\ g_{r4}\\
&\quad=\sum_{\eta\in Z^2}\mathscr{A}_r(\eta),\\
&q_r=\sum_{\eta\in\mathbb{Z}^2}\Big(64\pi^6\langle 2\eta-\Psi_r,Q\rangle^6\\
&\qquad-4\alpha\pi^2\langle 2\eta-\Psi_r,B\rangle^2\\
&\qquad+16\gamma\pi^4\langle 2\eta-\Psi_r,Q\rangle^3\\
&\quad\langle 2\eta-\Psi_r,B\rangle\Big)\mathscr{A}_r(\eta).
\end{aligned}
\tag{53, 54}
$$

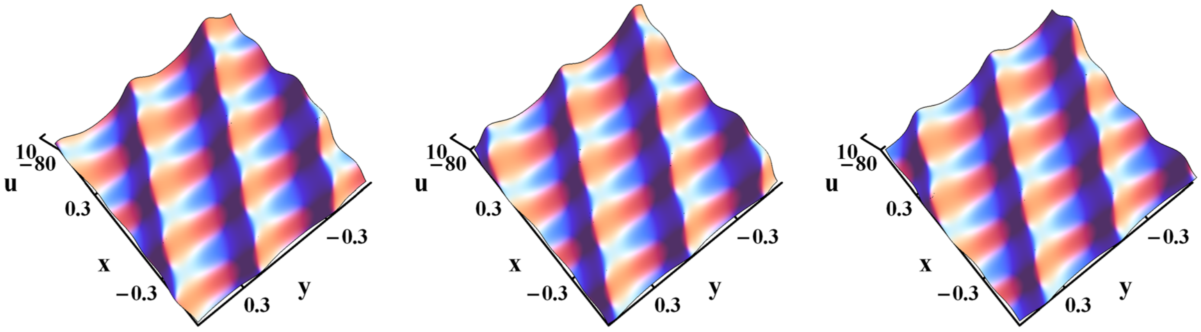
Solving System (52), we can derive the two-periodic wave[2] solutions for Eq. (1) as

$$
u=u_0+2\left[\ln\vartheta(\zeta_1,\zeta_2,\lambda)\right]_{xx}.
\tag{55}
$$

Figure 4 shows that the periodic behaviors for the two-periodic wave exist along the x and y directions, respectively. Similarly, the asymptotic behaviors of Two-Periodic Wave Solutions (55) will be studied.

---

[2] Two-periodic wave indicates a periodic wave formed by the superposition of two waves with the different periods in the x, y and t directions [63].



(a) t=-1 **(b)** t=0 **(c)** t=1

Fig. 4 Two-periodic wave via Solutions (55) with $\lambda_{11} = 0.6i,\ \lambda_{12} = 0.5i,\ \lambda_{22} = 2i,\ Q_1 = 1,\ Q_2 = -2.5,\ B_1 = 2,\ B_2 = 2.2$ and $\alpha = \gamma = u_0 = 1$

Expansions for the matrices in System (52) can be written as

$$
\begin{aligned}
G = &\begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}\\
&+ \begin{pmatrix} 0&0&0&0\\ -288\pi^2 Q_1 & 0 & -128\pi^6 Q_1^6 & 2\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \mathscr{J}_1\\
&+ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0 & -288\pi^2 Q_2 & -128\pi^6 Q_2^6 & 2\\ 0&0&0&0 \end{pmatrix} \mathscr{J}_2\\
&+ \begin{pmatrix} -1152\pi^2 Q_1 & 0 & -8192\pi^6 Q_1^6 & 2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \mathscr{J}_1^2\\
&+ \begin{pmatrix} 0 & -1152\pi^2 Q_2 & -8192\pi^6 Q_2^6 & 2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \mathscr{J}_2^2\\
&+ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ \beta_1 & -\beta_1 & \beta_2 & 2 \end{pmatrix} \mathscr{J}_1 \mathscr{J}_2\\
&+ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ \beta_3 & \beta_3 & \beta_4 & 2 \end{pmatrix} \mathscr{J}_1 \mathscr{J}_2 \mathscr{J}_3\\
&+ o\left(\mathscr{J}_1^r, \mathscr{J}_2^j, \mathscr{J}_3^l\right),\ r + j + l \geq 3,
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
\begin{pmatrix} R_1\\ R_2\\ u_0\\ c \end{pmatrix} = &\begin{pmatrix} R_1^{(00)}\\ R_2^{(00)}\\ u_0^{(00)}\\ c^{(00)} \end{pmatrix} + \begin{pmatrix} R_1^{(11)}\\ R_2^{(11)}\\ u_0^{(11)}\\ c^{(11)} \end{pmatrix} \mathscr{J}_1 + \begin{pmatrix} R_1^{(21)}\\ R_2^{(21)}\\ u_0^{(21)}\\ c^{(21)} \end{pmatrix} \mathscr{J}_2\\
&+ \begin{pmatrix} R_1^{(12)}\\ R_2^{(12)}\\ u_0^{(12)}\\ c^{(12)} \end{pmatrix} \mathscr{J}_1^2 + \begin{pmatrix} R_1^{(22)}\\ R_2^{(22)}\\ u_0^{(22)}\\ c^{(22)} \end{pmatrix} \mathscr{J}_2^2\\
&+ \begin{pmatrix} R_1^{(2)}\\ R_2^{(2)}\\ u_0^{(2)}\\ c^{(2)} \end{pmatrix} \mathscr{J}_1 \mathscr{J}_2 + \begin{pmatrix} R_1^{(3)}\\ R_2^{(3)}\\ u_0^{(3)}\\ c^{(3)} \end{pmatrix} \mathscr{J}_1 \mathscr{J}_2 \mathscr{J}_3\\
&+ o\left(\mathscr{J}_1^r, \mathscr{J}_2^j, \mathscr{J}_3^l\right),\ r + j + l \geq 3,
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
q = &\begin{pmatrix} 0\\ \rho_1\\ 0\\ 0 \end{pmatrix} \mathscr{J}_1 + \begin{pmatrix} 0\\ 0\\ \rho_2\\ 0 \end{pmatrix} \mathscr{J}_2 + \begin{pmatrix} \rho_3\\ 0\\ 0\\ 0 \end{pmatrix} \mathscr{J}_1^2\\
&+ \begin{pmatrix} \rho_4\\ 0\\ 0\\ 0 \end{pmatrix} \mathscr{J}_2^2 + \begin{pmatrix} 0\\ 0\\ 0\\ \rho_5 \end{pmatrix} \mathscr{J}_1 \mathscr{J}_2\\
&+ \begin{pmatrix} 0\\ 0\\ 0\\ \rho_6 \end{pmatrix} \mathscr{J}_1 \mathscr{J}_2 \mathscr{J}_3 + o\left(\mathscr{J}_1^r, \mathscr{J}_2^j, \mathscr{J}_3^l\right),\\
&r + j + l \geq 3,
\end{aligned}
\tag{58}
$$

with

$$
\begin{aligned}
&\beta_1 = -288\pi^2(Q_1 - Q_2),\ \beta_2 = -128\pi^6(Q_1 - Q_2)^6,\\
&\beta_3 = -288\pi^2(Q_1 + Q_2),\ \beta_4 = -128\pi^6(Q_1 + Q_2)^6,\\
&\rho_1 = 8\pi^2\left(16\pi^4 Q_1^6 - \alpha B_1^2 + 4\gamma\pi^2 Q_1^3 B_1\right),
\end{aligned}
$$

$$
\begin{aligned}
\rho_2 &= 8\pi^2\left(16\pi^4 Q_2^6 - \alpha B_2^2 + 4\gamma\pi^2 Q_2^3 B_2\right),\\
\rho_3 &= 32\pi^2\left(256\pi^4 Q_1^6 - \alpha B_1^2 + 16\gamma\pi^2 Q_1^3 B_1\right),\\
\rho_4 &= 32\pi^2\left(256\pi^4 Q_2^6 - \alpha B_2^2 + 16\gamma\pi^2 Q_2^3 B_2\right),\\
\rho_5 &= 8\pi^2\Big[16\pi^4(Q_1 - Q_2)^6 - \alpha(B_1 - B_2)^2\\
&\quad + 4\gamma\pi^2(Q_1 - Q_2)^3(B_1 - B_2)\Big],\\
\rho_6 &= 8\pi^2\Big[16\pi^4(Q_1 + Q_2)^6 - \alpha(B_1 + B_2)^2\\
&\quad + 4\gamma\pi^2(Q_1 + Q_2)^3(B_1 + B_2)\Big],
\end{aligned}
\tag{59}
$$

where $o\left(\mathscr{J}_1^r, \mathscr{J}_2^j, \mathscr{J}_3^l\right)$ denotes the infinitely small quantity.

Substituting Eqs. (56), (58) and (57) into System (52) and comparing the same order of $\mathscr{J}_1$, $\mathscr{J}_2$ and $\mathscr{J}_3$, we can obtain

$$
\begin{aligned}
&c^{(00)} = c^{(11)} = c^{(21)} = c^{(2)} = c^{(3)} = 0,\\
&-288\pi^2 Q_1 R_1^{(00)} - 128\pi^6 Q_1^6 u_0^{(00)} = \rho_1,\\
&-288\pi^2 Q_2 R_2^{(00)} - 128\pi^6 Q_2^6 u_0^{(00)} = \rho_2,\\
&c^{(12)} - 1152\pi^2 Q_1 R_1^{(00)} - 8192\pi^6 Q_1^6 u_0^{(00)} = \rho_3,\\
&c^{(22)} - 1152\pi^2 Q_2 R_2^{(00)} - 8192\pi^6 Q_2^6 u_0^{(00)} = \rho_4,\\
&\beta_1 R_1^{(00)} - \beta_1 R_2^{(00)} + \beta_2 u_0^{(00)} = \rho_5,\\
&\beta_3 R_1^{(00)} + \beta_3 R_2^{(00)} + \beta_4 u_0^{(00)} = \rho_6,\\
&288\pi^2 Q_2 R_2^{(11)} + 128\pi^6 Q_2^6 u_0^{(11)} = 0,\\
&288\pi^2 Q_1 R_1^{(21)} + 128\pi^6 Q_1^6 u_0^{(21)} = 0,\\
&288\pi^2 Q_1 R_1^{(11)} + 128\pi^6 Q_1^6 u_0^{(11)} = 0,\\
&288\pi^2 Q_2 R_2^{(21)} + 128\pi^6 Q_2^6 u_0^{(21)} = 0.
\end{aligned}
\tag{60}
$$

Combining Eqs. (57) and (60), and taking $u_0^{(00)} = 0$, we can notice that

$$
\begin{aligned}
u_0 &= o\left(\mathscr{J}_1, \mathscr{J}_2\right) \to 0,\ c \to 0,\\
R_1 &= \frac{16\pi^4 Q_1^6 - \alpha B_1^2 + 4\gamma\pi^2 Q_1^3 B_1}{-36 Q_1} + o\left(\mathscr{J}_1, \mathscr{J}_2\right)\\
&\to \frac{16\pi^4 Q_1^6 - \alpha B_1^2 + 4\gamma\pi^2 Q_1^3 B_1}{-36 Q_1},\\
R_2 &= \frac{16\pi^4 Q_2^6 - \alpha B_2^2 + 4\gamma\pi^2 Q_2^3 B_2}{-36 Q_2} + o\left(\mathscr{J}_1, \mathscr{J}_2\right)\\
&\to \frac{16\pi^4 Q_2^6 - \alpha B_2^2 + 4\gamma\pi^2 Q_2^3 B_2}{-36 Q_2},
\end{aligned}
\tag{61}
$$

when $\left(\mathscr{J}_1, \mathscr{J}_2\right) \to 0$, and assuming that

$$
\begin{aligned}
&u_0 = 0,\ Q_1 = \frac{k_1 + k_2}{2i\pi},\ Q_2 = \frac{k_3 + k_4}{2i\pi},\\
&B_1 = \frac{5k_1^3 + 5k_2^3}{2\gamma i\pi},\ B_2 = \frac{5k_3^3 + 5k_4^3}{2\gamma i\pi},\\
&\epsilon_1 = \frac{-i\pi\lambda_{11} + \ln A_1}{2i\pi},\ \epsilon_2 = \frac{-i\pi\lambda_{22} + \ln A_2}{2i\pi},\\
&\lambda_{12} = \frac{\ln A_{12}}{2i\pi},\ \alpha = \frac{\gamma^2}{5},
\end{aligned}
\tag{62}
$$

where $k_1, k_2, k_3, k_4, A_1, A_2, A_{12}, \alpha$ and γ are determined by Eq. (20). We can rewrite Eq. (46) as

$$
\begin{aligned}
\vartheta(\zeta_1, \zeta_2, \lambda) &= 1 + \left(e^{2\pi i\zeta_1} + e^{-2\pi i\zeta_1}\right)e^{i\pi\lambda_{11}}\\
&\quad + \left(e^{2\pi i\zeta_2} + e^{-2\pi i\zeta_2}\right)e^{i\pi\lambda_{22}}\\
&\quad + \left[e^{2\pi i(\zeta_1+\zeta_2)} + e^{-2\pi i(\zeta_1+\zeta_2)}\right]\\
&\qquad e^{i\pi(\lambda_{11}+2\lambda_{12}+\lambda_{22})} + \cdots\\
&= 1 + A_1 e^{\xi_1+\xi_2} + A_2 e^{\xi_3+\xi_4}\\
&\quad + A_{12} e^{\xi_1+\xi_2+\xi_3+\xi_4},
\end{aligned}
$$

when $\left(\mathscr{J}_1, \mathscr{J}_2\right) \to 0$. (63)

Thus, we notice that Two-Periodic Wave Solutions (55) approach to Two-Soliton Solutions (20) under the limiting conditions $\left(\mathscr{J}_1, \mathscr{J}_2\right) \to 0$ [$\mathscr{J}_1$ and $\mathscr{J}_2$ are defined in (53)].

4 Conclusions

Fluid mechanics has the applications in a wide range of disciplines, such as oceanography, astrophysics, meteorology, and biomedical engineering. In this paper, we have investigated the $(2+1)$-dimensional gCDGKS equation, i.e., Eq. (1), in fluid mechanics. Based on the Pfaffian technique and Constraint (11) on the real constant α, the Nth-Order Pfaffian Solutions (16) have been obtained. One- and two-soliton solutions, i.e., Solutions (19) and (20), have been derived via the Nth-Order Pfaffian Solutions (16). One- and two-periodic-wave solutions, i.e., Solutions (36) and (55), have been constructed via the Hirota–Riemann method. Results can be summarized as follows:

1. Amplitude of the one soliton is irrelevant to the real constant γ, the velocity along the x direction of the one soliton is independent of γ, while the velocity along the y direction of the one soliton is proportional to γ;

2. We show the propagation of the one soliton in Fig. 1 and the interaction between the two solitons in Fig. 2, and found that the one soliton keeps its amplitude and velocity invariant during the propagation and total amplitude of the two solitons in the interaction region is lower than that of any soliton;
3. One-periodic wave has been viewed as a superposition of the overlapping solitary waves, placed one period apart, as shown in Fig. 3;
4. Periodic behaviors for the two-periodic wave have existed along the x and y directions, respectively, as depicted in Fig. 4;
5. With the asymptotic behaviors of One-Periodic-Wave Solutions (36) and Two-Periodic-Wave Solutions (55), we have noticed that One-Periodic-Wave Solutions (36) approach to One-Soliton Solutions (19) under the limiting condition with respect to $\varDelta$ in (33), i.e., $\varDelta \to 0$, that Two-Periodic-Wave Solutions (55) approach to Two-Soliton Solutions (20) under the limiting conditions with respect to $\mathscr{J}_1$ and $\mathscr{J}_2$ in (53), i.e., $\left(\mathscr{J}_1, \mathscr{J}_2\right) \to 0$.

Acknowledgements The authors express their sincere thanks to the members of their discussion group for their valuable suggestions. This work has been supported by the National Natural Science Foundation of China under Grant No. 11772017, and by the Fundamental Research Funds for the Central Universities under Grant No. 50100002016105010.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Nakayama, Y., Boucher, R.F.: Introduction to Fluid Mechanics. Butterworth-Heinemann, Oxford (1999)
2. Maris, H.J.: Note on the history effect in fluid mechanics. Am. J. Phys. **87**, 643 (2019)
3. Wazwaz, A.M.: Two new integrable fourth-order nonlinear equations: multiple soliton solutions and multiple complex soliton solutions. Nonlinear Dyn. **94**, 2655–2663 (2018)
4. Lan, Z.Z., Su, J.J.: Solitary and rogue waves with controllable backgrounds for the non-autonomous generalized AB system. Nonlinear Dyn. **96**, 2535–2546 (2019)
5. Gao, X.Y.: Looking at a nonlinear inhomogeneous optical fiber through the generalized higher-order variable-coefficient Hirota equation. Appl. Math. Lett. **73**, 143–149 (2017)
6. Gao, X.Y.: Mathematical view with observational/experimental consideration on certain (2+1)-dimensional waves in the cosmic/laboratory dusty plasmas. Appl. Math. Lett. **91**, 165–172 (2019)
7. Zhao, X.H., Tian, B., Guo, Y.J., Li, H.M.: Solitons interaction and integrability for a (2+1)-dimensional variable-coefficient Broer-Kaup system in water waves. Mod. Phys. Lett. B **32**, 1750268 (2018)
8. Zhao, X.H., Tian, B., Xie, X.Y., Wu, X.Y., Sun, Y., Guo, Y.J.: Solitons, Backlund transformation and Lax pair for a (2+1)-dimensional Davey-Stewartson system on surface waves of finite depth. Wave. Random Complex **28**, 356–366 (2018)
9. Fogelson, A.L., Neeves, K.B.: Fluid mechanics of blood clot formation. Annu. Rev. Fluid Mech. **47**, 377–403 (2015)
10. Wazwaz, A.M.: Gaussian solitary wave solutions for nonlinear evolution equations with logarithmic nonlinearities. Nonlinear Dyn. **83**, 591–596 (2016)
11. Johnson, R.S.: Application of the ideas and techniques of classical fluid mechanics to some problems in physical oceanography. Philos. Trans. R. Soc. Lond. A **376**, 1 (2017)
12. Teyssier, R.: Grid-based hydrodynamics in astrophysical fluid flows. Annu. Rev. Astron. Astrophys. **53**, 325–364 (2015)
13. Benamou, J.D., Brenier, Y.: A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem. Numer. Math. **84**, 375–393 (2000)
14. Yuan, Y.Q., Tian, B., Liu, L., Wu, X.Y., Sun, Y.: Solitons for the (2+1)-dimensional Konopelchenko-Dubrovsky equations. J. Math. Anal. Appl. **460**, 476–486 (2018)
15. Yuan, Y.Q., Tian, B., Chai, H.P., Wu, X.Y., Du, Z.: Vector semirational rogue waves for a coupled nonlinear Schrödinger system in a birefringent fiber. Appl. Math. Lett. **87**, 50–56 (2019)
16. Yin, H.M., Tian, B., Chai, J., Wu, X.Y.: Stochastic soliton solutions for the (2+1)-dimensional stochastic Broer-Kaup equations in a fluid or plasma. Appl. Math. Lett. **82**, 126–131 (2018)
17. Lan, Z.Z., Hu, W.Q., Guo, B.L.: General propagation lattice Boltzmann model for a variable-coefficient compound KdV-Burgers equation. Appl. Math. Model. **73**, 695–714 (2019)
18. Liu, L., Tian, B., Yuan, Y.Q., Du, Z.: Dark-bright solitons and semirational rogue waves for the coupled Sasa-Satsuma equations. Phys. Rev. E **97**, 052217 (2018)
19. Xie, X.Y., Meng, G.Q.: Dark solitons for the (2+1)-dimensional Davey–Stewartson-like equations in the electrostatic wave packets. Nonlinear Dyn. **93**, 779–783 (2018)
20. Benjamin, T.B., Feir, J.E.: The disintegration of wave trains on deep water. Part 1. Theory. J. Fluid Mech. **27**, 417–430 (1967)
21. Jackiw, R., Pi, S.Y.: Soliton solutions to the gauged nonlinear Schrödinger equation on the plane. Phys. Rev. Lett. **64**, 2969 (1990)
22. Dai, C.Q., Wang, Y.Y., Fan, Y., Yu, D.G.: Reconstruction of stability for Gaussian spatial solitons in quintic-septimal nonlinear materials under PT-symmetric potentials. Nonlinear Dyn. **92**, 1351–1358 (2018)
23. Yin, H.M., Tian, B., Chai, J., Liu, L., Sun, Y.: Numerical solutions of a variable-coefficient nonlinear Schrödinger equation for an inhomogeneous optical fiber. Comput. Math. Appl. **76**, 1827–1836 (2018)
24. Du, Z., Tian, B., Chai, H.P., Sun, Y., Zhao, X.H.: Rogue waves for the coupled variable-coefficient fourth-order nonlinear Schrödinger equations in an inhomogeneous optical fiber. Chaos Soliton. Fract. **109**, 90–98 (2018)

25. Du, Z., Tian, B., Chai, H.P., Yuan, Y.Q.: Vector multi-rogue waves for the three-coupled fourth-order nonlinear Schrödinger equations in an alpha helical protein. Commun. Nonlinear Sci. Numer. Simulat. **67**, 49–59 (2019)
26. Zhang, C.R., Tian, B., Wu, X.Y., Yuan, Y.Q., Du, X.X.: Rogue waves and solitons of the coherently coupled nonlinear Schrödinger equations with the positive coherent coupling. Phys. Scr. **93**, 095202 (2018)
27. Wazwaz, A.M.: Construction of soliton solutions and periodic solutions of the Boussinesq equation by the modified decomposition method. Chaos Solitons Fractals **12**, 1549–1556 (2001)
28. Liu, J.G., Zhou, L., He, Y.: Multiple soliton solutions for the new (2+1)-dimensional Korteweg–de Vries equation by multiple exp-function method. Appl. Math. Lett. **80**, 71–78 (2018)
29. Osman, M.S., Wazwaz, A.M.: An efficient algorithm to construct multi-soliton rational solutions of the (2+1)-dimensional KdV equation with variable coefficients. Appl. Math. Comput. **321**, 282–289 (2018)
30. Zhang, C.R., Tian, B., Liu, L., Chai, H.P., Du, Z.: Vector breathers with the negatively coherent coupling in a weakly birefringent fiber. Wave Motion **84**, 68–80 (2019)
31. Du, X.X., Tian, B., Wu, X.Y., Yin, H.M., Zhang, C.R.: Lie group analysis, analytic solutions and conservation laws of the (3 + 1)-dimensional Zakharov-Kuznetsov-Burgers equation in a collisionless magnetized electron-positron-ion plasma. Eur. Phys. J. Plus **133**, 378 (2018)
32. Ahmed, I., Seadawy, A.R., Lu, D.C.: Mixed lump-solitons, periodic lump and breather soliton solutions for (2+1)-dimensional extended Kadomtsev–Petviashvili dynamical equation. Int. J. Mod. Phys. B **33**, 1950019 (2019)
33. Manukure, S., Zhou, Y., Ma, W.X.: Lump solutions to a (2+1)-dimensional extended KP equation. Comput. Math. Appl. **75**, 2414–2419 (2018)
34. Sun, Y., Tian, B., Xie, X.Y., Chai, J., Yin, H.M.: Rogue waves and lump solitons for a (3+1)-dimensional B-type Kadomtsev–Petviashvili equation in fluid dynamics. Wave Random Complex **28**, 544–552 (2018)
35. Qian, C., Rao, J.G., Liu, Y.B., He, J.S.: Rogue waves in the three-dimensional Kadomtsev–Petviashvili equation. Chin. Phys. Lett. **33**, 110201 (2016)
36. Hu, C.C., Tian, B., Wu, X.Y., Du, Z., Zhao, X.H.: Lump wave-soliton and rogue wave-soliton interactions for a (3+1)-dimensional B-type Kadomtsev-Petviashvili equation in a fluid. Chin. J. Phys. **56**, 2395–2403 (2018)
37. Hu, C.C., Tian, B., Wu, X.Y., Yuan, Y.Q., Du, Z.: Mixed lump-kink and rogue wave-kink solutions for a (3 + 1)-dimensional B-type Kadomtsev-Petviashvili equation in fluid mechanics. Eur. Phys. J. Plus **133**, 40 (2018)
38. Huang, Q.M., Gao, Y.T.: Wronskian, Pfaffian and periodic wave solutions for a (2+1)-dimensional extended shallow water wave equation. Nonlinear Dyn. **89**, 2855–2866 (2017)
39. Arkadiev, V.A., Pogrebkov, A.K., Polivanov, M.C.: Inverse scattering transform method and soliton solutions for Davey–Stewartson II equation. Physica D **36**, 189–197 (1989)
40. Ablowitz, M.J., Musslimani, Z.H.: Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation. Nonlinearity **29**, 915 (2016)
41. Khalique, C.M., Biswas, A.: A Lie symmetry approach to nonlinear Schrödinger's equation with non-Kerr law nonlinearity. Commun. Nonlinear Sci. Numer. Simul. **14**, 4033–4040 (2009)
42. Chen, S.S., Tian, B., Sun, Y., Zhang, C.R.: Generalized Darboux Transformations, Rogue Waves, and Modulation Instability for the Coherently Coupled Nonlinear Schrödinger Equations in Nonlinear Optics. Ann. Phys. (2019). https://doi.org/10.1002/andp.201900011
43. Chen, S.S., Tian, B., Liu, L., Yuan, Y.Q., Zhang, C.R.: Conservation laws, binary Darboux transformations and solitons fora higher-order nonlinear Schrödinger system. Chaos, Solitons Fractals **118**, 337–346 (2019)
44. Nakamura, A., Ohta, Y.: Bilinear, Pfaffian and Legendre function structuresof the Tomimatsu–Sato solutions of the Ernst equation in general relativity. J. Phys. Soc. Jpn. **60**, 1835–1838 (1991)
45. Kumar, S., Zhou, Q., Bhrawy, A.H., Zerrad, E., Biswas, A., Belic, M.: Optical solitons in birefringent fibers by Lie symmetry analysis. Rom. Rep. Phys. **68**, 341–352 (2016)
46. Lan, Z.Z.: Periodic, breather and rogue wave solutions for a generalized (3+1)-dimensional variable-coefficient B-type Kadomtsev–Petviashvili equation in fluid dynamics. Appl. Math. Lett. **94**, 126–132 (2019)
47. Ohta, Y.: Pfaffian solution for coupled discrete nonlinear Schrödinger equation. Chaos Solitons Fractals **11**, 91–95 (2000)
48. Wang, M., Tian, B., Sun, Y., Yin, H.M., Zhang, Z.: Mixed lump-stripe, bright rogue wave-stripe, dark rogue wave stripe and dark rogue wave solutions of a generalized Kadomtsev-Petviashvili equation in fluid mechanics. Chin. J. Phys. **60**, 440–449 (2019)
49. Xie, X.Y., Meng, G.Q.: Collisions between the dark solitons for a nonlinear system in the geophysical fluid. Chaos, Solitons Fractals **107**, 143–145 (2018)
50. Xie, X.Y., Meng, G.Q.: Dark solitons for a variable-coefficient AB system in the geophysical fluids or nonlinear optics. Eur. Phys. J. Plus **134**, 359 (2019)
51. Gilson, C.R.: Generalizing the KP hierarchies: Pfaffian hierarchies. Theor. Math. Phys. **133**, 1663–1674 (2002)
52. Ma, P.L., Tian, S.F., Zou, L., Zhang, T.T.: The solitary waves, quasi-periodic waves and integrability of a generalized fifth-order Korteweg–de Vries equation. Wave Random Complex **29**, 247–263 (2019)
53. Peng, W.Q., Tian, S.F., Zou, L., Zhang, T.T.: Characteristics of the solitary waves and lump waves with interaction phenomena in a (2+1)-dimensional generalized Caudrey–Dodd–Gibbon–Kotera–Sawada equation. Nonlinear Dyn. **93**, 1841–1851 (2018)
54. Meng, X.H.: The periodic solitary wave solutions for the (2 + 1)-dimensional fifth-order KdV equation. J. Appl. Math. Phys. **2**, 639–643 (2014)
55. Cao, C.W., Wu, Y.T., Geng, X.G.: On quasi-periodic solutions of the 2+1 dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada equation. Phys. Lett. A **256**, 59–65 (1999)
56. Fang, T., Gao, C.N., Wang, H., Wang, Y.H.: Lump-type solution, rogue wave, fusion and fission phenomena for the (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada equation. Mod. Phys. Lett. B **33**, 1950198 (2019)

57. Batwa, S., Ma, W.X.: Lump solutions to a (2+1)-dimensional fifth-order KdV-like equation. Adv. Math. Phys. **2018**, 2062398 (2018)
58. Gupta, A.K., Ray, S.S.: Numerical treatment for the solution of fractional fifth-order Sawada–Kotera equation using second kind Chebyshev wavelet method. Appl. Math. Model. **39**, 5121–5130 (2015)
59. Liu, C.F., Dai, Z.D.: Exact soliton solutions for the fifth-order Sawada–Kotera equation. Appl. Math. Comput. **206**, 272–275 (2008)
60. Naher, H., Abdullah, F.A., Mohyud-Din, S.T.: Extended generalized Riccati equation mapping method for the fifth-order Sawada–Kotera equation. AIP Adv. **3**, 052104 (2013)
61. Guo, Y.F., Li, D.L., Wang, J.X.: The new exact solutions of the fifth-order Sawada–Kotera equation using three wave method. Appl. Math. Lett. **94**, 232–237(2019)
62. Hirota, R.: The Direct Method in Soliton Theory. Cambridge Univ. Press, Cambridge (2004)
63. Xu, M.J., Tian, S.F., Tu, J.M., Ma, P.L., Zhang, T.T.: Quasi-periodic wave solutions with asymptotic analysis to the Saweda–Kotera–Kadomtsev–Petviashvili equation. Eur. Phys. J. Plus **130**, 174 (2015)
64. Furukawa, M., Tokuda, S.: Mechanism of stabilization of ballooning modes by toroidal rotation shear in tokamaks. Phys. Rev. Lett. **94**, 175001 (2005)