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Two new integrable fourth-order nonlinear equations: multiple soliton solutions and multiple complex soliton solutions

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Abstract In this paper, we develop two new fourth-order integrable equations represented by nonlinear PDEs of second-order derivative in time t. The new equations model both right- and left-going waves in a like manner to the Boussinesq equation. We will employ the Painlevé analysis to formally show the complete integrability of each equation. The simplified Hirota's method is used to derive multiple soliton solutions for this equation. We introduce a complex form of the simplified Hirota's method to develop multiple complex soliton solutions. More exact traveling wave solutions for each equation will be derived as well.

Keywords Fourth-order integrable equation · Painlevé test · Multiple soliton solutions · Multiple complex soliton solutions

1 Introduction

A major thrust of theoretical and experimental studies has been witnessed in the past few decades on nonlinear evolution equations and their applications in diverse areas, such as nonlinear dynamics, optical fibers, matter waves, plasma physics, electromagnetic waves. The nonlinear evolution equations describe a plethora of physical effects in fluid dynamics, condensed matter physics, optics, photonics, and nonlinear fiber optics. The work on these nonlinear equations has been flourishing in recent years to get an insight through qualitative and quantitative features of these nonlinear equations. One significant feature of most nonlinear equation is the perfect balance between nonlinearity and dispersion effects which results in soliton pulse.

The study of integrable properties for nonlinear evolution equations has formed a hot spot of research due to its rich physical structures and scientific features. During the past decades, much attention has been focused on the integrability of the nonlinear equations and particularly on deriving nonlinear integrable equations that describe various physical phenomena. Finding new integrable systems has been a major concern of research work on nonlinear phenomena and solitary waves theory.

Therefore, finding nonlinear integrable partial differential equations that incorporate higher-order terms is of great importance from both theoretical and experimental points of view. Researchers in [1–23] have invested a lot of work for deriving nonlinear integrable equations in $(n+1)$ dimensions, where $n = 1, 2, 3$. The study of integrable equations, which possess sufficiently large number of conservation laws, Lax pairs, bi-Hamiltonian and give rise to multiple soliton solutions, [8–15] plays a major role in scientific and engineering fields. Lot of research works has been invested to derive new integrable equations, where powerful works, such as the symmetries method, recursion oper-

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ators, have been employed to achieve this goal. The recursion operator has been used recently to establish more higher-dimensional integrable models [1–11]. The Painlevé analysis is applied to confirm the complete integrability of the newly developed equations.

Many reliable methods are used in the literature to investigate completely integrable equations that admit multiple soliton solutions [1–6]. The algebraic-geometric method [4–8], the inverse scattering method, the Bäcklund transformation method, the Darboux transformation method, the Hirota bilinear method [7–14], and other methods are used to make progress and new developments in this filed. The Hirota's bilinear method is rather heuristic and possesses significant features that make it practical for the determination of multiple soliton solutions and for multiple singular soliton solutions [7–20] for a wide class of nonlinear evolution equations in a direct method.

In [1], we introduced a new fifth-order nonlinear integrable equation of the form

$$u_{ttt} - u_{txxxx} - (u_x u_t)_{xx} - 4(u_x u_{xt})_x = 0. \tag{1}$$

In another work [2], we developed another generalized integrable form of Eq. (1) that reads

$$u_{ttt} - u_{txxxx} - \alpha(u_x u_t)_{xx} - \beta(u_x u_{xt})_x = 0. \tag{2}$$

where α and β are nonzero parameters.

Our aim in the present work is to explore two new nonlinear integrable equations by using the sense of Boussinesq equation and the idea presented in (1), (2). Following up on our earlier work in [1,2], we develop new integrable equations which take the forms

$$u_{tt} + u_{txxx} + \alpha(u_x u_t)_x = 0, \tag{3}$$

and

$$u_{tt} + u_{txxx} + \alpha(u_x u_t)_x + \beta u_{xx} = 0, \tag{4}$$

that will be named, for further use, first fourth-order integrable equation and second fourth-order integrable equation, respectively. The parameters α and β are nonzero real numbers. It is obvious that Eq. (3) can be derived from (4) by setting $\beta = 0$. However, the two equations will be examined independently due to the distinct features of these equations.

In this work, we plan to follow a twofold analysis. We first aim to show that these two equations pass the Painlevé test; hence, these are integrable equations with all merits and properties of integrable equations such as multiple soliton solutions. The simplified Hirota's method will be employed to develop multiple soliton solutions for each model. We second plan to introduce, to our knowledge for the first time, a new modified complex version of the simplified Hirota's method to show that Eqs. (3) and (4) give also multiple complex soliton solutions. We will finally use hyperbolic and trigonometric ansatze to derive more traveling wave solutions for each of the derived equations.

2 First fourth-order integrable equation

In this section, we will examine the first fourth-order integrable equation (3). The analysis will be conducted in a systematic manner where we will first use the Painlevé analysis to confirm the integrability of this equation. We then will employ the simplified Hirota's method to derive multiple soliton solutions. We finally will introduce a new complex simplified Hirota's method, which to our knowledge will be introduced for the first time, to develop multiple complex soliton solutions.

2.1 Painlevé analysis for the first equation

In this section, we will employ the Painlevė analysis [1–6] to formally confirm the integrability of the first developed equation (3). The Painlevė analysis is a powerful method, used heavily in the literature in addition to other techniques for testing the integrability of nonlinear partial differential equations. More concretely, we seek solutions of Eq. (3) in the form of the assumption that Eq. (3) has solution as a Laurent expansion about a singular manifold $\phi = \phi(x, t)$ given as

$$u(x, t) = \sum_{k=0}^{\infty} u_k(x, t)\phi^{k-\gamma}, \tag{5}$$

where $u_k(x, t)'s$ $(k = 0, 1, 2, \ldots)$ are the functions of x and t. On substitution of (5) in Eq. (3), then equating the most dominant terms we get, $\gamma = 1$ and

$$u_0(x, t) = \frac{6}{\alpha}\phi_x. \tag{6}$$

Putting this result in (5) yields

$$u(x,t) \cong \phi_x \phi^{-1} + u_k(x,t)\phi^{k-1}, \tag{7}$$

Further, using (7) and Eq. (6) in Eq. (3), characteristic equation for resonances has been obtained, which is subsequently solved to get one branch with four resonances at $k = -1, 1, 4$, and 6. As usual, the resonance at -1 corresponds to the arbitrariness of the singular manifold $\phi(x,t) = 0$. Now our task is to confirm the existence of sufficient number of arbitrary functions without introducing the movable critical manifold at the resonance values.

The next step is to determine the coefficients $u_2,\ u_3$ and u_5, from the recursion relation and to verify the compatibility conditions for the existence of the free functions u_1, u_4, and u_6. After detailed computations, we observed that u_1 comes out to be arbitrary function and corresponding expressions for u_2, u_3, and u_5 are found to be arbitrary functions as well. Moreover, we discovered that u_1, u_4, u_6 turn out to be arbitrary functions and also compatibility conditions, for $k = 1, 4, 6$, are satisfied identically which implies that Eq. (3) passes the Painlevé test for integrability. The consequent feature of integrability is the derivation of multiple soliton solutions which will be carried out in the next section.

2.2 Multiple soliton solutions of the first equation

It is now useful, for the purposes of our analysis, to show that the first fourth-order integrable equation

$$u_{tt} + u_{txxx} + \alpha(u_x u_t)_x = 0, \tag{8}$$

gives multiple soliton solutions. We plan in this section to determine the dispersion relation and the phase shift of the interaction of solitons and hence derive the multiple soliton solutions. Substituting

$$u(x,t) = \mathrm{e}^{k_i x - c_i t}, \tag{9}$$

into the linear terms of (8) gives the dispersion relation as

$$c_i = k_i^3, i = 1, 2, \ldots, N. \tag{10}$$

The following phase variables

$$\theta_i = k_i x - k_i^3 t, i = 1, 2, \ldots, N, \tag{11}$$

follow immediately. Using the transformation

$$u(x,t) = R(\ln f(x,t))_x, \tag{12}$$

into Eq. (8), where the auxiliary function $f(x,t)$, for the single-soliton solution, is given as

$$f(x,t) = 1 + \mathrm{e}^{\theta_1} = 1 + \mathrm{e}^{k_1 x - k_1^3 t}. \tag{13}$$

and solving for R we find

$$R = \frac{6}{\alpha},\ \alpha \neq 0. \tag{14}$$

This in turn leads to the single-soliton solution

$$u(x,t) = \frac{6k_1 \mathrm{e}^{k_1 x - k_1^3 t}}{\alpha(1 + \mathrm{e}^{k_1 x - k_1^3 t})}. \tag{15}$$

For the two-soliton solutions, we use the auxiliary function as

$$f(x,t) = 1 + \mathrm{e}^{k_1 x - k_1^3 t} + \mathrm{e}^{k_2 x - k_2^3 t} + a_{12}\mathrm{e}^{(k_1+k_2)x - (k_1^3 + k_2^3)t}, \tag{16}$$

where a_{12} is the phase shift of the interaction of solitons. To determine the phase shift a_{12}, we substitute (16) into (8), and solving for the phase shift a_{12}, we obtain

$$a_{12} = \frac{k_1^4 - k_1 k_2 (k_1^2 + k_2^2) + k_2^4}{k_1^4 + k_1 k_2 (k_1^2 + k_2^2) + k_2^4} = \frac{(k_1 - k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)}{(k_1 + k_2)^2 (k_1^2 - k_1 k_2 + k_2^2)}, \tag{17}$$

which can be generalized as

$$a_{ij} = \frac{k_i^4 - k_i k_j (k_i^2 + k_j^2) + k_j^4}{k_i^4 + k_i k_j (k_i^2 + k_j^2) + k_j^4} = \frac{(k_i - k_j)^2 (k_i^2 + k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 - k_i k_j + k_j^2)},\ 1 \le i < j \le 3. \tag{18}$$

The result (18) shows that the phase shifts do not depend on the parameters α. The two-soliton solutions are obtained by substituting (17) and (16) into (12), where $R = \frac{6}{\alpha}$.

For the three-soliton solutions, we apply the auxiliary function $f(x,t)$ as

$$f(x,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\mathrm{e}^{\theta_1+\theta_2} + a_{13}\mathrm{e}^{\theta_1+\theta_3} + a_{23}\mathrm{e}^{\theta_2+\theta_3} + b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}. \tag{19}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{20}$$

The three-soliton solutions are obtained by substituting (19) into (12). This also shows that N-soliton solutions can be obtained for finite N, where $N \geq 1$.

2.3 Multiple complex soliton solutions of the first equation

It remains now to show that the first fourth-order integrable equation

$$u_{tt} + u_{txxx} + \alpha(u_x u_t)_x = 0, \tag{21}$$

gives multiple complex soliton solutions. Proceeding as before, we obtain the dispersion relation as

$$c_i = k_i^3, i = 1, 2, \ldots, N, \tag{22}$$

and hence the phase variables

$$\theta_i = k_i x - k_i^3 t, i = 1, 2, \ldots, N, \tag{23}$$

follow immediately.

We now introduce a new complex form of the simplified Hirota's method. The proposed method for the determination of multiple complex soliton solutions consists of the following steps:

1. For the single-complex soliton solution, the auxiliary function $f(x,t)$ takes the form:

$$f(x,t) = I + e^{\theta_1}, I = \sqrt{-1}. \tag{24}$$

2. For the two-complex soliton solutions, we set the auxiliary function $f(x,t)$ as:

$$f(x,t) = I + e^{\theta_1} + e^{\theta_2} - Ia_{12}e^{\theta_1+\theta_2}, \tag{25}$$

where a_{12} is the phase shift for the interaction between the solitons.

3. For the three-complex soliton solutions, we set the auxiliary function $f(x,t)$ as:

$$\begin{aligned} f(x,t) = I &+ e^{\theta_1} + e^{\theta_2} + e^{\theta_3} - Ia_{12}e^{\theta_1+\theta_2} \\ &- Ia_{13}e^{\theta_1+\theta_3} - Ia_{23}e^{\theta_2+\theta_3} \\ &- b_{123}e^{\theta_1+\theta_2+\theta_3}, \end{aligned} \tag{26}$$

where the phase variable $\theta_i, i = 1, 2, \ldots, N$ is given before in (23). Note that for $b_{123} = a_{12}a_{13}a_{23}$, the equation gives three-complex soliton solutions and hence multiple complex soliton solutions as examined in the standard algorithm.

Using the transformation

$$u(x,t) = R(\ln f(x,t))_x, \tag{27}$$

into Eq. (21), where the auxiliary function $f(x,t)$, for the single-soliton solution, is given as

$$f(x,t) = I + e^{\theta_1} = I + e^{k_1 x - k_1^3 t}, I = \sqrt{-1}. \tag{28}$$

and solving for R we find

$$R = \frac{6}{\alpha}, \ \alpha \neq 0. \tag{29}$$

We now apply the proposed method to achieve our goal.

Using (24) into (27) gives the single-complex soliton solution as

$$u(x,t) = \frac{6k_1 e^{k_1 x - k_1^3 t}}{\alpha(I + e^{k_1 x - k_1^3 t})}, I = \sqrt{-1}. \tag{30}$$

For the two-soliton solutions, we use the auxiliary function as given in (25) to find that the phase shift a_{12} remains same as in the real case and given as

$$\begin{aligned} a_{12} &= \frac{k_1^4 - k_1k_2(k_1^2 + k_2^2) + k_2^4}{k_1^4 + k_1k_2(k_1^2 + k_2^2) + k_2^4} \\ &= \frac{(k_1 - k_2)^2(k_1^2 + k_1k_2 + k_2^2)}{(k_1 + k_2)^2(k_1^2 - k_1k_2 + k_2^2)}, \end{aligned} \tag{31}$$

which can be generalized as

$$\begin{aligned} a_{ij} &= \frac{k_i^4 - k_ik_j(k_i^2 + k_j^2) + k_j^4}{k_i^4 + k_ik_j(k_i^2 + k_j^2) + k_j^4} \\ &= \frac{(k_i - k_j)^2(k_i^2 + k_ik_j + k_j^2)}{(k_i + k_j)^2(k_i^2 - k_ik_j + k_j^2)}, \quad 1 \leq i < j \leq 3. \end{aligned} \tag{32}$$

The result (32) shows that the phase shifts do not depend on the parameter α as in the real case. The two-complex soliton solutions are obtained by substituting (31) and (25) into (27), where $R = \frac{6}{\alpha}$.

For the three-complex soliton solutions, we apply the auxiliary function $f(x,t)$ as given in (26), and by proceeding as before, we find

$$b_{123}=a_{12}a_{23}a_{13}. \tag{33}$$

The three-complex soliton solutions and hence the N-complex soliton solutions can be obtained in a like manner to our approach presented before, for finite N, where $N \geq 1$.

We will close this section by employing a variety of hyperbolic and trigonometric ansatze to derive new exact solutions for the first integrable equation (3). The ansatze that will be used will be applied in a systematic manner.

2.4 The tanh-coth method

First substituting

$$u(x,t)=a_0+a_1\tanh(kx-ct), \tag{34}$$

into (8), collecting the coefficients of $\tanh^{2i}(kx-ct)$, $i=0,1,2$, setting these coefficients to zeros, and by solving the resulting system, we find

$$\begin{aligned} &a_0=a_0,\ \text{left free parameter},\\ &a_1=\tfrac{6k}{\alpha},\\ &c=4k^3. \end{aligned} \tag{35}$$

This in turn gives the soliton solution

$$u(x,t)=a_0+\frac{6k}{\alpha}\tanh(kx-4k^3\,t). \tag{36}$$

In a like manner, we can derive the singular soliton solution

$$u(x,t)=a_0+\frac{6k}{\alpha}\coth(kx-4k^3\,t). \tag{37}$$

2.5 The tan–cot method

Proceeding as before, we can assume the solution in the form

$$u(x,t)=a_0+a_1\tan(kx-ct). \tag{38}$$

Substituting this assumption, equation (8), and proceeding as before, we obtain the periodic solution

$$u(x,t)=a_0-\frac{6k}{\alpha}\tan(kx+4k^3\,t), \tag{39}$$

and the singular solution

$$u(x,t)=a_0+\frac{6k}{\alpha}\cot(kx+4k^3\,t). \tag{40}$$

3 Second fourth-order integrable equation

In a like manner to the previous section, we will study the second fourth-order integrable equation (4). The analysis will be conducted where we will use the Painlevé analysis, followed by using the simplified Hirota's method to derive multiple soliton solutions. We then will apply the proposed complex simplified Hirota's method to determine multiple complex soliton solutions.

3.1 Painlevé analysis for the second equation

Following the analysis presented earlier, we use the Painlevé analysis [1–6] to examine the integrability of the second developed equation (4). We assume that Eq. (4) has solution as a Laurent expansion about a singular manifold $\phi=\phi(x,t)$ given as

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t)\phi^{k-\gamma}, \tag{41}$$

where $u_k(x,t)'s$ $(k=0,1,2,\ldots)$ are the functions of x and t. On substitution of (41) in Eq. (4), then equating the most dominant terms we get, $\gamma=1$ and

$$u_0(x,t)=\frac{6}{\alpha}\phi_x. \tag{42}$$

Putting this value of α in (41) yields

$$u(x,t)\cong\phi_x\phi^{-1}+u_k(x,t)\phi^{k-1}, \tag{43}$$

Further, using (43) and Eq. (42) in equation (4), characteristic equation for resonances has been obtained, which is subsequently solved to get one branch with four resonances at $k=-1,1,4,$ and 6. As usual, the resonance at -1 corresponds to the arbitrariness of the singular manifold $\phi(x,t)=0$. Proceeding as before, we determine the coefficients $u_2,\ u_3$ and u_5, from the recursion relation and verify the compatibility conditions for the existence of the free functions u_1, u_4, and u_6. After detailed computations, we observed that u_1 comes out to be arbitrary function and corresponding expressions for u_2, u_3, and u_5 are determined as well. Moreover, we discovered that u_1, u_4, u_6 turn out to be arbitrary functions and also compatibility conditions, for $k = 1, 4, 6$, are satisfied identically which implies that Eq. (4) passes the Painlevé test for integrability.

3.2 Multiple soliton solutions of the second equation

We showed that the new fourth-order equation,

$$u_{tt} + u_{txxx} + \alpha(u_x u_t)_x + \beta u_{xx} = 0, \tag{44}$$

is integrable. We follow the same analysis as presented earlier for the first equation; hence, substituting

$$u(x,t) = e^{k_i x - c_i t}, \tag{45}$$

into the linear terms of (44) gives the dispersion relation as

$$c_i = \frac{k_i^3 \pm k_i^2\sqrt{k_i^4 - 4\beta}}{2}, i = 1, 2, \ldots, N. \tag{46}$$

Unlike the dispersion relation of the first fourth-order equation where we found that $c_i = k_i^3$, the dispersion relation in (46) depends on the parameter β as well.

The following phase variables

$$\theta_i = k_i x - \frac{k_i^3 \pm k_i\sqrt{k_i^4 - 4\beta}}{2} t, i = 1, 2, \ldots, N, \tag{47}$$

follow immediately. Using the transformation

$$u(x,t) = R(\ln\ f(x,t))_x, \tag{48}$$

into Eq. (44), where the auxiliary function $f(x,t)$, for the single-soliton solution, gives the following:

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - \frac{k_1^3 \pm k_1\sqrt{k_1^4 - 4\beta}}{2} t}. \tag{49}$$

and solving for R we find

$$R = \frac{6}{\alpha},\ \alpha \neq 0. \tag{50}$$

This in turn leads to the single-soliton solution

$$u(x,t) = \frac{6k_1 e^{k_1 x - \frac{k_1^3 \pm k_1\sqrt{k_1^4 - 4\beta}}{2} t}}{\alpha(1 + e^{k_1 x - \frac{k_1^3 \pm k_1\sqrt{k_1^4 - 4\beta}}{2} t})}. \tag{51}$$

For the two-soliton solutions, we use the auxiliary function as

$$f(x,t) = 1 + e^{k_1 x - k_1^3 t} + e^{k_2 x - k_2^3 t} + a_{12} e^{(k_1+k_2)x - (k_1^3 + k_2^3)t}, \tag{52}$$

where a_{12} is the phase shift of the interaction of solitons. To determine the phase shift a_{12}, we substitute (52) into (44), and solving for the phase shift a_{12}, we obtain

$$a_{12} = \frac{4\beta - 3(k_1^2 + k_1k_2 + k_2^2)(k_1 - k_2)^2 - k_1^2k_2^2 - 3(k_1 - k_2)(k_1\mu_1 - k_2\mu_2) + \mu_1\mu_2}{4\beta - 3(k_1^2 - k_1k_2 + k_2^2)(k_1 + k_2)^2 - k_1^2k_2^2 - 3(k_1 + k_2)(k_1\mu_1 + k_2\mu_2) + \mu_1\mu_2}, \tag{53}$$

where

$$\mu_i = \sqrt{k_i^4 - 4\beta},\ i = 1, 2, 3. \tag{54}$$

The phase shift (53) can be generalized as presented earlier.

$$a_{ij} = \frac{4\beta - 3(k_i^2 + k_i k_j + k_j^2)(k_i - k_j)^2 - k_i^2 k_j^2 - 3(k_i - k_j)(k_i \mu_i - k_j \mu_j) + \mu_i \mu_j}{4\beta - 3(k_i^2 - k_i k_j + k_j^2)(k_i + k_j)^2 - k_i^2 k_j^2 - 3(k_i + k_j)(k_i \mu_i + k_j \mu_j) + \mu_i \mu_j}, 1 \le i \le j \le 3. \tag{55}$$

The result (55) shows that the phase shifts depend on the parameters β in addition to the coefficients of the spatial variable. The two-soliton solutions are obtained by substituting (53) and (52) into (48), where $R = \frac{6}{\alpha}$.

For the three-soliton solutions, we apply the auxiliary function $f(x, t)$ as

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1+\theta_2} + a_{13} e^{\theta_1+\theta_3} + a_{23} e^{\theta_2+\theta_3} + b_{123} e^{\theta_1+\theta_2+\theta_3}. \tag{56}$$

Proceeding as before, we find

$$b_{123} = a_{12} a_{23} a_{13}. \tag{57}$$

The three-soliton solutions are obtained by substituting (56) into (48). This also shows that N-soliton solutions can be obtained for finite N, where $N \ge 1$.

3.3 Multiple complex soliton solutions of the second equation

The dispersion relation for the second fourth-order integrable equation

$$u_{tt} + u_{txxx} + \alpha (u_x u_t)_x + \beta u_{xx} = 0, \tag{58}$$

is obtained by using substitute

$$u(x,t) = e^{k_i x - c_i t}, \tag{59}$$

that gives the dispersion relation as

$$c_i = \frac{k_i^3 \pm k_i^2 \sqrt{k_i^4 - 4\beta}}{2}, i = 1, 2, \dots, N, \tag{60}$$

and phase variables

$$\theta_i = k_i x - \frac{k_i^3 \pm k_i \sqrt{k_i^4 - 4\beta}}{2} t, i = 1, 2, \dots, N, \tag{61}$$

remains same as derived earlier. Using the transformation

$$u(x,t) = R(\ln f(x,t))_x, \tag{62}$$

into Eq. (58), where the auxiliary function $f(x, t)$, for the single-soliton solution, givens the following:

$$f(x,t) = I + e^{\theta_1} = I + e^{k_1 x - \frac{k_1^3 \pm k_1 \sqrt{k_1^4 - 4\beta}}{2} t}. \tag{63}$$

and solving for R we find

$$R = \frac{6}{\alpha}, \alpha \ne 0. \tag{64}$$

Using (62) gives the single-complex soliton solution as

$$u(x,t) = \frac{6 k_1 e^{k_1 x - \frac{k_1^3 \pm k_1 \sqrt{k_1^4 - 4\beta}}{2} t}}{\alpha \left(I + e^{k_1 x - \frac{k_1^3 \pm k_1 \sqrt{k_1^4 - 4\beta}}{2} t}\right)}. \tag{65}$$

For the two-complex soliton solutions, we use the auxiliary function as

$$f(x,t) = I + e^{k_1 x - k_1^3 t} + e^{k_2 x - k_2^3 t} - I a_{12} e^{(k_1+k_2)x - (k_1^3 + k_2^3) t}, \tag{66}$$

which gives the phase shift a_{12}

$$a_{12} = \frac{4\beta - 3(k_1^2 + k_1 k_2 + k_2^2)(k_1 - k_2)^2 - k_1^2 k_2^2 - 3(k_1 - k_2)(k_1 \mu_1 - k_2 \mu_2) + \mu_1 \mu_2}{4\beta - 3(k_1^2 - k_1 k_2 + k_2^2)(k_1 + k_2)^2 - k_1^2 k_2^2 - 3(k_1 + k_2)(k_1 \mu_1 + k_2 \mu_2) + \mu_1 \mu_2}, \tag{67}$$

where

$$\mu_i=\sqrt{k_i^4-4\beta},\ i=1,2,3. \tag{68}$$

The phase shift (67) can be generalized as presented earlier.

In a like manner, we can derive the singular soliton solution

$$u(x,t)=a_0+\frac{6k}{\alpha}\coth\left(kx-\left(2k^3\pm k\sqrt{4k^4-\beta}\right)t\right). \tag{74}$$

$$a_{ij}=\frac{4\beta-3(k_i^2+k_ik_j+k_j^2)(k_i-k_j)^2-k_i^2k_j^2-3(k_i-k_j)(k_i\mu_i-k_j\mu_j)+\mu_i\mu_j}{4\beta-3(k_i^2-k_ik_j+k_j^2)(k_i+k_j)^2-k_i^2k_j^2-3(k_i+k_j)(k_i\mu_i+k_j\mu_j)+\mu_i\mu_j},\ 1\le i\le j\le 3. \tag{69}$$

This shows that the phase shifts remain same as in the real case. For the three-complex soliton solutions, we proceed as before to find

$$b_{123}=a_{12}a_{23}a_{13}. \tag{70}$$

The three-complex soliton solutions are obtained by substituting (69) into (62). This also shows that N-soliton solutions can be obtained for finite N, where $N\ge 1$.

We will close this section by applying a variety of hyperbolic and trigonometric ansatze to derive new exact traveling wave solutions. The ansatze that will be used in a like manner to the previous section.

3.4 The tanh-coth method

We first substitute

$$u(x,t)=a_0+a_1\tanh(kx-ct), \tag{71}$$

into (44), collecting the coefficients of $\tanh^i(kx-ct)$, $i=0,1,2,3,4$, setting these coefficients to zeros, and by solving the resulting system we find

$$\begin{aligned} a_0&=a_0,\ \text{left free parameter},\\ a_1&=\tfrac{6k}{\alpha},\\ c&=2k^3\pm k\sqrt{4k^4-\beta}. \end{aligned} \tag{72}$$

This in turn gives the soliton solution

$$u(x,t)=a_0+\frac{6k}{\alpha}\tanh\left(kx-\left(2k^3\pm k\sqrt{4k^4-\beta}\right)t\right). \tag{73}$$

3.5 The tan–cot method

Proceeding as before, we can assume the solution in the form

$$u(x,t)=a_0+a_1\tan(kx-ct). \tag{75}$$

Substituting this assumption, equation (8), and proceeding as before, we obtain the periodic solution

$$u(x,t)=a_0-\frac{6k}{\alpha}\tan\left(kx-\left(-2k^3\pm k\sqrt{4k^4-\beta}\right)t\right), \tag{76}$$

and the singular solution

$$u(x,t)=a_0+\frac{6k}{\alpha}\cot\left(kx-\left(-2k^3\pm k\sqrt{4k^4-\beta}\right)t\right). \tag{77}$$

4 Discussion

In the present work, we have systematically constructed two fourth-order nonlinear integrable equations. This formulation revealed both the dispersion and the phase shifts structures. More importantly, the Painlevé analysis confirmed the integrability of these two equations. The findings that an integrable equation can be reduced to another integrable equation, as in the case of the KP to be reduced to the KdV equation, is rarely examined in the literature. The simplified Hirota's method was used to carry out our analysis to determine multiple soliton solutions for each of the two developed integrable equations.

More importantly, we introduced, for the first time, a new complex form of the simplified Hirota's method where we developed multiple complex soliton solutions for each of the developed models. The findings reported in this work will be useful to design more integrable systems.

Compliance with ethical standards

Conflicts of interest The author declares no conflict of interest.

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