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On a “deterministic” explanation of the stochastic resonance phenomenon

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Abstract The present paper concerns the analysis of the stochastic resonance phenomenon that previously has been thoroughly studied and found numerous applications in physics, neuroscience, biology, medicine, mechanics, etc. A novel “deterministic” explanation of this phenomenon is proposed that allows broadening the range of dynamical systems for which the phenomenon can be predicted and analysed. Our results indicate that stochastic resonance, similarly to vibrational resonance, arises due to deterministic reasons: it occurs when a system is excited with two (or more) vastly different frequencies, one of which is much higher than another. The effective properties of the system, e.g. stiffness or mass, change under the action of the high-frequency excitation; and the low-frequency excitation acts on this “modified” system leading to low-frequency resonances. In the case of a broadband random excitation, the high-frequency part of the excitation spectrum affects the effective properties of the system. The low-frequency part of the spectrum acts on this modified system. Thus by varying the noise intensity one can change properties of the system and attain resonances. This explanation allows using “deterministic” approach, i.e. replacing noise by high-frequency excitation, when studying the stochastic resonance phenomenon. Employing this approach, we demonstrate that linear and nonlinear stochastic systems with varying parameters, i.e. parametrically excited systems, can exhibit the phenomenon and determine the corresponding resonance conditions.

Keywords Stochastic resonance · Deterministic explanation · High-frequency excitation · Effective properties · Oscillatory strobodynamics · Parametric excitation

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1 Introduction

The stochastic resonance phenomenon implies “positive” changing of a (nonlinear) system behaviour when white noise is added to the system. It is widely used to increase responses and signal-to-noise ratios of nonlinear systems, e.g. sensors and amplifiers. Due to its high practical and theoretical importance, this phenomenon has been thoroughly studied in many papers, e.g. [1–10]. It was first introduced by Benzi [11–13] to describe periodically recurrent ice ages. In 1980s, the stochastic resonance phenomenon has been considered only within analysis of bi-stable systems under a periodic input signal and random noise. Later, it has been discussed and used also when studying non-bi-stable systems, e.g. excitable systems and static threshold systems [14–18]. The phenomenon has found numerous applications in physics, neuroscience, biology, medicine, mechanics and other fields, cf. e.g. [1–21].

Stochastic resonance is often described as a counterintuitive phenomenon. This is because for conventional linear signal-processing systems it is well known that output signal-to-noise ratio is maximized in the absence of noise. So, increase in the output signal-to-noise ratio due to added noise can seem to be highly counterintuitive. In this sense, stochastic resonance is similar to numerous counterintuitive phenomena arising in nonlinear dynamical systems under action of high-frequency vibration, e.g. stabilization of the upper position of a pendulum with vibrating pivot [22,23], sinking of gas bubbles in vibrating fluid-filled volumes [24,25], etc., see e.g. [26–28]. These phenomena were effectively employed to improve and intensify various technological processes, e.g. relevant for the mining industry [26–29].

The resemblance of stochastic and high-frequency excitations has been already noticed and discussed in several papers, cf. e.g. [30–33] and more recent publications [34–36]. In particular, it has been noted that these excitations can lead to similar effects when applied to nonlinear dynamical systems. More specifically, a phenomenon, called vibrational resonance, has been described [30,32–36] that resembles stochastic resonance, but with white noise replaced by a high-frequency harmonic signal. In the present paper, we continue and expand this research. We revisit the conventional bi-stable deterministic system studied in [30,32,33] and provide a more accurate theoretical model of the arising phenomenon. By contrast to the previous model, the new one very well corresponds to numerical results in a wide range of the system parameters. Most importantly, it explains why increasing the amplitude of the original input (low-frequency) signal, the resonant "noise" (i.e. high-frequency vibration) intensity decreases. This model emphasises the similarity between the considered deterministic bi-stable system under high-frequency excitation and the same system with added noise, i.e. between stochastic resonance and vibrational resonance. The maximum response of the system and signal-to-"noise" ratio are attained at a certain nonzero value of the high-frequency excitation (or noise) intensity. In the case of high-frequency excitation, the signal-to-"noise" ratio means the ratio between the low-frequency component of the output signal and the high-frequency component of this signal.

The obtained results indicate that stochastic resonance and vibrational resonance arise due to the same "deterministic" physical phenomenon. The phenomenon occurs for systems excited with two (or more) vastly different frequencies, one of which is much higher than another. The effective properties of the system, e.g. stiffness or mass, change under the action of the high-frequency excitation; and the low-frequency excitation acts on this "modified" system leading to low-frequency resonances. In the case of a broadband random excitation, the high-frequency part of the excitation spectrum affects the effective properties of the system. The low-frequency part of the spectrum acts on this modified system. Therefore, by varying the intensity of the random (high-frequency) excitation, one can achieve resonance in the system. This observation provides a novel "deterministic" explanation of the stochastic resonance phenomenon and allows replacing noise by high-frequency excitation when studying the phenomenon. Moreover, it indicates that systems exhibiting vibrational resonance will also exhibit stochastic resonance, which is validated by the other examples considered in the paper.

Conventionally (cf. [1–13]), stochastic resonance has been considered as a phenomenon that can occur only in nonlinear systems. In several papers [37–40], however, it has been shown that this phenomenon can also appear in linear systems, e.g. featuring multiplicative noise [37,38]. Using the proposed "deterministic" explanation of the stochastic resonance and replacing noise by high-frequency excitation, we show why this phenomenon occurs for the linear systems, e.g. the one studied in [38].

The "deterministic" approach is also used to study systems with an input signal that is not direct, but parametric. Parametrically excited systems are now widely used for signal sensing, filtering, and amplification, particularly in micro- and nanoscale applications [41–43]. And, as is known [42–44], noise and uncertainty can be essential for systems at this scale. Thus, these systems potentially can exhibit stochastic resonance, which can be of considerable importance for practical applications. Theoretical analysis of parametrically excited systems with noise, however, is a non-trivial matter and only few papers are devoted to it, cf. e.g. [45]. We show that stochastic resonance can occur in such systems and determine the corresponding resonance conditions; note that the stochastic resonance phenomenon hasn't been previously predicted or described for systems with parametric input signals.

The dynamical systems considered in the present paper involve combined excitations with different, aliquant frequencies and relate to an important class of dynamical systems, the quasi-periodic systems. The conventional methods of linear and nonlinear dynamics often appear to be not applicable for studying such systems due to the restrictions they impose on the systems parameter space [46–48]. Thus in the present paper, a novel analytical approach, the oscillatory strobodynamics (OS) approach [49,50], is used. The name is motivated by the fact that the considered system behaviour is perceived as under a stroboscopic light so that only the main, slow component of motions is captured. The approach broadens the concept of Vibrational Mechanics [26,27] by means of which several notable phenomena arising in mechanical systems due to high-frequency vibration were revealed and described [25–29].

The paper is structured as follows: In Sect. 2, a model bi-stable system conventional for studying the stochastic resonance phenomenon is considered, however, instead of adding white noise we introduce high-frequency excitation. The obtained results are compared with the classical ones for the system with the added white noise. Section 3 concerns analysis of the stochastic resonance phenomenon for linear systems using the proposed deterministic approach. In particular, we study systems with an input signal that is not direct, but parametric. In Sect. 4, a model nonlinear system featuring combined direct and parametric input signals is considered. In the last section, the conclusions of the paper are outlined.

2 Conventional first-order bi-stable system exhibiting stochastic resonance

2.1 Preliminary remarks

A conventional generic model illustrating the stochastic resonance phenomenon is the overdamped motion of a Brownian particle in a bi-stable potential in the presence of noise and periodic forcing [3,8]:

$$\dot{x}(t) = -V'(x) + A\cos(\Omega t + \varphi) + \xi(t), \tag{1}$$

where $V(x)$ denotes the reflection-symmetric quartic potential

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4. \tag{2}$$

In dimensionless equation (1), $\xi(t)$ denotes a zero-mean, Gaussian white noise with autocorrelation function

$$\langle \xi(t)\xi(0)\rangle = 2D\delta(t), \tag{3}$$

and intensity D. The dimensionless potential $V(x)$ is bi-stable with minima located at $\pm x_m = \pm 1$. The height of the potential barrier between the minima is $\Delta V = 0.25$.

Without noise $\xi(t)$, accounting for the periodic signal $A\cos(\Omega t + \varphi)$, the time-dependent potential of Eq. (1) can be written as [3]:

$$V_o(x,t) = V(x) - Ax\cos(\Omega t + \varphi). \tag{4}$$

As is seen, it is tilted back and forth, thereby raising and lowering alternately the potential barriers of the right and the left well, respectively, in an antisymmetric manner. During time, the potential (4) changes from $V(x) - Ax$ to $V(x) + Ax$; and $V_{\text{om}} = V(x) \pm Ax$ features either two minima and one maximum between them or just one minimum, governed by the equation:

$$V'_{\text{om}}(x) = x^3 - x \pm A = 0 \tag{5}$$

that has either three or one real roots. The threshold (or bifurcation point) between these cases corresponds to the following value of the amplitude A:

$$A = A_{\text{cr}0} = \frac{2}{\sqrt{27}} \approx 0.385 \tag{6}$$

If amplitude A of the input signal is below $A_{\text{cr}0}$, then V_{om} has two minima and one maximum between them. Thus the particle cannot overcome the potential barrier and oscillates near one of the minima. However, if amplitude A is larger than $A_{\text{cr}0}$, then at $t = (-\varphi + 2\pi n)/\Omega$, $n \in Z$, the system's potential, $V_o(x,t) = V(x) - Ax$, has only one minimum $x = x_{1m}$, while at $t = (-\varphi + \pi n)/\Omega$, $V_o(x,t) =$

$V(x) + Ax$ has another minimum at $x = x_{2m} = -x_{1m}$. Thus the particle is attracted to different minima during one period of the input signal and can hop from one minima to another. This is illustrated in Fig. 1 that shows the dependencies of $V_o(x, t)$ on x for (a) $A = 0.2$ and (b) $A = 0.6$. It is clear that for $A > A_{cr0}$ the amplitude of the particle oscillations is much larger than for $A < A_{cr0}$.

2.2 System under high-frequency excitation; solution by the OS approach

In this section, we consider dimensionless equation (1) with the white noise replaced by high-frequency excitation, $H \cos \omega t$:

$$\dot{x}(t) = -V'(x) + A\cos(\Omega t + \varphi) + H\cos\omega t. \tag{7}$$

Comparing (1) with (7), we note that the amplitude of the high-frequency excitation H directly corresponds to the noise intensity D; both of them represent the external (noise) energy supplied to the system. We also require the condition $\omega \gg \Omega$ to hold true, i.e. the frequency of the added excitation should be much higher than the frequency of the original input signal. Note that Eq. (7) has been already studied in [30,32,33]; in the present paper, however, a more accurate solution and novel interpretation of the results is given.

For solving the nonlinear dimensionless equation (7) that involves combined excitations with different, aliquant frequencies, we use the oscillatory strobodynamics (OS) approach [49,50] and the Method of Direct Separation of Motions (MDSM) [26–29]. The approach utilizes a general observation that motion of a dynamical system under high-frequency excitations can usually be separated into two components: slow $X(t)$ and fast $\psi(t)$ (notions "high-frequency", "fast" and "slow" can be formalized [26–29,49,50]). In essence, this observation merely means that the system under oscillating excitation exhibits oscillations. The main idea of the approach lies in the transition from initial governing equations of motions to equations describing only the slow component $X(t)$. This component is usually of primary interest; and equations describing it can be much simpler than the initial equations.

Following the method, we search a solution to (7) in the form:

$$x = X(t) + \psi(t, \omega t), \tag{8}$$

where X is slow, and ψ is fast 2π-periodic in time $\tau = \omega t$ variable, with average zero:

$$\langle\psi\rangle = \frac{1}{2\pi}\int_0^{2\pi} \psi(t, \tau)\mathrm{d}\tau = 0, \tag{9}$$

angular brackets designate averaging in the period 2π on time τ. Introducing (8) into (7) and averaging with respect to the fast time $\tau = \omega t$, we get the following equation that describes slow motions of the considered system:

$$\dot{X} = -\left\langle V'(X + \psi)\right\rangle + A\cos(\Omega t + \varphi). \tag{10}$$

For the initial equation (7) to be satisfied we also get:

$$\dot{\psi} = -V'(X + \psi) + \left\langle V'(X + \psi)\right\rangle + H\cos\omega t. \tag{11}$$

This equation describes fast motions of the system.

Note that Eqs. (10)–(11) are not simpler than the initial equation (7); however, they are much more convenient for the approximate solving. More specifically, as is shown in [26,27,49,50], since the slow variable $X(t)$ is of primary interest, it is sufficient to determine the fast variable ψ approximately, because it is present in Eq. (10) only under the averaging operator, and thus this approximation will not lead to any considerable errors in the resulting equation for the variable X. As one of the approximations, slow variables X and t are considered as constants ("frozen") when solving fast motions equations. Consequently, for the present problem, a solution to (11) is sought in the form of a series:

$$\psi = \sum_{n=1}^{\infty} B_{n1}\cos n\omega t + B_{n2}\sin n\omega t. \tag{12}$$

Taking into account only the first (primary) harmonic in (12), we get the following simple equations for the amplitudes B_{11} and B_{12}:

$$\begin{aligned} -B_{11} + \omega B_{12} + 3B_{11}\left(\frac{B_{11}^2}{4} + \frac{B_{12}^2}{4} + X^2\right) &= H \\ -B_{12} - \omega B_{11} + 3B_{12}\left(\frac{B_{11}^2}{4} + \frac{B_{12}^2}{4} + X^2\right) &= 0 \end{aligned} \tag{13}$$

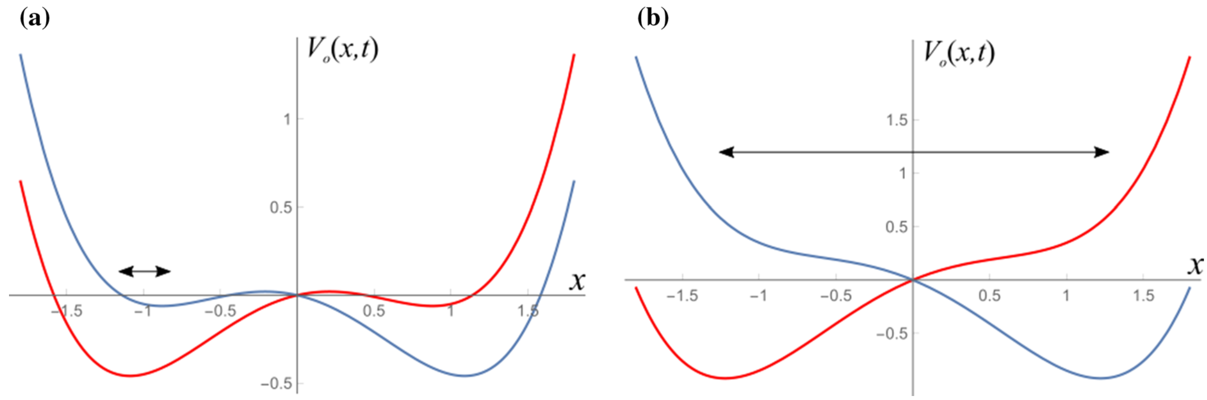

Fig. 1 The dependencies of the potential of the system (1) on x for **a** $A = 0.2$ and **b** $A = 0.6$. Blue lines correspond to $t = (-\varphi + 2\pi n)/\Omega$, while red lines to $t = (-\varphi + \pi n)/\Omega$, $n \in Z$. Black arrows indicate particle's oscillations amplitudes. (Color figure online)

These equations give three pairs of values for B_{11} and B_{12} the expressions for which are rather lengthy and thus not given here. For small B_{11} and B_{12}, Eq. (13) can be approximated as follows:

$$(3X^2 - 1)B_{11} + \omega B_{12} = H, \quad (3X^2 - 1)B_{12} - \omega B_{11} = 0, \tag{14}$$

with the solutions:

$$B_{11} = \frac{H(3X^2 - 1)}{(3X^2 - 1)^2 + \omega^2}, \quad B_{12} = \frac{H\omega}{(3X^2 - 1)^2 + \omega^2}. \tag{15}$$

Inserting (12) and (15) into (10), we get the following slow motions equation:

$$\dot{X} = \left[1 - \frac{3}{2}\frac{H^2}{(3X^2 - 1)^2 + \omega^2}\right]X - X^3 + A\cos(\Omega t + \varphi). \tag{16}$$

It should be noted that slow motions equation (16) differs from the one obtained in [32,33], since here we didn't use the inertial approximation when solving fast motions equation (11). In [32,33] it was concluded that the maximum output amplitude of the system (7) is obtained when the coefficient at X in the slow motions equation equals zero, i.e. $H^2/\omega^2 \approx 2/3$. Thus, the optimal value of the high-frequency excitation amplitude (i.e. "noise" intensity) doesn't depend on the amplitude A of the original (low-frequency) input signal. This conclusion, however, contradicts with the results of numerical experiments for Eq. (7), see Sect. 2.3. Moreover, as is known [1–21], the stochastic resonance phenomenon can also feature dependency of optimal D on A. In the following section, a new expression for the optimal value of the high-frequency excitation amplitude is obtained that depends explicitly on A and better conforms to the results of numerical experiments.

2.3 On the resonance phenomenon

Comparing Eq. (16) with the original equation (7), we immediately see that effective properties of the system with respect to slow motions change due to the added high-frequency excitation. More specifically, the effective time-dependent potential of the system (16) is:

$$\begin{aligned} V_{os}(X,t) &= V_s(X) - AX\cos(\Omega t + \varphi) \\ &= -\frac{1}{2}X^2 + \frac{1}{4}X^4 + \frac{H^2}{4\omega}\arctan\left(\frac{3X^2 - 1}{\omega}\right) \\ &\quad - AX\cos(\Omega t + \varphi), \end{aligned} \tag{17}$$

and its minima and the height of the potential barrier between the minima are different from those of (4). Similarly to (4), during time, the potential (17) changes from $V_s(X) - AX$ to $V_s(X) + AX$; and $V_{sm} = V_s(X) \pm AX$ features either two minima and one maximum between them or just one minimum that are determined from the equation:

$$V'_{sm}(X) = \left[1 - \frac{3}{2}\frac{H^2}{(3X^2 - 1)^2 + \omega^2}\right]X - X^3 \pm A = 0. \tag{18}$$

We note that taking into account only the first (primary) harmonic in (12) and approximation (15) are valid for small B_{11} and B_{12} implying that the following condition should hold true:

$$\omega^2 \gg H^2. \tag{19}$$

Using this condition, we get the following approximate relation for A that corresponds to the threshold (or bifurcation point) between the two different types of motions:

$$A_{\text{crs}} = \frac{2}{\sqrt{27}} \left(1 - \frac{3}{2}\frac{H^2}{\omega^2 + 4}\right)^{3/2}. \tag{20}$$

Similarly to the case without high-frequency excitation, if amplitude A of the input signal is below A_{crs}, then the system oscillates near one of the minima with a relatively small amplitude. For $A > A_{\text{crs}}$, the system is attracted to different minima during one period of the input signal and can hop from one minima to another, exhibiting relatively large amplitude oscillations. From relation (20) it is evident that introducing high-frequency excitation (or noise) leads to reducing the threshold value of the input signal amplitude A_{crs}. Consequently, adding such excitation allows to achieve relatively large output signal of the system even for weak input signal, $A < A_{\text{cr0}}$, with A_{cr0} defined by (6). Threshold value of the high-frequency excitation amplitude H corresponding to the bifurcation point, with relation (19) taken into account, can be approximated by:

$$H_{\text{cr}}^2 = \frac{4}{9}\left(1 - \frac{\sqrt{27}}{2}A\right)(\omega^2 + 4), \tag{21}$$

and for $H > H_{\text{cr}}$, we get large amplitude oscillations. In particular, from (21) it is evident that H_{cr} depends on the input signal amplitude A and relatively large output signal can be achieved even for small A. Note that for $A > A_{\text{cr0}} = 2/\sqrt{27}$, the system exhibits large amplitude oscillations even without any additional high-frequency excitation.

To further describe the resonance phenomenon in the system, we determine the minima of the potential $V_s(X)$ of the slow motions equation (16) without excitation $A\cos(\Omega t + \varphi)$. Taking into account (19), it is relevant to assume that these minima change only slightly as compared to those of (2). Consequently, we get:

$$X_{1\min} = 1 - \frac{3}{4}\frac{H^2}{4+\omega^2}, \quad X_{2\min} = -1 + \frac{3}{4}\frac{H^2}{4+\omega^2} \tag{22}$$

An important observation here is that $|X_{\min}|$ reduces with increasing the high-frequency excitation amplitude H.

If $H < H_{\text{cr}}$ (and $A < A_{\text{cr0}}$), then the system oscillates near one of the potential minima (22). Introducing $Y = X - X_{\min}$ and assuming Y to be small, we get the following approximate linear equation describing these oscillations:

$$\dot{Y} = -2\left(1 - \frac{3}{2}\frac{16+\omega^2}{(4+\omega^2)^2}H^2\right)Y + A\cos(\Omega t + \varphi). \tag{23}$$

Solving (23) is a trivial matter, giving the following steady-state solution:

$$Y = \frac{A}{C^2+\Omega^2}((C\cos\varphi + \Omega\sin\varphi)\cos\Omega t + (\Omega\cos\varphi - C\sin\varphi)\sin\Omega t), \tag{24}$$

here $C = 2\left(1 - \frac{3}{2}\frac{16+\omega^2}{(4+\omega^2)^2}H^2\right)$. Taking into account (19) and assuming $\omega \gg 1$, we note that $|C| > 1$ and magnitude of Y oscillations, $\frac{A}{\sqrt{C^2+\Omega^2}}$, can indeed be considered small; note that $A < A_{\text{cr0}}$. Consequently, for $H < H_{\text{cr}}$ the systems exhibits relatively low amplitude oscillations near one of the potential minima (22).

At $H = H_{\text{cr}}$, however, the behaviour of the system changes: it hops from one minima to another during one period of the input signal. These oscillations feature amplitude that can be roughly approximated as $|X_{\min}|$ and thus is relatively large, cf. relations (22). Further increasing the high-frequency excitation amplitude H leads to reducing of $|X_{\min}|$ as follows from (22); consequently, amplitude of the system oscillations, i.e. the output signal, reduces. This means that the system features resonance phenomenon: it's output signal depends non-monotonically on the high-frequency excitation amplitude H (or "noise" intensity); and the maximum output is achieved at $H = H_{\text{cr}}$. It should be noted that the signal-to-"noise" ratio, i.e. the ratio between the low- and high-frequency components of the output, also attains its maximum value at $H = H_{\text{cr}}$, since, as follows from (15), B_{11} and B_{12} increase with increasing H. For the conventional stochastic system described by Eq. (1), the stochastic resonance phenomenon exhibits itself as an abrupt jump of the output signal at a certain value of the noise intensity, similarly to the resonance phenomenon discussed in this section. This illustrates the validity of the "deterministic" approach to describe the stochastic resonance phenomenon and indicates direct relationship between the noise intensity and the amplitude of the high-frequency excitation. Note that in the case of combined noise and high-frequency excitations, the corresponding resonance phenomenon is also qualitatively similar [51].

As an illustration, Fig. 2 shows the dependencies of the amplitude Q of the low-frequency component of the system output signal on $H/\sqrt{4+\omega^2}$ for $\Omega = 0.1$ and different values of the input signal amplitude A. Markers represent results of direct numerical integration of Eq. (7), while lines correspond to the theoretical predictions described in the present section. A good agreement between numerical and theoretical results can be noted even when relation (19) is not fulfilled, however, as expected, the discrepancy increases with increasing $H/\sqrt{4+\omega^2}$. In all cases, the described resonance phenomenon is evident and the maximum output signal is achieved at $H = H_{\mathrm{cr}}$. And changing Ω doesn't qualitatively affect the results. As noted above, for $H > H_{\mathrm{cr}}$, the amplitude of oscillations is approximated as $|X_{\min}|$ and thus does not depend on the input signal amplitude A. Consequently, the upper portion of the curves shown in Fig. 2, corresponding to $H > H_{\mathrm{cr}}$, follow the same path. For $H < H_{\mathrm{cr}}$, low amplitude oscillations of the system depend on the input signal amplitude A, c.f. (24).

Results obtained in this section indicate qualitative similarities between the considered deterministic bistable system under high-frequency excitation and the same system with added noise, e.g. in the way they exhibit resonance phenomena. We believe that there are similarities not only in the way how vibrational resonance and stochastic resonance are exhibited by these systems, but also in the physical essence of these phenomena. Both of them occur for systems excited with two (or more) vastly different frequencies, one of which is much higher than another. The effective properties of the system, e.g. stiffness or mass, change under the action of the high-frequency excitation; and the low-frequency excitation acts on this "modified" system leading to low-frequency resonances. In the case of a broadband random excitation, the high-frequency part of the excitation spectrum affects the effective properties of the system. The low-frequency part of the spectrum acts on this modified system. This observation provides a novel "deterministic" explanation of the stochastic resonance phenomenon that allows replacing noise by high-frequency excitation when studying the phenomenon. There is a direct relationship between the noise intensity and the amplitude of the high-frequency excitation: both of them represent the external (noise) energy supplied to the system.

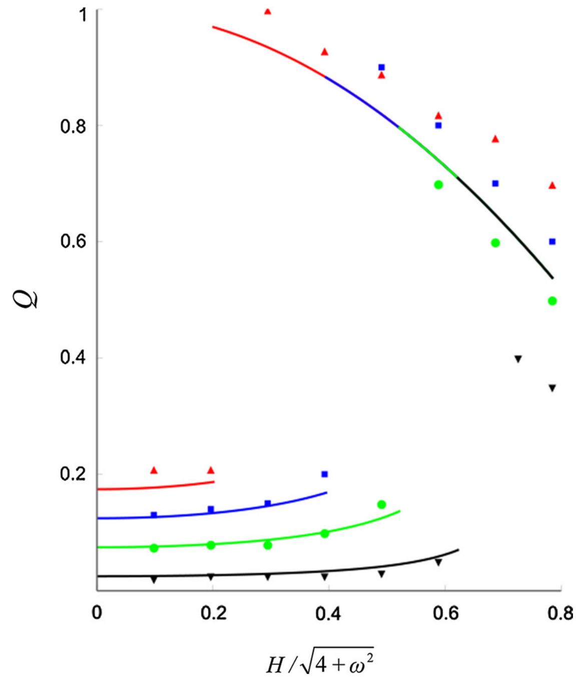

Fig. 2 Theoretical (lines) and numerical (markers) dependencies of the amplitude Q of the low-frequency component of the system output signal on $H/\sqrt{4+\omega^2}$ for $\Omega = 0.1$ and $A = 0.05$ (black line, down triangles), $A = 0.15$ (green line, circles), $A = 0.25$ (blue line, squares), $A = 0.35$ (red line, triangles). (Color figure online)

The conventional notion of "stochastic resonance" may thus be considered as deficient in the sense that it doesn't fully reflect the physical meaning of the phenomenon: stochasticity of the excitation is not essential. The notion of "vibrational resonance" seems to be inappropriate to describe the phenomenon, since conventionally term "vibration" implies oscillations in mechanical systems. However, the described resonance phenomenon occurs not only in mechanical, but also in electromagnetic, biological and other dynamical systems. As possible alternative names for the phenomenon, we propose: (1) Low-frequency resonance induced by high-frequency excitations (LRIHE); (2) Modulated resonance.

3 Linear systems exhibiting the stochastic resonance phenomenon

In this section, we employ the "deterministic" approach to describe the stochastic resonance phenomenon for linear systems. Gitterman [38] has shown that a forced, underdamped linear oscillator with random natural frequency described by:

$$\ddot{x} + \gamma \dot{x} + [\lambda^2 + \xi(t)]x = A \sin(\Omega t + \varphi), \quad (25)$$

can feature stochastic resonance; here the random force $\xi(t)$ is a Gaussian variable with zero mean. Following the "deterministic" approach, we replace $\xi(t)$ by a high-frequency excitation:

$$\ddot{x} + \gamma \dot{x} + [\lambda^2 + H \sin \omega t]x = A \sin(\Omega t + \varphi), \quad (26)$$

with amplitude H governing the excitation intensity and frequency ω satisfying relations $\omega \gg \Omega$ and $\omega \gg \lambda$. System (26) is the well-known forced Mathieu equation. Omitting the details, the corresponding approximate equation of slow motions is obtained in the form:

$$\ddot{X} + \gamma \dot{X} + \left[\lambda^2 + \frac{H^2}{2} \frac{\omega^2 - \lambda^2}{(\omega^2 - \lambda^2)^2 + \gamma^2 \omega^2}\right] X = A \sin(\Omega t + \varphi), \quad (27)$$

that can be further simplified as:

$$\ddot{X} + \gamma \dot{X} + \left[\lambda^2 + \frac{H^2}{2\omega^2}\right] X = A \sin(\Omega t + \varphi). \quad (28)$$

As is seen from (28), the effective properties of the system with respect to slow motions change due to the high-frequency excitation; more specifically, its effective natural frequency is:

$$\lambda_*^2 = \lambda^2 + \frac{H^2}{2\omega^2} \quad (29)$$

that depend on the high-frequency excitation amplitude H. Consequently, by changing the excitation intensity one can affect the natural frequency and attain resonance in the system.

Mathieu equation belongs to a wide class of parametrically excited systems. As has been shown, cf. e.g. [26,27,52], high-frequency parametric excitations can change effective properties of such systems with respect to slow motions leading to resonances. The proposed "deterministic" approach thus indicates that linear systems featuring parametric excitation that is not deterministic, but random, can feature the stochastic resonance phenomenon. Considering the linear systems for which this phenomenon has been previously observed, e.g. [37–40], we note that all of them feature random parametric excitation, i.e. some of their parameters are random. This supports the above conclusion and the validity of the proposed "deterministic" explanation of the stochastic resonance phenomenon.

Substituting random excitation by a deterministic high-frequency excitation allows broadening the range of systems for which the stochastic resonance phenomenon can be predicted and described. In particular, this phenomenon can be present also for stochastic systems with an input signal that is not direct, but parametric. Such systems are now widely used; however, their theoretical analysis is rather complicated, so that the stochastic resonance phenomenon hasn't been previously predicted or described for them.

As an example, consider the following stochastic system that is similar to (25), but with the input signal that is not direct, but parametric:

$$\ddot{x} + \gamma \dot{x} + [\lambda^2 + \xi(t) + A \sin(\Omega t + \varphi)]x = 0, \quad (30)$$

Employing the "deterministic" approach, we replace $\xi(t)$ by a high-frequency excitation:

$$\ddot{x} + \gamma \dot{x} + [\lambda^2 + H \sin \omega t + A \sin(\Omega t + \varphi)]x = 0, \quad (31)$$

with $\omega \gg \Omega$ and $\omega \gg \lambda$. The corresponding approximate equation of slow motions takes the form:

$$\ddot{X} + \gamma \dot{X} + \left[\lambda^2 + \frac{H^2}{2} \frac{\omega^2 - \lambda^2 - A \sin(\Omega t + \varphi)}{(\omega^2 - \lambda^2 - A \sin(\Omega t + \varphi))^2 + \gamma^2 \omega^2} + A \sin(\Omega t + \varphi)\right] X = 0, \quad (32)$$

which can be further approximated by:

$$\ddot{X} + \gamma \dot{X} + \left[\lambda^2 + \frac{H^2}{2\omega^2} + A \sin(\Omega t + \varphi)\right] X = 0, \quad (33)$$

As is seen, similarly to Eq. (28), by changing the excitation intensity we can affect the natural frequency of the system and attain resonance. This simple example indicates that linear systems that feature random parametric excitations can exhibit the stochastic resonance phenomenon irrespective of whether the input signal is direct or parametric.

As the last example, consider the following stochastic system that features both direct and parametric random excitations and direct and parametric input signals:

$$\begin{aligned}&\ddot{x}+\gamma\dot{x}+[\lambda^2+\xi_\mathrm{p}(t)+A_\mathrm{p}\sin(\Omega_1 t+\varphi)]x\\&\quad=\xi_\mathrm{d}(t)+A_\mathrm{d}\sin(\Omega_2 t+\delta),\end{aligned}\tag{34}$$

Following the "deterministic" approach, we replace $\xi_\mathrm{p}(t)$ and $\xi_\mathrm{d}(t)$ by high-frequency excitations:

$$\begin{aligned}&\ddot{x}+\gamma\dot{x}+[\lambda^2+H_\mathrm{p}\sin\omega t+A_\mathrm{p}\sin(\Omega_1 t+\varphi)]x\\&\quad=H_\mathrm{d}\sin(\omega t+\theta)+A_\mathrm{d}\sin(\Omega_2 t+\delta),\end{aligned}\tag{35}$$

with $\omega \gg \Omega_1, \Omega_2$ and $\omega \gg \lambda$. The corresponding equation of slow motions is:

$$\begin{aligned}&\ddot{X}+\gamma\dot{X}+\left[\lambda^2+\frac{H_\mathrm{p}^2}{2}\frac{\omega^2-\lambda^2-A_\mathrm{p}\sin(\Omega_1 t+\varphi)}{(\omega^2-\lambda^2-A_\mathrm{p}\sin(\Omega_1 t+\varphi))^2+\gamma^2\omega^2}+A_\mathrm{p}\sin(\Omega_1 t+\varphi)\right]X\\&\quad=A_\mathrm{d}\sin(\Omega_2 t+\delta)-\frac{H_\mathrm{d}H_\mathrm{p}}{2}\frac{\gamma\omega\sin\theta-\left(\omega^2-\lambda^2-A_\mathrm{p}\sin(\Omega_1 t+\varphi)\right)\cos\theta}{(\omega^2-\lambda^2-A_\mathrm{p}\sin(\Omega_1 t+\varphi))^2+\gamma^2\omega^2},\end{aligned}\tag{36}$$

which can be reduced to:

$$\begin{aligned}&\ddot{X}+\gamma\dot{X}+\left[\lambda^2+\frac{H_\mathrm{p}^2}{2\omega^2}+A_\mathrm{p}\sin(\Omega_1 t+\varphi)\right]X\\&\quad=A_\mathrm{d}\sin(\Omega_2 t+\delta)+\frac{H_\mathrm{d}H_\mathrm{p}}{2\omega^2}\cos\theta.\end{aligned}\tag{37}$$

As is seen from (37), the direct high-frequency excitation (with amplitude H_d) cannot affect the effective properties of the system with respect to slow motions in the absence of the parametric excitation, i.e. for $H_\mathrm{p}=0$. Thus, linear stochastic systems with only direct random excitations cannot feature the stochastic resonance phenomenon, which corresponds well to the previously known results.

4 On nonlinear parametrically excited systems exhibiting the stochastic resonance phenomenon

In this section, we briefly consider a model nonlinear parametrically and directly excited stochastic system and determine the corresponding resonance conditions:

$$\begin{aligned}&\ddot{x}+\gamma\dot{x}+[\lambda^2+\xi_\mathrm{p}(t)+A_\mathrm{p}\sin(\Omega_1 t+\varphi)]x+kx^3\\&\quad=\xi_\mathrm{d}(t)+A_\mathrm{d}\sin(\Omega_2 t+\delta).\end{aligned}\tag{38}$$

As noted in the previous section, the corresponding linear system can feature stochastic resonance only when the random parametric excitation is nonzero, i.e. $\xi_\mathrm{p}(t)\neq 0$. Here we study how resonance conditions change in the presence of the Duffing-type nonlinearity. Following the "deterministic" approach, we replace $\xi_\mathrm{p}(t)$ and $\xi_\mathrm{d}(t)$ by high-frequency excitations:

$$\begin{aligned}&\ddot{x}+\gamma\dot{x}+[\lambda^2+H_\mathrm{p}\sin\omega t+A_\mathrm{p}\sin(\Omega_1 t+\varphi)]x+kx^3\\&\quad=H_\mathrm{d}\sin(\omega t+\theta)+A_\mathrm{d}\sin(\Omega_2 t+\delta)\end{aligned}\tag{39}$$

with $\omega \gg \Omega_1, \Omega_2$ and $\omega \gg \lambda$. The equations of slow and fast motions are, respectively:

$$\begin{aligned}&\ddot{X}+\gamma\dot{X}+\left[\lambda^2+A_\mathrm{p}\sin(\Omega_1 t+\varphi)\right]X\\&\quad+H_\mathrm{p}\langle\psi\sin\omega t\rangle+k(X^3+3X\langle\psi^2\rangle+\langle\psi^3\rangle)\\&\quad=A_\mathrm{d}\sin(\Omega_2 t+\delta)\end{aligned}\tag{40}$$

$$\begin{aligned}&\ddot{\psi}+\gamma\dot{\psi}+[\lambda^2+H_\mathrm{p}\sin\omega t+A_\mathrm{p}\sin(\Omega_1 t+\varphi)]\psi\\&\quad+k(3X^2\psi+3X\psi^2+\psi^3)\\&\quad=H_\mathrm{d}\sin(\omega t+\theta)+H_\mathrm{p}\langle\psi\sin\omega t\rangle\\&\qquad+k(3X\langle\psi^2\rangle+\langle\psi^3\rangle)-H_\mathrm{p}X\sin\omega t\end{aligned}\tag{41}$$

A solution to (41) is sought in the form of series (12). Taking into account only the first (primary) harmonic, we get the following equations for the amplitudes B_{11} and B_{12}:

$$\begin{aligned}&\frac{3}{4}kB_{11}\left(B_{11}^2+B_{12}^2+4X^2\right)+B_{11}(\lambda^2-\omega^2\\&\quad+A_\mathrm{p}\sin(\Omega_1 t+\varphi))+\gamma\omega B_{12}=H_\mathrm{d}\sin\theta\\&\frac{3}{4}kB_{12}\left(B_{11}^2+B_{12}^2+4X^2\right)+B_{12}(\lambda^2-\omega^2\\&\quad+A_\mathrm{p}\sin(\Omega_1 t+\varphi))-\gamma\omega B_{11}+H_\mathrm{p}X=H_\mathrm{d}\cos\theta\end{aligned}\tag{42}$$

These equations give three pairs of values for B_{11} and B_{12} the expressions for which are rather lengthy and thus not given here. For small B_{11} and B_{12}, Eqs. (42) can be approximated as follows:

$$
\begin{aligned}
&3kB_{11}X^2 + B_{11}\left(\lambda^2 - \omega^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi)\right) \\
&\quad + \gamma\omega B_{12} = H_\mathrm{d}\sin\theta \\
&3kB_{12}X^2 + B_{12}\left(\lambda^2 - \omega^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi)\right) \\
&\quad - \gamma\omega B_{11} + H_\mathrm{p}X = H_\mathrm{d}\cos\theta
\end{aligned} \tag{43}
$$

with the solutions:

$$
\begin{aligned}
B_{11} &= \frac{-\gamma\omega H_\mathrm{d}\cos\theta + H_\mathrm{d}\sin\theta\left(\lambda^2 - \omega^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi) + 3kX^2\right) + \gamma\omega H_\mathrm{p}X}{\gamma^2\omega^2 + \left(\lambda^2 - \omega^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi) + 3kX^2\right)^2} \\
B_{12} &= \frac{\gamma\omega H_\mathrm{d}\sin\theta + \left(H_\mathrm{d}\cos\theta - H_\mathrm{p}X\right)\left(\lambda^2 - \omega^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi) + 3kX^2\right)}{\gamma^2\omega^2 + \left(\lambda^2 - \omega^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi) + 3kX^2\right)^2}
\end{aligned} \tag{44}
$$

Inserting (12) and (44) into (40), we get the following slow motions equation:

$$
\begin{aligned}
&\ddot{X} + \gamma\dot{X} + \left[\lambda^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi)\right]X \\
&\quad + \frac{H_\mathrm{p}}{2}\frac{\gamma\omega H_\mathrm{d}\sin\theta + \left(H_\mathrm{d}\cos\theta - H_\mathrm{p}X\right)\left(\lambda^2 - \omega^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi) + 3kX^2\right)}{\gamma^2\omega^2 + \left(\lambda^2 - \omega^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi) + 3kX^2\right)^2} \\
&\quad + k\left(X^3 + \frac{3}{2}X\frac{H_\mathrm{d}^2 + H_\mathrm{p}X\left(H_\mathrm{p}X - 2H_\mathrm{d}\cos\theta\right)}{\gamma^2\omega^2 + \left(\lambda^2 - \omega^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi) + 3kX^2\right)^2}\right) = A_\mathrm{d}\sin(\Omega_2 t + \delta)
\end{aligned} \tag{45}
$$

Taking into account only the first harmonic in (12) and assuming B_{11} and B_{12} to be small implies the following relations to hold true: $H_\mathrm{d}, H_\mathrm{p} \ll \omega$. Consequently, Eq. (45) can be farther simplified into:

$$
\begin{aligned}
&\ddot{X} + \gamma\dot{X} + \left[\lambda^2 + A_\mathrm{p}\sin(\Omega_1 t + \varphi) + \frac{H_\mathrm{p}^2}{2\omega^2} + \frac{3}{2}\frac{H_\mathrm{d}^2}{\omega^4}k\right]X \\
&\quad + k\left(X^3 - 9\frac{H_\mathrm{d}H_\mathrm{p}}{2\omega^4}X^2\cos\theta + 3\frac{H_\mathrm{p}^2}{\omega^4}X^3\right) \\
&\quad = A_\mathrm{d}\sin(\Omega_2 t + \delta) + \frac{H_\mathrm{p}H_\mathrm{d}}{2\omega^2}\cos\theta.
\end{aligned} \tag{46}
$$

As is seen, Eq. (46) is similar to (37), but with the nonlinear terms present (multiplied by k). By contrast to the linear case, the direct high-frequency excitation (with amplitude H_d) affects the effective properties of the system with respect to slow motions in the absence of the parametric excitation, i.e. for $H_\mathrm{p} = 0$. Note, however, that the corresponding change of the system's linearized effective natural frequency, $\frac{3}{2}\frac{H_\mathrm{d}^2}{\omega^4}k$, is much smaller than the one caused by the pure parametric excitation, $\frac{H_\mathrm{p}^2}{2\omega^2}$, since $H_\mathrm{d}, H_\mathrm{p} \ll \omega$.

The above model example indicates that nonlinear stochastic systems can feature the stochastic resonance phenomenon irrespective of whether the input signal and random excitations are direct or parametric. Specifically, by varying the noises intensities (represented by H_p and H_d) one can change the effective properties of the systems, e.g. their natural frequencies, with respect to the deterministic input signals. The example also shows that the stochastic resonance phenomenon can occur in nonlinear systems with single-well potentials, which agrees well with the known results [53]. As a novel application of the phenomenon, we note the possibility to maintain resonant mode of operation of a vibrating machine under a varying load by changing the intensity of stochastic (or high-frequency) excitations.

5 Conclusion

The present paper proposes a "deterministic" approach to describe and predict the stochastic resonance phenomenon, based on replacing noise by high-frequency excitations. Its applicability is illustrated by several examples, including the conventional first-order bistable system and linear systems for which this phenomenon has been previously observed. It is shown that due to random excitations the effective properties of the systems, e.g. their natural frequencies, change with respect to deterministic input signals. Therefore, by varying the intensity of the excitations resonances can be attained. It is explained, in particular, why for the considered linear stochastic systems, the phenomenon occurs only in the presence of parametric random excitations, i.e. when some parameters of the systems are random. We show that for stochastic systems, resonances can be attained by varying frequencies of deterministic input signals or by changing parameters of these systems for given random excitations, or, as mentioned above, by changing intensities of the random excitations for constant input signals, cf. also [33].

It is also shown that stochastic systems can exhibit the stochastic resonance phenomenon irrespective of whether the input signal is direct or parametric. Resonance conditions for model systems with parametric inputs, i.e. with varying parameters, are determined. A model nonlinear parametrically and directly excited stochastic system is studied and combined effects of nonlinearity and parametric excitation on the stochastic resonance conditions are revealed.

The above analysis shows that the conventional notion of "stochastic resonance" may be considered as deficient in the sense that it doesn't fully reflect the physical meaning of the phenomenon. Resonances in the considered stochastic systems occur due to multiple excitations with substantially different frequencies. The effective properties of the system, e.g. stiffness or mass, change under the action of the high-frequency excitation; and the low-frequency excitation acts on this "modified" system leading to low-frequency resonances. In the case of a broadband random excitation, the high-frequency part of the excitation spectrum affects the effective properties of the system. The low-frequency part of the spectrum acts on this modified system. The notion of "vibrational resonance" seems to be inappropriate to describe the phenomenon, since conventionally term "vibration" means oscillations in mechanical systems. However, the resonance phenomenon occurs not only in mechanical, but also in electromagnetic, biological and other dynamical systems.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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