

# The fastest, simplified method of Lyapunov exponents spectrum estimation for continuous-time dynamical systems

Marek Balcerzak · Danylo Pikunov ·  
Artur Dabrowski

Received: 18 May 2017 / Accepted: 27 August 2018 / Published online: 12 September 2018  
© The Author(s) 2018

**Abstract** Classical method of Lyapunov exponents spectrum estimation for a  $n$ -th-order continuous-time, smooth dynamical system involves Gram–Schmidt orthonormalization and calculations of perturbations lengths logarithms. In this paper, we have shown that using a new, simplified method, it is possible to estimate full spectrum of  $n$  Lyapunov exponents by integration of  $(n - 1)$  perturbations only. In particular, it is enough to integrate just one perturbation to obtain two largest Lyapunov exponents, which enables to search for hyperchaos. Moreover, in the presented algorithm, only very basic mathematical operations such as summation, multiplication or division are applied, which boost the efficiency of computations. All these features together make the new method faster than any other known by the authors if the order of the system under consideration is low. Correctness the method has been tested for three examples: Lorenz system, Duffing oscillator and three Duffing oscillators coupled in the ring scheme. Moreover, efficiency of the method has been confirmed by two practical tests. It has been revealed that for low-order systems, the presented method is faster than any other known by authors.

**Keywords** Lyapunov exponents · Spectrum · Continuous systems · Estimation method · Nonlinear dynamics

M. Balcerzak · D. Pikunov · A. Dabrowski (✉)  
Division of Dynamics, Lodz University of Technology,  
ul. Stefanowskiego 1/15, Lodz, Poland  
e-mail: Artur.Dabrowski@p.lodz.pl

## 1 Introduction

Depending on a dynamical system type and a kind of information that is useful for its investigations, different types of invariants characterizing system dynamics are applied. For instance, one may use Kolmogorov entropy [1,2], correlation dimension [3,4], Lyapunov energy function [5] to determine the stability of solutions and complexity of system dynamics [6]. However, in order to predict the influence of perturbations on solution of a system, Lyapunov exponents (LEs) are one of the most commonly applied tools. That is because these exponents determine the exponential convergence or divergence of trajectories that start close to each other. The existence of such numbers has been proved by Oseledec theorem [7], but the first numerical study of the system's behavior using Lyapunov exponents had been done by Henon and Heiles [8], before the Oseledec theorem publication. The most important algorithms for calculating Lyapunov exponents for continuous systems have been developed by Benettin et al. [9] and Shimada and Nagashima [10], later improved by Benettin et al. [11] and Wolf [12]. In order to estimate Lyapunov exponents from a scalar time series, the Takens procedure [13] can be applied. This approach can be utilized also in the cases when discontinuities or time delays exist in the analyzed system. Numerical algorithms for such estimation have been developed by Wolf et al. [14], Sano and Sawada [15] and later improved by Eckmann et al. [16], Rosenstein et al. [17] and Parlitz [18]. Alternative method appli-

cable to systems with discontinuities or time delays, based on synchronization phenomena, was elaborated by Stefanski [19–22].

Nowadays, Lyapunov exponents are employed in many different areas of scientific research such as: materials [23,24], electric power systems [25], non-continuous systems [20,22,26–29], systems with time delay [21], aerodynamics [30], time series analysis [31–34], optimal control [35,36], chaotic encryption and secure communication [37,38], multi-objective optimization [39], parametric oscillations and fluctuating parameters [19,40], neuronal models investigations [41].

Thus, there is still need to elaborate fast and simple methods of LE calculation. Recently, we have presented our simple and effective method of estimation of the largest Lyapunov exponent (LLE) from the perturbation vector and its derivative dot product. The method is based on a clear, geometrical reasoning. Its application involve only basic mathematical operations such as summing, multiplying, dividing. The method can be applied in different aspects of the nonlinear systems control. We investigated continuous systems [42], synchronization phenomena detection [43], time series in control systems [44–46].

The method presented previously was limited to calculation of the Largest Lyapunov exponent. In this paper, we have revealed that it is possible to apply it for estimation of the whole Lyapunov exponents spectrum too. Moreover, it has been shown that special features of the presented method enable to estimate the whole spectrum of  $n$  Lyapunov exponents by integration of  $(n - 1)$  perturbations only. The algorithm has been tested for three examples: Lorenz system, Duffing oscillator and three Duffing oscillators coupled in the ring scheme. Selection of the exemplary systems can be justified by the fact that they exhibit different types of dynamical behavior depending on the analyzed parameters. Therefore, correctness of the method has been confirmed for periodic, quasiperiodic and chaotic trajectories [47]. Computation times and convergence rates have been compared with the classical method [47] that involves Gram–Schmidt orthonormalization and calculations of perturbations lengths logarithms. The method presented in [47] uses the same approach as algorithms described in [9,11]. To authors' best knowledge, no algorithms of LE spectrum estimation for continuous-time, smooth dynamical systems faster than [9–11,47] have been published. In this paper, it

has been revealed that application of our new algorithm increases the efficiency of the calculations compared to the classical method [47] when the order of the system is low: 2, 4 or 6. Such acceleration has been achieved not only due to integration of  $(n - 1)$  perturbations instead of  $n$ , but also owing to the simplicity of the algorithm, which involves only the basic mathematical operations: addition, multiplication, division. Compared to the classical method, calculation of logarithms and normalization of perturbations after each iteration can be omitted. Therefore, authors claim that the method presented in this paper is the fastest one in the assumed range of applications if the order of the system is low enough.

## 2 The method

Assume that a dynamical system is described by the set of differential equations in the form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) \quad (1)$$

where  $\mathbf{x}$  is the state vector,  $t$  is the time and  $\mathbf{f}$  is a vector field that (in general) depends on  $\mathbf{x}$  and  $t$ . Evolution of any perturbation  $\mathbf{z}$  in such system can be found from the equation:

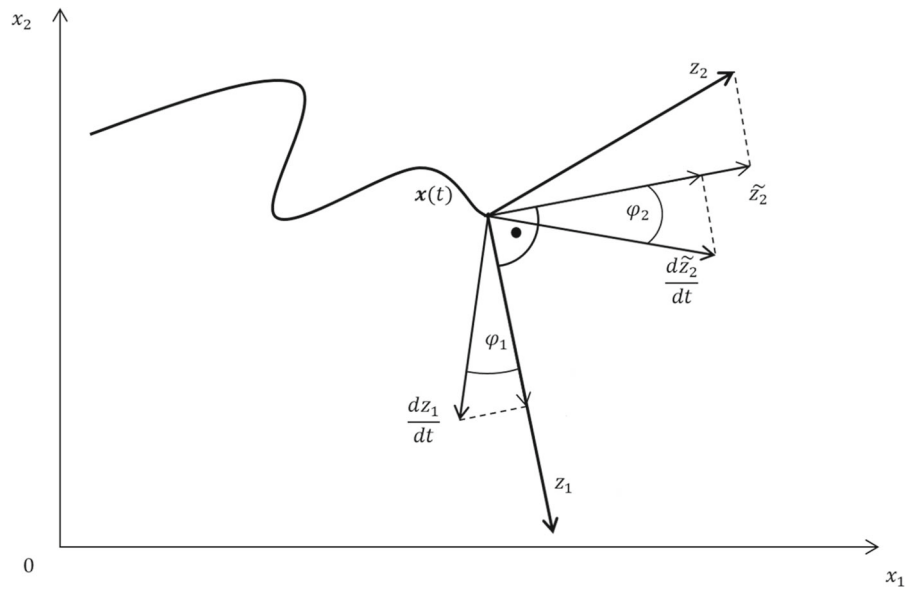
$$\frac{d\mathbf{z}}{dt} = \mathbf{U}(\mathbf{x}, t) \mathbf{z} \quad (2)$$

where  $\mathbf{U}(\mathbf{x}, t)$  is the Jacobi matrix obtained by differentiation of  $\mathbf{f}$  with respect to  $\mathbf{x}$ . As it has been shown in the previous paper [42], the value of LLE can be estimated from the following expression:

$$\bar{\lambda}_1^* = \frac{\frac{dz_1}{dt} \cdot \mathbf{z}_1}{|\mathbf{z}_1|^2} \quad (3)$$

where  $\mathbf{z}_1$  is a perturbation vector, whose evolution can be obtained by numerical integration of Eq. (2). The approximate value of LLE ( $\lambda_1$ ) is obtained by averaging values of  $\bar{\lambda}_1^*$  from subsequent computation steps. For time of integration long enough, the average value of  $\bar{\lambda}_1^*$  converges to the LLE. Please note that formula (3) involves only the most basic mathematical operations—addition, multiplication and division. On

**Fig. 1** Scheme of perturbation vectors and their derivatives



the other hand, the classical method involves calculation of perturbation length and its logarithm after each iteration.

According to [47], each LE  $\lambda_i$  is the average rate of perturbation contraction or expansion in a particular subspace near a particular limit set. The LLE— $\lambda_1$ —is the average rate of perturbation length change in (almost) whole space in which system (1) is defined. Let this whole space be named  $W_1$ .

For a system of order  $n$ , there are  $n$  nested subspaces:

$$W_1 \supset W_2 \supset \dots \supset W_n \tag{4}$$

such that the dimension of subspace  $W_i$  is  $n + 1 - i$ . In each subspace  $W_i$ , a perturbation evolves, on the average, according to  $\lambda_i$ . Therefore,  $i$ -th LE can be obtained from evolution of a perturbation in the subspace  $W_i$ .

The classical method of LE spectrum estimation involves Gram–Schmidt orthonormalization and calculations of perturbations lengths logarithms [47]. In this approach,  $n$  perturbations  $z_1, z_2, \dots, z_n$  are integrated. Orthonormalization procedure assures that each perturbation  $z_i$  evolves in the subspace  $W_i$ . Values of LE are obtained by calculating natural logarithm of perturbations lengths after subsequent integration steps and averaging these values over time. The crucial fact is that in order to calculate whole spectrum of  $n$  LE, integration of  $n$  perturbations is necessary.

In the presented method, complexity of calculations is reduced. In order to estimate the whole spectrum of

$n$  LE, it is enough to integrate  $(n - 1)$  perturbations. For clarity, in the next paragraph, the case of a second-order system is discussed. Authors demonstrate how integration of only one perturbation  $z_1$  enables to estimate both LE:  $\lambda_1, \lambda_2$ . The subsequent paragraph covers the general case, i.e., estimation of a full LE spectrum of a  $n$ th-order system using only  $(n - 1)$  integrated perturbations.

Let us assume that for a 2nd-order system, only one perturbation  $z_1$  is integrated. Obviously, using formula (3), LLE can be calculated. As  $z_1$  evolves according to (2), it tends to line up with the direction in which evolution of the perturbation is governed mainly by the value of  $\lambda_1$ , whereas the terms connected with  $\lambda_2$  become negligible [47]. Therefore, throughout its evolution,  $z_1$  tends to a direction orthogonal to  $W_2$ . Let us also take any random vector  $z_2$ , the length of which is nonzero and which is not parallel to  $z_1$  (Fig. 1).

Using the orthogonalization formula [47]:

$$\tilde{z}_2 = z_2 - z_1 * \frac{z_1 \cdot z_2}{|z_1|^2} \tag{5}$$

one obtains a vector  $\tilde{z}_2$  which is perpendicular to  $z_1$ . Therefore, to a good approximation,  $\tilde{z}_2 \in W_2$ . Notice that for a 2nd-order system,  $W_2$  is simply a line in  $W_1$  space. Thus,  $\tilde{z}_2$  must have the direction of a perturbation in  $W_2$  subspace, despite the fact that  $\tilde{z}_2$  has no connection with system (1) and the vector  $z_2$  was selected randomly. Moreover, it can be easily checked

that the length of a perturbation does not affect the value obtained from formula (2). Summing up, one can use formula (2) to find the 2nd LE. It is only needed to use the vector  $\tilde{z}_2$  which lies in  $W_2$  instead of  $z_1$ . The second LE can be found from the following expression:

$$\lambda_2^* = \frac{\frac{d\tilde{z}_2}{dt} \cdot \tilde{z}_2}{|\tilde{z}_2|^2} \tag{6}$$

The approximate value of  $\lambda_2$  is obtained by averaging values of  $\lambda_2^*$  from subsequent computation steps. For long enough time of integration, the average value of  $\lambda_2^*$  converges to the 2nd LE. Consequently, one does not need to integrate the 2nd perturbation; it is enough to find a correct direction of the vector  $\tilde{z}_2$ .

Let us now consider a system of order  $n$ . Suppose that  $(n - 1)$  perturbations  $z_1, z_2, \dots, z_{n-1}$  are integrated<sup>1</sup>. Their time derivatives can be found from formula (2). As  $z_1$  evolves, it tends to line up with the direction in which evolution of the perturbation is governed mainly by the value of  $\lambda_1$ , whereas the terms connected with subsequent LE  $\lambda_2, \lambda_3, \dots, \lambda_n$  become negligible [47]. Therefore, throughout its evolution,  $z_1$  tends to a direction orthogonal to  $W_2$ . The component  $\tilde{z}_2$  of the perturbation  $z_2$ , which is perpendicular to  $z_1$ , can be obtained by means of formula (5). To a good approximation  $\tilde{z}_2 \in W_2$ . Moreover,  $\tilde{z}_2$  tends to a direction which is orthogonal to  $W_3$ . Consequently, vectors  $\tilde{z}_3, \tilde{z}_4, \dots, \tilde{z}_{n-1}$  can be calculated using the following orthogonalization expression:

$$\begin{aligned} \tilde{z}_k &= z_k - z_1 * \frac{z_1 \cdot z_k}{|z_1|^2} \\ &\quad - \sum_{i=2}^{k-1} \tilde{z}_i * \frac{\tilde{z}_i \cdot z_k}{|\tilde{z}_i|^2}, \quad k = 3, 4, \dots, (n - 1) \end{aligned} \tag{7}$$

Formula (7) yields a vector  $\tilde{z}_k$  which has no components in directions of  $z_1, \tilde{z}_2, \dots, \tilde{z}_{k-1}$ . Each of the vectors  $\tilde{z}_k$ , estimated by means of (7), approximately fulfills the condition  $\tilde{z}_k \in W_k$  and tends to a direction orthogonal to  $W_{k+1}$ . Therefore, the LE  $\lambda_2, \lambda_3, \dots, \lambda_{n-1}$  can be estimated from the formula:

$$\lambda_k^* = \frac{\frac{d\tilde{z}_k}{dt} \cdot \tilde{z}_k}{|\tilde{z}_k|^2}, \quad k = 2, 3, \dots, (n - 1) \tag{8}$$

<sup>1</sup> In order to maintain orthogonality of the integrated perturbations, before each integration step the following substitution is performed:  $z_k := \tilde{z}_k, k = 2, 3, \dots, (n - 1)$ . The orthogonalized perturbations  $\tilde{z}_k$  are obtained from formula (7).

The approximate value of  $\lambda_k$  is obtained by averaging values of  $\lambda_k^*$  from subsequent computation steps. For long enough time of integration, the average value of  $\lambda_k^*$  converges to the  $k$ th LE. Now, let us take any random vector  $z_n$  such that its length is nonzero and it is linearly independent from  $z_1, \tilde{z}_2, \dots, \tilde{z}_{n-1}$ . Using formula (7), one can orthogonalize the random vector  $z_n$  with respect to  $z_1, \tilde{z}_2, \dots, \tilde{z}_{n-1}$ . In such a manner,  $\tilde{z}_n$  is obtained. The length of  $\tilde{z}_n$  is random, but owing to the orthogonalization process, it approximately fulfills the condition  $\tilde{z}_n \in W_n$ . Note that the order of the subspace  $W_n$  is equal to 1, so  $W_n$  is simply a line in the  $W_1$  space. Therefore,  $\tilde{z}_n$  must have the direction of a perturbation in the  $W_n$  space. Moreover, the value obtained from formula (8) is independent from the length of  $\tilde{z}_k$ . Concluding, one can apply formula (7) to orthogonalize a randomly selected random vector  $z_n$  with respect to the integrated perturbations  $z_1, \tilde{z}_2, \dots, \tilde{z}_{n-1}$  and then use the result  $\tilde{z}_n$  to estimate the last LE by means of formula (8). In such a manner, it has been shown that it is enough to integrate  $(n - 1)$  perturbations to estimate the full spectrum of LE of a  $n$ th-order system.

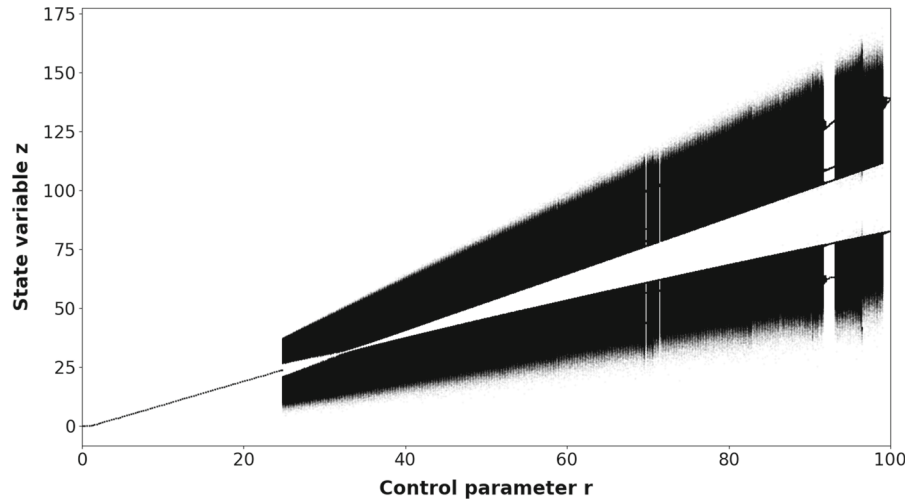
### 3 Numerical simulations

Simulation programs were written in C++ using Code-Blocks 16.01. Graphs and diagrams were built with *matplotlib* library as a tool of Python programming language. In order to integrate the system of differential Eqs. (1)–(2), Runge-Kutta method of the fourth order was used. The integration step was chosen experimentally for each system.

In order to estimate the whole spectrum of Lyapunov exponents, the method requires to simulate  $(n - 1)$  perturbations, where  $n$  is the order of the system. For example, for the second-order non-autonomous Duffing oscillator with external excitation [42], one perturbation was calculated. For the third-order autonomous system of Lorenz equations [48], two disturbances were observed. For three Duffing systems coupled in a ring scheme [50], which together constitute a system of the 6th order, five perturbations were integrated. After each integration step, subsequent perturbations were normalized and orthogonalized according to Eqs. (5), (7).

After every integration step, values  $\lambda_i^*$  are calculated for each perturbation. For the value of  $\lambda_1^*$ , formula (3) is used; for  $\lambda_2^*, \lambda_3^*, \dots, \lambda_n^*$ , formula (8).

**Fig. 2** Bifurcation diagram of the Lorenz system (9)



The simulation program remembered subsequent values of  $\lambda_i^*$  in  $n$  buffers of a fixed capacity. When buffers were full, the program calculated standard deviation of all the values in each of  $n$  buffers. If the standard deviation of values in any buffer was too high, buffers were cleared and computations were continued. Otherwise, if the value of standard deviation for each buffer was below a fixed threshold, the calculations were terminated. The final value of  $\lambda_i$  returned by the program was equal to the average of all values  $\lambda_i^*$  remembered in the  $i$ -th buffer.

**4 Results of numerical simulations**

In order to verify our method of the Lyapunov exponents spectrum estimation, three systems have been analyzed: Lorenz equations, Duffing oscillator with external periodic driving forcing and three Duffing systems coupled in a ring scheme. Firstly, the Lorenz system can be described with the following mathematical model [49]:

$$\begin{cases} \dot{x} = \sigma (y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = xy - \beta z; \end{cases} \tag{9}$$

Jacobi matrix was used to simulate evolution of perturbations according to Eq. (2). For the Lorenz system (9), the Jacobi matrix is described by formula (10).

$$U(x) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -\beta \end{pmatrix} \tag{10}$$

System (9) is nonlinear and autonomous. The spectrum of Lyapunov exponents of the system was computed with two parameters fixed,  $\sigma = 10, \beta = 8/3$ , while  $r$  was used as the control parameter in the range from 0 to 100. Initial conditions for  $r = 0$  were set as a random three-element vector of magnitude 1. Initial perturbations were also chosen randomly and normalized to 1.

The bifurcation diagram of the state variable  $z$  in system (9) is shown in Fig. 2. The corresponding graph of LE spectrum calculated using the new method is depicted in Fig. 3, whereas the LE spectrum graph obtained by means of the classical method is shown in Fig. 4.

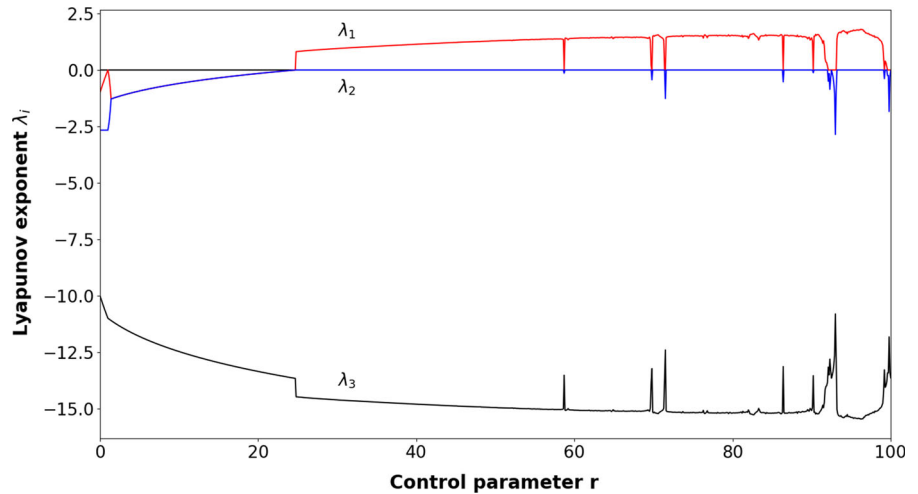
The bifurcation diagram presented in Fig. 2, together with the LE diagram presented in Figs. 3, 4, demonstrates that the Lorenz system (9) is periodic for all the values of  $r$  smaller or equal to 24.7, whereas it is chaotic for most values of  $r$  larger or equal to 24.8. However, for some values of  $r$  above 24.8, periodic windows exist. Detailed information on different kinds of limit sets in dynamical systems can be found in [47]. It can be noticed that the graphs in Fig. 3 (the new method) and in Fig. 4 (the classical method) are in complete agreement. Moreover, both graphs are consistent with [48], which confirms the correctness of the algorithm presented in this paper.

The Duffing oscillator can be described by the following mathematical model [42]:

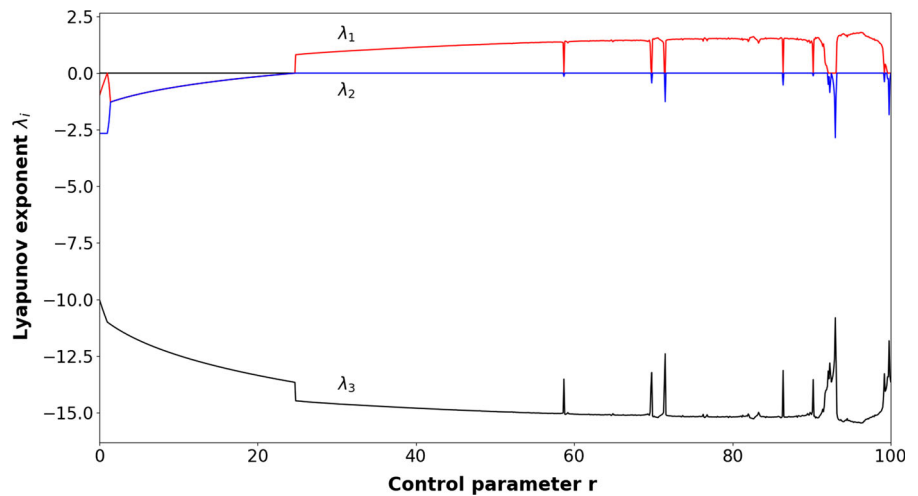
$$\ddot{x} + \beta\dot{x} + \alpha x^3 = q \cos(\eta t) \tag{11}$$

The linear stiffness in Eq. (11) was skipped as in [42], which enables to compare the results. Jacobi matrix was

**Fig. 3** LE spectrum of the Lorenz system (9), calculated using the new method



**Fig. 4** LE spectrum of the Lorenz system (9), calculated using the classical method



used to simulate evolution of perturbations according to Eq. (2). For the Duffing system (11), the Jacobi matrix is described by formula (12).

$$U(x) = \begin{pmatrix} 0 & 1 \\ -3\alpha x^2 & -\beta \end{pmatrix} \tag{12}$$

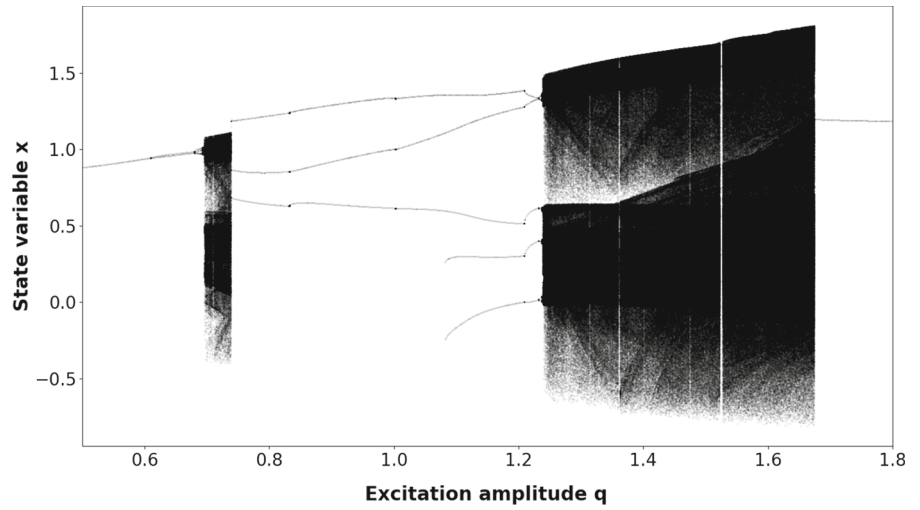
The spectrum of Lyapunov exponents for the Duffing oscillator was computed with such fixed parameters as  $\beta = 0.05, \alpha = 1, \eta = 0.5$ . Parameter  $q$  was used as the control parameter in the range from 0.5 to 1.8 [42]. In the simulation, a random vector of length 1 was used as initial condition. Initial perturbations were also random and normalized to 1.

The bifurcation diagram of the state variable  $x$  of the Duffing Eq. (11) is shown in Fig. 5. The corresponding graph of LE spectrum calculated using the new method

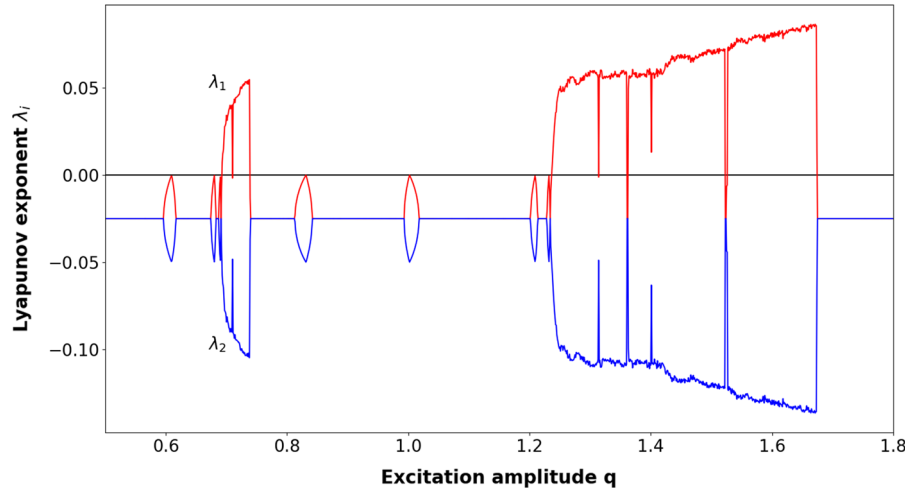
is depicted in Fig. 6. The LE spectrum graph obtained by the classical method is presented in Fig. 7. In both cases, red line corresponds to the largest LE, while blue line corresponds to the second Lyapunov exponent of system (11).

Figures 5, 6 and 7 show that the Duffing system is chaotic for most values of the parameter  $q$  in ranges from 0.693 to 0.738 and from 1.237 to 1.673. However, some periodic windows exist in these ranges too. For the values of  $r$  between 0.5 and 0.692, from 0.739 to 1.236 and from 1.674 to 1.8, the system is periodic. For the Duffing system, the generalized divergence of the flow is constant and equal to  $-\beta$  [49]. Sum of Lyapunov exponents has to be equal to the generalized divergence of the flow. Therefore, graphs of the first and second Lyapunov exponents should be symmetri-

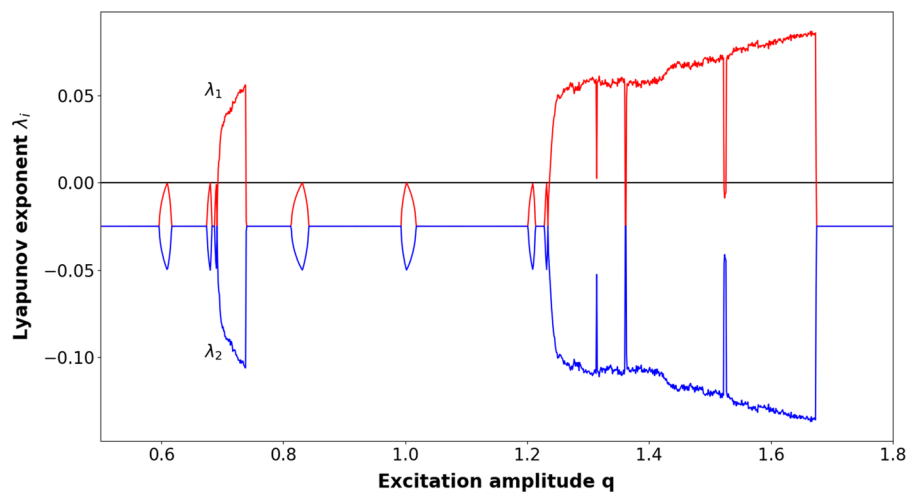
**Fig. 5** Bifurcation diagram of the Duffing Eq. (11)



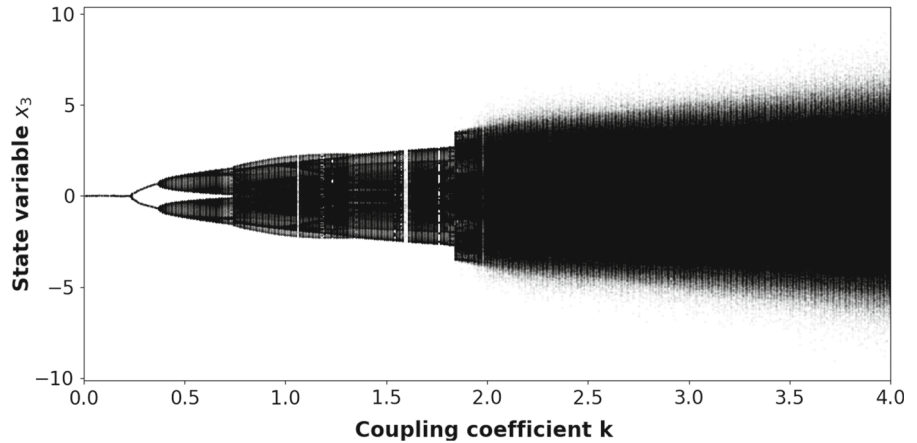
**Fig. 6** LE spectrum of the Duffing Eq. (11), calculated using the new method



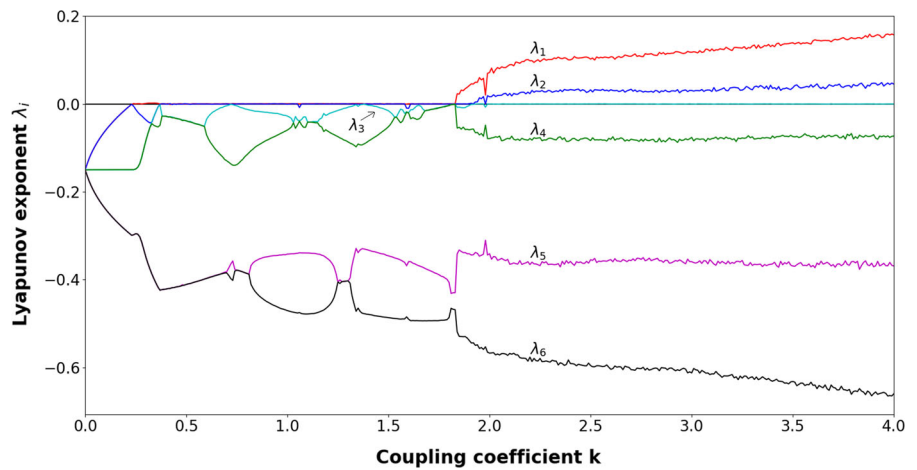
**Fig. 7** LE spectrum of the Duffing Eq. (11), calculated with the classical method



**Fig. 8** Bifurcation diagram of three Duffing systems in the ring scheme



**Fig. 9** LE diagram of three coupled Duffing systems (13) obtained with the new method



cal with respect to horizontal line  $\lambda = -\frac{\beta}{2}$ . This fact was confirmed in Fig. 6 and in Fig. 7.

Next, we consider a ring of three Duffing systems [50]. The resulting system is given by:

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = -dy_i - ax_i - x_i^3 + k(x_{p(i)} - x_i) \end{cases} \quad (13)$$

where  $i = 1, 2, 3$  and  $p(i)$  is the index of the previous oscillator in the ring, i.e.,  $p(3) = 2, p(2) = 1, p(1) = 3$ . Jacobi matrix of system (13) is described by formula (14):

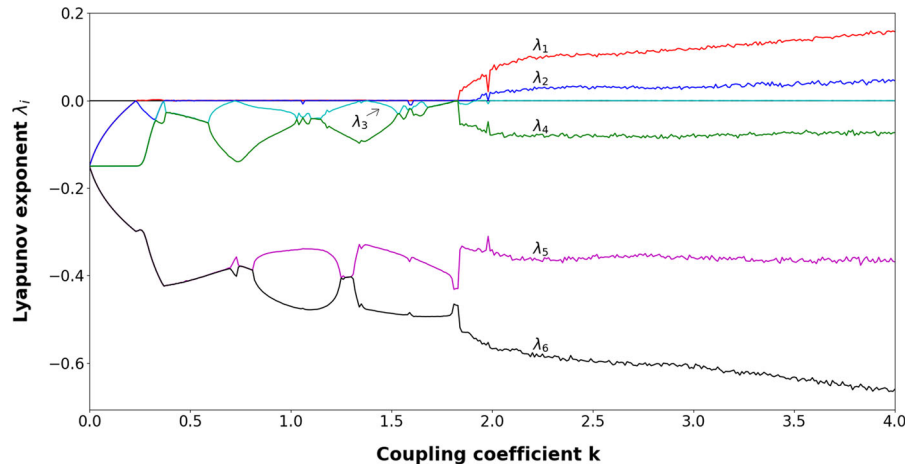
$$U(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ u_1 & -d & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k & 0 & u_2 & -d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & k & 0 & u_3 & -d \end{pmatrix} \quad (14)$$

where  $u_i = -a - 3x_i^2 - k$ . System (13) is nonlinear and autonomous. The spectrum of Lyapunov exponents of the system was computed with two parameters fixed,  $d = 0.3, a = 0.1$ , while the coupling coefficient  $k$  was used as the control parameter in the range from 0 to 4. Initial conditions for  $k = 0$  were set as a random six-element vector of magnitude 1. Initial perturbations were also chosen randomly and normalized to 1. Please note that names of parameters are different in formulas (11) and (13). This is due to the fact that results of both simulations can be verified in two different sources. Results of calculations for system (11) can be checked in [42], whereas system (13) can be verified in [50].

The bifurcation diagram of the state variable  $x_3$  in system (13) is depicted in Fig. 8. The corresponding graph of LE spectrum calculated using the new method is shown in Fig. 9. The LE spectrum graph obtained by the classical method is presented in Fig. 10.



**Fig. 10** LE diagram of three coupled Duffing systems (13) obtained with the classical method



Figures 8, 9 and 10 demonstrate how dynamics of three Duffing systems (13) changes as the coupling strength increases [50]. For the values of  $k$  smaller than 0.24, one can notice that all the LEs are negative and the system tends to a stable fixed point. For the values of  $k$  between 0.24 and 0.38, one LE is equal to 0, which corresponds to a periodic solution. In the range of parameter  $k$  from 0.38 to 1.79, more than one LEs are zero and a quasiperiodic solution is obtained. Transition to chaos takes place for  $k$  equal to 1.79, followed by transition to hyperchaos as coupling coefficient reaches 1.9. Please note that the full spectrum of six LEs was obtained by integration of five perturbation vectors. However, if the task is to just detect chaos and hyperchaos, only one perturbation is necessary when our new method is applied, whereas integration of two perturbations is required if the classical method is used.

## 5 Computation time comparisons

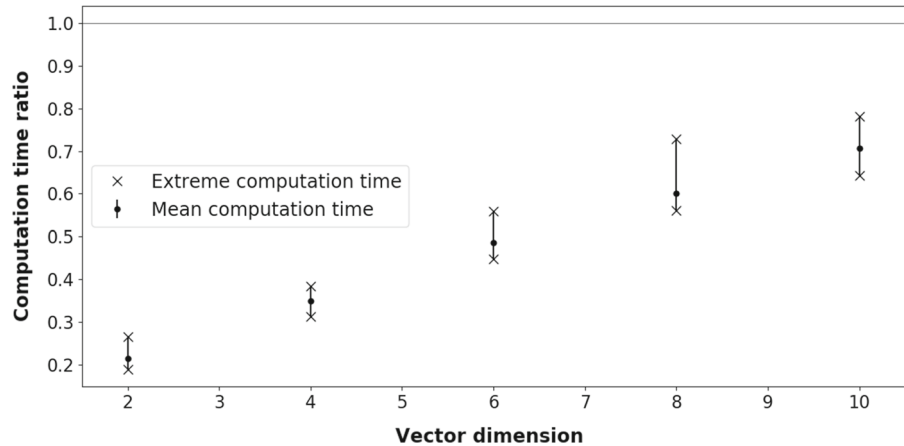
In the previous sections, the novel method has been explained and results of its application have been demonstrated. It has been confirmed that the novel method requires integration of only  $(n - 1)$  perturbations to estimate  $n$  Lyapunov exponents, which is expected to increase efficiency of calculations. Moreover, formulae (3) and (8), which are used for LE estimation in the new method, require only the most basic mathematical calculations—addition, multiplication and division. In such a manner, calculation of perturbation's norm logarithm, which is necessary in the classical method, can be avoided. Therefore, there are

two possible reasons for efficiency of the novel method. Firstly, size of the problem is reduced, because only  $(n - 1)$  perturbations are integrated. However, computation speed gain due to this feature may vary with the order of the system. Secondly, application of formulae (3) and (8) instead of estimation of the logarithm of the perturbation norm can additionally shorten the computation time.

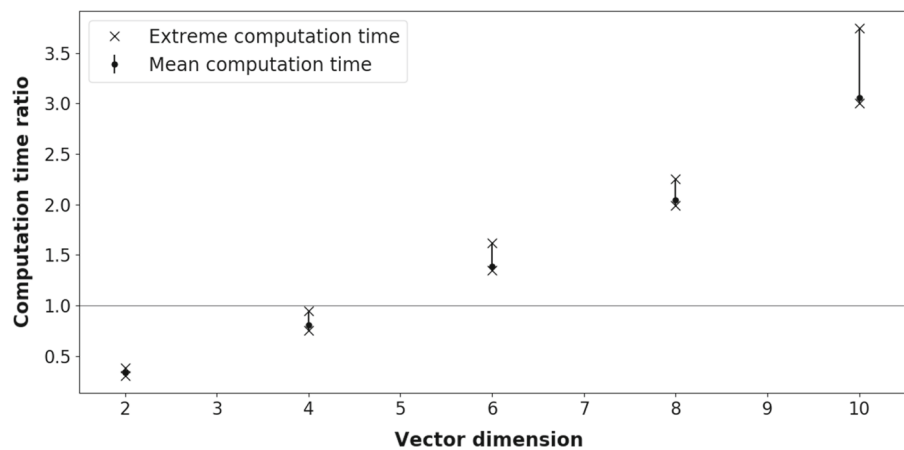
Firstly, influence of application of formulae (3) and (8) is analyzed without taking into account the number of perturbations to be integrated. When the classical method of LE estimation is applied, each computation step requires to calculate the norm of a perturbation. The norm can be treated as the square root of the dot product of the perturbation vector with itself. Then, the natural logarithm of the norm is calculated. On the other hand, formulae (3) and (8) of the new method require not only the perturbation vector, but also its time derivative. When both vectors are known, then according to (3) and (8), the dot product of the perturbation and its time derivative is divided by the dot product of the perturbation vector with itself.

Efficiency of both approaches has been tested by a C++ program. The program selects  $2 * 10^7$  vectors at random. Then, it checks how much time it takes to estimate natural logarithm of the norm of each of these vectors, just as it is done in the classical method of LE estimation [47]. Afterward, formula (3) is applied for each vector from the same set and time of these operations is also measured. In such a manner, efficiency of both methods of LE estimation can be compared. This test has been executed 100 times for different vector dimensions. The average, the minimum and the max-

**Fig. 11** Computation time of formula (3) divided by the average computation time of the classical method for the best case, when only one element of  $U(x)$  in each row is nonzero



**Fig. 12** Computation time of formula (3) divided by the average computation time of the classical method for the worst case, when all elements of  $U(x)$  are nonzero



imum computation time has been recorded for both methods.

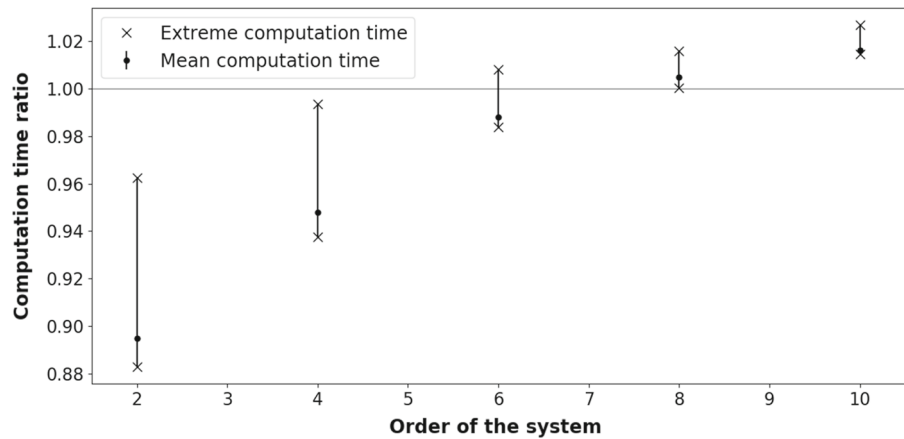
However, an important issue arises: Efficiency of the novel method of LE estimation can be largely influenced by estimation of the perturbation's derivative, which is required by the new method. If the Jacobi matrix  $U(x)$  contains only few nonzero elements, then calculation of the perturbation's derivative according to formula (2) is fast. When most elements of  $U(x)$  are nonzero, then calculation of the perturbation's derivative is slower. Therefore, two separate cases have been analyzed. In the best scenario, only one element in each row of  $U(x)$  is nonzero. In the worst case, all elements of  $U(x)$  are nonzero. Results for both cases with different vector dimensions are presented in Figs. 11 and 12. These figures represent the best case and the worst case, respectively. In each of these graphs, the computation time ratio is obtained by dividing computation time of formula (3) by the average time of calculations with the classical method, i.e., when the natural logarithm

of the vector's norm is estimated. Therefore, value 1 in the graph means that the computation time for both methods is equal. Values below 1 show that the new method is faster, and values above 1 indicate that the novel method is slower.

Figures 11 and 12 show that when order of the system is less or equal to 4, then evaluation of formula (3) or (8) is always faster than calculation of the natural logarithm of the perturbation's norm. However, for higher order systems, evaluation of the perturbation's norm logarithm can be faster if the Jacobi matrix  $U(x)$  is dense enough. On the other hand, if  $U(x)$  is sparse, then evaluation of formula (3) or (8) is faster than the classical formula for all vector dimensions under consideration.

Now, as the influence of using the simplified formulae (3) and (8) has been investigated, overall efficiency of the novel method can be checked on a practical example. To do so, the spectrum of Lyapunov exponents of Duffing systems coupled in ring scheme (13) has

**Fig. 13** LE spectrum estimation time for system (13) using the new method divided by the average computation time of the classical method



been estimated with both methods for different number of oscillators in the ring, from 1 to 5. Therefore, systems of orders 2, 4,  $\dots$ , 10 have been investigated. The same coupling strength  $k = 3.0$  has been used in all tests. For each number of oscillators, the Lyapunov exponents spectrum has been estimated 100 times using both methods. The orthonormalization process [47] has been performed after each iteration of both methods. The average, minimum and maximum computations times have been recorded. The results are presented in Fig. 13. The computation time ratio is obtained by dividing the calculations time of the novel method by the average time of calculations of the classical method. Therefore, just as in previous graphs, values below 1 indicate that the novel method is faster, whereas values above 1 show that the classical method performs better.

Figure 13 confirms that the novel method, presented in this paper, is faster than the classical method [47] in each test case, as long as order of system (13) is smaller or equal to 4. For the system with three oscillators and order 6, the novel method is faster in the average, but it is not faster in each trial. When more than three oscillators are coupled together, the classical method performs better.

## 6 Conclusions

The presented article shows that from dot products of perturbations vectors and their derivatives, one can easily extract the full spectrum of Lyapunov exponents. We have shown that full spectrum of LE for a system of order  $n$  can be obtained by integrating only  $(n - 1)$  perturbations. Moreover, due to the fact that

our method is based on very unsophisticated computations involving only basic mathematical operations such as summing, subtracting, multiplying, dividing, it is more efficient than the classical one when the order of the system is low enough. We have presented the theoretical description showing the mathematical simplicity of the method. The correctness and effectiveness of our method has been confirmed by simulations of autonomous and non-autonomous nonlinear dynamical systems of different orders. These results have been compared with the ones obtained using the classical algorithm. The efficiency of the new method has been compared to the method that involves the calculation of perturbations lengths logarithms and Gram–Schmidt orthonormalization. Firstly, it has been shown that application of the new, simple formulas for LE estimation is faster than calculation of the natural logarithm of perturbation’s norm for systems of dimension smaller or equal to 4, even if the Jacobi matrix of the system is dense. It has been revealed that if the Jacobi matrix is sparse, then the novel formulas can perform better even for systems of the dimension 10 or higher. Secondly, efficiency of the novel method has been verified for Duffing systems coupled in the ring scheme. It turned out that the novel method is in average faster if the dimension of the system is smaller or equal to 6. Obviously, efficiency of both methods may vary depending on properties of the system under consideration. Moreover, implementation of the method may influence the effectiveness. However, taking into account the obtained results, it can be stated that for low-dimensional, continuous-time, smooth systems, the presented method of LE spectrum estimation is the fastest one. The next step of the method’s develop-

ment can be considered in estimation of LE in systems with discontinuities, with time delays and others.

**Acknowledgements** This study has been supported by Polish Ministry of Science and Higher Education under the program “Diamond Grant,” Project No. D/2013 019743. This study has been supported by Polish National Centre of Science (NCN) under the program MAESTRO: “Multi-scale modelling of hysteretic and synchronous effects in dry friction process,” Project No. 2012/06/A/ST8/00356.

**Open Access** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

## References

- Bennettin, G., Froeschle, C., Scheidecker, J.P.: Kolmogorov entropy of a dynamical system with increasing number of degrees of freedom. *Phys. Rev. A* **19**, 2454–2460 (1979)
- Bennettin, G., Galgani, L., Strelcyn, J.M.: Kolmogorov entropy and numerical experiment. *Phys. Rev. A* **14**, 2338–2345 (1976)
- Grassberger, P., Procaccia, I.: Characterization of strange attractors. *Phys. Rev. Lett.* **50**, 346–349 (1983)
- Alligood, K.T., Sauer, T.D., Yorke, J.A.: *Chaos an Introduction to Dynamical Systems*. Springer, New York (2000)
- Zhang, Y., Chen, D., Guo, D., Liao, B., Wang, Y.: On exponential convergence of nonlinear gradient dynamics system with application to square root finding. *Nonlinear Dyn.* **79**, 983–1003 (2015)
- Eckmann, J.-P., Ruelle, D.: Ergodic theory of chaos and strange attractors. *Rev. Mod. Phys.* **57**, 617–656 (1985)
- Oseledec, V.I.: A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Mosc. Math. Soc.* **19**, 197–231 (1968)
- Henon, M., Heiles, C.: The applicability of the third integral of the motion: some numerical results. *Astron. J.* **69**, 77 (1964)
- Bennettin, G., Galgani, L., Giorgilli, A., Strelcyn, J.M.: Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, part I: theory. *Meccanica* **15**, 9–20 (1980)
- Shimada, I., Nagashima, T.: A numerical approach to ergodic problem of dissipative dynamical systems. *Prog. Theor. Phys.* **61**(6), 1605–1616 (1979)
- Bennettin, G., Galgani, L., Giorgilli, A., Strelcyn, J.M.: Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, part II: numerical application. *Meccanica* **15**, 21–30 (1980)
- Wolf, A.: Quantifying chaos with Lyapunov exponents. In: Holden, V. (ed.) *Chaos*, pp. 273–290. Manchester University Press, Manchester (1986)
- Takens, F.: Detecting strange attractors in turbulence. *Lect. Notes Math.* **898**, 366 (1981)
- Wolf, A., Swift, J.B., Swinney, H.L., Vastano, J.A.: Determining Lyapunov exponents from a time series. *Phys. D* **16**, 285–317 (1985)
- Sano, M., Sawada, Y.: Measurement of the Lyapunov spectrum from a chaotic time series. *Phys. Rev. Lett.* **55**, 1082–1085 (1985)
- Eckmann, J.P., Kamphorst, S.O., Ruelle, D., Ciliberto, S.: Lyapunov exponents from a time series. *Phys. Rev. Lett.* **34**(9), 4971–4979 (1986)
- Rosenstein, M.T., Collins, J.J., De Luca, C.J.: A practical method for calculating largest Lyapunov exponents from small data sets. *Phys. D* **65**(1–2), 117–134 (1993)
- Parlitz, U.: Identification of true and spurious Lyapunov exponents from time series. *Int. J. Bifurc. Chaos* **2**(1), 155–165 (1992)
- Stefanski, A.: Lyapunov exponents of the systems with noise and fluctuating parameters. *J. Theor. Appl. Mech.* **46**(3), 665–678 (2008)
- Stefanski, A.: Estimation of the largest Lyapunov exponent in systems with impacts. *Chaos Solitons Fractals* **11**(15), 2443–2451 (2000)
- Stefanski, A., Dabrowski, A., Kapitaniak, T.: Evaluation of the largest Lyapunov exponent in dynamical systems with time delay. *Chaos Solitons Fractals* **23**, 1651–1659 (2005)
- Stefanski, A., Kapitaniak, T.: Estimation of the dominant Lyapunov exponent of non-smooth systems on the basis of maps synchronization. *Chaos Solitons Fractals* **15**, 233–244 (2003)
- Iwaniec, J., Uhl, T., Staszewski, W.: Detection of changes in cracked aluminium plate determinism by recurrence analysis. *Nonlinear Dyn.* **70**(1), 125–140 (2012)
- Rybczak, M., Aniszewska, D.: Lyapunov type stability and Lyapunov exponent for exemplary multiplicative dynamical systems. *Nonlinear Dyn.* **54**, 345–354 (2008)
- Wadduwage, D.P., Qiong, Wu C., Annakkage, U.D.: Power system transient stability analysis via the concept of Lyapunov Exponents. *Electric Power Syst. Res.* **104**, 183–192 (2013)
- Serweta, W., Okolewski, A., Blazejczyk-Okolewska, B., Czolczynski, K., Kapitaniak, T.: Mirror hysteresis and Lyapunov exponents of impact oscillator with symmetrical soft stops. *Int. J. Mech. Sci.* **101–102**, 89–98 (2015)
- Serweta, W., Okolewski, A., Blazejczyk-Okolewska, B.: Lyapunov exponents of impact oscillators with Hertz’s and Newton’s contact models. *Int. J. Mech. Sci.* **89**, 194–206 (2014)
- Lamarque, C.H., Malasoma, J.M.: Analysis of nonlinear oscillations by wavelet transform: Lyapunov exponents. *Nonlinear Dyn.* **9**, 333–347 (1996)
- Yue, Y., Xie, J., Gao, X.: Determining Lyapunov spectrum and Lyapunov dimension based on the Poincare map in a vibro-impact system. *Nonlinear Dyn.* **69**, 743–753 (2012)
- Hu, D.L., Huang, Y., Liu, X.B.: Moment Lyapunov exponent and stochastic stability of binary airfoil driven by non-Gaussian colored noise. *Nonlinear Dyn.* **70**, 1847–1859 (2012)
- Yang, C., Wu, C.Q., Zhang, P.: Estimation of Lyapunov exponents from a time series for n-dimensional state space using nonlinear mapping. *Nonlinear Dyn.* **69**, 1493–1507 (2012)

32. Yang, C., Wu, C.Q.: On stability analysis via Lyapunov exponents calculated from a time series using nonlinear mapping—a case study. *Nonlinear Dyn.* **59**, 239–257 (2010)
33. Sun, Y., Wu, C.Q.: A radial-basis-function network-based method of estimating Lyapunov exponents from a scalar time series for analyzing nonlinear systems stability. *Nonlinear Dyn.* **70**, 1689–1708 (2012)
34. Yang, C., Wu, C.Q.: A robust method on estimation of Lyapunov exponents from a noisy time series. *Nonlinear Dyn.* **64**, 279–292 (2011)
35. Zhu, W.Q.: Feedback stabilization of quasi nonintegrable Hamiltonian systems by using Lyapunov exponent. *Nonlinear Dyn.* **36**(2), 455–470 (2004)
36. Zhu, W.Q., Huang, Z.L.: Stochastic stabilization of quasi-partially integrable Hamiltonian systems by using Lyapunov exponent. *Nonlinear Dyn.* **33**(2), 209–224 (2003)
37. Li, C., Wang, J., Hu, W.: Absolute term introduced to rebuild the chaotic attractor with constant Lyapunov exponent spectrum. *Nonlinear Dyn.* **68**(4), 575–587 (2012)
38. Li, S.Y., Huang, S.C., Yang, C.H.: Generating tri-chaos attractors with three positive Lyapunov exponents in new four order system via linear coupling. *Nonlinear Dyn.* **69**(3), 805–816 (2012)
39. Fraga, L.G., Tlelo-Cuautle, E.: Optimizing the maximum Lyapunov exponent and phase space portraits in multi-scroll chaotic oscillators. *Nonlinear Dyn.* **76**, 1503–1515 (2014)
40. Rong, H., Meng, G., Wang, X.: Invariant measures and Lyapunov exponents for stochastic Mathieu system. *Nonlinear Dyn.* **30**(4), 313–321 (2002)
41. Soriano, D.C., Fazanaro, F.I., Suyama, R.: A method for Lyapunov spectrum estimation using cloned dynamics and its application to the discontinuously-excited FitzHugh–Nagumo model. *Nonlinear Dyn.* **67**(1), 413–424 (2012)
42. Dabrowski, A.: Estimation of the largest Lyapunov exponent from the perturbation vector and its derivative dot product. *Nonlinear Dyn.* **67**(1), 283–291 (2012)
43. Dabrowski, A.: The largest transversal Lyapunov exponent and master stability function from the perturbation vector and its derivative dot product (TLEVDP). *Nonlinear Dyn.* **69**(3), 1225–1235 (2012)
44. Balcerzak, M., Dabrowski, A., Kapitaniak, T., Jach, A.: Optimization of the control system parameters with use of the new simple method of the largest Lyapunov exponent estimation. *Mech. Mech. Eng.* **17**(3), 225–239 (2013)
45. Pijanowski, K., Dabrowski, A., Balcerzak, M.: New method of multidimensional control simplification and control system optimization. *Mech. Mech. Eng.* **19**(2), 127–139 (2015)
46. Dabrowski, A.: Estimation of the largest Lyapunov exponent-like (LLEL) stability measure parameter from the perturbation vector and its derivative dot product (part 2) experiment simulation. *Nonlinear Dyn.* **78**(3), 1601–1608 (2014)
47. Parker, T.S., Chua, L.O.: *Practical Numerical Algorithms for Chaotic Systems*. Springer, Berlin (1989)
48. Moghtadaei, M., Hashemi Golpayenagi, M.R.: Complex dynamic behaviors of the complex Lorenz system. *Sci. Iran.* **19**(3), 733–738 (2012)
49. Shin, K., Hammond, J.K.: The instantaneous Lyapunov exponent and its application to chaotic dynamical systems. *J. Sound Vib.* **218**(3), 389–403 (1998)
50. Perlikowski, P., Yanchuk, S., Wolfrum, M., Stefanski, A., Mosiolek, P., Kapitaniak, T.: Routes to complex dynamics in a ring of unidirectionally coupled systems. *Chaos Interdiscip. J. Nonlinear Sci.* **20**(1), 013111 (2010)