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ORIGINAL PAPER

High-order rogue waves and their dynamics of the Fokas–Lenells equation revisited: a variable separation technique

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Abstract The Fokas–Lenells (FL) equation is an integrable higher-order extension of nonlinear Schrödinger equation. One approach to generating its breather solutions is based on Darboux transformation (DT) and iterations. However, the DT of FL equation contains negative powers of the spectral parameter, which can lead to very complicated expressions when N is large. In this paper, we avoid the negative powers by adopting a variable separation and Taylor expansion technique to solve the Lax pair of FL system. Furthermore, stability of the proposed technique is demonstrated in detail.

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1 Introduction

Rogue waves are instantaneous large-amplitude localized waves and have been extensively studied in many fields, including oceanic motion, optics, plasmas and super fluids (see for instance in [1–13]). The generation of rogue waves is a complex process, involving many factors such as dispersion enhancement of transient wave groups, geometrical focusing, wave-current interaction and modulation instabilities. A much-studied model is the integrable nonlinear Schrödinger (NLS) equation [10] and its breather solutions, especially the Peregrine breather [11]. There are several integrable reductions of the higher-order NLS models, such as the derivative NLS equation and Hirota and Sasa–Satsuma equations [14–27]. Here, we consider the Fokas–Lenells (FL) equation, which is closely linked to the derivative NLS model,

$$iq_{xt} - iq_{xx} + 2q_x - q_x q q^* + iq = 0, \tag{1}$$

where q is a complex wave amplitude. It is a higher-order integrable extension of NLS equation [28–31] and has been invoked in the context of optical fibers

[29]. The soliton solutions of the FL equation (1) were exhibited by [28,29] and some breather solutions by [30,31].

The soliton solutions (often identified as rogue waves) of NLS equation are usually obtained through Darboux transformations (DTs). That is, the first-order solitons are found from a pre-specified seed solution, and the Nth-order solitons are found through iterations of DT. One key step is to expand the specific solution in terms of the spectral parameter. However, DT of the FL equation contains negative powers of the spectral parameter, which can lead to very complicated expressions when N is large [31]. Here, we adopt a different approach by introducing a parameter matrix and then directly find the Nth-order breathers through a variable separation technique and Taylor series expansion, see [32–36] for use of similar methods for NLS equations.

The rest of the article is organized as follows. Section 2 provides some preliminaries related to FL equation and introduces our variable separation technique. In Sect. 3, we describe the expansion of the eigenfunction and obtain the formula for Nth-order soliton solutions in Sect. 4. In Sect. 5, we confirm the effectiveness of our method and a range of dynamic behaviors of rogue wave solutions are displayed graphically. Section 6 summarizes the stability of the proposed technique guarantees.

2 Variable separation for the eigenfunction Ψ

It is useful to recall that we can extend (1) into an FL system, [30,31]

$$iq_{xt} - iq_{xx} + 2q_x - q_x qr + iq = 0, \tag{2}$$

$$ir_{xt} - ir_{xx} - 2r_x + r_x rq + ir = 0. \tag{3}$$

Clearly, when $r = q^*$, the FL system (2, 3) reduces to the FL equation (1). The Lax pair of the FL system (2, 3) is

$$\Psi_x = U\Psi,\ U = J\lambda^2 + Q\lambda, \tag{4}$$

$$\Psi_t=V\Psi,\ V=J\lambda^2 + Q\lambda + V_0+V_{-1}\lambda^{-1} + \frac{1}{4}J\lambda^{-2}. \tag{5}$$

$$\Psi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix},\quad J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},\quad Q = \begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix},$$

$$V_0=\begin{pmatrix} i - \frac{1}{2}iqr & 0 \\ 0 & -i + \frac{1}{2}iqr \end{pmatrix},\quad V_{-1}=\begin{pmatrix} 0 & \frac{1}{2}iq \\ -\frac{1}{2}ir & 0 \end{pmatrix}.$$

Here, λ is the complex spectral parameter. $\Psi(x,t) = (\varphi, \phi)^T$ is a two-dimensional vector, the eigenfunction corresponding to λ. Applying the expression

$$U_t - V_x + UV - VU = 0$$

to Eqs. (4) and (5) yields the FL system (2, 3).

Now, we present a variable separation for the eigenfunction Ψ. First, note that the FL equation (1) has a periodic seed solution

$$q = c\exp\left\{axi + \left(\frac{(a+1)^2}{a} - c^2\right)ti\right\}. \tag{6}$$

For any λ, we expand Ψ as

$$\Psi = \begin{pmatrix} \varphi(x,t) \\ \phi(x,t) \end{pmatrix} = AFGZ, \tag{7}$$

$$F = \exp(i\Lambda x),\quad G = \exp(i\Omega t), \tag{8}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pm i\zeta}. \end{pmatrix}. \tag{9}$$

Here, we assume that in matrix A, ζ is linearly composed of x and t, i.e., $\zeta = kx + \tilde{c}t$, where k and $\tilde{c}$ are two constants. Similarly, Z is a two-dimensional constant vector. Next, suppose that the matrices Λ, Ω satisfy the commutator relationship,

$$[\Lambda, \Omega] = \Lambda\Omega - \Omega\Lambda = 0. \tag{10}$$

Plugging equation (7) into the Lax equations (4, 5) yields

$$A_x + iA\Lambda - UA = 0,\quad A_t + iA\Omega - VA = 0.$$

Hence, we solve that

$$A=\begin{pmatrix} 1 & 0 \\ 0 & e^{-i(ax+\frac{(a+1)^2}{a}t-c^2t)} \end{pmatrix},\ \Lambda=\begin{pmatrix} -\lambda^2 & ac\lambda \\ -ac\lambda & \lambda^2+a \end{pmatrix},$$

$$\Omega=\left(1+\frac{1}{2a}\lambda^{-2}\right)\Lambda+\left(1-\frac{1}{4}\lambda^{-2}+\frac{1}{2a}-\frac{c^2}{2}\right). \tag{11}$$

In order to obtain F, we find the eigenvalues of the matrix Λ

$$a_1 = \frac{a+\sqrt{a^2-4(-\lambda^4+(a^2c^2-a)\lambda^2)}}{2}, \tag{12}$$

$$a_2 = \frac{a-\sqrt{a^2-4(-\lambda^4+(a^2c^2-a)\lambda^2)}}{2}, \tag{13}$$

$$a_1 + a_2 = a,\quad a_1 \cdot a_2 = -\lambda^4 - a\lambda^2 + a^2c^2\lambda^2,$$

and the eigenvector matrix is

$$H=\begin{pmatrix} 1 & 1 \\ \frac{\lambda^2+a_1}{ac\lambda} & \frac{\lambda^2+a_2}{ac\lambda} \end{pmatrix},\quad H^{-1}=\begin{pmatrix} \frac{\lambda^2+a_2}{a_2-a_1} & \frac{-ac\lambda}{a_2-a_1} \\ \frac{-\lambda^2-a_1}{a_2-a_1} & \frac{ac\lambda}{a_2-a_1} \end{pmatrix}. \tag{14}$$

To note

$$F=\exp(i\Lambda x)=H\begin{pmatrix} e^{a_1 ix} & 0 \\ 0 & e^{a_2 ix} \end{pmatrix}H^{-1}.$$

Let $\eta=\frac{a_1-a_2}{2}$,

$$F=e^{\frac{a}{2}ix}\begin{pmatrix} \cos(\eta x)-si\sin(\eta x) & \frac{ac\lambda}{\eta}i\sin(\eta x) \\ -\frac{ac\lambda}{\eta}i\sin(\eta x) & \cos(\eta x)+si\sin(\eta x) \end{pmatrix}, \tag{15}$$

where $s=\frac{\lambda^2+\frac{a}{2}}{\eta}$. Similarly, the combination of (11) and $G=\exp(i\Omega t)$ derives

$$G=e^{Kit}\begin{pmatrix} \cos(\eta\varepsilon t)-si\sin(\eta\varepsilon t) & \frac{ac\lambda}{\eta}i\sin(\eta\varepsilon t) \\ -\frac{ac\lambda}{\eta}i\sin(\eta\varepsilon t) & \cos(\eta\varepsilon t)+si\sin(\eta\varepsilon t) \end{pmatrix}, \tag{16}$$

with

$$\varepsilon=1+\frac{1}{2a}\lambda^{-2},\ K=(1+\frac{1}{2a}-\frac{c^2}{2}+\frac{a}{2}).$$

3 Expansion of eigenfunction Ψ

In this section, we describe the expansion of eigenfunction Ψ. When $a^2+4\lambda^4-4(a^2c^2-a)\lambda^2\to 0$ and $\eta\to 0$, F and G become rational matrices. To take advantage of this, we choose λ_0 being one solution to the equation $a^2+4y^4-4(a^2c^2-a)y^2=0$ (concerning y) and set $\lambda=\lambda_0(1+\delta)$. Through the Taylor series expansions,

$$F|_{\lambda=\lambda_0(1+\delta)}=e^{\frac{a}{2}ix}\sum_{n=0}^{\infty}F_n\delta^n, \tag{17}$$

where

$$F_n=\begin{pmatrix} F_{n11} & F_{n12} \\ F_{n21} & F_{n22} \end{pmatrix}, \tag{18}$$

$$\begin{aligned}
F_{n11}&=\gamma_n-(\lambda_0^2+\frac{a}{2})i\tau_n-2\lambda_0^2 i\tau_{n-1}-\lambda_0^2 i\tau_{n-2},\\
F_{n12}&=ac\lambda_0 i(\tau_n+\tau_{n-1}),\\
F_{n21}&=-F_{n12},\\
F_{n22}&=\gamma_n+(\lambda_0^2+\frac{a}{2})i\tau_n+2\lambda_0^2 i\tau_{n-1}+\lambda_0^2 i\tau_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
\gamma_n=&\sum_{k=0}^{\lfloor\frac{3}{4}n\rfloor}\sum_{l=0}^{\lfloor\frac{k}{3}\rfloor}\sum_{m=0}^{\lfloor\frac{k-3l}{2}\rfloor}C_{n-k}^{l}C_{n-k-l}^{m}C_{n-k-l-m}^{k-3l-2m}(-1)^{n-k}\\
&\cdot 4^m\lambda_0^{2n-2k+2l+2m}(6\lambda_0^2-2a^2c^2+a)^{k-3l-2m}\\
&\cdot(4\lambda_0^2-2a^2c^2+2a)^{n+2l+m-2k}X_{2(n-k)},\\
\tau_n=&\sum_{k=0}^{\lfloor\frac{3}{4}n\rfloor}\sum_{l=0}^{\lfloor\frac{k}{3}\rfloor}\sum_{m=0}^{\lfloor\frac{k-3l}{2}\rfloor}C_{n-k}^{l}C_{n-k-l}^{m}C_{n-k-l-m}^{k-3l-2m}(-1)^{n-k}\\
&\cdot 4^m\lambda_0^{2n-2k+2l+2m}(6\lambda_0^2-2a^2c^2+a)^{k-3l-2m}\\
&\cdot(4\lambda_0^2-2a^2c^2+2a)^{n+2l+m-2k}X_{2(n-k)+1},\\
X_m=&\frac{x^m}{m!}.
\end{aligned}$$

Similarly, by expanding G,

$$G|_{\lambda=\lambda_0(1+\delta)}=e^{(1+\frac{1}{2a}-\frac{c^2}{2}+\frac{a}{2})it}\sum_{n=0}^{\infty}G_n\delta^n. \tag{19}$$

The calculation of G_n is a lengthy calculation, and we only outline some key steps and definitions here. First, note that

$$\cos(\eta\varepsilon t)=\sum_{k=0}^{\infty}(-1)^k\eta^{2k}\varepsilon^{2k}T_{2k},\quad T_m=\frac{t^m}{m!}.$$

If $\eta=\frac{\sqrt{a^2+4\lambda^4-4(a^2c^2-a)\lambda^2}}{2}$, $\lambda=\lambda_0(1+\delta)$, $\varepsilon=1+\frac{1}{2a}\lambda^{-2}$, then

$$\cos(\eta\varepsilon t)=\sum_{n=0}^{\infty}(-1)^n\lambda_0^{2n}\delta^n\alpha\frac{(2a\lambda_0^2(1+\delta)^2+1)^{2n}}{(1+\delta)^{4n}4^n a^{2n}\lambda_0^{4n}}T_{2k},$$

where

$$\alpha=(\lambda_0^2\delta^3+4\lambda_0^2\delta^2+(6\lambda_0^2-a^2c^2+a)\delta+(4\lambda_0^2-2a^2c^2+2a))^n.$$

Next, let

$$\begin{aligned}
\sum_{i=0}^{\infty}\pi_i\delta^i&=(-1)^k\lambda_0^{2k}\delta^k[\lambda^2\delta^3+4\lambda_0^2\delta^2+4\lambda_0^2-2a^2c^2\\
&\quad+(6\lambda^2-a^2c^2+a)\delta+2a]^k,\\
\sum_{j=0}^{\infty}\kappa_j'\delta^j&=[2a\lambda_0^2+1+4a\lambda_0^2\delta+2a\lambda_0^2\delta^2]^{2k},\\
\sum_{m=0}^{\infty}\upsilon_m'\delta^m&=\left((\delta+1)^{2k}4^k a^{2k}\lambda_0^{4k}\right)^{-1}.
\end{aligned}$$

$$\pi_i=\sum_{l=0}^{\lfloor\frac{i-k}{3}\rfloor}\sum_{m=0}^{\lfloor\frac{i-k-3l}{2}\rfloor}(-1)^k C_k^l C_{k-l}^m C_{k-l-m}^{i-k-3l-2m}$$

$$\cdot \lambda_0^{2k+2l+2m} 4^m (4\lambda_0^2 - 2a^2c^2 + 2a)^{2k+2l+m-i} \cdot (6\lambda_0^2 - a^2c^2 + a)^{i-k-3l-2m}, \tag{20}$$

$$\kappa'_j = \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} C_{2k+1}^{p} C_{2k+1-p}^{j-2p} 2^{2j-3p} a^{j-p} \lambda_0^{2j-2p} \cdot (2a\lambda_0^2 + 1)^{2k+p-j}, \tag{21}$$

$$\upsilon'_m = \frac{(-1)^m}{m!} (4k)_m \left(2a\lambda_0^2\right)^{-2k}, \tag{22}$$

where $(k)_n = k(k+1)\cdots(k+n-1),\ n > 0,\ (k)_0 = 1$. Here, when $i < k$ and $i > 4k$, define $\pi_i = 0$, and when $j > 4k+2$ define $\kappa_j = 0$. Next, let

$$\alpha_n = \sum_{k=0}^{n} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \pi_i \kappa'_j \upsilon'_{n-i-j} T_{2k}. \tag{23}$$

Finally, we obtain G_n,

$$G_n = \begin{pmatrix} G_{n11} & G_{n12} \\ G_{n21} & G_{n22} \end{pmatrix}, \tag{24}$$

$$G_{n11} = \alpha_n - (\lambda_0^2 + \frac{a}{2}) i\beta_n - 2\lambda_0^2 i\beta_{n-1} - \lambda_0^2 i\beta_{n-2},$$
$$G_{n12} = G_{n21} = ac\lambda_0 i(\beta_n + \beta_{n-1}),$$
$$G_{n22} = \alpha_n + (\lambda_0^2 + \frac{a}{2}) i\beta_n + 2\lambda_0^2 i\beta_{n-1} + \lambda_0^2 i\beta_{n-2},$$

where

$$\beta_n = \sum_{k=0}^{n} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \pi_i \kappa_j \upsilon_{n-i-j} T_{2k+1}, \quad \alpha_0 = 1,$$

$$\kappa_j = \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} C_{2k}^{p} C_{2k-p}^{j-2p} 2^{2j-3p} a^{j-p} \lambda_0^{2j-2p} (2a\lambda_0^2 + 1)^{2k+1+p-j},$$

$$\upsilon_m = \frac{(-1)^m}{m!} (4k+2)_m \cdot \frac{1}{(2a\lambda_0^2)^{2k+1}}.$$

Then, let

$$Z = \sum_{q=0}^{\infty} Z_q \delta^q,$$

where Z_j is a complex vector. Thus, we expand Ψ as

$$\Psi|_{\lambda=\lambda_0} = e^{Kit+a/2ix} A \sum_{n=0}^{\infty} \Psi_n \delta^n,$$

$$\Psi_n = \begin{pmatrix} \varphi_n \\ \phi_n \end{pmatrix} = \sum_{s=0}^{n} \sum_{t=0}^{n} F_s G_t Z_{n-s-t}. \tag{25}$$

In this manner, we have expanded Ψ around the point $\lambda = \lambda_0$ with only algebraic manipulations. Since the matrix F_n depends only on x and G_n depends only on t, we call this a variable separation method.

It is useful to note that the binomial expansion

$$\lambda^j \varphi(\delta) = (\lambda_0)^j (1+\delta)^j \varphi(\delta) = \sum_{i=0}^{\infty} \varphi[j,i] \delta^i,$$
$$\lambda^j \phi(\delta) = (\lambda_0)^j (1+\delta)^j \phi(\delta) = \sum_{i=0}^{\infty} \phi[j,i] \delta^i, \tag{26}$$

so that

$$\varphi[j,n] = \sum_{s=0}^{n} (\lambda_0)^j C_j^{n-s} \varphi_s,$$
$$\phi[j,n] = \sum_{s=0}^{n} (\lambda_0)^j C_j^{n-s} \phi_s, \quad j > 0,$$
$$\varphi[j,n] = \sum_{s=0}^{n} (-1)^{n-s} (\lambda_0)^j \frac{(-j)_{n-s}}{(n-s)!} \varphi_s, \quad j < 0,$$
$$\phi[j,n] = \sum_{s=0}^{n} (-1)^{n-s} (\lambda_0)^j \frac{(-j)_{n-s}}{(n-s)!} \phi_s, \quad j < 0. \tag{27}$$

4 Nth order rogue waves

In this section, we examine the Nth-order DT and derive the formula for breather solutions. When $r = q^*$, the FL system (2, 3) reduces to the FL equation (1). Performing this reduction, $r[N] = q[N]^*$ and $\lambda_k^* = \lambda_l$, which lead to

$$\varphi_k^* = \phi_l, \quad \phi_k^* = \varphi_l, \quad k \neq l.$$
$$\lambda_l = \lambda_{l+1}^*, \ \Psi_l = \begin{pmatrix} \varphi_l \\ \phi_l \end{pmatrix} = \begin{pmatrix} \phi_{l-1}^* \\ \varphi_{l-1}^* \end{pmatrix}, \ l = 1, 2, \ldots, n.$$

Combining (25) and the binomial expansion, we get

$$q[N] = q\left(1 + \frac{|E_{N2}|}{|E_{N1}|}\right), \tag{28}$$

where

$$E_{N2} = \left[\mathbf{W} \middle| \begin{matrix} -\varphi[-N,0] \\ -\phi[-N,0]^* \\ \vdots \\ -\varphi[-N,N-1] \\ -\phi[-N,N-1]^* \end{matrix} \right],$$

$$E_{N1} = \left[\mathbf{W} \middle| \begin{matrix} \phi[-N+1,0] \\ \varphi[-N+1,0]^* \\ \vdots \\ \phi[-N+1,N-1] \\ \varphi[-N+1,N-1]^* \end{matrix} \right].$$

And W is defined as

$$\begin{bmatrix} \varphi[N,0] & \phi[N-1,0] & \cdots & \varphi[-N+2,0] \\ \phi[N,0]^* & \varphi[N-1,0]^* & \cdots & \phi[-N+2,0]^* \\ \vdots & \vdots & & \vdots \\ \varphi[N,N-1] & \phi[N-1,N-1] & \cdots & \varphi[-N+2,N-1] \\ \phi[N,N-1]^* & \varphi[N-1,N-1]^* & \cdots & \phi[-N+2,N-1]^* \end{bmatrix}.$$

Thus, using the seed solution (6), $q[N]$ is an Nth-order breather solution, which has been obtained here using only algebraic and matrix manipulations.

5 Applications with $N = 1, 2, 3$

To verify our method, we display the first, second and third breather solutions both numerically and graphically. Fix $a = c = 1$ in the seed solution (6). With

$$F_0 = \begin{pmatrix} 1 - \frac{i-1}{2}x & \frac{i-1}{2}x \\ \frac{1-i}{2}x & 1 + \frac{i-1}{2}x \end{pmatrix}, G_0 = \begin{pmatrix} 1 - it & it \\ -it & 1 + it \end{pmatrix}.$$

And combine Eqs. (25, 27),

$$\Psi_0 = \begin{pmatrix} \varphi_0 \\ \phi_0 \end{pmatrix} = F_0 G_0 Z_0,$$
$$\varphi[0,0] = \varphi_0, \quad \phi[0,0] = \phi_0,$$
$$\varphi[1,0] = \frac{1+i}{2}\varphi_0, \quad \phi[1,0] = \frac{1+i}{2}\phi_0,$$
$$\varphi[-1,0] = (1-i)\varphi_0, \quad \phi[-1,0] = (1-i)\phi_0.$$

According to (28), we can express $q[1]$ as

$$q[1] = \exp i(x+3t) \cdot \left(1 + \frac{|E_{12}|}{|E_{11}}|\right),$$

where

$$|E_{12}| = \begin{vmatrix} \varphi[1,0] & -\varphi[-1,0] \\ \phi[1,0]^* & -\phi[-1,0]^* \end{vmatrix}, \quad |E_{11}| = \begin{vmatrix} \varphi[1,0] & \phi[0,0] \\ \phi[1,0]^* & \varphi[0,0]^* \end{vmatrix}.$$

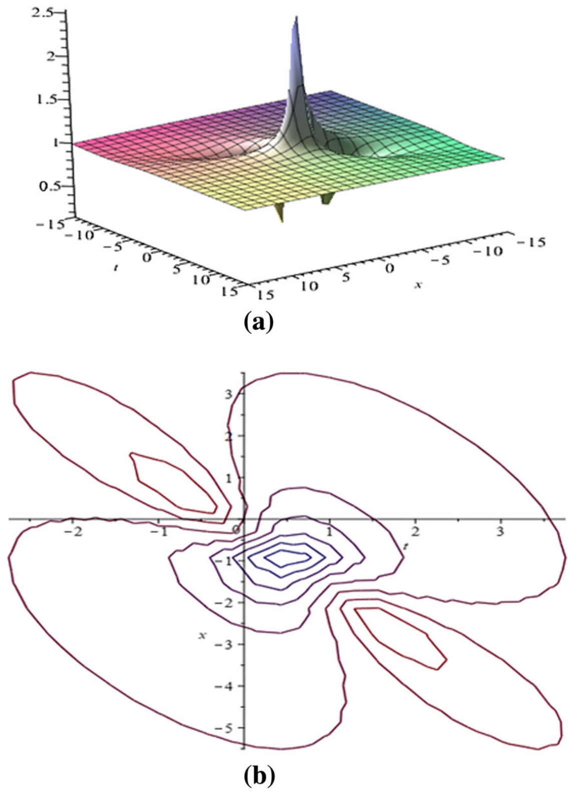


Fig. 1 The image of first-order rogue wave with specific parameters $a = 1, c = 1, Z_0 = (1,0)^T$. The maximum amplitude occurs at $t = 0.5$ and $x = -1$

With $Z_0 = (1,0)^T$, we plot the solution in Fig. 1. In the limit $x \to \infty, t \to \infty$, $|q[1]| = 1$. The maximum amplitude of $|q[1]|$ equals 3 and occurs at $t = 0.5$ and $x = -1$. Note that [30] Fig. 2 displays the plot of $|q[1]|^2$, with parameters $a = 1$ and $c = -1$. Hence, the maximum amplitude in their computation is the same as ours, and the performance of the waves is quite similar. The small difference, like the position of the apex, is due to the selection of Z_0.

For the second- or third-order rouge waves, we can similarly obtain explicit expressions given fixed parameters. Let $N = 2$ and (18, 24) yield,

$$F_1 = \begin{pmatrix} -\frac{i-1}{12}x^3 + \frac{x^2}{2} + x & \frac{i-1}{2}(x + \frac{x^3}{6}) \\ -\frac{i-1}{2}(x + \frac{x^3}{6}) & \frac{i-1}{12}x^3 + \frac{x^2}{2} - x \end{pmatrix},$$

$$G_1 = \begin{pmatrix} -\frac{t^3}{3} - it^2 + 2t & \frac{t^3}{3} - t \\ -\frac{t^3}{3} + t & \frac{t^3}{3} - it^2 - 2t \end{pmatrix},$$

and

$$\Psi_1 = \begin{pmatrix} \varphi_1 \\ \phi_1 \end{pmatrix} = F_1 G_0 Z_0 + F_0 G_1 Z_0 + F_0 G_0 Z_1,$$

$$\varphi[0,1] = \varphi_1,\ \phi[0,1] = \phi_1,\ \varphi[2,0] = \frac{i}{2}\varphi_0,\ \phi[2,0] = \frac{i}{2}\phi_0,$$
$$\varphi[1,1] = \frac{i+1}{2}(\varphi_0 + \varphi_1),\ \phi[1,1] = \frac{i+1}{2}(\phi_0 + \phi_1),$$
$$\varphi[-1,1] = (1-i)(-\varphi_0 + \varphi_1),$$
$$\phi[-1,1] = (1-i)(-\phi_0 + \phi_1),$$
$$\varphi[2,1] = \frac{i}{2}(2\varphi_0 + \varphi_1),\ \phi[2,1] = \frac{i}{2}(2\phi_0 + \phi_1),$$
$$\varphi[-2,0] = -2i\varphi_0,\ \phi[-2,0] = -2i\phi_0,$$
$$\varphi[-2,1] = -2i(-2\varphi_0 + \varphi_1),$$
$$\phi[-2,1] = -2i(-2\phi_0 + \phi_1).$$

Hence, from (28), we have the second-order rogue wave expression

$$q[2] = \exp i(x+3t) \cdot \left(1 + \frac{E_{22}}{E_{21}}\right),$$

where

$$E_{22} = \begin{vmatrix} \varphi[2,0] & \phi[1,0] & \varphi[0,0] & -\varphi[-2,0] \\ \phi[2,0]^* & \varphi[1,0]^* & \phi[0,0]^* & -\phi[-2,0]^* \\ \varphi[2,1] & \phi[1,1] & \varphi[0,1] & -\varphi[-2,1] \\ \phi[2,1]^* & \varphi[1,1]^* & \phi[0,1]^* & -\phi[-2,1]^* \end{vmatrix},$$

$$E_{21} = \begin{vmatrix} \varphi[2,0] & \phi[1,0] & \varphi[0,0] & \phi[-1,0] \\ \phi[2,0]^* & \varphi[1,0]^* & \phi[0,0]^* & \varphi[-1,0]^* \\ \varphi[2,1] & \phi[1,1] & \varphi[0,1] & \phi[-1,1] \\ \phi[2,1]^* & \varphi[1,1]^* & \phi[0,1]^* & \varphi[-1,1]^* \end{vmatrix}.$$

A typical plot of a second-order solution is shown in Fig. 2. Note that when $Z_0 = (1,1)^T$, $Z_1 = (0,0)^T$, this reduces to a first-order solution.

The third-order rogue wave ($N = 3$) can be found in the same way. From Eq. (28), we have

$$q[3] = \exp i(x+3t) \cdot \left(1 + \frac{E_{32}}{E_{31}}\right).$$

And similarly from (18, 24), we have

$$F_2 = \begin{pmatrix} F_{2,11} & F_{2,12} \\ F_{2,21} & F_{2,22} \end{pmatrix}, \quad G_2 = \begin{pmatrix} G_{2,11} & G_{2,12} \\ G_{2,21} & G_{2,22} \end{pmatrix},$$

where

$$F_{2,11} = \frac{1-i}{240}x^5 + \frac{1}{24}x^4 + \frac{7-3i}{24}x^3 + \frac{3}{4}x^2 + \frac{1}{2}x,$$
$$F_{2,12} = \left(\frac{i-1}{240}\right)(x^2+50)x^3,$$
$$F_{2,21} = -\left(\frac{i-1}{240}\right)(x^2+50)x^3,$$

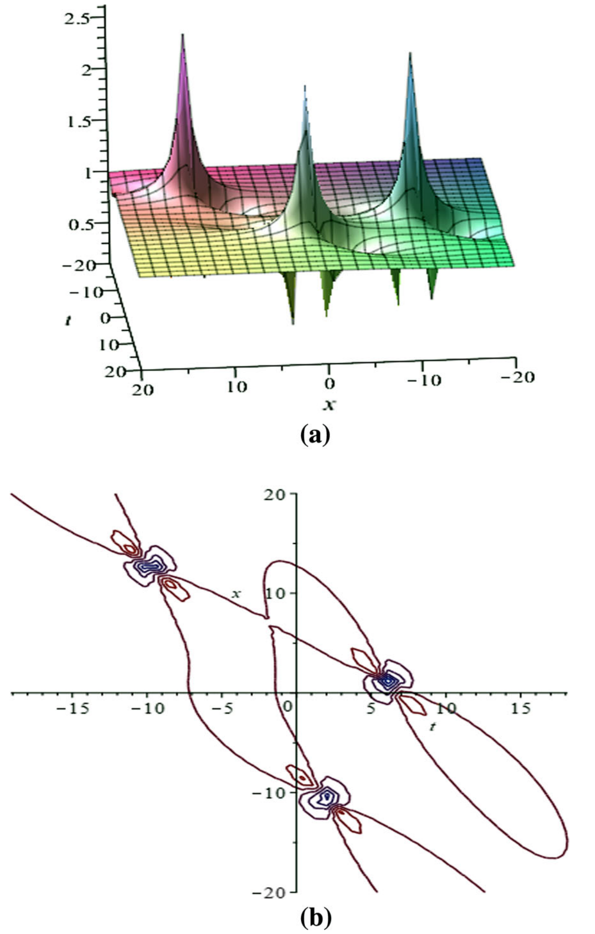
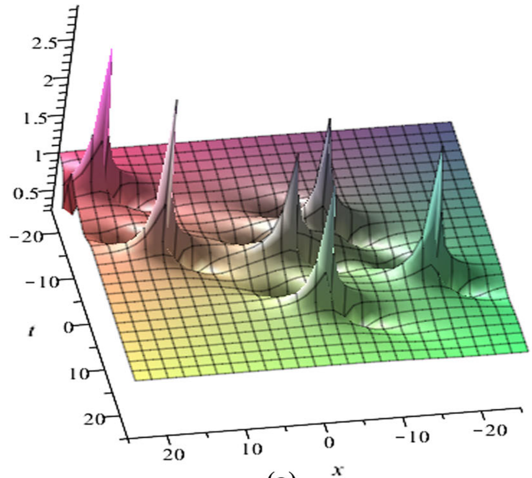


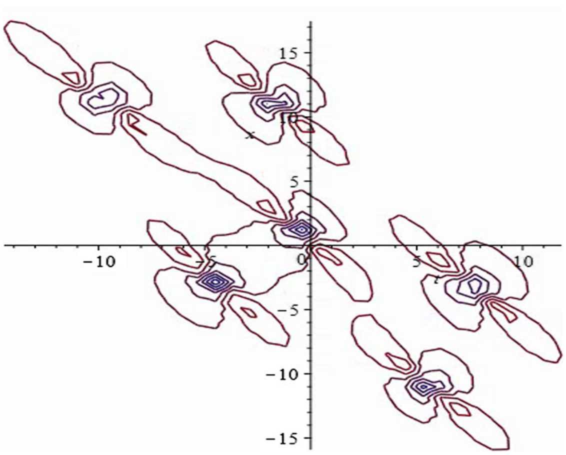
Fig. 2 The image of second-order rogue wave with specific parameters $a = 1, c = 1, Z_0 = (1,40)^T$, $Z_1 = (8000,1)^T$

$$F_{2,22} = -\frac{1-i}{240}x^5 + \frac{1}{24}x^4 - \frac{7-3i}{24}x^3 + \frac{3}{4}x^2 - \frac{1}{2}x,$$
$$G_{2,11} = \frac{1}{30}t(-30 + t^4 i - 5t^3 + (-40i+5)t^2 + (15i+60)t),$$
$$G_{2,12} = -\frac{1}{30}t(t^4 i + (-30i+5)t^2 - 15i - 15),$$
$$G_{2,21} = \frac{1}{30}t(t^4 i + (-30i+5)t^2 - 15i - 15),$$
$$G_{2,22} = -\frac{1}{30}t(-30 + t^4 i - 5t^3 + (-40i+5)t^2 + (15i+60)t).$$

A typical plot is shown in Fig. 3. We select $Z_0 = (1,2)^T$, $Z_1 = (50,1)^T$, $Z_2 = (2000,1)^T$ and draw the plot 3.



(a)



(b)

Fig. 3 The image of second-order rogue wave with specific parameters $a = 1, c = 1, Z_0 = (1,2)^T$, $Z_1 = (50,1)^T$, $Z_2 = (2000,1)^T$

6 Stability of the proposed technique

In this section, we would like to show that our propose technique would be stable. We give an error tolerance to the seed solution parameter a and c, to limit the residual between the after-perturbation solution and the original one. To clarify, we would let $||\cdot||$ denote the modulus of a complex number.

At first, we introduce two lemmas. For convenience, in the following discussion, we would focus on one root of λ_0 (12)

$$\lambda_0 = \sqrt{\frac{1}{2}(a^2c^2 - a + ac\sqrt{a^2c^2 - 2a})}. \tag{29}$$

In the first lemma, we consider the case where if given a tiny perturbation on a, and other parameters remain unchanged, the computation of λ_0 is stable.

Lemma 1 *Assume that c is fixed and $\|\lambda_0\| > 0$. For any $\delta > 0$, there exists an $\varepsilon > 0$, such that given $\left|\frac{a-\tilde{a}}{a}\right| < \varepsilon$, $\left\|\lambda_0 - \tilde{\lambda}_0\right\| < \delta$ holds, where $\tilde{\lambda}_0$ is the derivation of $\tilde{a}$ according to (29).*

Proof Fix c and set $\tilde{a} = a(1+\varepsilon_a)$. With the assumption $\|\lambda_0\| > 0$, $\varepsilon_a > 0$ and $\varepsilon_a \to 0$, we have

$$\left\|\lambda_0 - \tilde{\lambda}_0\right\| = \sqrt{\frac{1}{2}}\left\|\frac{N_1}{D_1}\right\|,$$

where N_1 equals

$$\left\|a^2c^2 - a + ac\sqrt{a^2c^2 - 2a} - \tilde{a}^2c^2 + \tilde{a} - \tilde{a}c\sqrt{\tilde{a}^2c^2 - 2\tilde{a}}\right\|,$$

and D_1 equals

$$\left\|\sqrt{a^2c^2 - a + ac\sqrt{a^2c^2 - 2a}} + \sqrt{\tilde{a}^2c^2 - \tilde{a} + \tilde{a}c\sqrt{\tilde{a}^2c^2 - 2\tilde{a}}}\right\|.$$

For the numerator N_1,

$$N_1 \le c^2a^2\left\|\varepsilon_a^2 + 2\varepsilon_a\right\| + \|a\|\|\varepsilon_a\| + \left\|ac\sqrt{a}\right\|\left\|\sqrt{ac^2 - 2} - \sqrt{\tilde{a}c^2 - 2}(1+\varepsilon_a)^{\frac{3}{2}}\right\|.$$

Here, we focus on the third term

$$\left\|\sqrt{ac^2 - 2} - \sqrt{\tilde{a}c^2 - 2}(1+\varepsilon_a)^{\frac{3}{2}}\right\|$$

and analyze it case by case.

If $ac^2 - 2 \ge 0$,

$$\begin{aligned}
&\left\|\sqrt{ac^2 - 2} - \sqrt{\tilde{a}c^2 - 2}(1+\varepsilon_a)\sqrt{1+\varepsilon_a}\right\| \\
&= \sqrt{ac^2 - 2 + a\varepsilon_a c^2}(1+\varepsilon_a)^{\frac{3}{2}} - \sqrt{ac^2 - 2}(1+\varepsilon_a)^{\frac{3}{2}} \\
&\quad + \sqrt{ac^2 - 2}((1+\varepsilon_a)^{\frac{3}{2}} - 1), \\
&\le (1+\varepsilon_a)^{\frac{3}{2}}(\sqrt{ac^2 - 2 + a\varepsilon_a c^2} - \sqrt{ac^2 - 2}) \\
&\quad + \sqrt{ac^2 - 2}((1+\varepsilon_a)^{\frac{3}{2}} - 1).
\end{aligned}$$

If $ac^2 - 2 < 0$, we can find ε_a small enough so that $ac^2 - 2 + \varepsilon_a c^2 < 0$.

$$\begin{aligned}
&\left\|\sqrt{ac^2 - 2} - \sqrt{\tilde{a}c^2 - 2}(1+\varepsilon_a)^{\frac{3}{2}}\right\| \\
&= \left\|\sqrt{2 - ac^2}i - \sqrt{2 - ac^2 - a\varepsilon_a c^2}(1+\varepsilon_a)^{\frac{3}{2}}i\right\|, \\
&= \left\|\sqrt{2 - ac^2} - \sqrt{2 - ac^2 - a\varepsilon_a c^2}(1+\varepsilon_a)^{\frac{3}{2}}\right\|,
\end{aligned}$$

$$\leq \left\| \sqrt{2-ac^2} - \sqrt{2-ac^2}(1+\varepsilon_a)^{\frac{3}{2}} \right\| + \left\| \sqrt{2-ac^2} - \sqrt{2-ac^2-a\varepsilon_a c^2} \right\| \left\| (1+\varepsilon_a)^{\frac{3}{2}} \right\|,$$
$$= \left\| \sqrt{2-ac^2} \right\| \left\| (1+\varepsilon_a)^{\frac{3}{2}} - 1 \right\| + \left\| \frac{\varepsilon_a c^2}{\sqrt{2-ac^2}+\sqrt{2-ac^2-\varepsilon_a c^2}} \right\|, \left\| (1+\varepsilon_a)^{\frac{3}{2}} \right\|.$$

When it comes to the denominator, we can choose the perturbation ε_a wisely to make sure $||D_1||$ is bigger than some positive constant K_0, which helps us to reach the conclusion

$$\left\| \lambda_0 - \tilde{\lambda}_0 \right\| \leq \frac{K_a}{K_0}\varepsilon_a = K_1\varepsilon_a,$$

where $K_1 \sim O(1)$. □

This completes the proof.

In the second lemma, we discuss the cases where perturbations on both a and c.

Lemma 2 *Assume* $\|\lambda_0\| > 0$. *For any* $\delta > 0$, *there exists an* $\varepsilon > 0$, *such that if given* $\left|\frac{a-\tilde{a}}{a}\right| < \varepsilon$ *and* $\left|\frac{c-\tilde{c}}{c}\right| < \varepsilon$, *we will also have* $\left\| \lambda_0 - \tilde{\lambda}_0 \right\| < \delta$.

Proof Now we assume $\tilde{a} = (1+\varepsilon_0)a$, $\tilde{c} = (1+\varepsilon_0)c$, $\varepsilon_0 > 0$, $\varepsilon_0 \to 0$. Similarly,

$$\left\| \lambda_0 - \tilde{\lambda}_0 \right\| = \sqrt{\frac{1}{2}} \left\| \frac{N_2}{D_2} \right\|,$$

where N_2 equals

$$\left\| a^2c^2 - a + ac\sqrt{a^2c^2-2a} - \tilde{a}^2\tilde{c}^2 + \tilde{a} - \tilde{a}\tilde{c}\sqrt{\tilde{a}^2\tilde{c}^2-2\tilde{a}} \right\|$$

and D_2 equals

$$\left\| \sqrt{a^2c^2 - a + ac\sqrt{a^2c^2-2a}} + \sqrt{\tilde{a}^2\tilde{c}^2 - \tilde{a} + \tilde{a}\tilde{c}\sqrt{\tilde{a}^2\tilde{c}^2-2\tilde{a}}} \right\|.$$

For the numerator N_2,

$$N_2 \leq \left\| a^2c^2 - a + ac\sqrt{a^2c^2-2a} - \tilde{a}^2c^2 + \tilde{a} - \tilde{a}c\sqrt{\tilde{a}^2c^2-2\tilde{a}}) \right\| + \left\| \tilde{a}^2c^2 - \tilde{a} + \tilde{a}c\sqrt{\tilde{a}^2c^2-2\tilde{a}} - (\tilde{a}^2\tilde{c}^2 - \tilde{a} + \tilde{a}\tilde{c}\sqrt{\tilde{a}^2\tilde{c}^2-2\tilde{a}}) \right\|,$$

$$\leq K_a\varepsilon_0 + \tilde{a}^2c^2\|\varepsilon_0^2 + 2\varepsilon_0\|, + \|\tilde{a}c\| \left\| \sqrt{\tilde{a}^2c^2-2\tilde{a}} - (1+\varepsilon_0)\sqrt{\tilde{a}^2c^2(1+\varepsilon_0)^2-2\tilde{a}} \right\|. \tag{30}$$

Similarly, we will focus on the last term. If $a^2c^2 - 2a > 0$, we can choose ε_0 small enough to let $\tilde{a}^2c^2 - 2\tilde{a} > 0$ and $\tilde{a}^2\tilde{c}^2 - 2\tilde{a} > 0$, which results in

$$\left\| \sqrt{\tilde{a}^2c^2-2\tilde{a}} - (1+\varepsilon_0)\sqrt{\tilde{a}^2\tilde{c}^2-2\tilde{a}} \right\|$$
$$\leq \left\| \varepsilon_0\sqrt{\tilde{a}^2\tilde{c}^2-2\tilde{a}} \right\| + \left\| \sqrt{\tilde{a}^2\tilde{c}^2-2\tilde{a}} - \sqrt{\tilde{a}^2c^2-2\tilde{a}} \right\|,$$
$$= \|\varepsilon_0\| \left\| \sqrt{\tilde{a}^2\tilde{c}^2-2\tilde{a}} \right\| + \left\| \frac{\tilde{a}^2(\tilde{c}^2-c^2)}{\sqrt{\tilde{a}^2\tilde{c}^2-2\tilde{a}}+\sqrt{\tilde{a}^2\tilde{c}^2-2a}} \right\|.$$

If $ac^2 - 2a < 0$, search an ε_0 so that $\tilde{a}^2\tilde{c}^2 - 2\tilde{a} < 0$ and $\tilde{a}^2c^2 - 2\tilde{a} < 0$.

$$\left\| \sqrt{\tilde{a}^2c^2-2\tilde{a}} - (1+\varepsilon_0)\sqrt{\tilde{a}^2\tilde{c}^2-2\tilde{a}} \right\|$$
$$= \left\| \sqrt{2\tilde{a}-\tilde{a}^2c^2} - (1+\varepsilon_0)\sqrt{2\tilde{a}-\tilde{a}^2\tilde{c}^2} \right\|,$$
$$\leq \left\| \sqrt{2\tilde{a}-\tilde{a}^2c^2} - (1+\varepsilon_0)\sqrt{2\tilde{a}-\tilde{a}^2c^2} \right\| + \|1+\varepsilon_0\| \left\| \sqrt{2\tilde{a}-\tilde{a}^2c^2} - \sqrt{2\tilde{a}-\tilde{a}^2c^2-O(\varepsilon_0)} \right\|,$$
$$= \|\varepsilon_0\| \left\| \sqrt{2\tilde{a}-\tilde{a}^2c} \right\| + \|1+\varepsilon_0\| \left\| O(\varepsilon_0)(\sqrt{2\tilde{a}-\tilde{a}^2c} + \sqrt{2\tilde{a}-\tilde{a}^2c^2-O(\varepsilon_0)})^{-1} \right\|.$$

Similar to what we have done in the last proof, we can search for some small ε_0 to control the denominator. Therefore,

$$\|\lambda_0 - \tilde{\lambda_0}\| \leq K\varepsilon_0, \tag{31}$$

where K is a constant. □

Using these two lemmas, we can then prove the theorem as follows:

Theorem 1 *Assume* $\|\lambda_0\| > 0$ *and* $|E_{N1}| > 0$. *For any* $\delta > 0$, *there always exists an* $\varepsilon > 0$, *s.t. given* $\left|\frac{a-\tilde{a}}{a}\right| < \varepsilon$ *and* $\left|\frac{c-\tilde{c}}{c}\right| < \varepsilon$, *we will have* $\|q[N] - \tilde{q}[N]\| < \delta$, *where* $\tilde{q}[N]$ *is the derivation of* $\tilde{a}$ *and* $\tilde{c}$ *according to* (28).

Proof Recall that in (18)

$$F_n = \begin{pmatrix} F_{n11} & F_{n12} \\ F_{n21} & F_{n22} \end{pmatrix}.$$

We now perform perturbations on a and c as discussed above. In order to present $\tilde{\lambda}$ in a similar pattern like $\tilde{a}$ and $\tilde{c}$, using the result of Lemma 1 and Lemma 2, we can redefine λ from the expression (31) so that

$$\tilde{\lambda}_0 = \lambda_0(1+\varepsilon),$$

From (19), we can derive that

$$\gamma_n = \lambda^{\frac{n}{2}}(C_\gamma + O(\lambda)),$$

where C_γ is in $O(1)$ order. Hence,

$$\widetilde{\gamma_n} = \lambda_0^{\frac{n}{2}}(1+\varepsilon)^{\frac{n}{2}}(C_\gamma + O(\lambda)) = \gamma_n + O(\varepsilon).$$

Similarly,

$$\widetilde{\tau_n} = \tau_n + O(\varepsilon).$$

Thus, $\widetilde{F_n} = F_n + \Delta F$, where all the elements of ΔF are in $O(\varepsilon)$ order.

From the expressions of $\pi_i, \kappa'_j, \upsilon'_m, \beta_n, \kappa_j, \upsilon_m$ ((20)-(25))

$$\begin{aligned}
\widetilde{\pi_i} &= \widetilde{\lambda}_0^{2k} O(1), \\
\widetilde{\kappa'_j} &= \sum_{p=0}^{\frac{j}{2}} \widetilde{\lambda}_0^{2j-2p} \tilde{a}^{\lceil \frac{j}{2} \rceil} O(1), \\
\widetilde{\kappa_j} &= \sum_{p=0}^{\frac{j}{2}} \widetilde{\lambda}_0^{2j-2p} \tilde{a}^{\lceil \frac{j}{2} \rceil} O(1), \\
\widetilde{\upsilon'_m} &= \frac{(-1)^m}{m!}(4k)_m \frac{1}{2^{2k}\tilde{a}^{2k}\widetilde{\lambda}_0^{4k}}, \\
\widetilde{\upsilon_m} &= \frac{(-1)^m}{m!}(4k+2)_m \frac{1}{2^{2k+1}\tilde{a}^{2k+1}\widetilde{\lambda}_0^{4k+2}}.
\end{aligned}$$

Thus, from (23) and (25)

$$\alpha_n = \begin{cases} \frac{1}{\lambda^{2k-1}} O(1), & j \text{ is odd} \\ \frac{1}{\lambda^{2k}} O(1), & j \text{ is even} \end{cases}, \tag{32}$$

$$\beta_n = \begin{cases} \frac{1}{\lambda^{2k+1}} O(1), & j \text{ is odd} \\ \frac{1}{\lambda^{2k+2}} O(1), & j \text{ is even} \end{cases}, \tag{33}$$

where $\tilde{a}$ is reduced from both the numerator and denominator.

What we have done in the steps above is to extract the factors which contain high-order λ_0. From G_n's expression (24), we notice that the term λ or λ^2 is multiplied by terms β_n, β_{n-1} and β_{n-2}. Since we have proved that $\tilde{\lambda}_0 \to \lambda_0$ and λ_0 is not a singular point, we can come to the conclusion $\widetilde{G}_n \approx G_n$. Thus,

$$\begin{aligned}
\widetilde{\Psi}_n &= \sum\sum \widetilde{F}_s G_t Z_{n-s-t} \\
&= \sum\sum (F_s + O(\varepsilon)) G_t Z_{n-s-t} = \Psi_n + O(\varepsilon),
\end{aligned}$$

which implies

$$\begin{pmatrix} \widetilde{\varphi}_n \\ \widetilde{\phi}_n \end{pmatrix} = \begin{pmatrix} \varphi_n + O(\varepsilon) \\ \phi_n + O(\varepsilon) \end{pmatrix}.$$

From what is shown in (27), we can then derive that

$$\begin{aligned}
\widetilde{\varphi}[j,n] &= \varphi[j,n] + O(\varepsilon), \\
\widetilde{\phi}[j,n] &= \phi[j,n] + O(\varepsilon).
\end{aligned}$$

Expression (28) shows $q[N]$ can be derived form E_{N1} and E_{N2}, whose elements are all $\varphi[j,n]$ and $\phi[j,n]$. Leibniz formula shows that the determinant of a matrix can be written as the linear combination of all its elements. Therefore, we can conclude that

$$\begin{aligned}
\widetilde{E}_{N1} &= E_{N1} + O(\varepsilon), \\
\widetilde{E}_{N2} &= E_{N1} + O(\varepsilon).
\end{aligned}$$

Finally, we have the stability of our algorithm. □

7 Conclusion

In this paper, we expand the Lax pair of Fokas–Lenells equation with a variable separation method and obtain Nth rogue wave expression. Compared to the more usual expansion methods [30,31], our method has several advantages. In particular, it is relatively easy to compute expressions and plot figures. Moreover, it is quite convenient for adjusting the parameters through selecting different Z_n's. The flexibility would hugely improve the efficiency in simulation and computation when the initial seed solution is given.

Similar to that in [30,31], we are inspired by the efficiency and structure of Darboux transformation to generate Nth-order solutions. The novel features presented by DT and the FL system are quite different from those generated from standard integrable systems like the AKNS and the KN systems. As shown in Figs. 1, 2 and 3, we obtain similar plots as in [30,31]. The maximum amplitudes in the examples are about three times to those when $x \to \infty$ and $t \to \infty$. We expect that our work may spark some research interests in generation of rogue waves and serve as a time saver applied to many much-studied methods.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest.

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