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The approximate Noether symmetries and approximate first integrals for the approximate Hamiltonian systems

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Abstract We provide the Hamiltonian version of the approximate Noether theorem developed for the perturbed ordinary differential equations (ODEs) (Govinder et al. in Phys Lett 240(3):127–131, 1998) for the approximate Hamiltonian systems. We follow the procedure adopted by Dorodnitsyn and Kozlov (J Eng Math 66(1–3):253–270, 2010) for the Hamiltonian systems of unperturbed ODEs. The approximate Legendre transformation connects the approximate Hamiltonian and approximate Lagrangian. The approximate Noether symmetries determining equation for the approximate Hamiltonian systems is defined explicitly. We provide a formula to establish an approximate first integral associated with an approximate Noether symmetry of the approximate Hamiltonian systems. We analyzed several physical models to elaborate the approach developed here.

Keywords Approximate Noether symmetries · Classification problem · Phase space

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1 Introduction

The non-trivial exact Lie symmetries do not exist for many differential equations arising in engineering, epidemics, economic growth theory and in several other fields. These type of differential equations are studied by the perturbation methods. Baikov et al. [3–5] provided the approximate version of Lie's theorems. The approximate version of Noether's theorem [6] was developed by Govinder et al. [1]. Feroze and Kara [7] analyzed a special group of second-order perturbed ODEs in approximate invariant and Lagrangian perspective. Naeem and Mahomed [8,9] developed the partial Lagrangian approach [10,11] for the perturbed ODEs. Naz et al. [12] presented a review of different techniques to compute first integrals/conservation laws for unperturbed differential equations. Dorodnitsyn et al. [13,14] developed new techniques to construct first integrals for the ordinary difference equations that do not have a variational formulation. In [15], new conservation laws were obtained directly from the equations of motion without invoking the Hamiltonians and Lagrangians.

A separate strand of literature studied the approximate Hamiltonian systems. The numerical analysis of the nearly integrable Hamiltonian systems [16] leads the foundations for the analytical studies of the approximate first integrals. Different perturbation techniques have been developed to establish the approximate first integrals, e.g., direct method [16], Birkhoff–

Gustavson normal form method [17] and other methods as described in [18]. Recently, Naz and Naeem [19] developed the approximate partial Noether approach to construct approximate first integrals for the approximate partial Hamiltonian systems.

Ünal [20] studied the approximate generalized symmetries, normal forms and approximate first integrals for the approximate Hamiltonian systems. The first step is to compute the exact Noether symmetry vector fields of the unperturbed part of an approximate Hamiltonian system and then combine it with the approximate symmetry vector field to obtain the approximate Noether symmetry of the approximate Hamiltonian system. Then, the total derivative of approximate first integral dI is the interior product of the approximate Noether symmetry and the simplectic form. Finally, on integration of dI one can get the approximate first integral I.

We provide the Hamiltonian version of the approximate Noether theorem developed earlier for the perturbed ODEs [1] for the approximate Hamiltonian systems. We follow the procedure adopted by Dorodnitsyn and Kozlov [2] for the Hamiltonian systems of unperturbed ODEs. The approximate Hamiltonian is associated with the approximate Lagrangian by utilizing the approximate Legendre transformation [21]. The approximate Noether symmetries determining equation for the approximate Hamiltonian system are defined explicitly. We provide a formula to establish an approximate first integral associated with an approximate Noether symmetry of the approximate Hamiltonian system. The approach developed here is applicable to a wide variety of approximate Hamiltonian systems including Economic growth theory, epidemics, physics, engineering and mechanics.

The detailed lay out of paper is as follows: In Sect. 2, the overview of approximate Noether's approach [1,7] is presented. The Hamiltonian version of approximate Noether theorem for the approximate Hamiltonian systems is provided in Sect. 3. We have explicitly provided the formulas for the approximate Noether symmetry determining equation and for the approximate first integrals of the approximate Hamiltonian systems. We analyzed several physical models to elaborate this approach in Sect. 4. We provide the concluding remarks in Sect. 5.

2 Approximate Noether's approach

First, we provide an overview of the approximate Noether approach for n-dependent variables case (see, e.g., Govinder et al. [1], Feroze and Kara [7]). Following definitions and Theorems are adopted from literature see, e.g., [1,7–9]

Let us consider the functional

$$\int_{t_0}^{t_1} L(t, q^j, \dot{q}^j; \epsilon)\mathrm{d}t, \quad j = 1, 2, \ldots, n, \tag{1}$$

where ϵ is a small parameter, t is the independent variable, q^j are n dependent variables, $\dot{q}^j$ are all first-order derivatives, and $L(t, q^j, \dot{q}^j; \epsilon)$ is a first-order approximate Lagrangian. Note that the derivatives of q^j with respect to t are given as $\dot{q}^j = q_1^j = D_t(q^j)$, $\ddot{q}^j = q_2^j = D_t^2(q^j)$ and so on.

The total derivative operator D_t with respect to t is defined as

$$D_t = \frac{\partial}{\partial t} + \dot{q}^j \frac{\partial}{\partial q^j} + \ddot{q}^j \frac{\partial}{\partial \dot{q}^j} + \ldots, \quad j = 1, 2, \cdots n, \tag{2}$$

and the Euler–Lagrange operator $\frac{\delta}{\delta q^j}$ is given by

$$\frac{\delta}{\delta q^j} = \frac{\partial}{\partial q^j} + \sum_{s \geq 1} (-D_t)^s \frac{\partial}{\partial q_s^j}, \quad j = 1, 2, \ldots, n. \tag{3}$$

Functional (1) reaches its extremal values when variables $q^j(t)$ satisfy following n approximate Euler–Lagrange equations:

$$\frac{\delta L}{\delta q^j} = \frac{\partial L}{\partial q^j} - D_t\left(\frac{\partial L}{\partial \dot{q}^j}\right) = O(\epsilon^2), \quad j = 1, 2, \ldots, n, \tag{4}$$

where $L(t, q^j, \dot{q}^j; \epsilon)$. Feroze and Kara [7] utilized the Lagrangian function of form $L(t, q^j, \dot{q}^j; \epsilon) = L_0(t, q^j, \dot{q}^j) + \epsilon L_1(t, q^j, \dot{q}^j)$.

Note that the system of n equations (4) is the system of second-order perturbed ODEs involving a small parameter ϵ.

Definition 1 (see, e.g., [8,9]) A differential function $I(t, q^j, \dot{q}^j; \epsilon) = I_0(t, q^j, \dot{q}^j) + \epsilon I_1(t, q^j, \dot{q}^j)$ is an approximate first integral of system (4) if

$$D_t(I_0 + \epsilon I_1) = O(\epsilon^2), \tag{5}$$

holds for every solution of system (4).

Theorem 1 (see, e.g., [7]) *Suppose* $L(t, q^j, \dot{q}^j; \epsilon) = L_0(t, q^j, \dot{q}^j) + \epsilon L_1(t, q^j, \dot{q}^j)$ *is a first-order Lagrangian corresponding to second-order ODEs* (4). *If the functional* $\int Ldt$ *is invariant under the one-parameter group of transformations with approximate Lie symmetry generator* $X = X_0 + \epsilon X_1$, *where* $X_0 = \xi_0 \frac{\partial}{\partial t} + \eta_0^j \frac{\partial}{\partial q^j}$ *and* $X_1 = \xi_1 \frac{\partial}{\partial t} + \eta_1^j \frac{\partial}{\partial q^j}$ *up to gauge* $B = B_0 + \epsilon B_1$, *then*

$$X_0^{[1]} L_0 + L_0 D_t(\xi_0) = D_t(B_0),$$
$$X_1^{[1]} L_0 + X_0^{[1]} L_1 + L_0 D_t(\xi_1) + L_1 D_t(\xi_0) = D_t(B_1), \tag{6}$$

where $\xi_0,\ \xi_1.\ \eta_0,\ \eta_1.\ B_0$, *depend on* t *and* q^j *only.*

Remark 1 It is important to mention here that Govinder et al. [1] provided approximate Noether symmetry determining equation (6) in following form:

$$\begin{aligned}
&(\xi_0 + \epsilon\xi_1)\frac{\partial L}{\partial t} + (\eta_0^j + \epsilon\eta_1^j)\frac{\partial L}{\partial q^j} \\
&\quad \times\left[D_t(\eta_0^j) + \epsilon D_t(\eta_1^j) - \dot{q}\Big(D_t(\xi_0) + \epsilon D_t(\xi_1)\Big)\right]\frac{\partial L}{\partial \dot{q}^j} \\
&\quad + \Big(D_t(\xi_0) + \epsilon D_t(\xi_1)\Big)L \\
&= D_t(B_0) + \epsilon D_t(B_1) + O(\epsilon^2).
\end{aligned} \tag{7}$$

Equation (7) can be re-written as

$$\begin{aligned}
&(X_0^{[1]} + \epsilon X_1^{[1]})L + L D_t(\xi_0 + \epsilon\xi_1) \\
&\quad = D_t(B_0 + \epsilon B_1) + O(\epsilon^2),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
X^{[1]} &= X_0^{[1]} + \epsilon X_1^{[1]} \\
&= (\xi_0 + \epsilon\xi_1)\frac{\partial}{\partial t} + (\eta_0^j + \epsilon\eta_1^j)\frac{\partial}{\partial q^j} \\
&\quad + \left[D_t(\eta_0^j + \epsilon\eta_1^j) - \dot{q}^j D_t(\xi_0 + \epsilon\xi_1)\frac{\partial}{\partial \dot{q}^j}\right], \\
j &= 1, 2, \ldots, n.
\end{aligned} \tag{9}$$

Feroze and Kara [7] utilized the Lagrangian function of form $L(t, q^j, \dot{q}^j; \epsilon) = L_0(t, q^j, \dot{q}^j) + \epsilon L_1(t, q^j, \dot{q}^j)$ in (8) and then separated the resulting equation by the powers of ϵ, up to order ϵ, to arrive at the approximate Noether symmetry determining equation (6). We will utilize approximate Noether symmetry determining equation (8) to establish results for the approximate Hamiltonian system.

Theorem 2 (see, e.g., [1,7]) *Corresponding to each approximate symmetry* $X = X_0 + \epsilon X_1$ *that fulfills the criterion provided in Theorem 1, there exists an approximate first integral* $I = I_0 + \epsilon I_1$ *given by*

$$\begin{aligned}
I_0 + \epsilon I_1 &= (\xi_0 + \epsilon\xi_1)(L_0 + \epsilon L_1) \\
&\quad + [\eta_0^j + \epsilon\eta_1^j - \dot{q}^j(\xi_0 + \epsilon\xi_1)]\frac{\delta}{\delta \dot{q}^j}(L_0 + \epsilon L_1) \\
&\quad - (B_0 + \epsilon B_1) + O(\epsilon^2).
\end{aligned} \tag{10}$$

3 Hamiltonian version of approximate Noether theorem for the approximate Hamiltonian systems

Let time and phase-space variables be t and (q^j, p^j), respectively. The variational operators $\delta/\delta q^i$ and $\delta/\delta p^j$ are defined as (see, e.g., [2,22])

$$\frac{\delta}{\delta q^j} = \frac{\partial}{\partial q^j} - D\frac{\partial}{\partial \dot{q}^j}, \tag{11}$$

and

$$\frac{\delta}{\delta p^j} = \frac{\partial}{\partial p^j} - D\frac{\partial}{\partial \dot{p}^j}. \tag{12}$$

The total derivative operator D satisfies

$$D = \frac{\partial}{\partial t} + \dot{q}^j\frac{\partial}{\partial q^j} + \dot{p}^j\frac{\partial}{\partial p^j} + \ldots, \tag{13}$$

and

$$\dot{p}^j = D(p^j),\ \dot{q}^j = D(q^j). \tag{14}$$

Definition 2 (*Approximate Legendre transformation*) The approximate Lagrangian $L(t, q^i, \dot{q}^i; \epsilon) = L_0(t, q^j, \dot{q}^j) + \epsilon L_1(t, q^j, \dot{q}^j)$ and the corresponding approximate Hamiltonian $H(t, q^j, p^j; \epsilon) = H_0(t, q^j, p^j) + \epsilon H_1(t, q^j, p^j)$ are related by the Legendre transformation (see, e.g., [19,21])

$$H_0 + \epsilon H_1 = p^j\dot{q}^j - L_0 - \epsilon L_1, \tag{15}$$

where $p^j = \frac{\partial}{\partial \dot{q}^j}(L_0 + \epsilon L_1)$ and $\dot{q}^j = \frac{\partial}{\partial p^j}(H_0 + \epsilon H_1)$. Relation (15) is an approximate version of Legendre transformation.

Remark 2 It is important to mention here that p^j are sometimes defined as the momenta in physics and mechanics. In Economic growth theory, p^j are the costate variables and interpreted as the shadow prices. In other fields of applied mathematics, the variables p^j can be interpreted in different ways.

Proposition 1 *Let $L(t,q^i,\dot{q}^i;\epsilon) = L_0(t,q^j,\dot{q}^j) + \epsilon L_1(t,q^j,\dot{q}^j)$ be an approximate Lagrangian and $H(t,q^j,p^j;\epsilon) = H_0(t,q^j,p^j) + \epsilon H_1(t,q^j,p^j)$ be the corresponding approximate Hamiltonian which are connected by the approximate Legendre transformation (15). If approximate Lagrangian L satisfies Euler–Lagrange equations (4) then the corresponding approximate Hamiltonian function H satisfies following system*

$$\dot{q}^j = \frac{\partial H}{\partial p^j} + O(\epsilon^2),$$
$$\dot{p}^j = -\frac{\partial H}{\partial q^j} + O(\epsilon^2), \;\; j = 1,\ldots,n, \tag{16}$$

and this is the first-order approximate Hamiltonian system.

Proof We employ the variational operator $\delta/\delta p_j$ on approximate Legendre transformation (15) which gives

$$\frac{\delta H}{\delta p^j} = \dot{q}^j - \frac{\delta L}{\delta p^j}, \tag{17}$$

as $\frac{\delta L}{\delta p^j} = 0$ and this yields the first equation of (16). We employ the variational operator $\delta/\delta q^j$ to (15) and we obtain

$$\frac{\delta H}{\delta q^j} = -\dot{p}^j - \frac{\delta L}{\delta q^j}. \tag{18}$$

Equation (18) reduces to the second equation of (16) as $\frac{\delta L}{\delta q^j} = O(\epsilon^2)$. This completes the proof. □

The canonical Hamiltonian equations for the unperturbed case studied in Dorodnitsyn and Kozlov [2] can be deduced by setting $\epsilon = 0$ in system (16).

3.1 Hamiltonian version of approximate Noether theorem

The approximate operators in the space of variables t, q^j and p^j are of the form $X = X_0 + \epsilon X_1$ where

$$X_0 + \epsilon X_1 = \left(\xi_0 + \epsilon\xi_1\right)\frac{\partial}{\partial t} + \left(\eta_0^j + \epsilon\eta_1^j\right)\frac{\partial}{\partial q^j} + \left(\zeta_0^j + \epsilon\zeta_1^j\right)\frac{\partial}{\partial p^j}, \tag{19}$$

and $\xi_0, \xi_1, \zeta_0^j, \zeta_1^j, \; \eta_0^j, \eta_1^j$ depend on t, q^j and p^j.

Definition 3 The approximate operator in (19) is an approximate point symmetry generator of the approximate Hamiltonian system (16) if (see, e.g., [22])

$$\begin{aligned}
&(\dot{\eta}_0^j + \epsilon\dot{\eta}_1^j) - \dot{q}^j(\dot{\xi}_0 + \epsilon\dot{\xi}_1) \\
&\quad - (X_0 + \epsilon X_1)\left(\frac{\partial}{\partial p^j}\right)(H_0 + \epsilon H_1) = O(\epsilon^2), \\
&(\dot{\zeta}^j + \epsilon\dot{\zeta}_1^j) - \dot{p}^j(\dot{\xi}_0 + \epsilon\dot{\xi}_1) \\
&\quad + (X_0 + \epsilon X_1)\left(\frac{\partial}{\partial q^j}\right)(H_0 + \epsilon H_1) = O(\epsilon^2), \\
&i = 1,\ldots,n
\end{aligned} \tag{20}$$

holds on system (16).

The point symmetry generator determining equation for the unperturbed case Dorodnitsyn and Kozlov [2] can be deduced by setting $\epsilon = 0$ in system (20).

The first prolongation of the generator X is

$$\begin{aligned}
X^{[1]} &= \xi\frac{\partial}{\partial t} + \eta^j\frac{\partial}{\partial q^j} + \zeta^j\frac{\partial}{\partial p^j} \\
&\quad + [D(\eta^j - \xi\dot{q}^j) + \xi\ddot{q}^j]\frac{\partial}{\partial\dot{q}^j} \\
&\quad + [D(\zeta^j - \xi\dot{p}^j) + \xi\ddot{p}^j]\frac{\partial}{\partial\dot{p}^j},
\end{aligned}$$

where $\xi(t,q^j,p^j) = \xi_0(t,q^j,p^j) + \epsilon\xi_1(t,q^j,p^j)$, $\zeta^j(t,q^j,p^j) = \zeta_0^j(t,q^j,p^j) + \epsilon\zeta_1^j(t,q^j,p^j)$, $\eta^j(t,q^j,p^j) = \eta_0^j(t,q^j,p^j) + \epsilon\eta_1^j(t,q^j,p^j)$.

The divergence invariance of approximate Hamiltonian action is established in the following proposition:

Proposition 2 *The approximate point symmetry generator $X = X_0 + \epsilon X_1$ of the approximate Hamiltonian system (16) is an approximate Noether symmetry corresponding to an approximate Hamiltonian $H(t,q^j,p^j;\epsilon)$, if there exists a function $B(t,q^j,p^j;\epsilon) = B_0(t,q^j,p^j) + \epsilon B_1(t,q^j,p^j)$ such that*

$$\begin{aligned}
&(\zeta_0^j + \epsilon\zeta_1^j)\frac{\partial}{\partial p^j}(H_0 + \epsilon H_1) + \; p^j D(\eta_0^j + \epsilon\eta_1^j) \\
&\quad -(X_0 + \epsilon X_1)(H_0 + \epsilon H_1) \\
&\quad - (H_0 + \epsilon H_1)D(\xi_0 + \epsilon\xi_1) \\
&= D(B_0 + \epsilon B_1) + O(\epsilon^2)
\end{aligned} \tag{21}$$

holds on system (16). *The function* $B(t, q^j, p^j; \epsilon) = B_0(t, q^j, p^j)+\epsilon B_1(t, q^j, p^j)$ *is an approximate gauge function.*

Proof The approximate Noether symmetry determining equation (8) with the aid of approximate Legendre transformation (15) yields

$$(X_0 + \epsilon X_1)[p^j \dot{q}^j - H_0 - \epsilon H_1] + [p^j \dot{q}^j - H_0 - \epsilon H_1] D(\xi_0 + \epsilon \xi_1) = D(B_0 + \epsilon B_1) + O(\epsilon^2). \quad (22)$$

On expanding Eq. (22), we have

$$p^j [D(\eta_0^j + \epsilon \eta_1^j) - \dot{q}^j D(\xi_0 + \epsilon \xi_1)] + \left(\zeta_0^j + \epsilon \zeta_1^j\right) \dot{q}^j - (X_0 + \epsilon X_1)(H_0 + \epsilon H_1) + p^j \dot{q}^j D(\xi_0 + \epsilon \xi_1) - (H_0 + \epsilon H_1) D(\xi_0 + \epsilon \xi_1) = D(B_0 + \epsilon B_1) + O(\epsilon^2). \quad (23)$$

With the aid of first equation from (16), Eq. (23) simplifies to

$$(\zeta_0^j + \epsilon \zeta_1^j) \frac{\partial}{\partial p^j}(H_0 + \epsilon H_1) + p^j D(\eta_0^j + \epsilon \eta_1^j) - (X_0 + \epsilon X_1)(H_0 + \epsilon H_1) - (H_0 + \epsilon H_1) D(\xi_0 + \epsilon \xi_1) = D(B_0 + \epsilon B_1) + O(\epsilon^2). \quad (24)$$

This completes the proof. □

The unperturbed part of the divergence invariance of approximate Hamiltonian action (21) yields the invariance of Hamiltonian action for the unperturbed case [2].

The question arises how the approximate Noether symmetry determining procedure proposed here for the approximate Hamiltonian system is different from the one given in Ünal [20].

Remark 3 The procedure adopted by Ünal [20] is to compute the exact Noether symmetry vector fields of the unperturbed part of an approximate Hamiltonian system and then combine it with the approximate symmetry vector field to obtain the approximate Noether symmetry of the approximate Hamiltonian system. We have provided approximate Noether symmetry determining equation (21) for the approximate Hamiltonian system which is analogue to the formula established by Emmy Noether [6]. The approximate Legendre transformation connects the approximate Hamiltonian function to the approximate Lagrangian function. It is utilized to establish approximate Noether symmetry determining equation (21) for the approximate Hamiltonian system. Dorodnitsyn and Kozlov [2] adopted this procedure to provide Noether symmetry determining equation for the unperturbed Hamiltonian systems.

Next, we provide a formula to construct the approximate first integrals for system (16) which is analogous to one provided in literature [2,19].

Proposition 3 *Corresponding to each approximate point symmetry generator* $X = X_0 + \epsilon X_1$ *of the approximate Hamiltonian system* (16) *that fulfills the criterion provided in Proposition 2, there exists an approximate first integral* $I = I_0 + \epsilon I_1$ *given by*

$$I_0 + \epsilon I_1 = p^j(\eta_0^j + \epsilon \eta_1^j) - (\xi_0 + \epsilon \xi_1)(H_0 + \epsilon H_1) - (B_0 + \epsilon B_1) + O(\epsilon^2). \quad (25)$$

Proof The formula for first integral (10) with the help of approximate Legendre transformation (15) and $p^j = \frac{\partial}{\partial \dot{q}^j}(L_0 + \epsilon L_1)$ yields

$$I_0 + \epsilon I_1 = (\xi_0 + \epsilon \xi_1)(p^j \dot{q}^j - H_0 - \epsilon H_1) + [\eta_0^j + \epsilon \eta_1^j - \dot{q}^j (\xi_0 + \epsilon \xi_1)] \frac{\delta}{\delta \dot{q}^j}(p^j \dot{q}^j - H_0 - \epsilon H_1) - (B_0 + \epsilon B_1) + O(\epsilon^2), \quad (26)$$

which after some simplifications provides formulas (25). This completes the proof. □

The unperturbed part of the approximate first integral formula (25) yields the first integrals for the unperturbed case [2].

Remark 4 Ünal [20] computed the total derivative of approximate first integral dI which is the interior product of the approximate Noether symmetry and the simplectic form. Finally, on integration of dI one can get the approximate first integral I. We have provided approximate first integral determining equation (26) for the approximate Hamiltonian system which is analogue to the formula established by Emmy Noether [6]. The approximate Legendre transformation is employed to establish approximate first integral formula (26) associated with each approximate Noether symmetry for the approximate Hamiltonian systems. Dorodnitsyn and Kozlov [2] adopted this procedure to provide first integral associated with each Noether symmetry for the unperturbed Hamiltonian systems. The Noether symmetry determining equation and first integrals for the unperturbed case [2] can be deduced by setting $\epsilon = 0$ in Eqs. (21) and (25).

Remark 5 It is worthy to mention here that for the sake of simplicity we have used expansions with respect to small parameter ϵ up to the first degree only. However, the theory developed in this section is valid for the higher powers of ϵ as well. One can utilize the approximate Lagrangian, approximate Hamiltonian, approximate point symmetry generator, gauge function and first integral involving the small parameter ϵ up to the degree $k \geq 1$ as follows:

$$L(t, q^i, \dot{q}^i; \epsilon) = L_0(t, q^j, \dot{q}^j) + \epsilon L_1(t, q^j, \dot{q}^j) + \cdots + \epsilon^k L_k(t, q^j, \dot{q}^j), \tag{27}$$

$$H(t, q^j, p^j; \epsilon) = H_0(t, q^j, p^j) + \epsilon H_1(t, q^j, p^j) + \cdots + \epsilon^k H_k(t, q^j, p^j), \tag{28}$$

$$X = X_0 + \epsilon X_1 + \cdots + \epsilon^k X_k, \tag{29}$$

$$\xi(t, q^j, p^j) = \xi_0(t, q^j, p^j) + \epsilon \xi_1(t, q^j, p^j) + \cdots + \epsilon^k \xi_k(t, q^j, p^j), \tag{30}$$

$$\zeta^j(t, q^j, p^j) = \zeta_0^j(t, q^j, p^j) + \epsilon \zeta_1^j(t, q^j, p^j) + \cdots + \epsilon^k \zeta_k^j(t, q^j, p^j), \tag{31}$$

$$\eta^j(t, q^j, p^j) = \eta_0^j(t, q^j, p^j) + \epsilon \eta_1^j(t, q^j, p^j) + \cdots + \epsilon^k \eta_k^j(t, q^j, p^j), \tag{32}$$

$$B(t, q^j, p^j; \epsilon) = B_0(t, q^j, p^j) + \epsilon B_1(t, q^j, p^j) + \cdots + \epsilon^k B_k(t, q^j, p^j), \tag{33}$$

$$I = I_0 + \epsilon I_1 + \cdots + \epsilon^k I_k. \tag{34}$$

4 Applications

In this section, we provide applications of theory presented in previous section. It is important to mention here that the approach developed here is applicable to a wide variety of problems from physics, engineering, mechanics, economics growth theory and many other fields of applied mathematics. First, we study the perturbed orbit equation governing one body problem [1,5] to re-derive previously established results to ensure that formulas derive here are correct. Second, we analyze a linear non-autonomous second-order ODE [7] for the approximated first integrals; this is a linear time-dependent model. Next, we show how the approach developed here can be utilized to establish approximate first integrals for more variables case. We have also shown that how to solve physical problems involving arbitrary functions and to determine the functional forms of these arbitrary functions such that the approximate Noether symmetry algebra in phase space is extended. The approximate Hamiltonian system discussed by Campoamor-Stursberg [23] is analyzed to look at this perspective.

4.1 The perturbed orbit equation governing one body problem

The perturbed orbit equation governing one body problem is given by (see, e.g., [1,5])

$$\ddot{q} + q = \epsilon F(q). \tag{35}$$

It is worthy to mention here that for the perturbed orbit equation (35) in order to admit approximate symmetries, Baikov et al. [5] computed four non-equivalent forms of F, in addition to F being arbitrary. We consider following special case of (35)

$$\ddot{q} + q - \epsilon A q^2 = 0. \tag{36}$$

Equation (36) is the perturbed version of harmonic oscillator equation. Dorodnitsyn and Kozlov [2] investigated the unperturbed harmonic oscillator for the discrete case.

The approximate Lagrangian for Eq. (36) is

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2 + \epsilon A \frac{q^3}{3}. \tag{37}$$

The approximate Hamiltonian by utilizing approximate Legendre transformation (15) is

$$H = p\dot{q} - L \tag{38}$$

which yields

$$H = \frac{p^2}{2} + \frac{q^2}{2} - \epsilon A \frac{q^3}{3}, \tag{39}$$

where $p = \frac{\partial L}{\partial \dot{q}}$. The approximate Hamiltonian system associated with approximate Hamiltonian given in Eq. (39) is

$$\dot{q} = p,$$
$$\dot{p} = -q + \epsilon A q^2. \tag{40}$$

Note that the approximate Hamiltonian (39) and approximate Hamiltonian system (40) reduce to the canonical Hamiltonian and Hamiltonian system for the unperturbed case [2] if we set $\epsilon = 0$. The approximate Noether symmetries determining equation (21) with the help of approximate Hamiltonian (39) and approximate Hamiltonian system (40) results in

$$\begin{aligned}
&p(\alpha_{0t} + p\beta_{0t} + \epsilon\alpha_{1t} + \epsilon p\beta_{1t}) \\
&\quad + p^2(\alpha_{0q} + p\beta_{0q} + \epsilon\alpha_{1q} + \epsilon p\beta_{1q}) \\
&\quad - (\alpha_0 + p\beta_0 + \epsilon\alpha_1 + \epsilon p\beta_1)(q - \epsilon A q^2) \\
&\quad - (\xi_{0t} + \epsilon\xi_{1t})\left(\frac{p^2}{2} + \frac{q^2}{2} - \epsilon A\frac{q^3}{3}\right) \\
&= B_{0t} + \epsilon B_{1t} + p(B_{0q} + \epsilon B_{1q}),
\end{aligned} \tag{41}$$

where $\xi(t) = \xi_0(t) + \epsilon\xi_1(t)$, $\eta(t, p, q) = \alpha_0(t, q) + p\beta_0(t, q) + \epsilon(\alpha_1(t, q) + p\beta_1(t, q))$, $B(t, q) = B_0(t, q) + \epsilon B_1(t, q)$. We separate Eq. (41) with respect to ϵ^0 and ϵ^1 to obtain approximate Noether symmetries determining equations for the zeroth-order and first-order approximations. After some simplifications, the zeroth-order approximation yields following systems of equations:

Zeroth-order approximation:

$$\begin{aligned}
&p^3 : \ \beta_{0q} = 0, \\
&p^2 : \ \beta_{0t} + \alpha_{0q} - \frac{1}{2}\xi_{0t} = 0, \\
&p : \ -2q\beta_0 + \alpha_{0t} = B_{0q}, \\
&p^0 : \ -q\left(\alpha_0 + \frac{q}{2}\xi_{0t}\right) = B_{0t}.
\end{aligned} \tag{42}$$

After some simplifications, the first-order approximation yields following systems of equations:

First-order approximation:

$$\begin{aligned}
&p^3 : \ \beta_{1q} = 0, \\
&p^2 : \ \beta_{1t} + \alpha_{1q} - \frac{1}{2}\xi_{1t} = 0, \\
&p : \ 2\beta_0 A q^2 - 2q\beta_1 + \alpha_{1t} = B_{1q}, \\
&p^0 : \ Aq^2\alpha_0 - q\alpha_1 - \frac{q^2}{2}\xi_{1t} + \frac{A}{3}q^3\xi_{0t} = B_{1t}.
\end{aligned} \tag{43}$$

The solution of systems of equations (42) and (43) for the zeroth- and first-order approximations yields following approximate Noether symmetries and the gauge terms in phase space:

$$\begin{aligned}
&X^1 = \frac{\partial}{\partial t}, \quad B^1 = 0, \\
&X^2 = 4\epsilon A\sin(t)\frac{\partial}{\partial t} + [3\cos t + 2\epsilon Aq\cos t]\frac{\partial}{\partial q}, \\
&B^2 = -3q\sin t - \epsilon Aq^2\sin t, \\
&X^3 = -4\epsilon A\cos t\frac{\partial}{\partial t} + [3\sin t + 2\epsilon Aq\sin t]\frac{\partial}{\partial q}, \\
&B^3 = 3q\cos t + \epsilon Aq^2\cos t, \\
&X^4 = \epsilon\frac{\partial}{\partial t}, \ B^4 = 0, \\
&X^5 = \epsilon\left(\sin(2t)\frac{\partial}{\partial t} + q\cos(2t)\frac{\partial}{\partial q}\right), \\
&B^5 = -\epsilon q^2\sin(2t), \\
&X^6 = \epsilon\left(\cos(2t)\frac{\partial}{\partial t} - q\sin(2t)\frac{\partial}{\partial q}\right), \\
&B^6 = -\epsilon q^2\cos(2t), \\
&X^7 = \epsilon\cos t\frac{\partial}{\partial q}, \ B^7 = -\epsilon q\sin(t), \\
&X^8 = \epsilon\sin t\frac{\partial}{\partial q}, \ B^8 = \epsilon q\cos t, \\
&X^9 = p\frac{\partial}{\partial q}, \ B^9 = -q^2 + \frac{2}{3}\epsilon Aq^3, \\
&X^{10} = \left(3\cos(t) + 2A\epsilon(q\cos(t) - p\sin(t))\right)\frac{\partial}{\partial q}, \\
&B^{10} = -3q\sin(t) + A\epsilon q^2\sin(t), \\
&X^{11} = \left(3\sin(t) + 2A\epsilon(q\sin(t) + p\cos(t))\right)\frac{\partial}{\partial q}, \\
&B^{11} = 3q\cos(t) - A\epsilon q^2\cos(t), \\
&X^{12} = \epsilon p\frac{\partial}{\partial q}, \ B^{12} = -\epsilon q^2, \\
&X^{13} = \epsilon\left(-2q\cos(2t) + p\sin(2t)\right)\frac{\partial}{\partial q}, \\
&B^{13} = \epsilon q^2\sin(2t), \\
&X^{14} = \epsilon\left(-2q\sin(2t) - p\cos(2t)\right)\frac{\partial}{\partial q}, \\
&B^{14} = -\epsilon q^2\cos(2t),
\end{aligned}$$

$$X^{\alpha} = 2\alpha(t)\frac{\partial}{\partial t} + p\alpha(t)\frac{\partial}{\partial q},$$
$$B^{\alpha} = \left(-q^2 + \epsilon\frac{2}{3}Aq^3\right)\alpha(t),$$
$$X^{\beta} = \epsilon\left(2\beta(t)\frac{\partial}{\partial t} + p\beta(t)\frac{\partial}{\partial q}\right),$$
$$B^{\beta} = -\epsilon\beta(t)q^2. \quad (44)$$

In [1], the Noether symmetries $X^1, X^2, \ldots, X^8$ were derived by utilizing the calculus of variation techniques and Lagrangian formulation. This shows that approach developed here is correct. We have obtained some additional Noether symmetries $X^9, X^{10}, \ldots, X^{14}, X^{\alpha}, X^{\beta}$ which are the Hamiltonian symmetries, but in Lagrangian picture these will be the generalized symmetries. Note that X^{α} and X^{β} yield infinite many Hamiltonian symmetries. The approximate Noether symmetries X^1, X^2, X^3, X^9, X^{10}, X^{11} and X^{α} of the approximate Hamiltonian system (40) are stable.

Next, we compare our results with those of the unperturbed harmonic oscillator investigated by Dorodnitsyn and Kozlov [2]. Setting $\epsilon = 0$ in (44) yields following Hamiltonian symmetries and gauge terms satisfying the invariance condition:

$$X^1 = \frac{\partial}{\partial t}, B^1 = 0,$$
$$X^2 = X^{10} = 3\cos t\frac{\partial}{\partial q} - 3\sin t\frac{\partial}{\partial p},$$
$$B^2 = B^{10} = -3q\sin t,$$
$$X^3 = X^{11} = 3\sin t\frac{\partial}{\partial q} + 3\cos t\frac{\partial}{\partial p},$$
$$B^3 = B^{11} = 3q\cos t,$$
$$X^9 = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}, \quad B^9 = -q^2,$$
$$X^{\alpha} = 2\alpha(t)\frac{\partial}{\partial t} + p\alpha(t)\frac{\partial}{\partial q} - (q\alpha(t) + p\alpha(t)')\frac{\partial}{\partial p},$$
$$B^{\alpha} = -q^2\alpha(t), \quad (45)$$

where the coefficient $\frac{\partial}{\partial p}$ term is computed with aid of $\zeta = D(\eta) - \dot{q}D(\xi)$. The Hamiltonian symmetries and gauge terms satisfying invariance condition given in Eq. (45) are exactly same as derived by Dorodnitsyn and Kozlov [2] except X^{α}.

Next, we utilize formula (25), to establish the approximate first integral $I = I_0 + \epsilon I_1$ corresponding each approximate Noether symmetry (44) in phase space. The approximate first integrals are

$$I^1 = \frac{p^2}{2} + \frac{q^2}{2} - \frac{\epsilon}{3}Aq^3 + O(\epsilon^2),$$
$$I^2 = 3p\cos t + 3q\sin t + \epsilon\left(2Apq\cos t - A(2p^2 + q^2)\sin t\right) + O(\epsilon^2),$$
$$I^3 = 3p\sin t - 3q\cos t + \epsilon\left(2Apq\sin t + A(2p^2 + q^2)\cos t\right) + O(\epsilon^2),$$
$$I^4 = \frac{1}{2}\epsilon(p^2 + q^2) + O(\epsilon^2), \quad (46)$$
$$I^5 = \epsilon\left(pq\cos(2t) - \frac{p^2}{2}\sin(2t) + \frac{q^2}{2}\sin(2t)\right) + O(\epsilon^2),$$
$$I^6 = \epsilon\left(-pq\sin(2t) - \frac{p^2}{2}\cos(2t) + \frac{q^2}{2}\cos(2t)\right) + O(\epsilon^2),$$
$$I^7 = \epsilon\left(p\cos t + q\sin t\right) + O(\epsilon^2),$$
$$I^8 = \epsilon\left(p\sin t - q\cos t\right) + O(\epsilon^2),$$
$$I^9 = I^1,\ I^{10} = I^2,\ I^{11} = I^3,$$
$$I^{12} = I^4,\ I^{13} = I^5,\ I^{14} = I^6,$$
$$I^{\alpha} = I^{\beta} = O(\epsilon^2).$$

In [1], the approximate first integrals $I^1, \ldots, I^8$ were derived by utilizing the calculus of variation techniques and Lagrangian formulation. This guarantees that formulas proposed here are correct. The approximate Noether symmetries X^{α}, X^{β} of the approximate Hamiltonian system (40) yield the trivial approximate first integrals which vanish on the solution of the approximate Hamiltonian system (40). The approximate Noether symmetries X^1 and X^9 yield same approximate first integral of the approximate Hamiltonian system (40). Similarly, the approximate first integrals $I^{10} = I^2$, $I^{11} = I^3$, $I^{12} = I^4$, $I^{13} = I^5$, $I^{14} = I^6$ and thus eight independent approximate first integrals $I^1, \ldots, I^8$ exist for the approximate Hamiltonian system (40). The stable approximate first integrals correspond to stable approximate symmetries and vice versa [20].

Next, we compare our results with those of the unperturbed harmonic oscillator investigated by Dorodnitsyn and Kozlov [2]. Setting $\epsilon = 0$ in (46) yields following three first integrals:

$$I^1 = I^9 = \frac{p^2}{2} + \frac{q^2}{2},$$
$$I^2 = I^{10} = 3p\cos t + 3q\sin t,$$
$$I^3 = I^{11} = 3p\sin t - 3q\cos t, \tag{47}$$

which are exactly same as derived by Dorodnitsyn and Kozlov [2].

4.2 A linear non-autonomous second-order ODE

Consider a linear non-autonomous second-order ODE (see, e.g., [7])

$$\ddot{q} - \epsilon(1-t^2)\dot{q} + q = 0. \tag{48}$$

Feroze and Kara [7] discussed the symmetries of the linear non-autonomous second-order ODE (48) that leave the invariant functional including a Lagrangian using the symmetry conservation laws theorem. The approximate Lagrangian for a linear non-autonomous second-order ODE given as in (48) is

$$L = \left(\frac{q^2}{2} - \frac{\dot{q}^2}{2}\right) e^{-\epsilon\left(t-\frac{t^3}{3}\right)}. \tag{49}$$

The approximate Hamiltonian by utilizing approximate Legendre transformation (15) is

$$H = \frac{p^2}{2} e^{\epsilon\left(t-\frac{t^3}{3}\right)} + \frac{q^2}{2} e^{-\epsilon\left(t-\frac{t^3}{3}\right)}, \tag{50}$$

and the corresponding approximate Hamiltonian system is given by

$$\dot{q} = p e^{\epsilon\left(t-\frac{t^3}{3}\right)},$$
$$\dot{p} = -q e^{-\epsilon\left(t-\frac{t^3}{3}\right)}. \tag{51}$$

Taylor series of $e^{\epsilon(t-\frac{t^3}{3})}$ and $e^{-\epsilon(t-\frac{t^3}{3})}$ up to order ϵ is given by

$$e^{\epsilon\left(t-\frac{t^3}{3}\right)} = 1 + \epsilon\left(t - \frac{t^3}{3}\right) + O(\epsilon^2),$$
$$e^{-\epsilon\left(t-\frac{t^3}{3}\right)} = 1 - \epsilon\left(t - \frac{t^3}{3}\right) + O(\epsilon^2). \tag{52}$$

The approximate Hamiltonian (50) and approximate Hamiltonian system (51), with the aid of (52), can be re-written as

$$H = \frac{p^2}{2} + \frac{q^2}{2} + \epsilon\left(t - \frac{t^3}{3}\right)\left(\frac{p^2}{2} - \frac{q^2}{2}\right), \tag{53}$$

and

$$\dot{q} = p\left[1 + \epsilon\left(t - \frac{t^3}{3}\right)\right],$$
$$\dot{p} = -q\left[1 - \epsilon\left(t - \frac{t^3}{3}\right)\right]. \tag{54}$$

The approximate Noether symmetries determining equation (21) with the aid of approximate Hamiltonian (53) and approximate Hamiltonian system (54) gives rise to

$$\begin{aligned}
&p\left[\eta_{0t} + \epsilon\eta_{1t} + p(\eta_{0q} + \epsilon\eta_{1q})\left(1 + \epsilon\left(t - \frac{t^3}{3}\right)\right)\right] \\
&\quad - \frac{\epsilon}{2}(\xi_0 + \epsilon\xi_1)(p^2 - q^2)(1 - t^2) \\
&\quad - (\eta_0 + \epsilon\eta_1)q\left[1 - \epsilon\left(t - \frac{t^3}{3}\right)\right] \\
&\quad - \left[\xi_{0t} + \epsilon\xi_{1t} + (\xi_{0q} + \epsilon\xi_{1q})p\left(1 + \epsilon\left(t - \frac{t^3}{3}\right)\right)\right] \\
&\quad \times \left[\frac{p^2}{2} + \frac{q^2}{2} + \epsilon\left(t - \frac{t^3}{3}\right)\left(\frac{p^2}{2} - \frac{q^2}{2}\right)\right] \\
&= B_{0t} + \epsilon B_{1t} + p(B_{0q} + \epsilon B_{1q})\left[1 + \epsilon\left(t - \frac{t^3}{3}\right)\right],
\end{aligned} \tag{55}$$

where $\xi_0 = \xi_0(t, q)$, $\xi_1 = \xi_1(t, q)$, $\eta_0 = \eta_0(t, q)$, $\eta_1 = \eta_1(t, q)$, $B_0 = B_0(t, q)$, $B_1 = B_1(t, q)$.

Separating (55), after expansion, with respect to ϵ^0 and ϵ^1 to obtain approximate Noether symmetries determining equations for the zeroth-order and first-order approximations. After some simplifications and separation with respect to different powers of p, the zeroth-order approximation yields following systems of equations:

Zeroth-order approximation:

$$p^3: \ \xi_{0q} = 0,$$
$$p^2: \ \eta_{0q} - \frac{1}{2}\xi_{0t} = 0,$$
$$p: \ -q\eta_{0t} = B_{0q},$$
$$p^0: \ -q\left(\eta_0 + \frac{q}{2}\xi_{0t}\right) = B_{0t}. \tag{56}$$

After some simplifications and separation with respect to different powers of p, the first-order approximation yields following systems of equations:

First-order approximation:

$$p^3: \ \xi_{1q} = 0,$$
$$p^2: \ \left(t - \frac{t^3}{3}\right)\eta_{0q} + \eta_{1q} - \left(\frac{t}{2} - \frac{t^3}{6}\right)\xi_{0t} - (1-t^2)\frac{\xi_0}{2} - \frac{1}{2}\xi_{1t} = 0,$$
$$p: \ -\left(t - \frac{t^3}{3}\right)B_{0q} + \eta_{1t} - \frac{q^2}{2}\xi_{1q} = B_{1q},$$
$$p^0: \ \frac{q^2}{2}(1-t^2)\xi_0 - q\eta_1 + q\left(t - \frac{t^3}{3}\right)\eta_0 + \frac{q^2}{2}\left(t - \frac{t^3}{3}\right)\xi_{0t} - \frac{q^2}{2}\xi_{1t} = B_{1t}. \quad (57)$$

The solution of systems of equations (56) and (57) yields the following approximate Noether symmetries and the gauge terms for the approximate Hamiltonian system (54):

$$X^1 = \left(1 + \frac{\epsilon}{2}t\right)\frac{\partial}{\partial t} - \frac{\epsilon}{2}q\left(t^2 - \frac{3}{2}\right)\frac{\partial}{\partial q}, \ B^1 = -\frac{\epsilon}{2}q^2 t,$$
$$X^2 = \left[\cos 2t + \epsilon\left(\left(\frac{t^2}{2} - \frac{1}{4}\right)\sin 2t + \frac{t}{2}\cos 2t\right)\right]\frac{\partial}{\partial t} + \left[-q\sin 2t + \frac{\epsilon}{2}q\cos 2t\right]\frac{\partial}{\partial q},$$
$$B^2 = -q^2\cos 2t + \epsilon\left[\frac{-2}{3}(-3t + t^3)q^2\cos^2 t - q^2\sin t\cos t + \frac{1}{12}(4t^3 - 12t)q^2\right],$$
$$X^3 = \left[\sin 2t + \epsilon\left(\frac{t}{2}\sin 2t + \frac{1}{4}(1-2t^2)\cos 2t\right]\frac{\partial}{\partial t} + \left[q\cos 2t + \frac{\epsilon}{2}q\sin 2t\right]\frac{\partial}{\partial q},$$
$$B^3 = -q^2\sin 2t + \epsilon\left[-\frac{q^2}{2} + q^2\cos^2 t - \frac{2}{3}q^2(-3t + t^3)\sin t\cos t\right],$$
$$X^4 = \left[\cos t + \epsilon\left(\frac{1}{12}(9t - 2t^3)\cos t + \frac{t^2}{4}\sin t\right)\right]\frac{\partial}{\partial q},$$
$$B^4 = -q\sin t + \epsilon\left(\frac{q}{4}(3-t^2)\cos t - \frac{q}{12}(2t^3 - 9t)\sin t\right)\Big],$$
$$X^5 = \left[\sin t + \epsilon\left(\frac{1}{12}(9 - 3t^2)\cos t + \frac{1}{12}(9t - 2t^3)\sin t\right)\right]\frac{\partial}{\partial q},$$
$$B^5 = q\cos t + \epsilon\left[\frac{q}{6}\left(t^3 - \frac{9}{2}t\right)\cos t - \frac{1}{4}t^2 q\sin t\right],$$
$$X^6 = \frac{\epsilon}{2}\sin 2t\frac{\partial}{\partial t} + \frac{\epsilon}{2}q\cos 2t\frac{\partial}{\partial q}, \ B^6 = -\frac{\epsilon}{2}q^2\sin 2t,$$
$$X^7 = -\frac{\epsilon}{2}\cos 2t\frac{\partial}{\partial t} + \frac{\epsilon}{2}q\sin 2t\frac{\partial}{\partial q}, \ B^7 = \frac{\epsilon}{2}q^2\cos 2t,$$
$$X^8 = \epsilon\frac{\partial}{\partial t}, \quad B^8 = 0,$$
$$X^9 = \epsilon\cos t\frac{\partial}{\partial q}, \ B^9 = -\epsilon q\sin t,$$
$$X^{10} = \epsilon\sin t\frac{\partial}{\partial q}, \ B^{10} = \epsilon q\cos t. \quad (58)$$

Next, we utilize formula (25), to establish the approximate first integrals $I = I_0 + \epsilon I_1$ associated with the approximate Noether symmetries (58) for the approximate Hamiltonian system (54). The approximate first integrals are

$$I^1 = \frac{\epsilon q}{4}\left(3p - 2pt^2 + 2tq\right) - \frac{1}{4}(2+\epsilon t)\left[p^2 + q^2 + \epsilon\left(t - \frac{t^3}{3}\right)(p^2 - q^2)\right] + O(\epsilon^2),$$
$$I^2 = p\left(-q\sin 2t + \frac{\epsilon q}{2}\cos 2t\right) - \frac{1}{2}\left[\cos 2t + \epsilon\left(\frac{1}{4}(2t^2 - 1)\sin 2t + \frac{t}{2}\cos 2t\right)\right] \times\left[p^2 + q^2 + \epsilon\left(t - \frac{t^3}{3}\right)(p^2 - q^2)\right] + O(\epsilon^2),$$
$$I^3 = q\left(p\cos 2t + q\sin 2t\right) + \frac{\epsilon q}{2}\left[\left(\left(-2t + \frac{2}{3}t^3\right)q + p\right)\sin 2t - q\cos 2t\right] - \frac{1}{2}\left[\sin 2t + \frac{\epsilon}{4}\left((1-2t^2)\cos 2t + 2t\sin 2t\right)\right] \times\left[p^2 + q^2 + \epsilon\left(t - \frac{t^3}{3}\right)(p^2 - q^2)\right] + O(\epsilon^2),$$
$$I^4 = p\cos t + p\epsilon\left(\frac{1}{12}(-2t^3 + 9t)\cos t + \frac{1}{4}t^2\sin t\right) + q\sin t - \epsilon\left(\frac{1}{4}(3-t^2)q\cos t - \frac{1}{12}(2t^3 - 9t)q\sin t\right) + O(\epsilon^2),$$
$$I^5 = p\sin t + p\epsilon\left(\frac{1}{12}(-2t^3 + 9t)\sin t + \frac{1}{12}(9 - 3t^2)\cos t\right) - q\cos t - \epsilon\left(\frac{1}{12}(2t^3 - 9t)q\cos t - \frac{1}{4}t^2 q\sin t\right) + O(\epsilon^2),$$

$$I^6 = \frac{\epsilon}{2}\Bigg[pq\cos 2t + q^2\sin 2t - \frac{1}{2}\sin 2t\Big(p^2+q^2+\epsilon(t-\frac{t^3}{3})(p^2-q^2)\Big)\Bigg] + O(\epsilon^2),$$

$$I^7 = \frac{\epsilon}{2}\Bigg[pq\sin 2t - q^2\cos 2t + \frac{1}{2}\cos 2t\Big(p^2+q^2+\epsilon\Big(t-\frac{t^3}{3}\Big)(p^2-q^2)\Big)\Bigg] + O(\epsilon^2),$$

$$I^8 = -\frac{\epsilon}{2}\Bigg[p^2+q^2+\epsilon\Big(t-\frac{t^3}{3}\Big)(p^2-q^2)\Bigg] + O(\epsilon^2),$$

$$I^9 = \epsilon\Big[p\cos t + q\sin t\Big] + O(\epsilon^2),$$

$$I^{10} = \epsilon\Big[p\sin t - q\cos t\Big] + O(\epsilon^2). \tag{59}$$

The approximate Noether approach in phase space provided five stable and five unstable first integrals. The stable approximate first integrals correspond to stable approximate symmetries and vice versa [20].

4.3 Classification problem

The physical systems arising in applications naturally contain small parameters or some arbitrary functions. We illustrate the advantages of this newly developed approach for the approximate Hamiltonian systems involving arbitrary functions. The equation of motion for the perturbed Lagrangian $L = \dot{q}_1\dot{q}_2 - \alpha q_2 q_1^{-3} + \epsilon\frac{F(r)}{q_1^2}$ is (see, e.g., Campoamor-Stursberg [23])

$$\ddot{q}_1 = \frac{3\alpha r}{q_1^3} - 2\epsilon\frac{F(r)}{q_1^3} - \epsilon\frac{rF'(r)}{q_1^3},$$

$$\ddot{q}_2 = -\frac{\alpha}{q_1^3} + \epsilon\frac{F'(r)}{q_1^3}, \tag{60}$$

where $r = \frac{q_2}{q_1}$. Campoamor-Stursberg [23] reported that the perturbed system (60) yields two stable approximate first integrals for arbitrary $F(r)$ and admits one additional approximate first integral when $F(r)$ is constant. Campoamor-Stursberg [23] studied the perturbed system (60) in approximate Lagrangian perspective. We will analyze the perturbed system (60) in approximate Hamiltonian perspective. The approximate Hamiltonian by utilizing approximate Legendre transformation (15) is

$$H = p_1p_2 + \frac{\alpha r}{q_1^2} - \epsilon\frac{F(r)}{q_1^2}, \tag{61}$$

and the associated approximate Hamiltonian system is given as follows:

$$\dot{q}_1 = p_2,$$
$$\dot{q}_2 = p_1,$$
$$\dot{p}_2 = -\frac{\alpha}{q_1^3} + \epsilon\frac{F'(r)}{q_1^3},$$
$$\dot{p}_1 = \frac{3\alpha r}{q_1^3} - 2\epsilon\frac{F(r)}{q_1^3} - \epsilon\frac{rF'(r)}{q_1^3}. \tag{62}$$

We utilize this newly developed approach to find the functional forms of arbitrary function $F(r)$ for which the associated approximate Hamiltonian system (61) becomes super-integrable. We assume $\xi(t,q_1,q_2)$, $\eta^1(t,q_1,q_2,p_1)$, $\eta^2(t,q_1,q_2,p_2)$ and $G(t,q_1,q_2)$ in the following form:

$$\xi(t,q_1,q_2) = \xi_0(t,q_1,q_2) + \epsilon\xi_1(t,q_1,q_2),$$
$$\eta^1(t,q_1,q_2,p_1) = \alpha_0(t,q_1,q_2) + \beta_0(t,q_1,q_2)p_1 + \epsilon\Big(\alpha_1(t,q_1,q_2) + \beta_1(t,q_1,q_2)p_1\Big),$$
$$\eta^2(t,q_1,q_2,p_2) = A_0(t,q_1,q_2) + B_0(t,q_1,q_2)p_2 + \epsilon\Big(A_1(t,q_1,q_2) + B_1(t,q_1,q_2)p_2\Big),$$
$$G(t,q_1,q_2) = G_0(t,q_1,q_2) + \epsilon G_1(t,q_1,q_2).$$

The approximate Noether symmetries determining equation (21) with the aid of approximate Hamiltonian (61) and approximate Hamiltonian system (62) yields following systems for ϵ^0 and ϵ:

ϵ^0 :

$$\beta_{0q_1} - \xi_{0q_2} = 0,\ \beta_{0q_2} = 0,\ B_{0q_1} = 0,$$
$$B_{0q_2} - \xi_{0q_1} = 0,$$
$$\beta_{0t} + \alpha_{0q_2} = 0,\ B_{0t} + A_{0q_1} = 0,$$
$$\alpha_{0q_1} + A_{0q_2} - \xi_{0t} = 0,$$
$$\alpha_{0t} + 6\alpha\beta_0\frac{q_2}{q_1^4} - \alpha\frac{q_2}{q_1^3}\xi_{0q_2} - G_{0q_2} = 0,$$
$$A_{0t} - 2B_0\frac{\alpha}{q_1^3} - \alpha\frac{q_2}{q_1^3}\xi_{0q_1} - G_{0q_1} = 0,$$
$$3\alpha_0\alpha\frac{q_2}{q_1^4} - A_0\frac{\alpha}{q_1^3} - \alpha\frac{q_2}{q_1^3}\xi_{0t} - G_{0t} = 0. \tag{63}$$

ϵ :

$$\beta_{1q_1} - \xi_{1q_2} = 0,\ \beta_{1q_2} = 0,\ B_{1q_1} = 0,$$
$$B_{1q_2} - \xi_{1q_1} = 0$$
$$\beta_{1t} + \alpha_{1q_2} = 0,\ B_{1t} + A_{1q_1} = 0,$$
$$\alpha_{1q_1} + A_{1q_2} - \xi_{1t} = 0,$$
$$\alpha_{1t} - 2\beta_0\left(\frac{2}{q_1^3}F(r) + \frac{q_2}{q_1^4}F'(r)\right) + 6\alpha\beta_1\frac{q_2}{q_1^4} - \alpha\frac{q_2}{q_1^3}\xi_{1q_2} + \frac{F(r)\xi_{0q_2}}{q_1^2} - G_{1q_2} = 0,$$
$$A_{1t} - 2\frac{\alpha}{q_1^3}B_1 + 2\frac{B_0}{q_1^3}F'(r) - \alpha\frac{q_2}{q_1^3}\xi_{1q_1} + \frac{F(r)\xi_{0q_1}}{q_1^2} - G_{1q_1} = 0,$$
$$-\alpha_0\left(\frac{2}{q_1^3}F(r) + \frac{q_2}{q_1^4}F'(r)\right) + 3\alpha_1\alpha\frac{q_2}{q_1^4} + \frac{A_0}{q_1^3}F'(r) - \frac{A_1\alpha}{q_1^3} + \frac{F(r)}{q_1^2}\xi_{0t} - \frac{\alpha q_2}{q_1^3}\xi_{1t} - G_{1t} = 0. \tag{64}$$

We arrive at different forms of $F(r)$ in search of solution of system of equations (63) and (64). The solution of systems of equations (63) and (64) yields different forms of $F(r)$. We provide as follows the details of approximate Noether symmetries, the gauge terms and approximate first integrals in phase space for these different forms of $F(r)$:

Case 1 $F(r)$ is arbitrary.
For arbitrary $F(r)$, we obtain the following approximate Noether symmetries and the gauge terms in phase space:

$$X^1 = \frac{\partial}{\partial t},\ G^1 = 0,$$
$$X^2 = q_1q_2\frac{\partial}{\partial t} + \frac{1}{2}q_1^2p_1\frac{\partial}{\partial q_1} + \frac{1}{2}q_2^2p_2\frac{\partial}{\partial q_2},$$
$$G^2 = \alpha\frac{q_2^2}{q_1^2} - \epsilon\frac{q_2}{q_1}F(r),$$
$$X^3 = 2t\frac{\partial}{\partial t} + q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2},\ G^3 = 0,$$
$$X64 = t^2\frac{\partial}{\partial t} + tq_1\frac{\partial}{\partial q_1} + tq_2\frac{\partial}{\partial q_2},\ G^4 = q_1q_2,$$
$$X^5 = \epsilon\frac{\partial}{\partial t},\ G^5 = 0,$$
$$X^6 = \epsilon q_1q_2\frac{\partial}{\partial t} + \frac{\epsilon}{2}q_1^2p_1\frac{\partial}{\partial q_1} + \frac{\epsilon}{2}q_2^2p_2\frac{\partial}{\partial q_2},$$
$$G^6 = \epsilon\alpha\frac{q_2^2}{q_1^2},$$
$$X^7 = \epsilon\big(2t\frac{\partial}{\partial t} + q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}\big),\ G^7 = 0,$$
$$X^8 = \epsilon\big(t^2\frac{\partial}{\partial t} + tq_1\frac{\partial}{\partial q_1} + tq_2\frac{\partial}{\partial q_2}\big),\ G^8 = \epsilon q_1q_2,$$
$$X^9 = \epsilon p_2\frac{\partial}{\partial q_2},\ G^9 = \frac{\epsilon\alpha}{q_1^2},$$
$$X^{10} = \epsilon(tp_2 - q_1)\frac{\partial}{\partial q_2},\ G^{10} = \epsilon\frac{t\alpha}{q_1^2},$$
$$X^{11} = \epsilon\left(\frac{t^2p_2}{2} - tq_1\right)\frac{\partial}{\partial q_2},\ G^{11} = \frac{\epsilon}{2}\left(\frac{\alpha t^2}{q_1^2} - q_1^2\right), \tag{65}$$

For arbitrary $F(r)$, four stable and seven unstable approximate Noether symmetries exist in phase space.

The first integrals of approximate Hamiltonian system (62) for arbitrary $F(r)$ are as follows:

$$I^1 = p_1p_2 + \alpha\frac{q_2}{q_1^3} - \epsilon\frac{F(r)}{q_1^2} + O(\epsilon^2),$$
$$I^2 = \frac{1}{2}(p_1q_1 - p_2q_2)^2 - 2\alpha\frac{q_2^2}{q_1^2} + 2\epsilon\frac{q_2}{q_1}F(r) + O(\epsilon^2),$$
$$I^3 = p_1q_1 + p_2q_2 - 2t\left(p_1p_2 + \alpha\frac{q_2}{q_1^3} - \epsilon\frac{F(r)}{q_1^2}\right) + O(\epsilon^2)$$
$$I^4 = t(p_1q_1 + p_2q_2) - t^2\big(p_1p_2 + \alpha\frac{q_2}{q_1^3} - \epsilon\frac{F(r)}{q_1^2}\big) - q_1q_2 + O(\epsilon^2)$$
$$I^5 = \epsilon\left(p_1p_2 + \alpha\frac{q_2}{q_1^3}\right) + O(\epsilon^2),$$
$$I^6 = \epsilon\left(\frac{1}{2}(p_1q_1 - p_2q_2)^2 - 2\alpha\frac{q_2^2}{q_1^2}\right) + O(\epsilon^2)$$
$$I^7 = \epsilon\left(p_1q_1 + p_2q_2 - 2t(p_1p_2 + \alpha\frac{q_2}{q_1^3})\right) + O(\epsilon^2),$$
$$I^8 = \epsilon\left(tp_1q_1 + tp_2q_2 - t^2(p_1p_2 + \alpha\frac{q_2}{q_1^3}) - q_1q_2\right) + O(\epsilon^2),$$
$$I^9 = \epsilon\left(p_2^2 - \frac{\alpha}{q_1^2}\right) + O(\epsilon^2),$$
$$I^{10} = \epsilon\left(p_2(tp_2 - q_1) - \frac{t\alpha}{q_1^2}\right) + O(\epsilon^2),$$
$$I^{11} = \epsilon\left(p_2\left(\frac{t^2p_2}{2} - tq_1\right)\frac{\partial}{\partial q_2} - \frac{1}{2}\left(\frac{\alpha t^2}{q_1^2} - q_1^2\right)\right) + O(\epsilon^2). \tag{66}$$

For arbitrary $F(r)$, four stable and seven unstable approximate first integrals exist in phase space. Campoamor-Stursberg [23] reported existence of only two stable approximate first integrals I^1 and I^2. The two additional stable approximate first integrals I^3 and I^4 arise by utilizing this newly developed approach.

Case 2 $F(r) = b_1$.
For $F(r) = b^1$, we have following three additional stable approximate Noether symmetry and approximate first integral in phase space:

$$X^{12} = p_2\frac{\partial}{\partial q_2},\ G^{12} = \frac{\alpha}{q_1^2},$$

$$I^{12} = \left(p_2^2 - \frac{\alpha}{q_1^2}\right) + O(\epsilon^2), \tag{67}$$

$$X^{13} = (tp_2 - q_1)\frac{\partial}{\partial q_2},\ G_{13} = \frac{t\alpha}{q_1^2},$$

$$I^{13} = \left(p_2(tp_2 - q_1) - \frac{t\alpha}{q_1^2}\right) + O(\epsilon^2), \tag{68}$$

$$X^{14} = \left(\frac{t^2p_2}{2} - tq_1\right)\frac{\partial}{\partial q_2},\ G_{14} = \frac{1}{2}\left(\frac{\alpha t^2}{q_1^2} - q_1^2\right),$$

$$I^{14} = p_2\left(\frac{t^2p_2}{2} - tq_1\right) - \frac{1}{2}\left(\frac{\alpha t^2}{q_1^2} - q_1^2\right) + O(\epsilon^2). \tag{69}$$

Campoamor-Stursberg [23] reported existence of only one stable approximate first integrals I^{12}. For this case, the two additional stable approximate first integrals I^{13} and I^{14} arise by utilizing this newly developed approach.

Case 3 $F(r) = b_2r$.
For $F(r) = b_2r$, we have following three additional stable approximate Noether symmetries and approximate first integrals in phase space:

$$X^{12} = p_2\frac{\partial}{\partial q_2},\ G_{12} = \frac{\alpha}{q_1^2} - \epsilon\frac{b_2}{q_1^2},$$

$$I^{12} = \left(p_2^2 - \frac{\alpha}{q_1^2} + \epsilon\frac{b_2}{q_1^2}\right) + O(\epsilon^2), \tag{70}$$

$$X^{13} = (tp_2 - q_1)\frac{\partial}{\partial q_2},\ G_{13} = \frac{t\alpha}{q_1^2} - \epsilon\frac{tb_2}{q_1^2},$$

$$I^{13} = \left(p_2(tp_2 - q_1) - \frac{t\alpha}{q_1^2}\right) + \epsilon\frac{tb_2}{q_1^2} + O(\epsilon^2), \tag{71}$$

$$X^{14} = \left(\frac{t^2p_2}{2} - tq_1\right)\frac{\partial}{\partial q_2},$$

$$G_{14} = \frac{1}{2}\left(\frac{\alpha t^2}{q_1^2} - q_1^2\right) - \epsilon\frac{t^2b_2}{2q_1^2},$$

$$I^{14} = p_2\left(\frac{t^2p_2}{2} - tq_1\right) - \frac{1}{2}\left(\frac{\alpha t^2}{q_1^2} - q_1^2\right) + \epsilon\frac{t^2b_2}{2q_1^2} + O(\epsilon^2). \tag{72}$$

Case 4 $F(r) = b_3r^2$.
When $F(r) = b_3r^2$, there are no additional approximate Noether symmetries and approximate first integrals in phase space.

Case 5 $F(r) = b_4r^3$.
For $F(r) = b_4r^3$, we have following three additional stable approximate Noether symmetries and approximate first integrals in phase space:

$$X^{12} = p_2\frac{\partial}{\partial q_2} - \frac{\epsilon}{2\alpha}b_4p_1\frac{\partial}{\partial q_1},\ G^{12}=\frac{\alpha}{q_1^2}-\epsilon\frac{3}{2}\frac{b_4q_2^2}{q_1^4},$$

$$I^{12} = p_2^2 - \frac{\alpha}{q_1^2} + \frac{\epsilon b_4}{2}\left(-\frac{p_1^2}{\alpha} + 3\frac{q_2^2}{q_1^4}\right) + O(\epsilon^2), \tag{73}$$

$$X^{13} = (tp_2 - q_1)\frac{\partial}{\partial q_2} + \frac{\epsilon b_4}{2\alpha}(q_2 - p_1t)\frac{\partial}{\partial q_1},$$

$$G^{13} = \frac{t\alpha}{q_1^2} - \epsilon\frac{3}{2}\frac{tb_4q_2^2}{q_1^4},$$

$$I^{13} = p_2(tp_2 - q_1) - \frac{t\alpha}{q_1^2} + \epsilon\left(\frac{tb_4}{2}\left(-\frac{p_1^2}{\alpha} + 3\frac{q_2^2}{q_1^4}\right) + \frac{1}{2\alpha}p_1b_4q_2\right) + O(\epsilon^2), \tag{74}$$

$$X^{14} = \left(\frac{t^2p_2}{2} - tq_1\right)\frac{\partial}{\partial q_2} + \frac{\epsilon b_4t}{2\alpha}(q_2 - \frac{1}{2}p_1t)\frac{\partial}{\partial q_1},$$

$$G^{14} = \frac{1}{2}\left(\frac{\alpha t^2}{q_1^2} - q_1^2\right) - \epsilon\frac{3}{4}\frac{t^2b_4q_2^2}{q_1^4} + \frac{\epsilon b_4q_2^2}{4\alpha},$$

$$I^{14} = p_2\left(\frac{t^2p_2}{2} - tq_1\right) - \frac{1}{2}\left(\frac{\alpha t^2}{q_1^2} - q_1^2\right)$$

$$+\epsilon\left(\frac{t^2b_4}{4}\left(-\frac{p_1^2}{\alpha}+3\frac{q_2^2}{q_1^4}\right)+\frac{t}{2\alpha}p_1b_4q_2 -\frac{b_4q_2^2}{4\alpha}\right)+O(\epsilon^2). \quad (75)$$

We have provided new forms of $F(r) = b_2r$ and $F(r) = b_4r^3$ for which additional stable approximate Noether symmetries and approximate first integrals exist.

5 Conclusions

The approximate Noether theorem proposed for the perturbed ODEs [1] was generalized in phase space for approximate Hamiltonian systems. We followed the procedure adopted by Dorodnitsyn and Kozlov [2] for the Hamiltonian systems of unperturbed ODEs. The approximate Legendre transformation connecting the approximate Hamiltonian and approximate Lagrangian was provided. The approximate Noether symmetries determining equation for the approximate Hamiltonian system was defined explicitly. We provided a formula to establish an approximate first integral associated with an approximate Noether symmetry of the approximate Hamiltonian system.

First, we analyzed the perturbed orbit equation governing one body problem to re-derive previously established results. The approximate Noether approach provided three stable and five unstable first integrals. In [1], the same approximate first integrals were derived by utilizing the calculus of variation techniques and Lagrangian formulation. We also compared our results with the unperturbed case studied by Dorodnitsyn and Kozlov [2]. This guarantees that formulas proposed here are correct. Next, we analyzed a linear non-autonomous second-order ODE [7] for the approximate first integrals. The approximate Noether approach provided five stable and five unstable first integrals.

This approach is applicable for the physical systems arising in applications containing small parameters or some arbitrary functions. We illustrated the advantages of this newly developed approach to determine the functional forms of an approximate Hamiltonian system involving arbitrary function. Campoamor-Stursberg [23] studied the perturbed system of ODEs in Lagrangian perspective, and we have investigated the same system in approximate Hamiltonian perspective. For arbitrary $F(r)$, four stable and seven unstable approximate first integrals were derived in phase space. Campoamor-Stursberg [23] reported existence of only two stable approximate first integrals for the case when $F(r)$ is arbitrary. The two additional stable approximate first integrals are determined here by utilizing this newly developed approach. Campoamor-Stursberg [23] reported existence of only one stable approximate first integrals for the case when $F(r)$ is constant, whereas we have provided three stable approximate first integrals for this case. We have also determined two additional functional forms $F(r) = b_2r$ and $F(r) = b_4r^3$ for which additional stable approximate Noether symmetries and approximate first integrals exist.

The approximate Noether approach [1] is applicable only for the calculus of variation problems. The approach developed here is applicable for the optimal control problems as well as calculus of variation problems, and this aspect of approach developed here will be considered in a future work.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this article.

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