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Parameter-splitting perturbation method for the improved solutions to strongly nonlinear systems

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Abstract A procedure named parameter-splitting perturbation method for improving the perturbation solutions to the forced vibrations of strongly nonlinear oscillators is proposed. The idea of the proposed procedure is presented in general first. After that, it is applied to optimize the solutions obtained by the multiple-scales method which is one of well-known perturbation methods. The harmonically forced Duffing oscillator, the harmonically forced oscillator with both nonlinear restoring force and nonlinear inertial force, the harmonically forced purely nonlinear oscillator and harmonically forced two-degree-of-freedom system with cubic nonlinearity are analyzed in various cases to show the advantages of the proposed method. The ratio of nonlinear stiffness coefficient to linear stiffness coefficient is chosen to be larger than one to highlight the validity of the propose method when dealing with strongly nonlinear oscillators. The validity of the proposed procedure is examined by comparing the frequency–response curves obtained by the proposed method, conventional multiple-scales method and numerical continuation method. Moreover, the errors corresponding to the results obtained by multiple-scales method are compared with those obtained by the proposed method to examine the performance of the proposed method. The results show that the proposed method can give much improved solutions in comparison with those obtained by multiple-scales method.

Keywords Nonlinear oscillator · Parameter splitting · Multiple-scales method · Strong nonlinearity · Optimum solution

1 Introduction

Nonlinearity exists in many real-world problems which can be approximately or exactly expressed by nonlinear ordinary differential equations (ODEs) or nonlinear partial differential equations (PDEs) [1,2]. However, it is difficult, if possible, to find the exact solutions of them. With the increasing interest in the applications of nonlinear problems, various approximate analytical methods for finding the approximate analytical solutions to those differential equations have been developed in recent years. The methods for the approximate analytical solutions to nonlinear ODEs or PDEs are normally classified as (1) perturbation method, (2) Adomian decomposition method (ADM), (3) homotopy analysis method (HAM), (4) harmonic balance method (HBM), (5) various modified perturbation methods and (6) hybrid perturbation-Galerkin method. The perturbation method breaks a nonlinear equation down to some linear differential equations for which exact solu-

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tions are obtainable. The regular perturbation method not only suffers from the limitation of small perturbation assumption but also the limitation of secular terms. To overcome the limitation of secular terms, multiple-scales (MS) method is proposed by introducing multiple timescales and setting the coefficients of the secular terms to be zeros [3]. However, the MS solution to strongly nonlinear ODE is still invalid because the asymptotic expansion of the solution is assumed to be a power series of perturbation parameter and this series can converge only when the perturbation parameter is much smaller than one. The ADM, which is capable for solving ODEs as well as PDEs, was developed by Adomian in 1986 [4,5]. The ADM is based on a decomposition of the solution of nonlinear operator equations in appropriate function spaces into a series of functions. It is valid for strongly nonlinear ODEs and PDEs. The convergence of the method was examined and studied in [6]. Recent applications in nano-systems by ADM were presented [7,8]. Based on the concept that the homotopy from topology can generate a convergent series solution to nonlinear problems, the HAM was developed to seek for approximate solutions to strongly nonlinear ODEs by Liao in 1992 [9]. Some modifications of the method can be found in [10,11]. The HAM with different orders is adopted to analyze the frequency response of micro-/nano-electromechanical system resonators in [12]. However, the selection of 'auxiliary parameter' can influence the convergence of the solution given by HAM [13]. The HBM was developed to obtain the solutions to nonlinear ODEs. This method was firstly implemented by Bailey and Lindnenlaub in 1968 and 1969, respectively [14,15]. Some improved HBM can be founded in many applications with higher accuracy [16,17]. A Newtonian HBM not limited to small response amplitude of oscillation was proposed by Lim et al. [18]. A two-degree-of-freedom quarter car model with a piecewise leaf spring for the rear suspension of a truck was analyzed by the incremental HBM. The effects of the leaf spring's stiffness, mass ratio and damping were investigated [19]. There are two well-known problems with HBM. The first one is that the HBM is not able to give transient response. The second one is that it is hard to find the initial conditions for solving the converted nonlinear algebraic equations for all solution branches [20]. The MS method and Lindstedt–Poincaré (LP) method are two improved perturbation methods developed for analyzing the nonlinear oscillators which can eliminate secular terms arising in the derivation of analytical solutions. However, the small perturbation parameter assumption cannot be avoided by this two methods. Burton and Rahman proposed a modified MS method and applied it to solve the forced vibration of oscillator with cubic nonlinearity or oscillator with quintic nonlinearity [21]. Burton and Rahman's method is valid for the oscillators with odd nonlinear terms. Cheung et al. introduced a new expanding parameter to LP method [22]. A strongly nonlinear oscillator with large perturbation parameter is transformed into an oscillator with small parameter. Chen et al. proposed a modified LP method for the analytical approximate solution of limit cycles in three-dimensional nonlinear autonomous dynamical systems [23]. Hu and Xiong applied the modified LP method to a Duffing equation and compared the obtained results with those from classical LP method [24]. In 2009, Pakdemirli provided a new way for finding the expansion parameter α in Burton's method [21] by combining the MS method and LP method [25], which has been applied to analyze various oscillators [26]. It is seen that the perturbation methods [21–25] are based on the different description of the nearness between the natural frequency and the excitation frequency according to the type of nonlinearity. It is noted that different expanding parameters were introduced for different oscillators with different nonlinearities as mentioned in [27]. Geer and Andersen proposed a hybrid perturbation-Galerkin technique by choosing the perturbation solutions as the trial basis functions in the Bubnov–Galerkin method [28] to accelerate the convergence of the Bubnov–Galerkin method [29,30]. The asymptotic methods are combined with various variational approaches in [31]. In order to increase the precision of the solution or the number of trial basis functions, higher-order perturbation solutions are needed.

A parameter-splitting method is proposed in this paper for improving the solutions obtained by the perturbation methods. The strategy of this method is that some parameters in the nonlinear system are split by introducing some unknown parameters. After that, the solution of the system is obtained by a perturbation method. Based on the solution obtained by a perturbation method, an optimization objective is formulated and the introduced unknown parameters are determined by minimizing the cumulative residual error of the original nonlinear system. This procedure is applied to improve the solutions given by MS method.

Therefore, the whole solution procedure is named parameter-splitting–multiple-scales (PSMS) method in the following analysis. The frequency–response curves (FRCs) obtained by MS method and PSMS method are compared to those obtained by numerical continuation method (NCM) with the software MATCONT [32,33]. Moreover, the errors in the results obtained by MS method are compared with those obtained by PSMS method to show the improvement in the results obtained by PSMS method. Totally four examples about, respectively, the harmonically forced strongly nonlinear Duffing oscillator, the harmonically forced oscillator with the coexistence of strongly nonlinear restoring force and strongly nonlinear inertial force, the harmonically forced purely nonlinear oscillator with strongly nonlinearity and the harmonically forced two-degree-of-freedom (TDOF) system with strong cubic nonlinearity are analyzed to highlight the validity of the proposed method in analyzing strongly nonlinear systems. The results show that the solutions obtained by PSMS method are much improved comparing to those obtained by MS method.

2 Strategy for improving the solutions obtained by perturbation methods

Consider the following second-order nonlinear oscillator

$$\ddot{y} + c\varepsilon^2\dot{y} + \omega_0^2 y + \varepsilon g(y, \dot{y}, \ddot{y}) = \varepsilon^2 F\cos(\Omega t), \quad (1)$$

where y is displacement, t is time, c is damping coefficient, ω_0 is the natural frequency of oscillator, ε is perturbation parameter, F is excitation amplitude, Ω is excitation frequency and $g(y, \dot{y}, \ddot{y})$ is a polynomial function of y, $\dot{y}$ and $\ddot{y}$. $g(y, \dot{y}, \ddot{y})$ is given as

$$g(y, \dot{y}, \ddot{y}) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_{ijk}y^i\dot{y}^j\ddot{y}^k, \quad (2)$$

where η_{ijk} are the nonlinear parameters which reflect the degrees of nonlinearity and $\sum i + j + k \geq 2$. The linear natural frequency ω_0 and the nonlinear parameters η_{ijk} are split and expressed as follows.

$$\omega_0^2 = \omega_{00}^2 + \omega_{01}^2\varepsilon + \omega_{02}^2\varepsilon^2 \quad (3)$$

$$\eta_{ijk} = \eta_{ijk,1} + \eta_{ijk,2}\varepsilon. \quad (4)$$

With the above procedure, Eq. (1) can be written as

$$\ddot{y} + c\varepsilon^2\dot{y} + \omega_{00}^2 y + \varepsilon\omega_{01}^2 y + \varepsilon^2\omega_{02}^2 y + \varepsilon g_1(y, \dot{y}, \ddot{y}) + \varepsilon^2 g_2(y, \dot{y}, \ddot{y}) = \varepsilon^2 F\cos(\Omega t), \quad (5)$$

where

$$g_1(y, \dot{y}, \ddot{y}) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_{ijk,1}y^i\dot{y}^j\ddot{y}^k \quad (6)$$

$$g_2(y, \dot{y}, \ddot{y}) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_{ijk,2}y^i\dot{y}^j\ddot{y}^k. \quad (7)$$

By perturbation methods, the oscillator response can be generally expressed as

$$y = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + O(\varepsilon^3). \quad (8)$$

Substituting Eq. (8) into Eq. (5) leads to

$$\begin{aligned}&(\ddot{y}_0 + \varepsilon\ddot{y}_1 + \varepsilon^2\ddot{y}_2) + c\varepsilon^2(\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2)\\&+ (\omega_{00}^2 + \varepsilon\omega_{01}^2 + \varepsilon^2\omega_{02}^2)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2)\\&+ \varepsilon\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_{ijk,1}(y_0 + \varepsilon y_1 + \varepsilon^2 y_2)^i(\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2)^j\\&(\ddot{y}_0 + \varepsilon\ddot{y}_1 + \varepsilon^2\ddot{y}_2)^k\\&+ \varepsilon^2\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_{ijk,2}(y_0 + \varepsilon y_1 + \varepsilon^2 y_2)^i(\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2)^j\\&(\ddot{y}_0 + \varepsilon\ddot{y}_1 + \varepsilon^2\ddot{y}_2)^k + O(\varepsilon^3) = \varepsilon^2 F\cos(\Omega t).\end{aligned} \quad (9)$$

Equating the coefficients of ε^s ($s = 0, 1, 2$) to zero gives the following equations.

$$O(\varepsilon^0): \ \ddot{y}_0 + \omega_{00}^2 y_0 = 0 \quad (10)$$

$$O(\varepsilon^1): \ \ddot{y}_1 + \omega_{01}^2 y_1 = -\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_{ijk,1}y_0^i\dot{y}_0^j\ddot{y}_0^k - \ldots \quad (11)$$

$$O(\varepsilon^2): \ \ddot{y}_2 + \omega_{02}^2 y_2 = F\cos(\Omega t) - c\dot{y}_0 - \eta_{110,1}y_1\dot{y}_0 - \ldots - \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_{ijk,2}y_0^i\dot{y}_0^j\ddot{y}_0^k. \quad (12)$$

The expressions of y_1 and y_2 can be easily determined as y_0 is obtainable which is the solution to a second-order homogeneous differential equation. Once y_0, y_1 and y_2 are determined, the approximate analytical steady-state response y_a can be expressed as

$$y_a = y_0 + \varepsilon y_1 + \varepsilon^2 y_2, \tag{13}$$

where the subscript a stands for approximation. Substituting Eq. (13) into Eq. (1) gives the following residual error function.

$$f(\ddot{y}_a, \dot{y}_a, y_a, t, \boldsymbol{\Theta_s}) = \ddot{y}_a + c\varepsilon^2\dot{y}_a + \omega_0^2 y_a + \varepsilon g(y_a, \dot{y}_a, \ddot{y}_a) - \varepsilon^2 F\cos(\Omega t), \tag{14}$$

where $\boldsymbol{\Theta_s}$ is a set of system parameters and split parameters. The following cumulative residual error R_e is introduced.

$$R_e = \int_0^T f^2(\ddot{y}_a, \dot{y}_a, y_a, t, \boldsymbol{\Theta_s})\mathrm{d}t, \tag{15}$$

where $T = 2\pi/\Omega$. Since the function f consists of periodic functions with periods $\frac{2\pi}{n\Omega}$ $(n = 1, 2, 3\ldots)$ where Ω is the excitation frequency, the integration upper limit is hence selected as T to cumulate all the errors induced by each periodic function. The unknown splitting parameters are determined by minimizing R_e. The details and advantages of this solution procedure are presented in the following when it is combined and applied with MS method.

3 Parameter-splitting–multiple-scales method and numerical analysis

The MS method was proposed decades ago [34] and many of its applications can be found [35,36]. Due to the small parameter assumption with MS method, the solution obtained by MS method is not valid if the nonlinear parameters in the governing equations are not small. In the following, the procedure presented above is combined and applied with MS method to improve the solutions obtained by MS method. Totally four types of oscillators are analyzed, which are the Duffing oscillator, an oscillator with both nonlinear restoring force and nonlinear inertial force, a purely nonlinear oscillator and a TDOF system with cubic nonlinearity. Hence, the whole solution procedure is named parameter-splitting–multiple-scales (PSMS) method in the following analysis. For each oscillator, the solution procedure of PSMS method is presented first. After that, the accuracy and the effectiveness of PSMS method is tested numerically. The FRCs obtained by PSMS method and MS method are compared with those obtained by NCM. In the last, the results are discussed.

3.1 Duffing oscillator and an oscillator with both nonlinear restoring force and nonlinear inertial force

Consider the following nonlinear oscillator

$$\ddot{y} + \alpha\varepsilon y^2\ddot{y} + c\varepsilon^2\dot{y} + \omega_0^2 y + \beta\varepsilon y\dot{y}^2 + \eta\varepsilon y^3 = F\varepsilon^2\cos(\Omega t). \tag{16}$$

This oscillator is a damped and forced Duffing oscillator if $\alpha = \beta = 0$. When $\alpha = \beta \neq 0$, this oscillator describes the forced vibration of an inextensible single-mode cantilever beam carrying an intermediate lumped mass with a rotary inertia [37]. The second and fifth terms in Eq. (16) are inertial nonlinearity arising from the inextensibility assumption in analyzing the vibration of cantilever beam.

3.1.1 Parameter splitting

The natural frequency ω_0 and the nonlinear parameters α, β and η are split and expressed as

$$\omega_0^2 = \omega_{00}^2 + \omega_{01}^2\varepsilon + \omega_{02}^2\varepsilon^2 \tag{17}$$

$$\alpha = \alpha_1 + \alpha_2\varepsilon \tag{18}$$

$$\beta = \beta_1 + \beta_2\varepsilon \tag{19}$$

$$\eta = \eta_1 + \eta_2\varepsilon. \tag{20}$$

Then the nonlinear oscillator can be rewritten as

$$\ddot{y} + \alpha_1\varepsilon y^2\ddot{y} + \alpha_2\varepsilon^2 y^2\ddot{y} + c\varepsilon^2\dot{y} + \omega_{00}^2 y + \omega_{01}^2\varepsilon y + \omega_{02}^2\varepsilon^2 y + \beta_1\varepsilon y\dot{y}^2 + \beta_2\varepsilon^2 y\dot{y}^2 + \eta_1\varepsilon y^3 + \eta_2\varepsilon^2 y^3 = F\varepsilon^2\cos(\Omega t). \tag{21}$$

3.1.2 The solution by MS method

With MS method, the response of the oscillator is assumed to be

$$y = y_0(T_0, T_1, T_2) + \varepsilon y_1(T_0, T_1, T_2) + \varepsilon^2 y_2(T_0, T_1, T_2) + O(\varepsilon^3), \quad (22)$$

where T_0, T_1 and T_2 are the fast and slow timescales expressed by

$$T_0 = t, \quad T_1 = \varepsilon t \quad \text{and} \quad T_2 = \varepsilon^2 t. \quad (23)$$

By chain rule, the operators of time derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \quad (24)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \cdots, \quad (25)$$

where $D_n = \partial/\partial T_n$ and $D_n^2 = \partial^2/\partial T_n^2$. Substituting Eqs. (22), (24) and (25) into Eq. (21) and equating the coefficients of $\varepsilon^m (m = 0, 1, 2)$ to zero lead to the following equations.

$$O(\varepsilon^0) \ : D_0^2(y_0) + \omega_{00}^2 y_0 = 0, \quad (26)$$

$$O(\varepsilon^1) \ : D_0^2(y_1) + \omega_{00}^2 y_1 = -\omega_{01}^2 y_0 - 2D_0 D_1(y_0) - \alpha_1 y_0^2 D_0^2(y_0) - \beta_1 y_0[D_0(y_0)]^2 - \eta_1 y_0^3, \quad (27)$$

$$O(\varepsilon^2) \ : D_0^2(y_2) + \omega_{00}^2 y_2 = F\cos(\Omega t) - \omega_{01}^2 y_1 - \omega_{02}^2 y_0 - 2D_0 D_1(y_1) - D_1^2(y_0) - 2D_0 D_2(y_0) - cD_0(y_0) - 2\beta_1 y_0 D_0(y_0) D_0(y_1) - 2\beta_1 y_0 D_0(y_0) D_1(y_0) - \beta_1 y_1 [D_0(y_0)]^2 - \beta_2 y_0 [D_0(y_0)]^2 - \alpha_1 D_0^2 y_1 y_0^2 - 2\alpha_1 y_0^2 D_0 D_1(y_0) - 2\alpha_1 y_0 y_1 D_0^2(y_0) - \alpha_2 y_0^2 D_0^2(y_0) - 3\eta_1 y_0^2 y_1 - \eta_2 y_0^3. \quad (28)$$

The $O(\varepsilon^0)$ equation is a homogenous differential equation and the solution to it is

$$y_0 = C(T_1, T_2)e^{i\omega_{00}T_0} + \bar{C}(T_1, T_2)e^{-i\omega_{00}T_0}, \quad (29)$$

where C is a function of timescales T_1 and T_2 which can be determined by omitting the secular terms in the $O(\varepsilon^1)$ equation. Substituting Eq. (29) into the right-hand side of the $O(\varepsilon^1)$ equation and eliminating the secular terms yield

$$D_1(C) = \frac{3\alpha_1 C^2 \bar{C}\omega_{00}}{2i} - \frac{3\eta_1 C^2 \bar{C}}{2i\omega_{00}} - \frac{\beta_1 C^2 \bar{C}\omega_{00}}{2i} - \frac{C\omega_{01}^2}{2i\omega_{00}} \quad (30)$$

and

$$y_1 = \Lambda e^{3i\omega_{00}T_0} + \bar{\Lambda} e^{-3i\omega_{00}T_0}, \quad (31)$$

in which

$$\Lambda = \frac{\eta_1 C^3}{8\omega_{00}^2} - \frac{\beta_1 C^3}{8} - \frac{\alpha_1 C^3}{8}. \quad (32)$$

Substituting the expressions of y_0 and y_1 into the $O(\varepsilon^2)$ equation, eliminating the secular terms and using the expression $\Omega = \omega_{00} + \varepsilon^2\sigma$ where σ is a detuning parameter that can be determined if Ω is given, it gives

$$D_2(C) = \frac{Fe^{i\sigma T_2}}{4i\omega_{00}} - \frac{cC}{2} - \frac{C\omega_{02}^2}{2i\omega_{00}} + \frac{C\omega_{01}^4}{8i\omega_{00}^3} + \frac{7\beta_1\eta_1 i C^3\bar{C}^2}{8\omega_{00}} + \frac{\beta_1^2 i\omega_{00}C^3\bar{C}^2}{16} + \frac{65\alpha_1^2 i\omega_{00}C^3\bar{C}^2}{16} + \frac{\beta_2 i\omega_{00}C^2\bar{C}}{2} - \frac{3\alpha_2 i\omega_{00}C^2\bar{C}}{2} - \frac{\beta_1\omega_{01}^2 C^2\bar{C}}{4i\omega_{00}} + \frac{3\alpha_1\omega_{01}^2 C^2\bar{C}}{4i\omega_{00}} - \frac{15\alpha_1\beta_1 i\omega_{00}C^3\bar{C}^2}{8} - \frac{15i\eta_1^2 C^3\bar{C}^2}{16\omega_{00}^3} - \frac{25\alpha_1\eta_1 i C^3\bar{C}^2}{8\omega_{00}} - \frac{3\eta_2 C^2\bar{C}}{2i\omega_{00}} + \frac{3\eta_1 C^2\bar{C}\omega_{01}^2}{4i\omega_{00}^3} \quad (33)$$

and

$$q_2 = \Gamma_1 e^{3i\omega_{00}T_0} + \Gamma_2 e^{5i\omega_{00}T_0} + \bar{\Gamma}_1 e^{-3i\omega_{00}T_0} + \bar{\Gamma}_2 e^{-5i\omega_{00}T_0}, \quad (34)$$

in which

$$\Gamma_1 = -\frac{\beta_2 C^3}{8} - \frac{\alpha_2 C^3}{8} - \frac{\beta_1^2 C^4 \bar{C}}{64} + \frac{19\alpha_1^2 C^4 \bar{C}}{64} + \frac{9\alpha_1\beta_1 C^4 \bar{C}}{32} - \frac{5\beta_1\eta_1 C^4 \bar{C}}{32\omega_{00}^2} + \frac{\alpha_1\eta_1 C^4 \bar{C}}{32\omega_{00}^2} + \frac{\eta_2 C^3}{8\omega_{00}^2} - \frac{\eta_1 C^3 \omega_{01}^2}{8\omega_{00}^4} - \frac{21\eta_1^2 C^4 \bar{C}}{64\omega_{00}^4} \tag{35}$$

and

$$\Gamma_2 = \frac{11\alpha_1^2 C^5}{192} + \frac{7\beta_1^2 C^5}{192} + \frac{3\alpha_1\beta_1 C^5}{32} - \frac{7\alpha_1\eta_1 C^5}{96\omega_{00}^2} - \frac{5\beta_1\eta_1 C^5}{96\omega_{00}^2} + \frac{\eta_1^2 C^5}{64\omega_{00}^4}. \tag{36}$$

The time derivative of C can be expressed as

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \varepsilon D_1(C) + \varepsilon^2 D_2(C) + O(\varepsilon^3). \tag{37}$$

The polar form of C is assumed to be

$$C = \frac{1}{2} A e^{ib}, \tag{38}$$

where A is the response amplitude and b is the phase of oscillator response. Substituting Eqs. (30), (33) and (38) into Eq. (37) and separating the real and imaginary parts yield

$$\dot{A} = \frac{F\varepsilon^2}{2\omega_{00}} \sin\gamma - \frac{cA\varepsilon^2}{2} \tag{39}$$

and

$$\dot{\gamma} = \Omega - \omega_{00} - \frac{\varepsilon\omega_{01}^2}{2\omega_{00}} - \frac{\varepsilon^2\omega_{02}^2}{2\omega_{00}} + \frac{\varepsilon^2\omega_{01}^4}{8\omega_{00}^3} - \frac{\beta_1^2\omega_{00}\varepsilon^2 A^4}{256} - \frac{65\alpha_1^2\omega_{00}\varepsilon^2 A^4}{256} - \frac{\beta_1\omega_{00}\varepsilon A^2}{8} + \frac{3\alpha_1\omega_{00}\varepsilon A^2}{8} - \frac{\beta_2\omega_{00}\varepsilon^2 A^2}{8} + \frac{3\alpha_2\omega_{00}\varepsilon^2 A^2}{8} + \frac{15\alpha_1\beta_1\omega_{00}\varepsilon^2 A^4}{128} - \frac{7\beta_1\eta_1\varepsilon^2 A^4}{128\omega_{00}} + \frac{25\alpha_1\eta_1\varepsilon^2 A^4}{128\omega_{00}} - \frac{\beta_1\omega_{01}^2\varepsilon^2 A^2}{16\omega_{00}} + \frac{3\alpha_1\omega_{01}^2\varepsilon^2 A^2}{16\omega_{00}} - \frac{3A^2\eta_2\varepsilon^2}{8\omega_{00}} + \frac{15A^4\eta_1^2\varepsilon^2}{256\omega_{00}^3} - \frac{3A^2\eta_1\varepsilon}{8\omega_{00}} + \frac{\varepsilon^2 F\cos(\gamma)}{2A\omega_{00}} + \frac{3A^2\eta_1\varepsilon^2\omega_{01}^2}{16\omega_{00}^3}, \tag{40}$$

where $\gamma = \sigma T_2 - b$.
At steady state, $\dot{A}$ and $\dot{\gamma}$ are equal to zero. Then the FRC can be obtained by eliminating γ and σ in Eq. (40). The relation between the excitation frequency and the response amplitude at steady state is then obtained to be

$$\Omega = \omega_{00} + \frac{\varepsilon\omega_{01}^2}{2\omega_{00}} + \frac{\varepsilon^2\omega_{02}^2}{2\omega_{00}} - \frac{\varepsilon^2\omega_{01}^4}{8\omega_{00}^3} + \frac{\beta_1^2\omega_{00}\varepsilon^2 A^4}{256} + \frac{65\alpha_1^2\omega_{00}\varepsilon^2 A^4}{256} + \frac{\beta\omega_{00}\varepsilon A^2}{8} - \frac{3\alpha\omega_{00}\varepsilon A^2}{8} - \frac{15\alpha_1\beta_1\omega_{00}\varepsilon^2 A^4}{128} + \frac{7\beta_1\eta_1\varepsilon^2 A^4}{128\omega_{00}} - \frac{25\alpha_1\eta_1\varepsilon^2 A^4}{128\omega_{00}} + \frac{\beta_1\omega_{01}^2\varepsilon^2 A^2}{16\omega_{00}} - \frac{3\alpha_1\omega_{01}^2\varepsilon^2 A^2}{16\omega_{00}} - \frac{15A^4\eta_1^2\varepsilon^2}{256\omega_{00}^3} + \frac{3A^2\eta_1\varepsilon}{8\omega_{00}} - \frac{\varepsilon^2 F\cos(\gamma)}{2A\omega_{00}} - \frac{3A^2\eta_1\varepsilon^2\omega_{01}^2}{16\omega_{00}^3}. \tag{41}$$

The approximate response of the oscillator is obtained to be

$$y_a = \mathbb{A}_1 \cos\left(\Omega t - \gamma\right) + 2\mathbb{A}_3 \cos\left(3\Omega t - 3\gamma\right) + 2\mathbb{A}_5 \cos\left(5\Omega t - 5\gamma\right), \tag{42}$$

in which

$$\mathbb{A}_1 = A, \tag{43}$$

$$\mathbb{A}_3 = \frac{\eta\varepsilon A^3}{64\omega_{00}^2} - \frac{\alpha\varepsilon A^3}{64} - \frac{\beta\varepsilon A^3}{64} - \frac{\beta_1^2\varepsilon^2 A^5}{2048} + \frac{19\alpha_1^2\varepsilon^2 A^5}{2048} - \frac{21\eta_1^2\varepsilon^2 A^5}{2048\omega_{00}^4} - \frac{5\beta_1\eta_1\varepsilon^2 A^5}{1024\omega_{00}^2} + \frac{9A^5\alpha_1\beta_1\varepsilon^2}{1024} + \frac{\alpha_1\eta_1\varepsilon^2 A^5}{1024\omega_{00}^2} - \frac{A^3\varepsilon^2\eta_1\omega_{01}^2}{64\omega_{00}^4} \tag{44}$$

and

$$\mathbb{A}_5 = \frac{11\alpha_1^2\varepsilon^2 A^5}{6144} + \frac{7\beta_1^2\varepsilon^2 A^5}{6144} + \frac{\eta_1^2\varepsilon^2 A^5}{2048\omega_{00}^4} + \frac{3\alpha_1\beta_1\varepsilon^2 A^5}{1024} - \frac{5\beta_1\eta_1\varepsilon^2 A^5}{3072\omega_{00}^2} - \frac{7\alpha_1\eta_1\varepsilon^2 A^5}{3072\omega_{00}^2}. \quad (45)$$

3.1.3 Optimization objective

From Eq. (42) it is seen that the expression of y_a can be considered as a function $f(\omega_{01}, \omega_{02}, \alpha_1, \beta_1, \eta_1)$ of the variables ω_{01}, ω_{02}, α_1, β_1 and η_1 once the system parameters $\omega_0, c, \alpha, \beta, \eta, \Omega$ and F are given. Select an interval $\mathbf{\Omega} = [\Omega_l, \Omega_u]$ on the positive frequency axis, in which most of the FRC falls. Then for a given value of excitation frequency within $\mathbf{\Omega}$, the values of ω_{01}, ω_{02}, α_1, β_1 and η_1 can be determined by minimizing the value of the residual error R_e expressed by Eq. (15). The complete FRC can be obtained by repeating this procedure and varying Ω from Ω_l to Ω_u.

3.1.4 Numerical analysis

(a) Duffing oscillators 1 and 2 with viscous damping and harmonic force

When α and β in Eq. (16) equal zeros, Eq. (16) expresses a Duffing oscillator excited by harmonic force. This oscillator can be found in many areas such as the forced vibrations of pendulum, isolator, electrical circuit [38–42] and so on. The Duffing oscillator was also frequently analyzed for testing new solution procedures. Two strongly nonlinear Duffing oscillators whose ratios of nonlinear restoring force to linear restoring force $\left(\frac{\eta\varepsilon y^3}{\omega_0^2 y}\right)$ are equal to y^2 and $1.5y^2$ are analyzed, respectively. The parameter values of these two oscillators are listed in Table 1. The FRCs obtained by PSMS method, MS method and numerical continuation method are shown and compared in Figs. 1 and 2 for oscillators 1 and 2, respectively. A large deviation can be observed in each FRC obtained by MS method in comparison with those obtained by NCM as the value of $\eta\varepsilon y^3/(\omega_0^2 y)$ increases. Even when the response amplitude is small, MS method still cannot predict the results with enough accuracy, which can be observed from the enlarged local details shown in Figs. 1 and 2. On the other hand, PSMS method can give accurate solution to the Duffing oscillator in the whole frequency domain in comparison with those obtained by NCM.

Table 1 Parameters in the Duffing oscillators 1 and 2

Oscillator no.	ε	ω_0	c	η	F
1	0.1	20	20	4000	1000
2	0.1	20	20	6000	2000

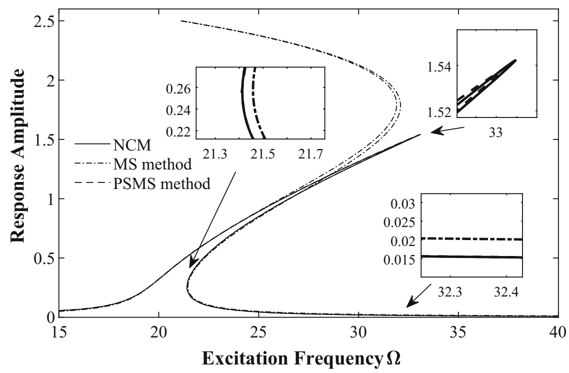

Fig. 1 Oscillator 1's FRCs by MS method, PSMS method and NCM

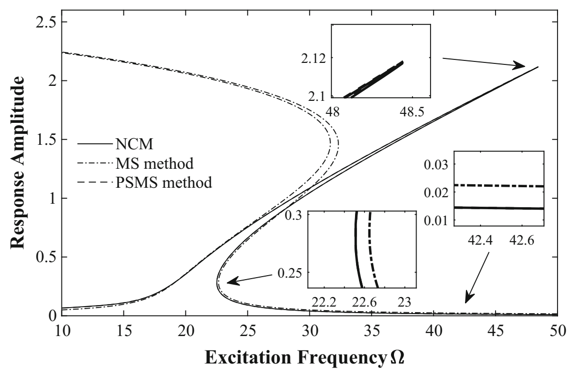

Fig. 2 Oscillator 2's FRCs by MS method, PSMS method and NCM

(b) Oscillators 3 and 4 with nonlinear inertial force and nonlinear restoring force

When $\alpha = \beta \neq 0$, Eq. (16) expresses the equation of motion governing the forced vibration of a cantilever beam with large deflection. It can be found in many applications [43,44]. Two cases are considered for this strongly nonlinear oscillators which are analyzed by means of PSMS method, MS method and NCM, respectively. The first case represents the forced vibration of the cantilever beam when the nonlinear restoring force dominates the oscillator nonlinearity, which can

Table 2 Parameters in oscillators 3 and 4 with restoring force and inertial force

Oscillator no.	ε	ω_0	c	α	β	η	F
3	0.1	1	4	5	5	10	5
4	0.1	1	4	10	10	0.1	5

be found in the first-mode vibration of the beam. The second case represents the forced vibration of the cantilever beam when the nonlinear inertial force dominates the oscillator nonlinearity, which can be found in the second-mode or higher-mode vibration of the beam [37,45]. The parameter values are listed in Table 2. The FRCs obtained by PSMS method, MS method and numerical continuation method are shown and compared in Figs. 3 and 4 for oscillators 3 and 4, respectively. From the numerical analysis on the FRCs of oscillator 3, it is observed that the term $\eta\varepsilon y^3$ ($\eta > 0$) dominates the response of the oscillator, which can be found in the application of the first-mode vibration of a cantilever beam [37,45]. The deviation between the FRC obtained by MS method and the FRC obtained by NCM increases as the response amplitude increases. From the numerical analysis on the FRCs of oscillator 4, it is observed that the terms $\alpha\varepsilon y^2\ddot{y}$ and $\beta\varepsilon\dot{y}^2 y$ ($\alpha = \beta > 0$) dominate the response of the oscillator, which can be found in the applications of the second- or higher-mode vibrations of a cantilever beam [37,45]. The FRC obtained by MS method starts to turn right near $\Omega = 0.9$, which not agrees with the behavior of the FRC obtained by numerical continuation method. The peak of the FRC obtained by MS method is obviously different from that obtained by numerical continuation method. On the contrary, the FRCs obtained by PSMS method agree well with the FRCs obtained by numerical continuation method no matter which kind of nonlinearity dominates the response. What's more, even when the response amplitude is small, the response amplitudes obtained by PSMS method are still improved in comparison with those obtained by MS method as shown in the enlarged local details in Figs. 3 and 4. The results presented and discussed above show that PSMS method can give accurate solutions to the forced vibrations of the oscillators with nonlinear restoring force and inertial force no matter which term, either the nonlinear restoring force or the nonlinear inertial force, dominates the nonlinearity of the oscillator.

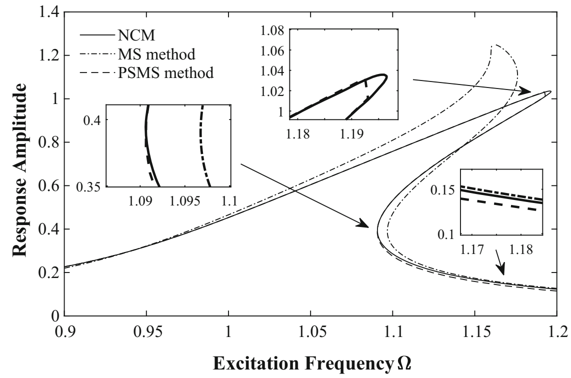

Fig. 3 Oscillator 3's FRCs by MS method, PSMS method and NCM

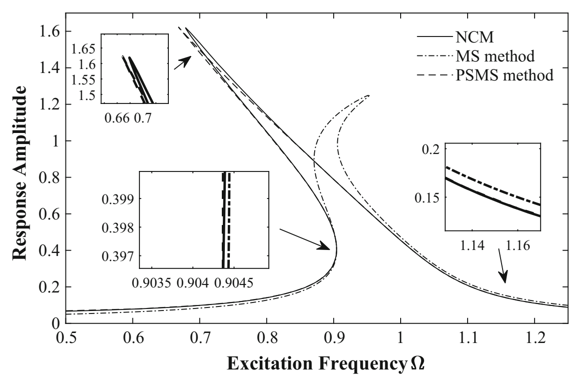

Fig. 4 Oscillator 4's FRCs by MS method, PSMS method and NCM

3.2 PSMS solution procedure and numerical analysis of a purely nonlinear oscillator

In this section, a harmonically forced oscillator without linear restoring force but with nonnegative geometric nonlinearity is considered. This kind of oscillator can be found the applications in purely nonlinear material properties [46,47]. The oscillator is given as

$$\ddot{y} + 2\zeta\varepsilon\dot{y} + \kappa\varepsilon\mathrm{sgn}(y)|y|^{\alpha} = F\varepsilon\cos(\Omega t), \qquad (46)$$

where κ and α are oscillator parameters and ε is perturbation parameter. As there is no linear restoring force, classical perturbation method cannot be applied as the nearness between the natural frequency and the excitation frequency cannot be described. Kovacic made a transformation and some solutions were obtained by Lindstedt–Poincare method and MS method [46]. As in [46], adding a zero linear stiffness term to the oscillator yields

$$\ddot{y} + 2\zeta\varepsilon\dot{y} + \omega_0^2 y + \kappa\varepsilon \operatorname{sgn}(y)|y|^\alpha = F\varepsilon\cos(\Omega t), \tag{47}$$

where $\omega_0 = 0$.

3.2.1 Parameter splitting

The natural frequency ω_0 and the nonlinear parameter κ are split and expressed as

$$\omega_0^2 = \omega_{00}^2 + \omega_{01}^2\varepsilon \tag{48}$$

$$\kappa = \kappa_0 + \kappa_1\varepsilon. \tag{49}$$

After that, the nonlinear oscillator can be rewritten as

$$\ddot{y}+2\zeta\varepsilon\dot{y}+\omega_{00}^2 y+\omega_{01}^2 y+\kappa_0\varepsilon \operatorname{sgn}(y)|y|^\alpha = F\varepsilon\cos(\Omega t) \tag{50}$$

3.2.2 The solution by MS method

A two-term expansion of the response is assumed to be

$$y = y_0(T_0, T_1) + \varepsilon y_1(T_0, T_1) + O(\varepsilon^2), \tag{51}$$

where the definition of T_0, T_1 and the operation of time derivatives are the same as those in Eqs. (23)–(25). The nearness between natural frequency and excitation frequency is assumed to be $\Omega = \omega_{00} + \sigma\varepsilon$. The natural frequency can be rewritten as $\omega_{00}^2 = \Omega^2 - 2\sigma\Omega\varepsilon + O(\varepsilon^2)$. As y is a power series of perturbation parameter ε, the nonlinear term can be expanded by using Taylor series expansion as

$$\operatorname{sgn}\left(\sum_{i=0}^{\infty} y_i\varepsilon^i\right)\left|\sum_{i=0}^{\infty} y_i\varepsilon^i\right|^\alpha = \operatorname{sgn}(y_0)|y_0|^\alpha + \varepsilon^1\alpha y_1|y_0|^{\alpha-1} + \cdots \tag{52}$$

Substituting this along with $\omega_{00}^2 = \Omega^2 - 2\sigma\Omega\varepsilon + O(\varepsilon^2)$, Eqs. (51) and (23)-(25) into Eq. (50) and equating the coefficients of ε^m $(m = 0, 1)$ to zero lead to the following equations.

$$O(\varepsilon^0)\ : D_0^2(y_0) + \Omega^2 y_0 = 0, \tag{53}$$

$$O(\varepsilon^1)\ : D_0^2(y_1) + \Omega^2 y_1 = F\cos(\Omega t) - 2D_0D_1y_0 - 2\zeta D_0 y_0 + 2\sigma\Omega y_0 - \omega_{01}^2 y_0 - \kappa_0 \operatorname{sgn}(y_0)|y_0|^\alpha. \tag{54}$$

The $O(\varepsilon^0)$ equation is a homogenous differential equation and the solution to it is

$$y_0 = C(T_1)e^{i\Omega T_0} + \bar{C}(T_1)e^{-i\Omega T_0}, \tag{55}$$

where C is assumed to be

$$C = \frac{1}{2}Ae^{i\gamma}, \tag{56}$$

where A is the response amplitude and γ is the response phase angle. Therefore, y_0 can be rewritten as

$$y_0 = A\cos(\Omega t + \gamma). \tag{57}$$

Since y_0 is a cosine function, the nonlinear term $\operatorname{sgn}(y_0)|y_0|^\alpha$ can be expressed by a Fourier expansion as follows as in [39].

$$\operatorname{sgn}(y_0)|y_0|^\alpha = |A|^\alpha[b_{1\alpha}\cos(\Omega t + \gamma) + b_{3\alpha}\cos(3\Omega t + 3\gamma)], \tag{58}$$

where

$$b_{1\alpha} = \frac{2\Gamma\left(1+\frac{\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3+\alpha}{2}\right)}, \quad b_{3\alpha} = \frac{(\alpha-1)\Gamma\left(1+\frac{\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{5+\alpha}{2}\right)}, \tag{59}$$

where $\Gamma(.)$ is the gamma function defined as $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x$. Substituting Eqs. (55) and (56) into the right-hand side of the $O(\varepsilon^1)$ equation and eliminating the secular terms yield

$$D_1(C) = \frac{F}{4i\Omega} - \frac{\zeta Ae^{ib}}{2} + \frac{\sigma Ae^{ib}}{2i} - \frac{\omega_{01}^2 Ae^{ib}}{4i\Omega} - \frac{\kappa_0 A^\alpha b_{1\alpha}e^{ib}}{4i\Omega}. \tag{60}$$

With elimination of the secular terms, the solution y_1 to the $O(\varepsilon^1)$ is determined to be

$$y_1 = \frac{\kappa_0 A^\alpha b_{3\alpha}}{8\Omega^2}\cos(3\Omega t + 3\gamma). \tag{61}$$

The time derivative of C can be expressed as

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \varepsilon D_1(C) + O(\varepsilon^2). \tag{62}$$

With the polar form of C, separating the real and imaginary parts yields

Table 3 Parameters in the purely nonlinear oscillators 5, 6 and 7

Oscillator no.	ε	ω_0	ζ	α	κ	F
5	0.1	0	1	1/3	100	100
6	0.1	0	1	2	100	100
7	0.1	0	1	3	100	100

$$\frac{1}{2}\dot{A} = -\frac{F\varepsilon}{4\Omega}\sin(\gamma) - \frac{\zeta A\varepsilon}{2} \tag{63}$$

and

$$\frac{1}{2}A\dot{\gamma} = \frac{-F\varepsilon}{4\Omega}\cos(\gamma) - \frac{\sigma A\varepsilon}{2} + \frac{\omega_{01}^2 A\varepsilon}{4\Omega} + \frac{\kappa_0 A^\alpha b_{1\alpha}\varepsilon}{4\Omega}. \tag{64}$$

At steady state, $\dot{A}$ and $\dot{\gamma}$ are equal to zero. Then the FRC can be obtained by eliminating γ and σ. The relationship between the excitation frequency and the response amplitude at steady state is then obtained to be

$$\left[2A\Omega\left(\frac{\omega_{00}-\Omega}{\varepsilon}\right) + \omega_{01}^2 A + \kappa_0 A^\alpha b_{1\alpha}\right]^2 + 4\zeta^2 A^2\Omega^2 = F^2. \tag{65}$$

For a given excitation frequency Ω, the value of A can be solved numerically.
The approximate response of the oscillator is obtained by combining y_0 and y_1 as

$$y_a = A\cos(\Omega t+\gamma) + \frac{\kappa_0 A^\alpha b_{3\alpha}\varepsilon}{8\Omega^2}\cos(3\Omega t+3\gamma). \tag{66}$$

The optimum FRC can then be obtained in the same way as stated in Sect. 3.1.3.

3.2.3 Numerical analysis of a purely nonlinear oscillator

Consider three oscillators with nonlinearity be of a fractional power ($\alpha = 1/3$) and nonlinearity with $\alpha = 2$ and 3. The first one is a 'under-linear' (softening) oscillator, and the second and third ones are 'over-linear' (hardening) oscillators [47]. The responses of both the softening-type and hardening-type oscillators are adopted to examine the validity of PSMS method. The parameter values in 'Fig. 5' of [46] are adopted for numerical analysis, which correspond to strongly nonlinear oscillators and are listed in Table 3. The FRCs

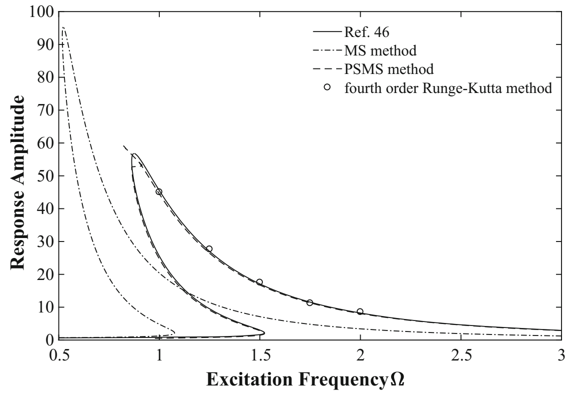

Fig. 5 Oscillator 5's FRCs by MS method, the PSMS method, Ref. [46] and fourth-order Runge–Kutta method

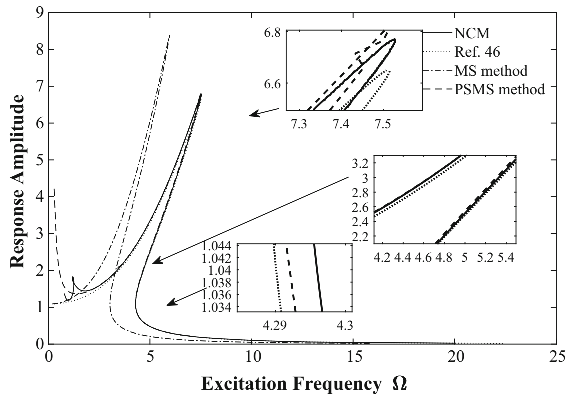

Fig. 6 Oscillator 6's FRCs by MS method, PSMS method, Ref. [46] and NCM

obtained by PSMS method, MS method and NCM are shown and compared in Figs. 5, 6 and 7. It is noted that when $\alpha = 1/3$, the NCM implemented in 'MATCONT' fails to give a solution. Therefore, the perturbation solution obtained in [46] and the fourth-order Runge–Kutta method are both adopted to examine the validity of PSMS method when $\alpha = 1/3$. In this case, good agreement between the FRC obtained by PSMS method and that obtained by the perturbation method in [46] can be observed. However, the FRC obtained by the conventional MS method differs a lot from the FRC obtained by the perturbation method in [46]. In particular, the peak amplitude of the FRC obtained by the conventional MS method is about 1.65 times larger than that obtained by the perturbation method in [46]. Moreover, the occurrence excitation frequency corresponding to the lower turning point of MS solution differs a lot from that of the solution in [46]. In the

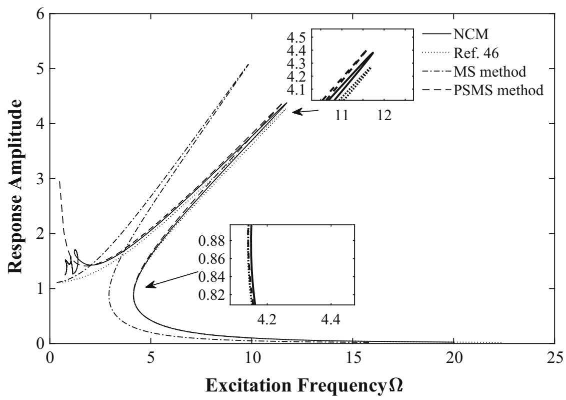

Fig. 7 Oscillator 7's FRCs by MS method, PSMS method, Ref. [46] and NCM

cases of $\alpha = 2$ and $\alpha = 3$, the PSMS solution agrees well with the NCM solution. On the contrary, the MS solution differs a lot from the NCM solution.

3.3 TDOF system with cubic nonlinearity

The primary resonance with the coexistence of internal resonance of a TDOF system with cubic nonlinearity is considered in the following.

The two-degree-of-freedom system with cubic nonlinearity is given in general as

$$\ddot{y}_1 + 2\mu_1\varepsilon^2\dot{y}_1 + \omega_1^2 y_1 + \alpha_1 y_1^3 + \alpha_2 y_1^2 y_2 + \alpha_3 y_1 y_2^2 + \alpha_4 y_2^3 = F_1\varepsilon^3\cos(\Omega t) \quad (67)$$

$$\ddot{y}_2 + 2\mu_2\varepsilon^2\dot{y}_2 + \omega_2^2 y_2 + \alpha_5 y_1^3 + \alpha_6 y_1^2 y_2 + \alpha_7 y_1 y_2^2 + \alpha_8 y_2^3 = F_2\varepsilon^3\cos(\Omega t). \quad (68)$$

This system can be found in the application of a forced geometrically nonlinear beam with simple support [48].

3.3.1 Parameter splitting

The natural frequencies ω_i $(i = 1, 2)$ and the nonlinear parameters $\alpha_j (j = 1, \ldots, 8)$ are split and expressed as

$$\omega_i^2 = \omega_{i1}^2 + \omega_{i3}^2\varepsilon^2 \quad (i = 1, 2) \quad (69)$$

$$\alpha_j = \alpha_{j1} + \alpha_{j3}\varepsilon^2 \quad (j = 1, \ldots, 8). \quad (70)$$

After that, the system can be rewritten as

$$\ddot{y}_1 + 2\mu_1\varepsilon^2\dot{y}_1 + \omega_{11}^2 y_1 + \omega_{13}^2\varepsilon^2 y_1 + \alpha_1 y_1^3 + \alpha_2 y_1^2 y_2 + \alpha_3 y_1 y_2^2 + \alpha_4 y_2^3 = F_1\varepsilon^3\cos(\Omega t) \quad (71)$$

$$\ddot{y}_2 + 2\mu_2\varepsilon^2\dot{y}_2 + \omega_{21}^2 y_2 + \omega_{23}^2\varepsilon^2 y_1 + \alpha_5 y_1^3 + \alpha_6 y_1^2 y_2 + \alpha_7 y_1 y_2^2 + \alpha_8 y_2^3 = F_2\varepsilon^3\cos(\Omega t). \quad (72)$$

3.3.2 The solution by MS method

The responses are assumed to be

$$y_1 = y_{11}(T_0, T_2)\varepsilon + y_{13}(T_0, T_2)\varepsilon^3 \quad (73)$$

$$y_2 = y_{21}(T_0, T_2)\varepsilon + y_{23}(T_0, T_2)\varepsilon^3, \quad (74)$$

where the definition of T_0, T_2 and the operation of time derivatives are the same as those in Eqs. (23)–(25). It is noted that the terms $O(\varepsilon^2)$ and the timescale T_1 are missing in Eqs. (73) and (74) since the effect of nonlinearity appears at $O(\varepsilon^3)$ as stated in [1]. The responses are therefore expressed as it is in [1] to examine whether it is valid for strongly nonlinear system. In order to examine the performance of PSMS method by using only two-term expansions of the responses, the two-term expansions of the responses are adopted in the following. Substituting Eqs. (73) and (74) into Eqs. (71) and (72) and equating the coefficients of $\varepsilon^m (m = 1, 3)$ to zero lead to the following equations.

$$O(\varepsilon) : D_0^2(y_{11}) + \omega_{11}^2 y_{11} = 0 \quad (75)$$

$$D_0^2(y_{21}) + \omega_{21}^2 y_{21} = 0 \quad (76)$$

$$O(\varepsilon^3) : D_0^2(y_{13}) + \omega_{11}^2 y_{13} = F_1\cos(\Omega t) - 2D_0D_2(y_{11}) - \omega_{13}^2 y_{11} - \mu_1 D_0(y_{11}) - \alpha_{11}y_{11}^3 - \alpha_{21}y_{11}^2 y_{21} - \alpha_{31}y_{11}y_{21}^2 - \alpha_{41}y_{21}^3. \quad (77)$$

$$D_0^2(y_{23}) + \omega_{21}^2 y_{23} = F_2\cos(\Omega t) - 2D_0D_2 y_{21} - \omega_{23}^2 y_{21} - \mu_2 D_0(y_{21}) - \alpha_{51}y_{11}^3 - \alpha_{61}y_{11}^2 y_{21} - \alpha_{71}y_{11}y_{21}^2 - \alpha_{81}y_{21}^3. \quad (78)$$

Equations (75) and (76) are homogenous differential equations. The solutions of them can be expressed as

$$y_{11} = C_{10}(T_2)e^{i\omega_{11}T_0} + \bar{C}_{10}(T_2)e^{-i\omega_{11}T_0} \quad (79)$$

$$y_{21} = C_{20}(T_2)e^{i\omega_{21}T_0} + \bar{C}_{20}(T_2)e^{-i\omega_{21}T_0}. \quad (80)$$

As we restrict our attention to the case of internal resonances, an internal resonance is assumed to occur when $\omega_2 \approx 3\omega_1$. A detuning parameter σ_1 is therefore introduced to describe the internal resonance as

$$\omega_{21} = 3\omega_{11} + \sigma_1\varepsilon^2. \quad (81)$$

Here we consider the case of the primary resonance around ω_1 and $F_2 = 0$. Hence, a second detuning parameter σ_2 is introduced to describe the nearness between the excitation frequency and the natural frequency ω_1 as

$$\Omega = \omega_{11} + \sigma_2 \varepsilon^2. \tag{82}$$

Substituting the expressions for y_{11} and y_{21} and Eqs. (81) and (82) into the right-hand side of Eqs. (77) and (78) yields the coefficients of secular terms as

$$\begin{aligned} &2i\omega_{11} D_2(C_{10}) + \mu_1 i\omega_{11} C_{10} \\ &\quad + \omega_1 3^2 C_{10} + 3\alpha_{11} C_{10}^2 \bar{C}_{10} \\ &\quad + 2\alpha_{31} C_{20} \bar{C}_{20} C_{10} \\ &\quad + \alpha_{21} C_{20} \bar{C}_{10}^2 e^{i\sigma_1 T_2} - \frac{F_1}{2} e^{i\sigma_2 T_2} = 0 \end{aligned} \tag{83}$$

$$\begin{aligned} &2i\omega_{21} D_2(C_{20}) + \mu_2 i\omega_{21} C_{20} \\ &\quad + \omega_2 3^2 C_{20} + 3\alpha_{81} C_{20}^2 \bar{C}_{20} \\ &\quad + 2\alpha_{61} C_{10} \bar{C}_{10} C_{20} + \alpha_{51} C_{10}^3 e^{-i\sigma_1 T_2} = 0. \end{aligned} \tag{84}$$

After eliminating the secular terms, y_{13} and y_{23} can be obtained as

$$\begin{aligned} y_{13} = {} & D_{(0,1)} e^{i\omega_{21} T_0} + D_{(3,0)} e^{3i\omega_{11} T_0} + D_{(0,3)} e^{3i\omega_{21} T_0} \\ & + D_{(2,1)} e^{i(2\omega_{11}+\omega_{21}) T_0} + D_{(1,2)} e^{i(\omega_{11}+2\omega_{21}) T_0} \\ & + D_{(1,-2)} e^{i(\omega_{11}-2\omega_{21}) T_0} + c.c. \end{aligned} \tag{85}$$

$$\begin{aligned} y_{23} = {} & E_{(1,0)} e^{i\omega_{11} T_0} + E_{(0,3)} e^{3i\omega_{21} T_0} \\ & + E_{(2,1)} e^{i(2\omega_{11}+\omega_{21}) T_0} \\ & + E_{(1,2)} e^{i(\omega_{11}+2\omega_{21}) T_0} + E_{(-2,1)} e^{i(-2\omega_{11}+\omega_{21}) T_0} \\ & + E_{(1,-2)} e^{i(\omega_{11}-2\omega_{21}) T_0} + c.c, \end{aligned} \tag{86}$$

where $c.c$ represents the complex conjugate and the values of $D_{(.,.)}$ and $E_{(.,.)}$ are expressed as

$$D_{(0,1)} = \frac{3\alpha_{41} C_{20}^2 \bar{C}_{20} + 2\alpha_{21} C_{10} \bar{C}_{10} C_{20}}{\omega_{21}^2 - \omega_{11}^2} \tag{87}$$

$$D_{(3,0)} = \frac{\alpha_{11} C_{10}^3}{8\omega_{11}^2} \tag{88}$$

$$D_{(0,3)} = \frac{\alpha_{41} C_{20}^3}{8\omega_{21}^2} \tag{89}$$

$$D_{(2,1)} = \frac{\alpha_{21} C_{10}^2 C_{20}}{3\omega_{11}^2 + 4\omega_{11}\omega_{21} + \omega_{21}^2} \tag{90}$$

$$D_{(1,2)} = \frac{\alpha_{31} C_{10} C_{20}^2}{4\omega_{11}\omega_{21} + 4\omega_{21}^2} \tag{91}$$

$$D_{(1,-2)} = \frac{\alpha_{31} C_{10} \bar{C}_{20}^2}{4\omega_{21}^2 - 4\omega_{11}\omega_{21}} \tag{92}$$

$$E_{(1,0)} = \frac{3\alpha_{51} C_{10}^2 \bar{C}_{10} + 2\alpha_{71} C_{10} C_{20} \bar{C}_{20}}{\omega_{11}^2 - \omega_{21}^2} \tag{93}$$

$$E_{(0,3)} = \frac{\alpha_{81} C_{20}^3}{8\omega_{21}^2} \tag{94}$$

$$E_{(2,1)} = \frac{\alpha_{61} C_{20} C_{10}^2}{4\omega_{11}^2 + 4\omega_{11}\omega_{21}} \tag{95}$$

$$E_{(-2,1)} = \frac{\alpha_{61} C_{20} \bar{C}_{10}^2}{4\omega_{11}^2 - 4\omega_{11}\omega_{21}} \tag{96}$$

$$E_{(1,2)} = \frac{\alpha_{71} C_{10} C_{20}^2}{\omega_{11}^2 + 4\omega_{11}\omega_{21} + 3\omega_{21}^2} \tag{97}$$

$$E_{(1,-2)} = \frac{\alpha_{71} C_{10} \bar{C}_{20}^2}{\omega_{11}^2 - 4\omega_{11}\omega_{21} + 3\omega_{21}^2}. \tag{98}$$

Inserting $C_{m0} = \frac{1}{2} A_m e^{ib_m}$ into Eqs. (79), (80), (85) and (86) yields

$$\begin{aligned} y_1 = {} & A_1 \varepsilon \cos(\Omega t - \gamma_1) \\ & + \frac{3\alpha_{41} A_2^3 \varepsilon^3}{4\omega_{21}^2 - 4\omega_{11}^2} \cos(3\Omega t - 3\gamma_1 + \gamma_2) \\ & + \frac{\alpha_{21} A_1^2 A_2 \varepsilon^3}{2\omega_{21}^2 - 2\omega_{11}^2} \cos(3\Omega t - 3\gamma_1 + \gamma_2) \\ & + \frac{\alpha_{11} A_1^3 \varepsilon^3}{32\omega_{11}^2} \cos(3\Omega t - 3\gamma_1) \\ & + \frac{\alpha_{41} A_2^3 \varepsilon^3}{32\omega_{21}^2} \cos(9\Omega t - 9\gamma_1 + 3\gamma_2) \\ & + \frac{\alpha_{21} A_1^2 A_2 \varepsilon^3}{12\omega_{11}^2 + 16\omega_{11}\omega_{21} + 4\omega_{21}^2} \\ & \times \cos(5\Omega t - 5\gamma_1 + \gamma_2) \\ & + \frac{\alpha_{31} A_1 A_2^2 \varepsilon^3}{16\omega_{11}\omega_{21} + 16\omega_{21}^2} \cos(7\Omega t - 7\gamma_1 + 2\gamma_2) \\ & + \frac{\alpha_{31} A_1 A_2^2 \varepsilon^3}{16\omega_{21}^2 - 16\omega_{11}\omega_{21}} \cos(5\Omega t - 5\gamma_1 + 2\gamma_2) \end{aligned} \tag{99}$$

$$\begin{aligned} y_2 = {} & A_2 \varepsilon \cos(3\Omega t - 3\gamma_1 + \gamma_2) \\ & + \frac{3\alpha_{51} A_1^3 \varepsilon^3}{4\omega_{11}^2 - 4\omega_{21}^2} \cos(\Omega t - \gamma_1) \\ & + \frac{\alpha_{71} A_1 A_2^2 \varepsilon^3}{2\omega_{11}^2 - 2\omega_{21}^2} \cos(\Omega t - \gamma_1) \\ & + \frac{\alpha_{81} A_2^3 \varepsilon^3}{32\omega_{21}^2} \cos(9\Omega t - 9\gamma_1 + 3\gamma_2) \end{aligned}$$

Table 4 Parameters in the TDOF system with cubic nonlinearity

ε	ω_1	ω_2	μ_1	μ_2	$\alpha_{1,4,5,8}$	$\alpha_{2,3,6,7}$	F_1	F_2
0.1	1	3.01	2	2	1	0	50	0

$$+\frac{\alpha_{61}A_1^2A_2\varepsilon^3}{16\omega_{11}^2+16\omega_{11}\omega_{21}}\cos(5\Omega t-5\gamma_1+\gamma_2)$$
$$+\frac{\alpha_{61}A_1^2A_2\varepsilon^3}{16\omega_{11}^2-16\omega_{11}\omega_{21}}$$
$$\times\cos(\Omega t-\gamma_1+\gamma_2)$$
$$+\frac{\alpha_{71}A_1A_2^2\varepsilon^3}{4\omega_{11}^2+16\omega_{11}\omega_{21}+12\omega_{21}^2}$$
$$\times\cos(7\Omega t-7\gamma_1+2\gamma_2)$$
$$+\frac{\alpha_{71}A_1A_2^2\varepsilon^3}{4\omega_{11}^2-16\omega_{11}\omega_{21}+12\omega_{21}^2}$$
$$\times\cos(5\Omega t-5\gamma_1+2\gamma_2), \quad (100)$$

where $\gamma_1=\sigma_2T_2-b_1$ and $\gamma_2=\sigma_1T_2+b_2-3b_1$. Again, inserting $C_{m0}=\frac{1}{2}A_me^{ib_m}$ into Eqs. (83) and (84) and separating the real and imaginary terms yield

$$4F_1\sin(\gamma_1)-4\mu_1A_1\omega_{11}$$
$$-\alpha_{21}A_1^2A_2\sin(\gamma_2)=0 \quad (101)$$
$$-4F_1\varepsilon^2\cos(\gamma_1)+4\omega_{13}^2\varepsilon^2A_1$$
$$+3\alpha_{11}\varepsilon^2A_1^3+2\alpha_{31}\varepsilon^2A_1A_2^2$$
$$+\alpha_{21}\varepsilon^2A_1^2A_2\cos(\gamma_2)$$
$$+8A_1\omega_{11}^2-8A_1\Omega\omega_{11}=0 \quad (102)$$
$$\alpha_{51}A_1^3\sin(\gamma_2)-4\mu_2\omega_{21}A_2=0 \quad (103)$$
$$4\omega_{23}^2\varepsilon^2A_2+3\alpha_{81}\varepsilon^2A_2^3$$
$$+2\alpha_{61}\varepsilon^2A_1^2A_2+\alpha_{51}\varepsilon^2A_1^3\cos(\gamma_2)$$
$$-8\omega_{21}A_2(3\Omega-\omega_{21})=0. \quad (104)$$

3.3.3 Numerical analysis of a TDOF system with cubic nonlinearity

One case is considered with parameters given in Table 4. The comparisons of FRCs of y_1 and y_2 obtained by MS method, PSMS method and NCM are shown in Figs. 8 and 9, respectively. It is observed that MS method over-evaluates the nonlinear stiffness of the system as the FRCs obtained by MS method bend to the right-hand side too much. However, the FRCs obtained by PSMS method can still agree well with those obtained by NCM for this TDOF system.

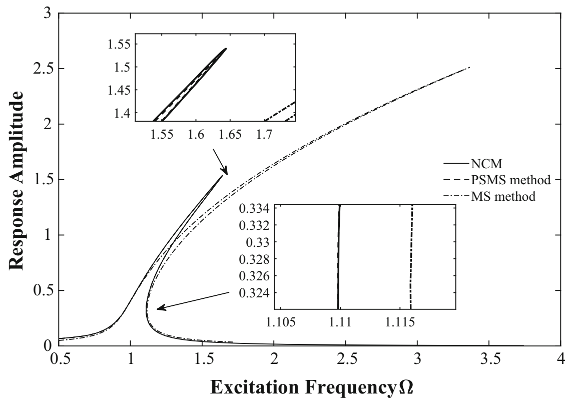

Fig. 8 FRCs of y_1 in a TDOF system by MS method, PSMS method and NCM

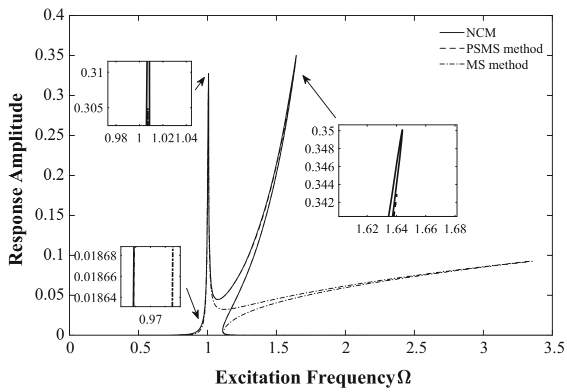

Fig. 9 FRCs of y_2 in a TDOF system by MS method, PSMS method and NCM

4 Error analysis

In this section, the values of cumulative error reflected by R_e corresponding to the results obtained by MS method and PSMS method are compared to reveal the improvement in the results obtained by PSMS method. Besides for the values of R_e, the values of the splitting parameters that optimized the error function at some specific excitation frequencies are also presented. They are shown in Tables 5, 6, 7, 8 and 9 for the two Duffing oscillators, the two oscillators with nonlinear restor-

Table 5 Some values of splitting parameters and R_e by PSMS method and corresponding R_e values by MS method about oscillator 2

Ω	η_1	η_2	ω_{00}^2	ω_{01}^2	ω_{02}^2	$R_e^{(MS)}$	$R_e^{(PSMS)}$
10	6.0000E+03	− 0.0359	236.5169	1.6348E+03	0.2624	7.5844	0.0021
20	− 5.5648E+03	1.1565E+05	381.7393	402.7605	− 2.2015E+03	3.8428E+06	4.2951E−04
30	− 6.3713E+03	1.2371E+05	897.4618	5.3236E+03	− 1.0298E+05	1.9365E+04	0.0738
40	6.0000E+03	− 0.0256	959.3436	−5.5935E+03	0.6758	7.8441	7.1257E−05

Table 6 Some values of splitting parameters and R_e by PSMS method and corresponding R_e values by MS method about oscillator 4

Ω	α_1	α_2	β_1	β_2	η_1	η_2	ω_{00}^2	ω_{01}^2
0.6118	− 1.0672	110.6721	12.6285	− 26.2851	50.9961	−508.9611	0.3742	− 0.1055
0.8444	− 64.1045	741.0452	11.2600	− 12.6003	− 40.1042	402.0420	0.7117	−0 .1913
0.9011	− 37.8506	478.5061	−47.4620	574.6200	− 62.7801	628.8010	0.7920	3.9204
1.0002	17.7165	−77.1653	10.7419	−7.4190	15.9403	−158.4032	1.0001	−0.2609

Ω	ω_{02}^2	$R_e^{(MS)}$	$R_e^{(PSMS)}$
0.6118	63.6402	4.3563E−04	1.9235E−10
0.8444	30.7420	5.1087E−05	2.0154E−08
0.9011	− 18.4069	3.1062	3.3372E−07
1.0002	2.5946	5.8143E−05	2.0291E−06

Table 7 Some values of splitting parameters and R_e by PSMS method and corresponding R_e values by MS method about oscillator 5

Ω	κ_0	κ_1	ω_{00}^2	ω_{01}^2	$R_e^{(MS)}$	$R_e^{(PSMS)}$
1.0004	100	0	1.0008	−10.0079	2.4747E+05	2.9123E−02
1.2516	100	0	1.5665	−15.6647	2.1155E+05	2.1643E−02
1.5032	100	0	2.2595	−22.5952	1.8405E+05	1.7517E−02
2.0009	100	0	4.0036	−40.0365	1.4547E+05	1.2892E−02

Table 8 Some values of splitting parameters and R_e by PSMS method and corresponding R_e values by MS method about oscillator 6

Ω	κ_0	κ_1	ω_{00}^2	ω_{01}^2	$R_e^{(MS)}$	$R_e^{(PSMS)}$
2.5270	100	0	17.4207	− 174.2094	7.2892E+03	6.1090E−03
3.0420	100	0	22.9657	− 229.6661	4.4526E+03	5.4270E−03
4.0205	100	0	33.1027	− 331.0270	3.5350E+03	4.6569E−03
5.0505	100	0	42.0294	− 420.3014	2.6265E+03	4.1221E−03

ing force and inertial force, the three purely nonlinear oscillators and a TDOF system. It is seen that the errors induced by PSMS method are much less than the errors induced by MS method for all the given oscillators and TDOF system.

5 Conclusions

A method named parameter-splitting method is proposed in this paper to improve the solutions given by the perturbation methods. The idea of this method is that some parameters in the system are split and some

Table 9 Some values of splitting parameters and R_e by PSMS method and corresponding R_e values by MS method about y_1 and y_2 of the TDOF system

Ω	α_{11}	α_{41}	α_{51}	α_{81}	α_{13}	α_{43}	α_{53}	α_{83}
0.75	1.0079	1.0000	1.4990	1.0001	– 0.7887	3.5230E-05	– 49.8975	– 0.0114
1.00	0.9993	1.0569	1.0340	0.9585	0.0744	– 5.6893	– 3.3952	4.1498
1.25	1.0179	7.0378	0.9504	1.3185	– 1.7854	– 603.7783	4.9613	– 31.8547
1.50	1.0233	8.0834	0.8555	0.7008	– 2.3320	– 708.3367	14.4534	29.9175
Ω	ω_{11}^2	ω_{21}^2	ω_{13}^2	ω_{23}^2	$R_e^{(MS)}$	$R_e^{(PSMS)}$		
0.75	0.5571	13.2908	44.2933	– 423.0712	0.0874	3.2324E−08		
1.00	0.9754	9.3124	2.4576	– 25.2281	0.0784	9.9160E−06		
1.25	1.5196	8.5275	– 51.9639	53.2641	0.0770	4.5193E−04		
1.50	2.1761	7.8885	– 117.6119	117.1550	0.746	0.0078		

unknown parameters are introduced to the system. Then the solution of the system with unknown parameters is obtained by a perturbation method. The cumulative error of the equation of motion due to the approximate solution obtained by the perturbation method is formulated and considered as objective function. The optimum values of the unknown parameters can then be determined by minimizing the objective function. This solution procedure is named parameter-splitting–multiple-scales (PSMS) method when it is applied to improve the solutions obtained by MS method. The FRCs of the nonlinear oscillators are analyzed by PSMS method, MS method and NCM, respectively. Given an excitation frequency, the values of the introduced unknown parameter and the response amplitude are determined by minimizing the cumulative equation error expressed by Eq. (15). The forced vibrations of the Duffing oscillators, the oscillators with both nonlinear inertial force and nonlinear restoring force, a purely nonlinear oscillator and a TDOF system with cubic nonlinearity are analyzed to examine the effectiveness of PSMS method. Through numerical study, it is observed that the results obtained by PSMS method are much improved in comparison with the results obtained by MS method even if the oscillator or system nonlinearity is strong. The effectiveness of this method is not limited to the presented oscillators and system and the presented procedure is not limited to improving the solutions obtained by MS method. Other oscillators can also be studied by PSMS method, and the proposed parameter-splitting method can also be applied to improve the solutions obtained by other perturbation methods.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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