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ORIGINAL PAPER

Nonlinear dynamics in mechanics and engineering: 40 years of developments and Ali H. Nayfeh's legacy

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Abstract Nonlinear dynamics of engineering systems has reached the stage of full maturity in which it makes sense to critically revisit its past and present in order to establish an historical perspective of reference and to identify novel objectives to be pursued. This paper makes a first step in this direction, focusing on the mechanics of machines, solids and structures, with applications in both classical and novel technological areas. This is accomplished first by identifying some main stages of scientific development over the last four decades, with the characterizing features as regards addressed systems, underlying mathematical tools and phenomenological aspects. These are substantiated in terms of topics and involved people with also an archival, and tentatively comprehensive, list of related activities. The second part of the study deals with Ali Nayfeh's contributions as a scientist and a scholar, embedding his research achievements within the identified four stages of development and highlighting some qualifying methodological features of his activity as a book author. His legacy is framed within an overall scenario of expected future developments of nonlinear dynamics.

Keywords Nonlinear dynamics · Applied mechanics · Engineering · Historical development · Ali Nayfeh · Future perspectives

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1 Introduction

Interest toward nonlinear oscillations in mechanics started with Huygens' studies on pendulum dynamics and with the n-body problem in celestial mechanics (which goes back to Kepler, Newton, Lagrange and Poincaré), and continued with the observation of nonlinear phenomena in a number of nineteenth-century industrial applications, for which particular methods fitted to the analytical solution of specific problems were elaborated. In the early twentieth century, there was an important phase of growth, marked by the achievements of two eminent engineers. Georg Düffing [1] is the mechanical engineer who, moved by the interest to solve practical vibration problems, formulated a nonlinear equation later on generalized to represent archetypal oscillators of reference for the analysis of a great variety of dynamical systems [2]. In turn, Balthasar van der Pol is the electrical engineer who obtained important results on self-sustained, and in particular relaxation, oscillations [3] in connection with radio engineering applications, where he also observed "an irregular noise" in certain frequency ranges, likely making the first experimental observation of deterministic chaos. Van der Pol's equation has become another classical equation in nonlinear vibrations.

In parallel and more general terms, dynamical system theory originated in the late nineteenth century with Henri Poincaré [4], who is considered the father of modern nonlinear dynamics, and later on developed mostly within the mathematical community. Another early giant was Alexander Lyapunov, who gave the basis of the theory of motion stability [5] which is important in all critical situations (i.e., bifurcations) of nonlinear dynamics marking the passage between different qualitative behaviors. Following Poincaré's insight into local and global analysis of nonlinear differential equations, and his earlier 'detection' of chaos in 'simple' mechanical systems, fundamentals of the science of nonlinear and complex systems were established in the first half of the twentieth century by the efforts of a great number of outstanding mathematicians. Besides Birkhoff's [6] contributions (e.g., on classification of all possible types of dynamic motion), the role played by two Soviet schools of thought has to be mentioned. Both approached problems in nonlinear dynamics drawing on examples of nonlinear oscillations from physical sciences and engineering, yet "with no direct interest in the study of a particular phenomenon of a given scientific field, and in search of analogies of behaviors concerning different branches of dynamics" [7]. The solution of equations of nonlinear systems was pursued in the Andronov and followers school (at Gorki—Moscow) via *qualitative* methods [8], with a view to applications mostly in electrical circuitry, radiophysics and electrical engineering, where van der Pol's independent contribution was also acknowledged. The Krylov–Bogoliubov–Mitropolski school (at Kiev) searched for the solution of equations of nonlinear systems via *analytical* (i.e., quantitative) methods, mostly dealing with problems in nonlinear mechanics [9].

Around the middle of the twentieth century and mostly in the 1960s and 1970s, novel theoretical ideas and perspectives (e.g., the topological one), and the innovative contributions of computer science, determined an 'explosion' of dynamical system theory, with the strong affirmation of the role of models and the importance of the nonlinear domain, along with intense interactions developed throughout physical and mathematical sciences. Distinct, yet interconnected, theories were developed (of bifurcation, catastrophe, complexity, chaos, fractals, turbulence), with applications to a wide variety of disciplines including not only physics and engineering but also chemistry, biology, neurology, astronomy, geophysics, meteorology and economics. Relevant historical outlines can be found in [7,10,11], all of these references widely highlighting mathematical and physical developments, and the latter also providing a wider perspective accounting for cultural and sociological aspects. Dynamical system approach to problems in fluid mechanics, as an area of confluence between mathematics and engineering, was also addressed in those surveys, mostly concerned with the development of models for the description of turbulence.

In contrast, to the best of the author's knowledge, historical aspects of nonlinear dynamics within applied mechanics and engineering of machines, solids and structures have not yet been addressed in the literature in a possibly comprehensive perspective, although topics of interest in nonlinear dynamics and vibrations of mechanical systems have been overviewed in a paper [12] also dwelling on the utility of nonlinear dynamics for analysis and design.

Independently of theoretical results obtained in the dynamical system environment, yet along the analytical lines paved by the Krylov–Bogoliubov–Mitropolski approach, nonlinear oscillations of engineering systems flourished within the mechanics community toward the middle of the twentieth century, with major attention paid to application aspects. Actually, no clear distinction in scope can be made between scientific contributions coming from applied mathematics or engineering, being the two areas quite strongly linked with each other, mostly at that time. And, indeed, many comprehensive books on nonlinear vibrations of single-degree-of-freedom oscillators appeared in those decades [13–16], to name just a few, consist of a valuable combination of analytical techniques necessary to deal with them and of nonlinear phenomena characterizing the behavior of mechanical or electrical systems in the background of the considered oscillators.

In this respect, a preferential view to either the mathematical aspects of the underlying ordinary differential equations or the physical aspects of the relevant nonlinear oscillations can be instead recognized in the first two books written by Ali H. Nayfeh in the 1970s, according to a more clearly distinct perspective. As a matter of fact, his first published book Perturbation Methods [17] settled the general framework for addressing nonlinear dynamical systems via an advanced asymptotic technique (the method of multiple time scales), while his later coauthored book Nonlinear Oscillations [18] ensued from their intense and extended applications to obtain the solution of archetypal oscillators (directly representing discrete systems or being simplified reduced-order models of continuous systems) and some multi-degree-of-freedom models.

With no attempt of being rigorous from the historical viewpoint, we choose to refer to the publication date of the latter book (which goes back just 40 years from now) as to the starting date of a more diffuse interest to nonlinear dynamics problems within the community of scientists working in solid mechanics and mechanical or structural engineering, thus justifying the otherwise questionable starting date of development of nonlinear dynamics in mechanics and engineering referred to in the title of this paper. Such interest, already intense and pervasive over the 1980s in the ‘classical’ sense of *nonlinear oscillations*, became much richer and varied later on, widening the perspectives of the area to what is presently called *nonlinear dynamics* to mark a substantially wider and more comprehensive context. This was due to the awareness (attained in few years) of the tremendous increase in methodological and phenomenological perspectives opened, on the one side, by the achievements occurred within the dynamical system community by the strong development of computational techniques, with the ensuing increase in geometrical description and understanding of nonlinear phenomena, and, on the other side, by the nearly parallel advancements in the conception/implementation of sophisticated experimental techniques for the analysis of nonlinear systems, occurred within the engineering community.

Upon forty and more years of impetuous development, nonlinear dynamics of engineering systems has reached the stage of full maturity in which it makes sense, and is indeed suitable, to critically revisit its past and present not only to establish an historical perspective of reference but also to build on it and identify novel objectives to be pursued in the future. This paper aims at making a first step in this direction, focusing on the area of solid mechanics and thus looking at nonlinear dynamics of machines and structures to be used for applications in both classical (mechanical, aeronautical, civil) and novel (bio-/micro-/nano-mechanics) technological areas, with the associated increasing importance of variably involved multiphysics aspects.

This is accomplished first by identifying some main historical stages of scientific development of nonlinear dynamics in mechanics and engineering, each one of them being characterized in the sequel in terms of topics of prevailing interest as to the addressed mechanical and structural systems/models, of techniques mostly employed to obtain solutions, of involved phenomenological aspects and of specific tools used for characterizing and representing the latter. This development identification level is complemented, and suitably substantiated in terms of topics and involved people, with an archival, tentatively comprehensive, list of events dealing with research, education and confrontation/diffusion of knowledge and information on nonlinear dynamics, held over the last four decades.

In the second part of the study, the established framework of development is conveniently referred to for discussing the achievements of Ali H. Nayfeh, whom the present special issue is dedicated to, and his legacy. He can indeed be considered as the most influential, worldwide recognized, scholar and scientist of the era of nonlinear dynamics in mechanics and engineering. This is accomplished by shortly looking at his life, by dwelling on his scientific contributions and achievements and by discussing the methodological features and novel aspects of his research activity and his books.

The paper is organized as follows. Section 2 summarizes the advancements of nonlinear dynamics in mechanics through the identification of four main stages of development, with their characterizing features in terms of addressed systems, underlying mathematical tools and phenomenological aspects. Scientific activities pursued in the area of nonlinear dynamics at the worldwide level over the last 40 years are summarized in Sect. 3, highlighting not only the richness and variety of the initiatives undertaken by the most representative international societies of mechanics but also the vitality of a number of national or local groups of research in nonlinear dynamics. Section 4 deals with Nayfeh’s activity as a scientist and a scholar, embedding his research achievements within the identified four stages of development and highlighting some qualifying common features of his activity as a book author. In the final section, Ali Nayfeh’s legacy is framed within an overall scenario of ongoing and expected developments of nonlinear dynamics.

2 Stages of development of nonlinear dynamics in mechanics

Attention is focused on the wide area of mechanics of solids, machines and structures, and their engineering applications, into which the effects of the theoretical accomplishments of the mathematical physics community have spread with a physiological time shift.

Starting at about the time (mid-1970s) when the theoretical accomplishments of the dynamical system community were ready to be transferred to the engineering realm, four main stages of development of nonlinear dynamics in mechanics are distinguished (through different colors) in the chronological chart in Fig. 1, along with a general modeling theme somehow affecting and encompassing all of them. For each stage, the time window over which those specific scientific developments were so intense to provide nonlinear dynamics with its peculiar and prevailing characterization is schematically indicated.

The first three fundamental stages (*Nonlinear Oscillations*, *Bifurcations and Complex Dynamics*, *Experimental Nonlinear Dynamics*) reflect the role played in the overall historical development by the three aspects (analytical, geometrical/computational and experimental) which presently characterize a possibly comprehensive approach to the nonlinear dynamics of a system or structure; and, indeed, they are given titles (or labels) which refer, on the one side, to the set of topics which attracted most of the interest of scholars over that period of time, with the ensuing advancements and, on the other side, to the general kind of solution techniques whose development stage actually enabled researchers to successfully afford those topics. Concerning the indicated time windows, the corresponding starting times have to be considered just as nominal ones of major diffusion within the community. Moreover, owed to the theoretical and application potential embedded in both the research topics of each stage and the techniques to tackle them, the development of the first three stages has continued (dashed blue, green and red underscores in Fig. 1) well beyond their nominal period of major importance (corresponding continuous underscores), with novel or revisited advancements, both theoretical and practical, which are still going on and are indeed expected to further enhance the knowledge in the area.

The identified fourth stage of development (*Hybridizing Nonlinear Dynamics with Other Areas*) is the one faithfully characterizing the present state of the art. In particular, spreading the by now 'traditional' themes of nonlinear dynamics in mechanics toward companion or novel areas of research and technological development, it also paves the way to a renewed—and actually hybridized—character of the whole area.

In the sequel, to give a taste of what was going on in the research on nonlinear dynamics over each time period, the sets of topics and techniques which characterize the first three stages of development are elaborated by grouping the relevant main items under four categories (systems; techniques/solutions; phenomena; and tools) for the sake of a better understanding. Reference is implicitly made to two sets of problems. The first concerns the study of equations not directly tied with practice but having the lowest dimension and the simplest structure (archetypal systems), which permits to isolate mathematical phenomena in their purest form, by eliminating the involved effects of a more complicated structure; the other is related to issues directly suggested by the practice of mechanical and structural engineering.

For each stage, general references (mostly selected within the mechanics and engineering realm) are also provided. Of course, the list of themes of each development stage has to be considered incremental, meaning that only the new theoretical and operational items coming into play and becoming increasingly popular within the novel stage are reported, with nearly all of the items listed in a previous one being still under consideration and common use.

In contrast, for the fourth stage of development whose character is substantially cross-disciplinary, no comparable categorization of topics/techniques typical of nonlinear dynamics makes sense and, given the variety of hybridizing relationships between nonlinear dynamics and other theoretical and application environments, just a tentative and incomplete list of the existing, and further developing, connections with companion areas is reported, along with some relevant basic references.

1. *Nonlinear Oscillations*, addressed mostly through *analytical* techniques [18–30].

    - Systems: Archetypal mathematical oscillators (Helmholtz, Duffing, van der Pol, pendulum, piecewise linear, impact), representative of discrete mechanical (or electrical) systems or being reliable discretized single-mode models of continuous systems (structures) suitable to investigate some fundamental aspects of nonlinear dynamic behavior. Self-excited oscillations. Two-degree-of-freedom systems. Rotating systems.

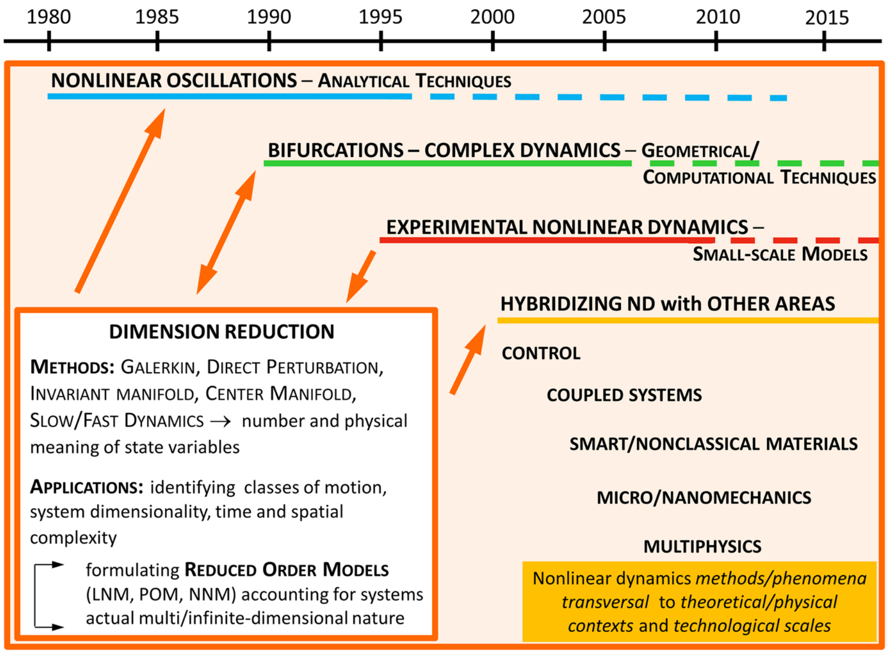

Fig. 1 Four stages of development of nonlinear dynamics (blue, green, red and yellow, with the continuous underscores denoting corresponding time periods of major importance), along with a general modeling theme affecting and encompassing all stages (orange). (Color figure online)

- Techniques/solutions: Asymptotics for problems with small nonlinearities: perturbation (Lindstedt–Poincaré; multiple time scales), averaging (generalized method, Krylov–Bogoliubov–Mitropolski); Melnikov method and its generalization. Normal forms theory. Method of harmonic balance, homotopy analysis method, for problems with also large nonlinearities. Exact and approximate solutions.
- Phenomena: Primary and secondary (superharmonic, subharmonic, ultra-subharmonic, combination) nonlinear resonances, under external and/or parametric excitation; hardening/softening behavior. Weakly nonlinear dynamics: regular (i.e., periodic) response; first hints/problems on nonlinear modal interaction. Nonstationary vibrations. Homoclinic bifurcations.
- Tools (for the characterization/representation of nonlinear dynamic response): Time history; state plane; frequency response spectra. Backbone curve; frequency–response and force–response curves. Stable and unstable manifolds.

2. *Bifurcations and Complex Dynamics*, addressed through *geometrical/topological* theory and *computational* techniques [22,31–54].

- Systems: Maps and flows. Multimode models (two/four degrees of freedom) of smooth continuous systems. Piecewise-smooth and non-smooth systems. Systems with time delay (machining, manufacturing). Multibody systems. Vehicle systems.
- Techniques/solutions: Invariant manifold theory, center manifold reduction; nonlinear normal modes. Normal form theory. Unfolding local behavior. Structural stability. Continuation (path following) of fixed points and periodic solutions (arc-length, shooting). Numerical simulation, brute-force analysis. System dimension calculation. Cell mapping.
- Phenomena: Nonlinear multimodal interaction in regular dynamics. Slow-/fast-scale phenomena with energy transfer between modes. Local bifurcations. Global bifurcations; crises. Invariant manifolds; homo/heteroclinic tangles. Quasiperiodicity; chaos; routes to chaos. Resonance capture. Synchronization. Stick–slip oscillations. Bifurcations of nonsmooth systems.
- Tools: State space, pseudo-state space; Poincaré maps; power spectra. Quantitative measures of nonregular (vs. regular) dynamics: Lyapunov exponents, fractal dimensions. Stable and unstable manifolds. Basins of attraction; attractor–basin phase portraits.

3. *Nonlinear Dynamics* of *Small-scale Physical Models*, addressed through *experimental techniques* [55–59], widely used in the new millennium for detecting complex/new phenomena unmodeled in theoretical analysis, for validating theoretical models, for nonlinear system identification.

   - Systems: Discrete rigid models. Flexible models of continuous systems. Also real systems/structures, mostly in a system identification/health monitoring perspective.
   - Techniques/solutions: Nonlinear time series analysis. Reconstruction of dynamical properties from experimental measurements: delay embedding, phase space reconstruction. Modeling and prediction. Noise reduction. Spatiotemporal analysis of complex signals: proper orthogonal decomposition.
   - Phenomena: Experimental scenarios of transition to chaos with respect to canonical ones of dynamical system theory. Proper orthogonal modes, spatial properties of nonlinear response. System dimensionality. Hints toward reduced/minimal theoretical models for describing experimental complex dynamics.
   - Tools: Pseudo-state space, spectral analysis, autocorrelation functions. Invariant measures: attractor dimension, Lyapunov exponents, local topological dimension. Experimental eigenfunctions.

4. *Hybridizing Nonlinear Dynamics* with other *Theoretical and Application Areas* (with some general references by scholars mostly in the area of mechanics).

   - *Control*: Vibration control/suppression. Control of bifurcations: open-loop and feedback. Control of chaos: OGY and delayed feedback methods. Targeted energy transfer and nonlinear energy sinks [60–66].
   - *From macro- to micro-/nano-systems*: Nonlinear dynamics of cables, beams, plates, shells, composites, rotating systems, shape memory materials, functionally graded materials, with applications in structural mechanics, machinery, vehicles, system identification, structural health monitoring, atomic force microscopy, micro-/nano-electromechanical systems, carbon nanotube [12,67–71].
   - *Coupled systems*: Chains of nonlinear oscillators, periodic systems. Wave propagation, localization. Structural instabilities. Metamaterials, granular media [72–77].
   - *Multiphysics*: Thermoelasticity; fluid–structure interaction; piezoelectricity; magnetoelasticity; biomechanics. Intelligent systems [78–82].

This list makes apparent how the transversal nature of nonlinear dynamics is concerned not only with phenomenological aspects common to a variety of application areas and with the methods necessary to tackle them, but also with a possibly creative remix of the underlying theoretical frameworks, of the reference physical contexts and of different technological scales.

One main feature of this fourth stage of development with respect to previous ones is the considerably higher attention paid to application aspects in all areas, which marks the ever-increasing awareness of the nonlinear dynamics community about the need of a progressive transition from strong conceptual achievements (which are of course anyway necessary) to a finalized technological context wherein exploiting them for also societal advancements. The matter will be resumed in Sect. 5.

Toward the last decade of the past century, the research items and the outcomes of the geometric/computational stage and of the experimental stage have made increasingly apparent the importance of the fundamental theme of *dimension reduction* of dynamical systems in science and engineering [83,84] and of its cross-correlation with all of the identified stages of development. It involves developing and implementing reliable (i) *methods* (Galerkin, direct perturbation on PDEs of motion, inertial manifolds, center manifold, slow/fast dynamics) for the identification of the number and the physical meaning of the mathematical state variables actually governing the system response, and pursuing (ii) *applications* aimed at properly evaluating the system dimensionality and the main features of its spatiotemporal dynamic response. All of this ending up to the identification and formulation of suitable *reduced-order models* (based on assumed expansions making use of linear normal modes or proper orthogonal modes or nonlinear normal modes) to be employed for a reliable description of the complicated nonlinear dynamics of engineering systems, which properly accounts for their actual multi- or infinite-dimensional nature; a description to be possibly pursued with also semi-analytical techniques, without hiding the main dynamical aspects behind an undue amount of numerical details.

3 An overview of scientific activities and events in nonlinear dynamics

In the contemporary society of knowledge communication and exchange, the above summary of main topics and techniques has to be suitably complemented with a synthetic, yet possibly comprehensive, description of the scientific activities and events through which nonlinear dynamics has developed within the applied mechanics community of engineering science over the considered time window.

3.1 Conferences

Consistent with the indicated nominal starting time of 'autonomous' development of nonlinear dynamics in mechanics, the first scientific events of international significance within the corresponding community took place in the 1980s. Before that time, as already mentioned, the major—if not even the sole—environment for meaningful exchange of knowledge on nonlinear dynamics was restricted to smaller, though extremely active, communities of more theoretical scientists including, but not limited to, scholars of mechanics. In this respect, the most important one at the international level was established in the East European countries under the sphere of influence of the former Soviet Union, though being able to attract a number of scientists also from capitalist countries. This was, on the one side, the natural consequence of the high quality of theoretical research on nonlinear oscillations via analytical methods conducted in that geographic area since the early twentieth century, which generally exploited the strong mathematical foundations of the relevant Academies of Science, and, on the other side, the apparent manifestation (as regards nonlinear dynamics) of the long lasting separation between socialist and capitalist societies occurring also in applied science and engineering. The first event of the prestigious series of *International Conferences on Nonlinear Oscillations* (ICNO) was organized in Kiev in 1961 by Yu. Mitropolski, the third scientist referred to in the KBM acronym of the powerful method(s) for the analysis of nonlinear oscillations initiated by N. Krylov and N. Bogoliubov; and the series of ICNO events held every 3 years in different cities of those countries lasted for 30 years, until the last one organized in 1990 in Krakow by W. Gutowski.

It was not by chance that the start of the first series of conferences on topics of nonlinear dynamics in the western countries was in the year 1986, when Ali H. Nayfeh—who had already published his fundamental books on perturbation methods and nonlinear oscillations—organized the first conference on *Nonlinear Vibrations, Stability and Dynamics of Structures and Mechanisms* at Virginia Tech, USA. But this time, the reference scientific environment was indeed mechanics, as per the meaningful expected consequences in the area that the general achievements on nonlinear oscillations had already envisaged since the 1970s, also in western countries. The new series of conferences (always held at Virginia Tech, which swiftly became a center of excellence for nonlinear dynamics in mechanics) had a definitely 'localized' character, as per the strong scientific personality of its founder, but it rapidly attracted an increasing number of scholars in nonlinear dynamics from all over the world. It lasted with about the same title until 2010, namely until about the time of Ali Nayfeh's retirement from active academic commitments.

Of course, the aim here is not to provide a detailed account of the huge amount of scientific events organized within more or less structured series of conferences of professional societies on a great variety of topics of nonlinear dynamics and chaos, as well as systems exhibiting nonlinear/complex responses. This is the case, in particular, of the American Society of Mechanical Engineers (ASME), whose two distinct series of International Design Engineering Technical Conferences (IDETC) on *Mechanical Vibration and Noise* and, later, on *Multibody Systems, Nonlinear Dynamics, and Control* always include a meaningful number of minisymposia on specific nonlinear dynamics items. Yet, it is worth dwelling on some sequences of events which have marked the subsequent progress of the research in the area.

At an actually international level from also a scientific societies' viewpoint, the first meaningful event gathering scholars of nonlinear dynamics in mechanics from different countries was the *IUTAM*

Table 1 IUTAM symposia in the area of nonlinear dynamics

Year	Location	Title	Chairs
1989	Stuttgart	*Nonlinear Dynamics in Engineering Systems*	W. Schiehlen
1992	London	*Nonlinearity and Chaos in Engineering Dynamics*	J.M.T. Thompson, S.R. Bishop
1996	Eindhoven	*Interaction Between Dynamics and Control in Advanced Mechanical Systems*	D.H. van Campen
1997	Ithaca, NY	*New Applications of Nonlinear and Chaotic Dynamics in Mechanics*	F.C. Moon
1999	Hanoi	*Recent Developments in Nonlinear Oscillations of Mechanical Systems*	N. Van Dao, E. Kreuzer
2003	Rome	*Chaotic Dynamics and Control of Systems and Processes in Mechanics*	G. Rega, F. Vestroni
2006	Nanjing	*Dynamics and Control of Nonlinear Systems with Uncertainty*	H.Y. Hu, E. Kreuzer
2008	Hamburg	*Fluid-Structure Interaction in Ocean Engineering*	E. Kreuzer
2010	Aberdeen	*Nonlinear Dynamics for Advanced Technologies and Engineering Design*	M. Wiercigroch, G. Rega
2011	Kyoto	*50 Years of Chaos*	T. Hikihara, T. Kambe
2015	Frankfurt	*Analytical Methods in Nonlinear Dynamics*	P. Hagedorn
2016	Nanjing	*Nonlinear and Delayed Dynamics of Mechatronic Systems*	Z. Wang, G. Stépan
2018	Novi Sad	*Exploiting Nonlinear Dynamics for Engineering Systems*	I. Kovacic, S. Lenci

Symposium on Nonlinear Dynamics in Engineering Systems organized by W. Schiehlen in 1989 at the University of Stuttgart, Germany, under the umbrella of the International Association of Theoretical and Applied Mechanics (IUTAM), namely the worldwide and most comprehensive reference society for scholars of mechanics. That was not the formal start of a new series of events on nonlinear dynamics, since IUTAM decides every 2 years about the organization of a small number of selected new symposia on a variety of topics of mechanics, based on advancements in specific areas and their potential to bring cutting edge problems to the attention of mechanicians. Yet, it was the substantial start of a new series of events, which made apparent the onset and the progressive increase in a well-connected community of international scholars (from Germany, UK, Poland, the Netherlands, Italy, USA, Austria, Denmark, Russia, but also increasingly involving non-European countries like Japan, Brazil, China and others) with clear interests in periodically exchanging ideas to face new scientific and technological issues of nonlinear dynamics. This originated a sequence of IUTAM Symposia (listed in Table 1 and covering 30 years, up to now) held at university departments playing some leading role in nonlinear dynamics within different application areas of mechanics (mechanical, civil, aeronautical, electrical engineering), with a variable (2–4 years) periodicity which reflects the need for further new advancements, and with variable titles making apparent the changing state of the art in the area and the need of connections with companion areas.

Characterizing topics (clearly reflected in the titles) include the meaningfulness of nonlinear dynamics for engineering systems—whose perspectives range from the analysis, description and understanding of phenomena variably addressed within different symposia, up to the current investigation of the usefulness of nonlinear dynamics for engineering design—the role played by chaos and complexity, the possible presence of uncertainties and the often unavoidable interaction between different areas (nonlinear dynamics and control) or different physical environments (fluids and structures, mechanics and electricity).

To a more general and wider level of diffusion of nonlinear dynamics, the series of *European Nonlinear Oscillations Conferences* (ENOC) of EUROMECH (European Society of Mechanics) has definitely to be mentioned. Following the crash down of the Berlin wall, the collapse of the East European cooperation also entailed the end of the above-mentioned series of ICNO events, which were indeed a direct expression of that world. Based on a recommendation of G. Schmidt and E. Kreuzer to the chairman of the European Mechanics Council D. Crighton to include ICNO into the society's conference activities, with the full support of Yu. Mitropolski, the relevant scientific tradition and the underlying patrimony of knowledge were inherited by EUROMECH, which started the new series of ENOC events at Hamburg, 1993. Its declared willing

Table 2 EUROMECH: European Nonlinear Dynamics Conferences (ENOC)

	Location	Chair	Participants	Countries	Minisymposia
1993	Hamburg	E. Kreuzer	228	33	
1996	Prague	L. Pust (F. Peterka)	205	36	
1999	Copenhaghen	H. True	153	27	
2002	Moscow	D.M. Klimov	101	21	
2005	Eindhoven	D. van Campen	346	42	22
2008	St. Petersburg	A. Fradkov	302	38	15
2011	Rome	G. Rega	384	41	19
2014	Wien	H. Ecker	≈ 400	37	20
2017	Budapest	G. Stépan	433	43	21
2020	Lyon	C.H. Lamarque			

was to become a meeting place for nonlinear dynamics scientists from all over the world, where in particular "East meets West." Consistent with this statement and under the guidance of a dedicated Conference Committee (ENOCC, [85]), ENOC events have been held every 3 years in different West and East Europe locations, by attracting an increasing number of scholars (Table 2). Besides hosting a small number of plenary lectures, they were articulated mostly through minisymposia concerned with an always increasing number and variety of topics, accurately selected in such a way to reduce the relevant overlapping to a minimum and organized by specifically invited expert scientists from different countries/institutions. In order to reflect the meaningful enlargement of scientific perspectives of nonlinear dynamics already emerged over the last decade of the twentieth century, starting with Eindhoven 2005 the title of the conference was modified to *European Nonlinear Dynamics Conference*, while still keeping the brand acronym of ENOC which had gained international visibility. Also, the last few events in the series have registered a meaningful opening toward new topics of interest in nonlinear dynamics, reflected in the organization of new dedicated minisymposia (or in slight modifications of classical ones), as well as a substantial increase in attendance from also non-European countries, so to be now considered by the international community of reference as the most important scientific conference on nonlinear dynamics in mechanics (and not only) of worldwide interest.

Since about beginning of the 1990s, smaller EUROMECH Colloquia on nonlinear dynamics topics have also been held with a variable periodicity, in addition to the wider and comprehensive European Conferences. A selection of them focused on general or specific aspects of nonlinear dynamics is reported in Table 3. Yet, many others devoted to companion items (mechatronics, multibody systems, nonsmooth systems, time-periodic systems, stochastic dynamics, with applications to vehicles, structures subject to moving loads, rotating machinery, fluid–structure interaction, micro-/nano-electromechanics, granular metamaterials, wave mechanics) and also including nonlinear dynamics aspects can be found in the EUROMECH website [86].

Of course, there are other more recent series of international conferences on topics of nonlinear dynamics, either focused or general. They include, among others, DSTA (*Dynamical Systems: Theory and Applications*) conferences based in Lodz, Poland; NNM (*Nonlinear Normal Modes*, also including dimension reduction, energy transfer and localization) conferences based in the Mediterranean area; RANM (*Recent Advances in Nonlinear Mechanics*) conferences initially based in Aberdeen; (ND) *Nonlinear Dynamics* conferences based in Kharkov, Ukraine; CSNDD (Conference on *Structural Nonlinear Dynamics and Diagnosis*) based in Morocco; ICDVC (International Conferences on *Dynamics, Vibration and Control*) based in China; but this is not the place to dwell on them.

3.2 Journals

It is instead important to shortly address two further kinds of scientific events specifically concerned with

Table 3 EUROMECH Colloquia clearly in the area of nonlinear dynamics

Year	Location	Title	Chairs
1993	Lodz	*Chaos and Noise in Dynamical Systems*	T. Kapitaniak, J. Brindley
1994	L'Aquila	*Bifurcation and Chaos in Solid and Structural Dynamics*	G. Rega, F.R. Pfeiffer
2001	Aberdeen	*Nonlinear Dynamics, Control and Condition Monitoring of Engineering Systems and Structures*	M. Wiercigroch A.A. Rodger, E. Kreuzer
2004	Frejus	*Nonlinear Modes of Vibrating Systems*	C.H. Lamarque, B. Cochelin
2007	Porto	*Geometrically Nonlinear Vibrations of Structures*	P.L. Ribeiro, M. Amabili
2008	Kazimierz Dolny	*Nonlinear Dynamics of Composites and Smart Structures*	J. Warminski, M.P. Cartmell
2009	Frascati	*Nonlinear Normal Modes, Dimension Reduction and Localization in Vibrating Systems*	G. Rega, A. Vakakis
2012	Frankfurt	*Time-periodic Systems: Current Trends in Theory and Application*	F. Dohnal, J.J. Thomsen, P. Hagedorn
2013	Senigallia	*New Advances in the Nonlinear Dynamics and Control of Composites for Smart Engineering Design*	S. Lenci, J. Warminski
2015	Sperlonga	*Stability and Control of Nonlinear Vibrating Systems*	A. Luongo, S. Casciati

nonlinear dynamics, namely (i) the appearance of new journals and (ii) the organization of advanced schools for PhD students and postdocs.

The first issue of *Nonlinear Dynamics*, the new journal timely founded by Ali Nayfeh to make apparent both the transition from classical nonlinear oscillations to the more modern and comprehensive concept and the leading role played by the group of Virginia Tech at the intersection between mechanics and nonlinear dynamics, was published in 1990. In turn, *Chaos, Solitons and Fractals* and *International Journal of Bifurcation and Chaos*, founded by M. El Naschie and G.R. Chen, respectively, first appeared in 1991. While the former journal aimed at filling a void of editorial presence in the area of mechanics (a feature which in the last few years has become less apparent), the latter two were rather conceived in a more cross-disciplinary perspective covering the wide area at the intersection between physics, chemistry, engineering, economy and other sciences, as per the characterizing features of complexity. All three journals—and in particular the first one—rapidly became references of archival interest in the area of nonlinear dynamics, though the second one also passed through a critical time somehow linked to the distorting effect increasingly produced in the new millennium by the bibliometric performance of scientific products. Further successful journals variably concerned with also nonlinear dynamics in connection with companion areas (*Journal of Vibration and Control, ASME Journal of Computational and Nonlinear Dynamics*) have appeared later on, up to some recent ones (*International Journal of Dynamics and Control, Journal of Applied Nonlinear Dynamics*).

Topics of nonlinear dynamics have of course been the subject of a huge number of Special Issues also in general-purpose journals in the area of mechanics.

3.3 Advanced schools

End of the 1980s was also the right time to hold at the International Center of Mechanical Sciences (CISM), Udine, Italy, the first Advanced Course (*Chaotic Motions in Nonlinear Dynamical Systems*, 1988) on topics of nonlinear dynamics, organized by W. Szemplinska Stupnicka, G. Iooss and F.C. Moon, and properly combining expertise in classical nonlinear oscillations within mechanics, bifurcation theory from the dynamical system environment, and the emerging field of experimental nonlinear dynamics. Upon further courses (e.g., *Engineering Applications of the Dynamics of Chaos* organized by H. Troger at CISM, 1991 and TU-Wien, 1994), it is worth mentioning the coordinated series of schools organized in 2007–2009 in different places (L'Aquila, Leuven, Liége, Vaulx-en-Velin, Wien, Athens, Rome) under the umbrella of the Marie Curie Action *Stability, Identification and Control in Nonlinear Structural Dynamics* (SICON), coordinated by A. Luongo and F. Vestroni, and combining nonlinear dynamics problems and systems with topics in companion areas.

In the current decade, new Advanced Schools on a variety of novel perspectives of nonlinear dynamics have been held again at CISM [87], always attracting a meaningful number of young attendees: nonlinear strategies for vibration mitigation and system identification [88], exploiting nonlinear behavior in structural dynamics [89], nonlinear modal analysis [90], global dynamics for engineering design [91].

4 Ali Nayfeh: the scientist and the scholar

Ali H. Nayfeh was born December 21, 1933, in Shuwaikah, Palestine, and passed away March 27, 2017, in Amman, Jordan. Upon working 10 years as a teacher of mathematics in remote villages and towns of Palestine due to the relevant harsh conditions and lack of higher education institutes, at the age of 26 he got a scholarship to study in the USA, where in 5 years (1959–1964) he moved from a junior college to a Ph.D. in Aeronautics and Astronautics at Stanford University. Upon working 7 years in the aerospace industry (1964–1970), he moved as a Professor (1971) and later as a University Distinguished Professor (1976) at Virginia Tech, where he spent all academic career. Nayfeh was extremely active on the international scene, taking part and meaningfully contributing to many of the scientific events mentioned in Sect. 3. A selection of pictures of Ali together with colleagues from all over the world, taken on the occasion of a variety of conferences, is reported in Fig. 2.

Summaries on Nayfeh's life, his incredible number of achievements and awards, and his human being qualities can be found in the obituaries [92–94] written by a number of his former students at Virginia Tech (now brilliant professors elsewhere), and in a tribute paper given in a recent IUTAM Symposium [95].

Here, an attempt is made to outline some aspects of his scientific personality, even though the lack of an adequate historical perspective prevents from reliably assessing the extent and depth of his contributions, as well as the legacy embedded in them.

Revisiting concepts from a recent obituary presentation (by a lecturer I do not remind), it can be said that Ali Nayfeh embodied a rare mixture of scientific training and expertise/interests developed over five decades of hardworking and untiring activity along which,

- in the 1960s and early 1970s, he started as a *fluid dynamicist* and, more generally, as an *applied mathematician*;
- in the 1970s and 1980s, he became a combined *applied mathematician*, *dynamicist* and *physicist*;
- in the 1990s and the new millennium, he evolved into a comprehensive *dynamicist*, with renewed and increased attention to a multitude of aspects in *structural mechanics.*

4.1 Research activity

Nayfeh's research activity was concerned with a huge variety of topics in different areas, which include perturbation methods, nonlinear oscillations, aerodynamics, flight mechanics, structural dynamics, acoustics, ship motion, hydrodynamic stability, nonlinear waves, experimental dynamics, linear and nonlinear control, and micromechanics.

His scientific achievements can be discussed in the background of four main characterizing aspects.

- The fundamental and worldwide recognized contributions given in the field of classical *asymptotic methods* and their application to *nonlinear dynamics* of a multitude of *engineering systems.*
- The progressive awareness of the importance to complement classical expertise with intensive use of modern and comprehensive approaches developed within the dynamical system theory, which include advanced *geometrical* and *computational techniques* and the associated tools for dealing with *complex dynamics*.
- The capability to revisit his basic nature of a theoretical scientist with a continuous—and ever-increasing—involvement with meaningful *experimental dynamics*, considered as an essential approach for catching the *actual features* of *nonlinear behavior.*

Fig. 2 Ali Nayfeh at various conferences, with colleagues listed from left to right (without my name, I do apologize for being always present). **a** ENOC 1993, Hamburg (with Francis Moon, Hans Troger, Vladimir Beletsky). **b** IUTAM Symposium 1996, Eindhoven (with A.nil Bajaj, Christine and Werner Schiehlen, Raouf Ibrahim, Robin Sharp, Dick van Campen, Edwin Kreuzer). **c** VPI Conference 1999 (with Fabrizio Vestroni, Dick van Campen, Edwin Kreuzer). **d** Campos do Jordao Conference 2000 (with Gabriela Hagedorn, Tomasz Kapitaniak, Peter Hagedorn and Jan Awrejcewicz (on the knees)). **e** IUTAM Symposium 2003, Rome (with Fabrizio Vestroni, Giorgio Novati, David Chelidze and his daughter). **f** NNM Colloquium 2004, Frejus (with Hans Troger (back), Claude Lamarque, Yuri Mikhlin). **g** ISVCS 2005, Berchstesgaden (with his wife Samira). **h** last VPI Conference 2010 [with Subhash Sinha, Khaled Ashfar, an unidentified colleague (sitting), Walter Lacarbonara, Hiroshi Yabuno, Bala Balachandran, Mohammed Younis, Josè Balthazar, another unidentified colleague (standing up)]. (Color figure online)

- The open mind allowing him to grasp the importance of entering *new research fields* ranging from *control* of nonlinear dynamic behavior of *real engineering systems* to modeling, analysis and simulation of *new materials* and *new application fields*, thus meaningfully *widening the perspectives* of the whole nonlinear dynamics area.

Such aspects are made apparent by framing the list of specific topics which he gave important contributions to within the four main stages of general development of nonlinear dynamics in mechanics previously identified in Sect. 2 (Figs. 3, 4, 5, 6). Of course, not all achievements obtained by Nayfeh (and coauthors, mostly in the stages from second to fourth) can be given the same credit of exceptionality.

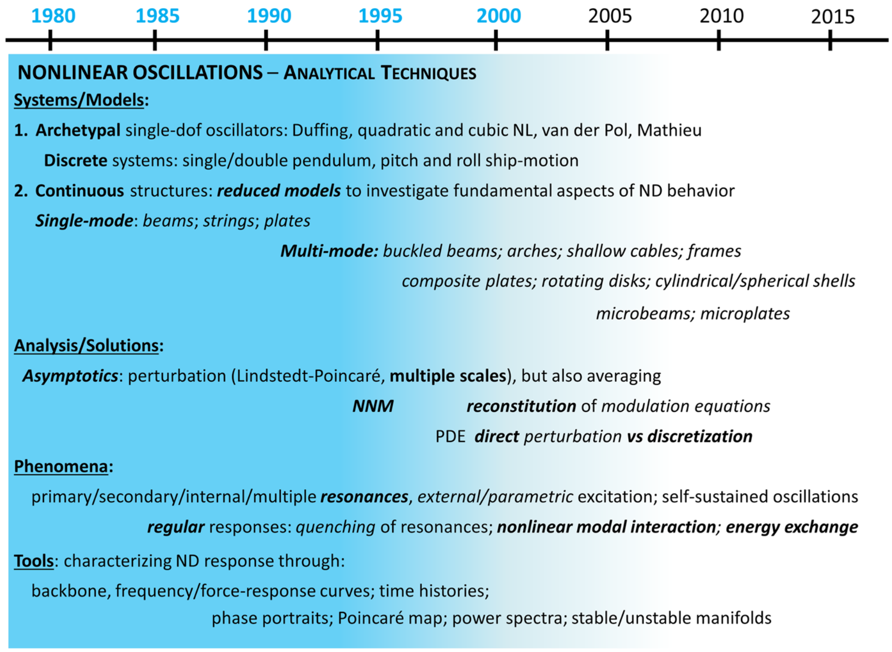

Fig. 3 Framing Nayfeh's addressed topics within the development of nonlinear dynamics in mechanics: the *Nonlinear Oscillations* stage. (Color figure online)

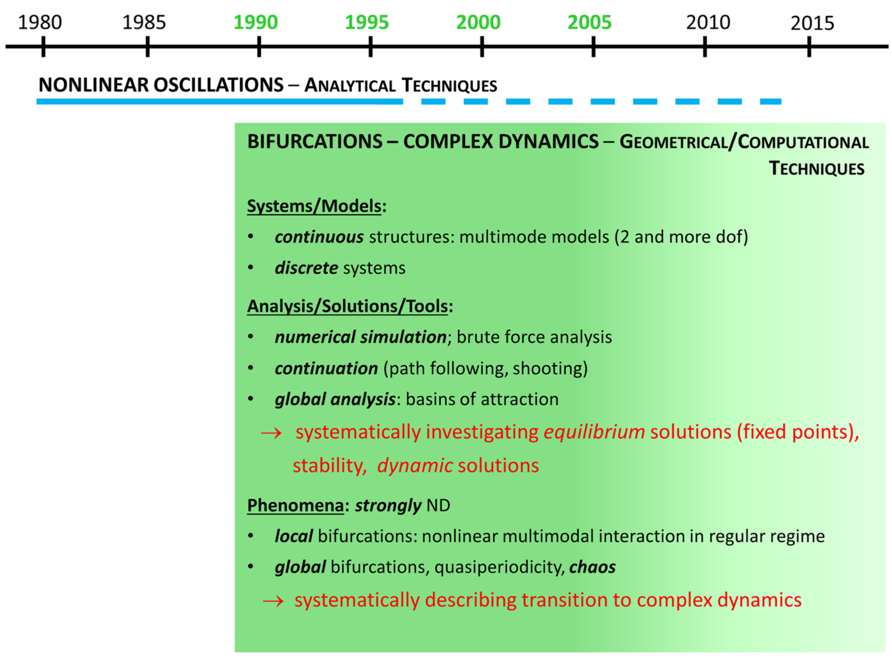

Fig. 4 Framing Nayfeh's addressed topics within the development of nonlinear dynamics in mechanics: the *Bifurcation and Complex Dynamics* stage. (Color figure online)

Building on the first [17] of his many books (Fig. 7), where "different *perturbation techniques are described using examples* which start with model simple ordinary equations that can be solved exactly and *progress toward complex* partial differential equations, ..., with examples drawn from *different branches of physics and engineering*," he attained an exceptional, worldwide recognized, mastery and unrivaled leadership in the use of the method of multiple time scales for solving nonlinear dynamics problems. This originated his companion second book [18] focused on the "*physical mechanisms and effects*, restricting the attention to uniform continuous systems with simple boundary conditions whose linear natural modes can be obtained analytically, and then using the method of multiple scales to solve the equations describing the

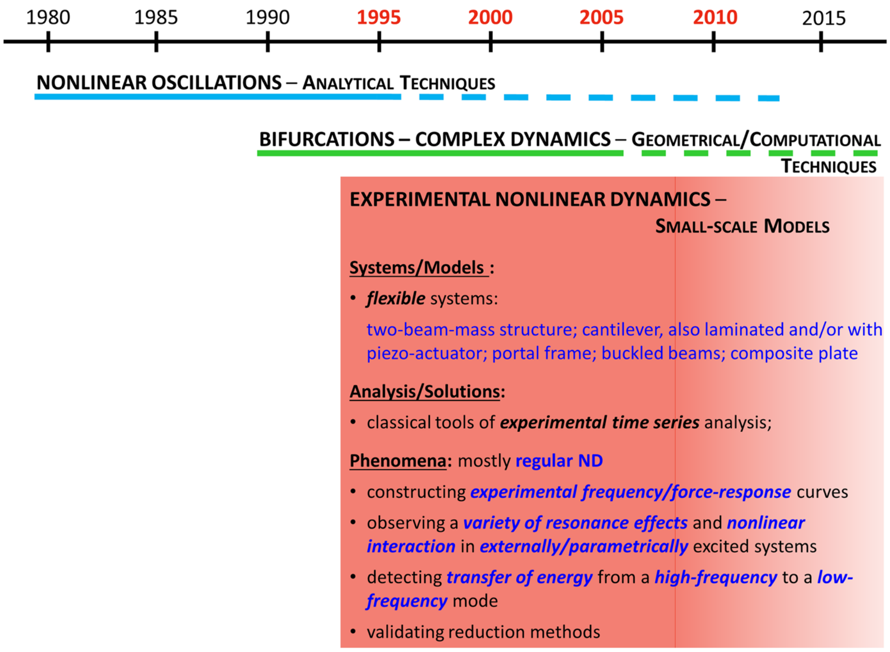

Fig. 5 Framing Nayfeh's addressed topics within the development of nonlinear dynamics in mechanics: the *Experimental Nonlinear Dynamics* stage. (Color figure online)

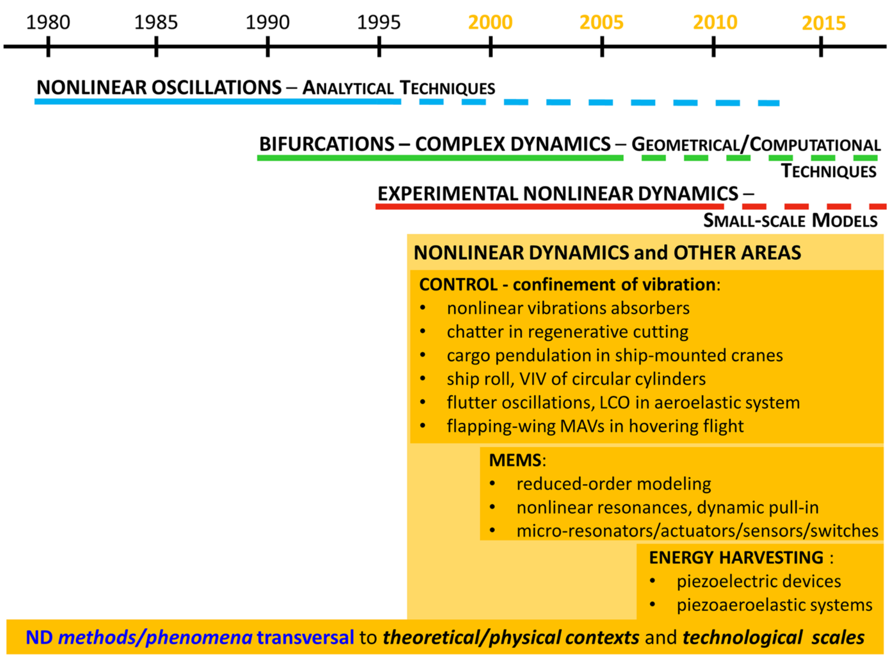

Fig. 6 Framing Nayfeh's addressed topics within the development of nonlinear dynamics in mechanics: the stage of hybridization of *Nonlinear Dynamics* with *Other Areas*. (Color figure online)

temporal functions." The actually pioneering character of this second book (swiftly assumed as a classroom textbook all over the world, with 9225 citations out of a total of 50032 according to Google Scholar, as of February 06, 2019) in mechanics consisted of aiming to "*fill the void of previously published books which emphasized*, and some exclusively treated, systems having a *single degree of freedom*." Using analytical techniques for solving nonlinear oscillation problems, Nayfeh's fundamental achievements in the 1970s and 1980s regarded archetypal oscillators and single- or two-mode models of continuous systems, for which he obtained an enormous amount of fundamental results (later on referred to in thousands of papers) in terms of regular response to external/parametric excitations in condition of primary, secondary, internal and/or multi-

Fig. 7 Ali Nayfeh's books. (Color figure online)

	ANALYTICAL METHODS	NONLINEAR PHYSICAL ASPECTS	STRUCTURAL MECHANICS
1970-79	*Perturbation Methods* (1973) companion	*Nonlinear Oscillations* (1979) with D.T. Mook	
1980-89	*Introduction to Perturbation Techniques* (1980) *Problems in Perturbations* (1985)		
1990-99	*Method of Normal Forms* (1993) *Perturbation Methods with Mathematica, Maple* (1999) with C.M. Chin	*Applied Nonlinear Dynamics* *Analytical, Computational and Experimental Methods* (1995) with B. Balachandran	
2000-09		*Nonlinear Interactions* *Analytical, Computational and Experimental Methods* (2000)	*Linear and Nonlinear Structural Mechanics* (2004) with P.F. Pai

ple resonances, self-sustained oscillations and quenching of resonances, by using dynamic tools such as backbone and frequency–response/force–response curves, time histories, phase portraits, Poincaré maps, power spectra and stable/unstable manifolds (Fig. 3).

In the 1990s, such results were updated, expanded and enriched in a variety of directions. Among them, it is worth mentioning: (i) the effective implementation of the direct perturbation approach for the solution of partial differential equations (as contrasted to the Galerkin-based discretization approach), which also provides a constructive procedure for obtaining nonlinear normal modes of continuous systems [96]; (ii) the further attention paid to higher asymptotic approximations, with the ensuing resolution of earlier controversies as to the proper reconstitution of modulation equations in the case of high-order multiple-scale solutions [97]; (iii) the consideration of more realistic internally resonant multimode models of systems and structures, with the many involved phenomena of nonlinear interaction and transfer of energy, also including a newly observed experimental transfer from a high-frequency mode to a low-frequency mode according to a novel mechanism entailing a slow modulation of the former [29].

Possibly more important in general terms is the circumstance that at about the same time, while not giving specific foundational contributions to the two stages of bifurcations and complex dynamics (Fig. 4) and of experimental nonlinear dynamics (Fig. 5), Nayfeh realized the significance of the corresponding, relatively novel (in the mechanics area), concepts and techniques and, together with his younger collaborators, became an extremely smart and effective user of the combination of advanced analytical, geometric/computational, and physical techniques which are jointly necessary to reliably and exhaustively characterize the dynamics of multi-/infinite-dimensional systems. A realm in which he established the comprehensive theoretical and operational framework which is necessary for in-depth investigations of nonlinear systems, later adopted by hundreds of scientists all over the world. This expanded interest toward physical aspects of nonlinear dynamics brought to the publication of two new books (Fig. 7). The first one [47], specifically dealing with geometrical concepts, local and global bifurcations, continuation methods, tools to characterize motions, quasiperiodicity, chaos and its control, fully fits to an *applied* nonlinear dynamics perspective. The second book [29], built around the general idea that "understanding of dynamic characteristics of a structural system is essential for its design and control," appears as a *worth blend of remarkable author's expertise* on *classical perturbation* methods and *nonlinear oscillations* theory—revisited through modern symbolic algebra—

with knowledge and tools from *dynamical system theory* and results from *experimental investigations*. Representing in some sense a *continuation* and an *update* of the 1979 book on Nonlinear Oscillations, it is "more *in the genuine author's attitude of mind* than others of his recent books" [98].

In about the last 20 years of his research activity, with inexhaustible energy, curiosity and open mind toward innovation, Nayfeh was likely the first of his generation to realize the increased importance of companion and new research areas to which nonlinear dynamics could have contributed a whole basket of unknown and/or unexploited resources in terms of general knowledge, understanding and interpretation of experimental phenomena, and ensuing technological advancements (Fig. 6). Expanding his earlier (from the 1980s) interest about quenching of nonlinear vibrations, general control and confinement of vibrations of systems and structures became one new area of interest, with creative attention paid to a variety of problems and systems, ranging from nonlinear vibration absorbers of cargo pendulation in ship-mounted cranes, up to ship roll, limit cycle oscillations in aeroelastic systems (both meaningfully addressed already in the past), and vortex-induced vibrations. A scientific and technological perspective of combination of expertise from companion areas which also led him to found the immediately successful new *Journal of Vibration and Control*. Yet, even more important in terms of exploitation of nonlinear dynamics knowledge and expertise was Nayfeh's capability to approach and authoritatively enter the exploding area of microelectromechanical systems, wherein classical paradigms of linear dynamics were still utilized in both analysis and design. Along with younger collaborators, Nayfeh paved the way to making apparent to the MEMS and NEMS communities the meaningful change of perspectives, and the ensuing technological advancements, made possible by a nonlinear approach to the relevant vibration problems.

Generally speaking, in the long last period of his scientific activity, Nayfeh resumed and updated his always underlying interest toward structural mechanics. In his last book [67] (see Fig. 7), motivated by the challenges posed to the researcher/designer by the nonlinear modeling and dynamic analysis of composite or ultra-lightweight structures, as well as of structures made by smart and/or micro-/nano-materials, mathematically consistent and systematic derivations of comprehensive structural theories (for strings, cables, beams, plates, shells, laminates) are presented, by also incorporating piezoelectric materials and thermoelasticity. Therein, working in a substantially three-dimensional approach perspective, modeling is aimed at accomplishing accurate and consistent nonlinear dynamics analyses and at showing the capability of a considered (exact or approximate) model to correctly highlight meaningful features of nonlinear behavior; this combination of rigorous formulation of motion equations and ensuing nonlinear dynamics analyses being certainly a distinctive and for the time novel feature of the book [99].

4.2 Some common features of Nayfeh's books

As already discussed in previous sections and comprehensively reported in Fig. 7, over his long career Ali Nayfeh wrote a considerable amount of generally important books, overall quite well distinct from each other. The interest here is in highlighting some of his underlying *methodological features* as a book author, and in illustrating them through some examples. Four main charactering points can be identified in his production.

1. Addressing and presenting topics according to an *incremental/additive* (and generally *inductive*) approach, through a *series of case studies.*
2. Constantly aiming to embed specific outcomes (concerned with *analytical techniques, different methods, different systems*) within a possibly *unified, comprehensive* and *comparative* framework.
3. Illustrating concepts with *numerous worked-out examples* and many *exercises* to be used for *reinforcing understanding* and *assessing progress* of students.
4. Providing an *extended* and *updated bibliography*: in each of his books, there are 70–80 pages of References.

The *incremental/additive* character of Nayfeh's approach can be easily recognized by looking at the contents of any of his books. Two examples taken from [18], respectively, relevant to the forced oscillations of systems with a single-degree-of-freedom and to systems with finite degrees of freedom, are reported in Fig. 8. The underlying criterion is to "treat simple systems that exhibit the essential ideas, instead of treating general systems for which the algebra is involved." This originates lists of case studies of possibly increasing (and anyway distinct) difficulty. In the

4. Forced Oscillations of Systems Having a Single Degree of Freedom

4.1. Systems with Cubic Nonlinearities
4.1.1. Primary Resonances, $\Omega \approx \omega_0$
4.1.2. Nonresonant Hard Excitations
4.1.3. Superharmonic Resonances, $\Omega \approx 1/3\ \omega_0$
4.1.4. Subharmonic Resonances, $\Omega \approx 3\omega_0$
4.1.5. Combination Resonances for Two-Term Excitations
4.1.6. Simultaneous Resonances: The Case in Which $\omega_0 \approx 3\omega_1$ and $\omega_0 \approx 1/3\ \Omega_2$
4.1.7. An Example of a Combination Resonance for a Three-Term Excitation
4.2. Systems with Quadratic and Cubic Nonlinearities
4.2.1. Primary Resonances
4.2.2. Superharmonic Resonances
4.2.3. Subharmonic Resonances
4.2.4. Combination Resonances
4.3. Systems with Self-Sustained Oscillations
4.3.1. Primary Resonances
4.3.2. Nonresonant Excitations
4.3.3. Superharmonic Resonances
4.3.4. Subharmonic Resonances
4.3.5. Combination Resonances

6. Systems Having Finite Degrees of Freedom

6.1. Examples
6.1.1. The Spherical Pendulum
6.1.2. The Spring Pendulum
6.1.3. A Restricted Ship Motion
6.1.4. Self-sustaining Oscillators
6.1.5. The Stability of the Triangular Points in the Restricted Problem of Three Bodies
....
6.5. Forced Oscillations of Systems Having Quadratic Nonlinearities
6.5.1. The Case of Ω Near ω_2
6.5.2. The Case of Ω Near ω_1
6.5.3. The Case of Nonresonant Excitations
6.5.4. The Case of 2Ω Near ω_1
6.5.5. The Case of Ω Near $\omega_1 + \omega_2$
6.6. Forced Oscillations of Systems Having Cubic Nonlinearities
6.6.1. The Case of Ω Near ω_1
6.6.2. The Case of Ω Near ω_2
6.7. Parametrically Excited Systems
6.7.1. The Case of Ω Near $2\omega_1$
6.7.2. The Case of Ω Near $2\omega_2$
6.7.3. The Case of Ω Near $\omega_1 + \omega_2$

Fig. 8 Incremental/additive approach in the contents of *Nonlinear Oscillations* (1979). (Color figure online)

first example, cubic nonlinearities, quadratic and cubic nonlinearities, and self-sustained oscillations are separately addressed, for each one of them dealing with conditions of primary, superharmonic, subharmonic, combination or simultaneous resonances, and nonresonant excitations; in the second example, distinct physical systems are considered and, for the subsets with quadratic nonlinearities, cubic nonlinearities or parametrically excited, various cases of resonance between excitation frequency and system natural frequencies are addressed. The scheme is certainly quite repetitive, unveiling in this respect a somehow encyclopedic underlying perspective. Yet, it also allows to more easily point out differences and peculiarities associated with the presence of distinct nonlinear aspects concerned with the analytical treatment or with the physical features of the underlying system/excitation. Overall, this gives rise to a mine of information illustrating richness and variety of nonlinear phenomena, as well as the theory behind them.

Another example (Fig. 9) is reorganized from the contents and text of [100], where—under the general criterion according to which "perturbation techniques are best explained using examples", already adopted in [17]—the techniques therein presented in a "*coincise* and *advanced*" way are revisited "in an *elementary*" way, as per the newly assumed textbook perspective, by selecting a limited number of them and amplifying their description considerably, to the benefits of students' understanding.

Two more examples (Fig. 10) also highlighting the *incremental/inductive* character of Nayfeh's approach refer to topics addressed about 20 years later or more and are concerned with the form of the modulation equations for systems belonging to a certain group of symmetry, and to the ready extension of methods to different and/or more complex structures.

Although in the prefaces of all his books Nayfeh declared in one way or another that the material is not presented within a mathematically rigorous framework (starting with his more mathematically oriented book

Fig. 9 Incremental/additive approach followed in *Introduction to Perturbation Techniques* (1980)

1. Free oscillations
   - Techniques:
     - Straightforward expansion (non-uniform for large times)
     - Exact solution (to show that system frequency is a function of the nonlinearity)
     - Lindstedt-Poincarè technique (uniform expansion)
     - Method of renormalization
     - Method of multiple scales
     - Method of averaging
     - Krylov-Bogoliubov-Mitropolski technique
   - Oscillators: moving from specific (Duffing conservative, linear damped, self-excited, with quadratic and cubic nonlinearities) to general (weakly nonlinear)
2. Forced oscillations with multiple scales (or averaging)
   - External excitation, multifrequency excitation
     Resonance cases: secondary ($\omega \approx 3$, 1/3, 2, 1/2), primary, combination ($\omega \approx \omega_2 \pm \omega_1$)
   - Parametric excitation (the Mathieu equation)
     Methods: straightforward expansion, strained parameters, Whittaker's, multiple scales, averaging

Fig. 10 Incremental/inductive approach followed in two more recent books. (Color figure online)

Nonlinear Interactions **(2000):**

- *2:1, 1:1, or 3:1 Internal Resonances*: primary resonance, principal parametric resonance (of first mode, of second mode), fundamental parametric resonance
  Combination Resonances: parametric, external, internal; of the additive or difference type
- *Nonlinear Normal Modes*: a symmetric 2-dof system; an asymmetric 2-dof system; cubic and quintic nonlinearities; quadratic and cubic nonlinearities; multi-dof systems; systems with internal resonances; continuous systems

Inductive: for systems with 1:1 internal resonance:
"the modulation equations have the same form for all systems with O_2 symmetry, irrespective of their physical origin and/or the source of the nonlinearity"

Linear and Nonlinear Structural Mechanics **(2004):**

- *Plates*: linear classical; linear shear-deformable; nonlinear classical; general nonlinear classical; nonlinear shear-deformable; nonlinear layerwise shear-deformable

Incremental/Inductive:
"as a result, the reader can readily extend the methods to formulate and analyze different and/or more complex structures"

[17], where he states that "the different techniques are described as formal procedures without any attempt at justifying them rigorously"), this characterizing aspect of his scientific personality originated criticisms from more mathematically oriented (and certainly more rigorous) scientists, nurtured within the dynamical system environment. But this feature was indeed associated with Nayfeh's main nature of a scholar in engineering sciences, fundamentally interested in understanding the nonlinear behavior of more and more involved engineering (mechanical/structural) systems, by exploiting his well-founded knowledge of fundamentals of applied mathematics. Although possibly questionable for having been brought to its extreme consequences, such an inductive perspective is somehow in the line of the one adopted by the majority of scientists previously active in nonlinear dynamics (including mathematicians), who "were not led to their discoveries by a process of deduction from general postulates or principles, but rather by a thorough examination of properly chosen particular cases," with the generalization coming later "because it is far easier to generalize an established result than to discover a new line of argument" [7]. In this sense, Nayfeh's approach showed to be not too far away from the purpose of studying *concrete nonlinear systems* with their "natural effects," which also inspired more theoretical research on nonlinear dynamics made within the earlier Andronov and Krylov–Bogoliubov schools.

Fig. 11 A unified framework. (Color figure online)

- *Perturbation Methods* (1973):
  "Presents in ***a unified*** way an account of most of the ***perturbation*** techniques, pointing out their ***similarities, differences, and advantages,*** as well as their ***limitations***"
- *Applied Nonlinear Dynamics* (1995):
  "Unlike most other texts, which emphasize either classical methods, experiments and physics, geometrical methods, computational methods, or applied mathematics, provides a ***coherent and unified*** treatment of ***analytical, computational, and experimental*** methods and concepts of ***nonlinear dynamics***"
- *Nonlinear Interactions* (2000):
  "Provides a *coherent and* ***unified*** treatment of ***analytical, computational, and experimental*** methods and concepts of ***modal interactions***"
  As an obvious extension of *Applied Nonlinear Dynamics* (1995), the relevant "methods are used to explore and unfold in a *unified* manner the ***fascinating complexities*** in nonlinear dynamical systems"
- *Linear and Nonlinear Structural Mechanics* (2004):
  "A ***unique unified*** approach, more general than those found in most structural mechanics books, is used to model ***geometric nonlinearities*** of structures"

Fig. 12 Worked-out examples and exercises. (Color figure online)

- *Perturbation Methods* (1973):
  "*coincise and advanced* material, intended for *researchers and advanced graduate students*"
- *Introduction to Perturbation Techniques* (1980):
  "material presented in an *elementary* way that makes it accessible to *advanced undergraduates and first-year graduate students* in a wide variety of scientific and engineering fields"
- *Problems in Perturbation* (1985):
  "detailed solutions of all the problems (360) in *Introduction to Perturbation Techniques* and about an equal number of unsolved supplementary problems" ……… "a book *ideal for self-study*"
- *Nonlinear Oscillations* (1979):
  "the book emphasizes the *physical aspects* of the systems …… many examples worked out completely, with explanations couched in physical terms……….. many exercises included at the end of each chapter, *progressing in complexity*; many contain intermediate steps to *help the reader*"
- *Applied Nonlinear Dynamics* (1995):
  "presenting *mathematical concepts* in a manner *comprehensible to engineers and applied scientists*"
- *Linear and Nonlinear Structural Mechanics* (2004):
  "the major goal is to *close the gap between* the *practicing engineer* and the *applied mathematician* in the modeling and analysis of *geometrically nonlinear structures*"

It is, however, important to notice that such a 'list-looking' aspect of the material addressed in his various books did not prevent Nayfeh from constantly pursuing, and indeed successfully achieving, a *unified, comprehensive* and *comparative* framework into which presenting the various case studies and understanding the relevant differences and peculiarities. Figure 11 witnesses the care always paid to the matter, by quoting proper sentences from the prefaces of some of his books.

Constantly illustrating concepts with worked-out examples and exercises to be used for reinforcing understanding and assessing progress of students was also on top of Nayfeh's care, in a fully effective educational—and also professional—perspective; it is summarized in Fig. 12, quoting again from some of his books' prefaces.

Last but not least, the very rich and (for their time) updated bibliographies listed at the end of each book are still invaluable sources of information and knowledge for scholars and scientists in nonlinear dynamics as regards both 'classical' books or journal/conference papers and more recent ones. Something which is even more important in a time as the present one where several scientific problems are somehow 'rediscovered,' in the context of generally low care paid to the historical evolution and advancement of science by younger scientists strongly (if not even solely) aimed at attaining specific, and not rarely limited, goals.

5 Nayfeh's legacy and the future of nonlinear dynamics

All of Nayfeh's achievements as a scientist and a scholar, summarized in the previous section, have to be kept in mind when trying to tentatively outline his general and more specific legacy for the community of people active in nonlinear dynamics within mechanics and engineering.

Notwithstanding some mathematical warnings about possible pitfalls occurring in the asymptotics with the method of multiple time scales, Nayfeh's methodological and operational contributions on multiple scales in the first outstanding part of his scientific career, yet continued all over his activity, are certainly a fundamental part of his legacy and are worldwide utilized for solving nonlinear dynamics problems. Pitfalls are mostly associated with the anticipated choice of timescales, as contrasted with methods of averaging and renormalization which could be preferred because of timescales emerging by the nonsecularity conditions with no a priori assumptions [101]. This is particularly true in the presence of resonances or bifurcations, where naturally ensuing algebraic timescales (including some with fractional powers of the small parameter ε) can play an important role in obtaining reliable asymptotic estimates on long intervals of time. Yet, Nayfeh was most likely well aware of those pitfalls, as witnessed by the stress constantly made on the need to properly select timescales (including fractional ones) depending on the dynamic problem at hand, as well as by the many performed comparisons of equivalent approximations obtained by multiple scales and by averaging. The sole possible element of reservation in this respect is likely the circumstance that Nayfeh's enormous skill and experience about how properly selecting a priori the timescales necessary for a reliable asymptotic solution of a given problem denote a feature of personal exceptionality which is not often encountered in also good scientists.

Again, this corresponds to a kind of *engineering approach* to nonlinear dynamic systems—as contrasted to a *mathematical approach*—which makes uses of mathematical/mechanical skill and expertise for grasping the nonlinear behavior of engineering systems. Indeed, starting with his earlier achievements on the nonlinear oscillations of relatively simple models, the whole set of interests and outcomes which marked the second part of Nayfeh's academic life highlight his further *distinguishing* and *influential features* just in the realm of engineering sciences and applied mechanics:

- A strong will and capability to pursue an *overall* and *comprehensive approach* to the treatment of relevant problems.
- Interests and accomplishments ranging from the adequate *modeling* of *nonlinear systems* in different fields (aeronautics, macro-/micromechanics, structures, ship engineering) for the study of their *dynamical behavior*, to *formulation* and *solution* of the relevant *mathematical problems*, to exhaustive understanding and description of *phenomenological aspects*, up to the identification of *problems* and *features common* to *different fields* of technical application.

Nayfeh's strongly founded knowledge and expertise in classical aspects of the theory of nonlinear oscillations was constantly updated, paralleled and enriched by the novelties brought into the scientific arena by the substantial widening of perspectives occurred in nonlinear dynamics at the end of the previous millennium and by fruitful confrontations with a number of companion research areas. In a general context of great specialization of the research activity up to its extreme consequences, this *global*—though very *penetrating* in each area—*feature* of his activity deserves a *special mention*, and can be considered as another fundamental aspect of his legacy in scientific terms.

Linked with this aspect, yet more specific in terms of contributions given to the advancement of nonlinear dynamics, is Nayfeh's unique attitude to pay attention to both *innovation* and *synthesis*. Mostly in the last few decades, he showed a tremendous capability to catch *novel advancements* and *research trends*, resulting in a research activity which has been a *highly remarkable blend* of:

1. *exploitment* of an *unrivaled expertise* on *asymptotic methods* and *nonlinear oscillations* theory, revisited through modern symbolic algebra;
2. *interpretation* and *organization* of both in-house and literature *outcomes* of intense *experimental investigations*;
3. *smart use* of *knowledge* and *tools* from *modern dynamical system* theory, also in view of entering and meaningfully contributing to novel scientific and technological areas.

All of this being always complemented with an enormous capability to timely make advanced scientific knowledge available to people involved in research, as well as to practicing engineers dealing with challenging problems in applied mechanics.

This last, but certainly not least, aspect of Nayfeh's legacy has to be embedded within a more general discussion on, and an ensuing vision of, the future of nonlinear dynamics in mechanics and engineering, which is herein just sketched.

Upon forty and more years of strong development, summarized in Sects. 2 and 3 in terms of conceptual and organizational aspects, nonlinear dynamics in mechanics has approached a turning point beyond which an unexplored land occurs and a novel mission has to be identified. The following main issues are seen to characterize the current research context: (i) the need to more clearly overcome limitations inherent to consideration of archetypal single- or few-degree-of-freedom models, and to deal with *more complicated engineered systems* from macro- to nano-mechanics; (ii) the parallel awareness of the theoretical, computational and experimental difficulties of local and, mostly, global dynamics aspects of real multidimensional systems, with the ensuing need to further refer to *reliable reduced-order models* to be identified within a fully consistent and controllable perspective; (iii) the ever-increasing interest toward *exploiting nonlinear and global dynamics* modeling and analysis for more *effective description, design and control* of engineering systems.

In the new millennium, and in the framework of the hybridized fourth stage of development discussed in Sect. 2, nonlinear dynamics has undergone a meaningful merging with novel areas at the frontier between engineering and other sciences, which are by now well represented by papers published, e.g., in *Nonlinear Dynamics*. They encompass, among others, nonlinear waves (including rogues waves), solitons (dark/bright), soliton–soliton interactions entailing dramatic new scenarios of singularities/localization in ocean engineering, wave propagation in optics/fluids/solids, advanced cryptography based on chaotic maps, advanced control and synchronization of complex networks (machines, vehicles, data processors, etc.), ecosystem dynamics, social media dynamics, multiagent systems.

In this context, the added value and the suitability of nonlinear dynamics to allow a significant enhancement of the performance, effectiveness, reliability and safety of systems in different physical contexts and at various technological scales have clearly emerged within the dynamical community, with interesting developments enjoyed in a number of application areas. With reference to the sole mechanical aspects, and definitely with no attempt to be exhaustive, a number of issues where nonlinear dynamics has already been successfully utilized [12] can be mentioned :

- Improving *understanding* of behavior in *macro-* to *micro-/nano-systems*
- *Mitigating vibration* in a variety of structures
- *Harvesting vibration energy*
- Exploiting *multistability* of nonlinear systems (structures and metamaterials)
- Using *nonlinear interaction* and bifurcation phenomena in *multiphysics* contexts
- *Controlling* local/global *bifurcations* and possible ensuing *chaos*
- *Harnessing/tailoring geometric* or *mechanical nonlinearities* for a variety of purposes (optimal design, system identification, structural health monitoring, etc.)
- Developing *novel technologies* and *applications* in macromechanics and in micro-/nano-/biomechanics
- Exploiting *global dynamics* for *safe engineering design*

In the last few years, a variety of scientific, educational and professional initiatives have been taken as regards exploiting nonlinear dynamics, with symposia and advanced schools (see Sect. 3) on novel perspectives held, journal special issues and surveys on specific themes (e.g., [102–106]) published, and even a new series of conferences on dynamics for design launched (within ASME), apart from subsequently aborting for reasons which denote the complicated growing stage faced by the new mission.

Indeed, much has yet to be done to take full advantage of the attained fundamental understanding of dynamic phenomena which produce nonlinear interactions, bifurcations and complex response. The richness of behavior generally offered by nonlinear systems [107] has to be exploited for conceiving and developing novel design criteria and related technologies, this being a new emerging field which may indeed be considered as the fifth, most recent and open stage of development of nonlinear dynamics in mechanics and engineering. Properly accounting for nonlinearities inherent to the system or artificially created may be expected to meaningfully influence current paradigms of technological systems, along with their design and control, in a magnitude of contexts, thus bridging the gap still existing between theory and applications.

It is important to stress how Ali Nayfeh meaningfully contributed to throw fruitful seeds in the direction of future developments of nonlinear dynamics. Building on his attitude toward innovation and his awareness of the need to embed nonlinear dynamics achievements within a novel vision, around the turn of the millennium Ali Nayfeh was among the very first to grasp this general matter and to make nontrivial operational steps toward exploitation of nonlinear dynamics in his everyday scientific life. Quoting from the Nayfeh's obituary written by one of his former students who was also my student [92], "Ali's work is far more than merely theoretical, with applications encompassing devices, structures and systems. He established a *new paradigm in higher education and engineering practice* called *Nonlinear Dynamics for design*, which seeks to *exploit advantageously nonlinear phenomena and principles to enhance the performance of engineering systems*."

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Compliance with ethical standards

Conflicts of interest The author declares that there is no conflict of interest regarding the publication of this paper.

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