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ORIGINAL PAPER

Nonstationary solutions of nonlinear dynamical systems excited by Gaussian white noise

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Abstract Nonlinear dynamical systems to random excitations may fail long before stationarity is achieved. Transient state has to be taken into account. A novel approximate technique for determining nonstationary probability density function (PDF) of randomly excited nonlinear Oscillator is developed. Specifically, it expresses the PDF approximation in terms of polynomial functions with time-dependent coefficients. By applying statistical linearization and weighted residual method, residual error of the FPK equation associated with approximated solution is reduced to a series of nonlinear first-order ordinary differential equations, which can be solved by the numerical method. Finally, a class of nonlinear vibrating systems with additive excitations or/and parametric excitations are considered. The obtained PDF has tail regions of logarithmic form, which are important for reliability and failure analysis, and agrees very well with the simulated ones. In particular, the computational time spent by the proposed procedure is a very small fraction of the one taken by the MCS method. This technique can be used as a convenient tool for assessing the accuracy of alternative, more general, approximate solution methods.

Keywords Stochastic processes · Random vibration · Fokker–Planck–Kolmogorov (FPK) equation · Monte Carlo simulation · Nonstationary solutions

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1 Introduction

Transient probability density function (PDF) solution is of importance in estimating structure reliability since failure may occur long before stationarity is achieved. Therefore, using only stationary solutions to evaluate structures is conservative.

As we all know, PDF solutions for nonlinear dynamical systems under white noise excitations are governed by Fokker–Planck–Kolmogorov (FPK) equations. When the FPK equation is in the reduced form without time derivative term, stationary PDF solutions can be obtained [1]. Much more attention has been paid to it, and thousands of papers have been published. There have been many approximate methods and numerical methods developed for the stationary solutions. On the other hand, when there is a time derivative term in the FPK equation, transient solutions have to be obtained. It is especially difficult to obtain transient solutions. Exact transient solutions are available only for linear systems and some special first-order nonlinear systems [2,3]. Much research effort is still needed, and approximate methods have to be developed. Stochastic linearization has been proved to be one of the most useful approximate techniques for nonlinear systems since it does not demand small parameter assumption and extensive computational cost [4–8]. However, since this technique is based on the hypothesis of Gaussianity of response process, only the response moments up to second order are predicted with reasonable accuracy, whereas higher-order moments and reliability estimates can lead to erroneous results, particularly for cases of strong nonlinearity and parametric excitations. Moment differential equation method or cumulant-neglect closure method enables to predict deviation from Gaussian distribution [9–11]. Based on the It$\hat{o}$ stochastic differential calculus, this method leads to an infinite hierarchy of coupled linear moment differential equations, from which a soluble set of nonlinear equations can be obtained by introducing a non-Gaussian closure approximation. Stochastic averaging method, combined generalized Galerkin method, can be used for nonstationary response envelope probability densities of nonlinear oscillators [12–15]. The main feature of this method is that it enables replacement of the original system by a lower-dimensional one through a combination of time averaging and ensemble averaging. However, since this method relies on perturbation, it yields accurate results only for small values of the nonlinearity parameter. Maximum entropy method, originated in its simple classical form in statistical physics, when suitably generalized, allows complicated stochastic systems to be solved successfully for stationary and nonstationary solutions using information contained in the equations for statistical moments of the responses [16]. Due to the difficulty in solving FPK equation, many numerical methods also have been proposed. Monte Carlo simulation (MCS) is a versatile technique for numerical solutions of stochastic differential equations. However, it associates with numerical convergence, stability, round-off error, and especially requirements for large computational effort in simulating small PDFs in the tail regions. Besides, A-type Gram–Charlier expansion of approximate PDF in time-dependent Hermite series is proposed to reduce the FPK equation to a series of first-order differential equations by means of weighted residual method. However, this method is connected with slow convergence and negative PDF values [17,18]. In order to improve such situations, C-type Gram–Charlie series is proposed with a log transformation to ensure positivity. This method provides extremely good results for systems governed by polynomial nonlinearities [19]. Additionally, there are other numerical methods, such as cell mapping method [20], path integration method [21–23], finite element method and finite difference method [24–27], smoothed particle hydrodynamic (SPH) method [28], meshless methods like partition of unity finite element method (PUFEM) [29]. Traditional numerical approaches have been plagued by several difficult issues like being positivity, domain identification or boundary conditions, curse of dimensionality, and time-consuming computation. The recently introduced PUFEM seems to overcome these difficulties. It represents a possibility to obtain good results with coarse mesh size. Considerable reduction in the memory storage requirements is expected due to coarse meshes employed, which makes it available to be applicable for higher-dimensional systems.

The exponential polynomial closure (EPC) method, which is originally proposed for stationary PDF solutions of FPK equation, has been proved to be a useful tool for analyzing systems governed by polynomial nonlinearities [30–33]. It is with relatively simple process, but high computation efficiency of only 2 or 3 seconds taken for the whole process with a standard desktop PC. A main feature of this method is that it reduces the FPK equation to a series of nonlinear algebraic equations, which can be solved easily by available numerical methods. A significant advantage of this method is the choice of the weighting functions, which releases the domain identification and simplifies the computation process. Furthermore, the accuracy of this method can be improved by changing the polynomial orders n. In particular, if the interested nonlinear system belongs to stationary potential, the solution obtained by the EPC method coincides with the exact one. What is more, the EPC method can be extended to multi-degree-of-freedom (MDOF) systems, namely state-space-split EPC (3S-EPC) method [34,35]. The investigation on the EPC method can be fundamental for analyzing MDOF systems. However, the EPC method has rarely been used for transient state of stochastic dynamic systems. In this paper, the EPC method is improved for approximate transient response of nonlinear system under Gaussian white noise excitations. Specifically, it expresses the PDF approximation in terms of polynomial functions with time-dependent coefficients. By applying statistical linearization and weighted residual method, residual error of the FPK equation associated with EPC approximated solution is reduced to a series of nonlinear first-order ordinary differential equations, where the unknowns are the time-dependent coefficients of the polynomial functions. In particular, the selected weighting functions provide substantial computational advantages. Finally, nonlinear vibrating systems with additional excitations or/and parametric excitations are considered to illustrate the application of the proposed procedure. It is shown that the results obtained for the examples by using the proposed procedure agree well with those from MCS.

2 Problem formulation

In this section, preliminary work concerning nonstationary solutions of nonlinear systems subjected to external and/or parametric stationary Gaussian white noise excitations is briefly summarized, and appropriate notations are introduced.

Consider the following nonlinear stochastic oscillator

$$\ddot{X} + f_0(X, \dot{X}, t) = g_j(X, \dot{X}, t)W_j(t),\quad j = 1, 2, \ldots, m. \tag{1}$$

where $\dot{()}$ denotes time derivative; $f_0(X, \dot{X}, t)$ and $g_j(X, \dot{X}, t)$ are generally nonlinear functions of the response process $\boldsymbol{X} = (X, \dot{X})$ and time t; W_j are zero-mean stationary Gaussian white noise excitations characterized as

$$E[W_i(t_1)W_j(t_2)] = S_{ij}\delta(t_1 - t_2), \tag{2}$$

in which $E[\cdot]$ means stochastic average, $\delta(\cdot)$ is Dirac's delta function and S_{ij} are constants representing cross-spectral densities of white noises W_i and W_j. It is assumed that $f_0(X, \dot{X}, t)$ is antisymmetrical such that the mean of $\boldsymbol{X}$ is also zero. Moveover, if functions $g_j(X, \dot{X}, t)$ depend only on time t, the nonlinear stochastic oscillator in Eq. (1) contains additive excitations only.

By setting $X = X_1, \dot{X} = X_2$, Eq. (1) can be converted into Ito's form

$$\begin{aligned}\dot{X}_1 &= X_2\\ \dot{X}_2 &= -f(\boldsymbol{X}, t) + g_j(\boldsymbol{X}, t)W_j(t),\end{aligned} \tag{3}$$

where

$$f(\boldsymbol{X}, t) = f_0(\boldsymbol{X}, t) + \frac{1}{2}S_{ij}\frac{\partial g_j(\boldsymbol{X}, t)}{\partial X_2}g_i(\boldsymbol{X}, t), \tag{4}$$

where the second terms represent the so-called Wong–Zakai or Stratonovich correction terms. They are vanished in the case of purely external excitations.

Following Ito differential rule, moment equations $M_k = x_1^{k_1}x_2^{k_2}$ of different orders can be obtained from Eq. (3)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[M_k] &= E[x_2\frac{\partial M_k}{\partial x_1}] - E\left[f(\boldsymbol{x})\frac{\partial M_k}{\partial x_2}\right]\\ &\quad+\frac{1}{2}S_{ij}E\left[g_i(\boldsymbol{x}, t)g_j(\boldsymbol{x}, t)\frac{\partial^2 M_k}{\partial x_2^2}\right],\\ k &= k_1 + k_2.\end{aligned} \tag{5}$$

It follows that response moments of nonlinear systems are governed by an infinite hierarchy of linear differential equations, whose exact solution is impossible. Approximate procedures, commonly called closure schemes, are needed in order to reduce the infinity hierarchy to a finite one. If the closure level is set as $n = 4$, such as cumulant-neglect closure method with truncation level $n = 4$, deviation of moments from Gaussianity can be explicitly described. When the closure scheme $n = 2$, Gaussian moments of different orders at any time instant can be obtained with statistical variances $[\sigma_1^2(t), \sigma_2^2(t)]$. For clarity's sake, Isserlis's theorem for zero-mean multivariate normal random vector at different time instant is adopted

$$\begin{aligned}E[x_1x_2\ldots x_{2n}; t] &= \sum E[x_ix_j; t],\\ &\qquad i, j = 1, 2, \ldots 2n.\\ E[x_1x_2\ldots x_{2n-1}; t] &= 0,\end{aligned} \tag{6}$$

which are needed in the following solution procedure.

The response process $\boldsymbol{X}$ governed by Eq. (1) is a Markov process, which is completely characterized by the transition PDF $p(\boldsymbol{x}, t|\boldsymbol{x_0}, t_0)$, defined as the PDF of $\boldsymbol{x}$ at time t, subjected to the initial condition $\boldsymbol{x} = \boldsymbol{x_0}$ at $t = t_0$. The transition PDF is ruled by the FPK equation

$$\frac{\partial p}{\partial t}+x_2\frac{\partial p}{\partial x_1}+\frac{\partial}{\partial x_2}[f(\boldsymbol{x}, t)p]-\frac{1}{2}\frac{\partial^2}{\partial x_2^2}[G(\boldsymbol{x}, t)p] = 0, \tag{7}$$

where $G(\boldsymbol{x}, t)$ is

$$G(\boldsymbol{x}, t) = S_{ls}g_l(\boldsymbol{x}, t)g_s(\boldsymbol{x}, t). \tag{8}$$

The appropriate initial condition associated with the FPK equation is

$$p(\boldsymbol{x}, t|\boldsymbol{x_0}, t) = \delta(\boldsymbol{x} - \boldsymbol{x_0}). \tag{9}$$

And it is assumed that $p(\boldsymbol{x}, t|\boldsymbol{x_0}, t)$ fulfills the following constraints

$$p(\boldsymbol{x}, t|\boldsymbol{x_0}, t) \geq 0$$

$$\lim_{x_i\to\pm\infty} p(\boldsymbol{x},t|\boldsymbol{x_0},t)=0,\quad i=1,2$$
$$\int_{R^2} p(\boldsymbol{x},t|\boldsymbol{x_0},t)d\boldsymbol{x}=1 \tag{10}$$

3 The developed EPC solution procedure

Since exact solution to Eq. (7) is usually not obtainable, approximate methods have to be adopted. Herein exponential polynomial closure (EPC) method is used. In order to describe the time evolution of response PDF, time variable t has to be introduced into the previous EPC approximation. As a result, approximate transient solution is assumed to be of the following form

$$\tilde{p}(\boldsymbol{x},t)=C\exp[Q_n(\boldsymbol{a},\boldsymbol{x},t)], \tag{11}$$

where C is a normalization constant and $Q_n(\boldsymbol{a},\boldsymbol{x},t)$ are n-degree polynomial functions. Hereto, positivity and normalization constraints in Eq. (10) for PDF solutions are automatically satisfied. In order to guarantee the second constraint in Eq. (10), coefficients of the highest polynomial order term have to be negative. Polynomial functions $Q_n(\boldsymbol{a},\boldsymbol{x},t)$ are expressed as

$$Q_n(\boldsymbol{x},\boldsymbol{a},t)=\sum_{k=1}^{N_p} Q_{nk}=\sum_{k=1}^{N_p} a_k(t)x_1^{h_1[k]}x_2^{h_2[k]}$$
$$h_1[k]=i-j,\quad h_2[k]=j$$
$$i=1,2,\ldots,n,\quad j=0,1,\ldots,i \tag{12}$$

where n are polynomial orders, k mean the sequence of polynomials, $h_1[k]$ and $h_2[k]$ represent polynomial orders of the kth term, which can be obtained by the relation between notations i and j. Notations i, j and k are completely independent. $a_k(t)$ are unknown time-dependent coefficients needed to be specified. The total number of unknown parameters is N_p.

Substituting approximate representation, Eq. (11) into the FPK equation Eq. (7), residual error is inevitably produced

$$R(\boldsymbol{x},\boldsymbol{a},t)=\frac{\partial\tilde{p}}{\partial t}+x_2\frac{\partial\tilde{p}}{\partial x_1}+\frac{\partial f}{\partial x_2}\tilde{p}+f\frac{\partial\tilde{p}}{\partial x_2}$$
$$-\frac{1}{2}\left(\frac{\partial^2 G}{\partial x_2^2}\tilde{p}+2\frac{\partial G}{\partial x_2}\frac{\partial\tilde{p}}{\partial x_2}+G\frac{\partial^2\tilde{p}}{\partial x_2^2}\right). \tag{13}$$

By using the following relations

$$\frac{\partial\tilde{p}}{\partial t}=\tilde{p}\frac{\partial Q_n}{\partial t}=\tilde{p}\sum_{k=1}^{N_p}\dot{a}_k(t)x_1^{h_1[k]}x_2^{h_2[k]}$$
$$\frac{\partial\tilde{p}}{\partial x_1}=\tilde{p}\frac{\partial Q_n}{\partial x_1}=\tilde{p}\sum_{k=1}^{N_p}h_1[k]a_k(t)x_1^{h_1[k]-1}x_2^{h_2[k]}$$
$$\frac{\partial\tilde{p}}{\partial x_2}=\tilde{p}\frac{\partial Q_n}{\partial x_2}=\tilde{p}\sum_{k=1}^{N_p}h_2[k]a_k(t)x_1^{h_1[k]}x_2^{h_2[k]-1}$$
$$\frac{\partial^2\tilde{p}}{\partial x_2^2}=\tilde{p}\left[\frac{\partial^2 Q_n}{\partial x_2^2}+\left(\frac{\partial Q_n}{\partial x_2}\right)^2\right]$$
$$=\tilde{p}\left[\sum_{k=1}^{N_p}h_2[k](h_2[k]-1)a_k(t)x_1^{h_1[k]}x_2^{h_2[k]-2}\right.$$
$$\left.+\sum_{k=1}^{N_p}\sum_{m=1}^{N_p}h_2[k]h_2[m]a_k(t)a_m(t)x_1^{h_1[k]+h_1[m]}x_2^{h_2[k]+h_2[m]-2}\right], \tag{14}$$

Residual error can be further expressed as

$$R(\boldsymbol{x},\boldsymbol{a})=\Delta(\boldsymbol{x},\boldsymbol{a},t)\tilde{p}, \tag{15}$$

where $\Delta(\boldsymbol{x},\boldsymbol{a},t)$ is

$$\Delta(\boldsymbol{x},\boldsymbol{a},t)=\left\{\sum_{k=1}^{N_p}\dot{a}_k(t)x_1^{h_1[k]}x_2^{h_2[k]}\right.$$
$$+\sum_{k=1}^{N_p}h_1[k]a_k(t)x_1^{h_1[k]-1}x_2^{h_2[k]+1}$$
$$+\sum_{k=1}^{N_p}fh_2[k]a_k(t)x_1^{h_1[k]}x_2^{h_2[k]-1}$$
$$-\frac{1}{2}\sum_{k=1}^{N_p}Gh_2[k](h_2[k]-1)a_k(t)x_1^{h_1[k]}x_2^{h_2[k]-2}$$
$$-\sum_{k=1}^{N_p}\frac{\partial G}{\partial x_2}h_2[k]a_k(t)x_1^{h_1[k]}x_2^{h_2[k]-1}$$
$$-\frac{1}{2}\sum_{k=1}^{N_p}\sum_{m=1}^{N_p}Gh_2[k]h_2[m]a_k(t)a_m(t)x_1^{h_1[k]+h_1[m]}x_2^{h_2[k]+h_2[m]-2}$$
$$\left.+\frac{\partial f}{\partial x_2}-\frac{1}{2}\frac{\partial^2 G}{\partial x_2^2}\right\} \tag{16}$$

According to the method of weighted residuals, N_p unknown coefficients $a_k(t)$ can be evaluated by imposing that the projection of the residual error $\Delta(\boldsymbol{x},\boldsymbol{a},t)$ on a properly selected functions $w_m(\boldsymbol{x},t)$ is zero, that is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(\boldsymbol{x},\boldsymbol{a},t)w_m(\boldsymbol{x},t)d\boldsymbol{x}=0. \tag{17}$$

This equation means that the FPK equation is fulfilled with $\tilde{p}$ in the average sense if $\Delta(\boldsymbol{x},\boldsymbol{a},t)w_k(\boldsymbol{x},t)$ is integrable in R^2. The weighting function $w_m(\boldsymbol{x},t)$ can be selected as

$$w_m(\boldsymbol{x},t)=x_1^{h_1[m]}x_2^{h_2[m]}f(\boldsymbol{x},t)\quad m=1,\ldots,N_p, \tag{18}$$

where an effective choice for $f(\boldsymbol{x}, t)$ is represented by the following normal distribution

$$f(\boldsymbol{x}, t) = \frac{1}{2\pi\sigma_1(t)\sigma_2(t)\sqrt{(1-\rho^2(t)}} \exp \times\Big(-\frac{1}{2(1-\rho^2(t))}\Big[\frac{(x_1(t)-\mu_1(t))^2}{\sigma_{x_1}^2(t)} + \frac{(x_2(t)-\mu_2(t))^2}{\sigma_{x_2}^2(t)} - \frac{2\rho(t)(x_1(t)-\mu_1(t))(x_2(t)-\mu_2(t))}{\sigma_{x_1}(t)\sigma_{x_2}(t)}\Big]\Big) \tag{19}$$

where time variants $\mu_i(t)$, $\sigma_i(t)$, and $\rho(t)$ are parameters obtained from moment equations in Eq. (5).

As a result, corresponding residual error can be re-expressed as

$$A(t)\dot{\boldsymbol{a}}(t) + B(t)\boldsymbol{a}(t) + C(t)\boldsymbol{a}^2(t) = F(t), \tag{20}$$

where $A(t)$, $B(t)$ are $M \times M$ time-dependent matrices. Elements of matrix $A(t)$ are statistical moments expressed as

$$A_{ij}(t) = E\Big[x_1^{h_1[j]+h_1[i]} x_2^{h_2[j]+h_2[i]}; t\Big], \tag{21}$$

and elements of $B(t)$ matrix are

$$\begin{aligned} B_{ik}(t) &= h_1[k]E\Big[x_1^{h_1[k]+h_1[i]-1} x_2^{h_2[k]+h_2[i]+1}; t\Big] \\ &+ h_2[k]E\Big[f(\boldsymbol{x}, t)x_1^{h_1[k]+h_1[i]} x_2^{h_2[k]+h_2[i]-1}; t\Big] \\ &- \frac{1}{2}h_2[k](h_2[k]-1)E\Big[G(\boldsymbol{x}, t)x_1^{h_1[k]+h_1[i]} x_2^{h_2[k]+h_2[i]-2}; t\Big] \\ &- h_2[k]E\Big[\frac{\partial G(\boldsymbol{x}, t)}{\partial x_2} x_1^{h_1[k]+h_1[i]} x_2^{h_2[k]+h_2[i]-1}; t\Big]. \end{aligned} \tag{22}$$

In Eq. (20), elements of matrix $C(t)$ is a matrix of order $M \times M^2$, whose ith row is defined as

$$\begin{aligned} &-\frac{1}{2}Gh_2[k]h_2[m]E\Big[x_1^{h_1[k]+h_1[m]+h_1[i]} x_2^{h_2[k]+h_2[m]+h_2[i]-2}; t\Big] \\ &= -\frac{1}{2}E\Big[Gx_1^{h_1[i]} x_2^{h_2[i]} Q_{nk} \otimes Q_{nm}; t\Big], \end{aligned} \tag{23}$$

where the symbol $\otimes$ denotes Kronecker product. And $\boldsymbol{a}^2$

$$a_{km}^2 = a_k(t) \times a_m(t). \tag{24}$$

Finally, $F(t)$ is an M vector with ith element given by

$$\begin{aligned} F_i(t) &= \frac{1}{2}E\Big[\frac{\partial^2 G}{\partial x_2^2} x_1^{h_1[i]} x_2^{h_2[i]}; t\Big] \\ &- E\Big[\frac{\partial f}{\partial x_2} x_1^{h_1[i]} x_2^{h_2[i]}; t\Big] \end{aligned} \tag{25}$$

In Eqs. (21)–(25), all the coefficients are related to Gaussian moments of different orders at different time instants, which can be simply computed by using Isserlis's theorem in Eq. (6). By integrating the system of nonlinear ordinary differential equations (20) with initial condition, then the time evolution of the N_p coefficients $a(t)$ at any time instant can be obtained. Herein the most basic numerical method Euler's method is adopted to solve these differential equations. It should be noticed that coefficients associated with moments at different time instants have to be updated at every iteration step. Then, evaluating C through the normalization condition, the PDF of the response process can be approximated according to Eqs. (11) and (12).

After exposure to the white noise excitations for a sufficiently long time, the system response attains the state of statistical stationary, at which time Eq. (20) reduces to the algebraic equation.

$$B\boldsymbol{a} + C\boldsymbol{a}^2 = F, \tag{26}$$

where matrices B, C, and F are the same matrices and vector as those defined in Eqs. (21)–(25), respectively, but have time-independent elements and without time evolution of moments. The solution procedure is greatly simplified. For the stationary PDF solution procedure, please refer to papers [30–32].

On the other hand, simulation procedure for the transient PDFs is conducted as follows: (1) Gaussian white noise is generated with the Gaussian variable generating function. Initial values of x_1 and x_2 are randomly generated in terms of Eq. (19); (2) the fourth-order Runge–Kutta algorithm is conducted for Eq. (3) with time increment; (3) the time history responses x_1 and x_2 in the duration t are obtained, and the values at time instant t are recorded; (4) the procedure from step 1 to step 3 is repeated N times, and then, samples of N for x_1 and x_2 are obtained; and (5) the transient PDFs of x_1 and x_2 at time instant t are statistically evaluated based on samples. In this paper, the sample size N is 10^7 for the transient PDFs.

4 Numerical examples and discussion on the results

In order to verify the efficiency of the developed EPC method, three examples of nonlinear oscillators under external and/or parametric excitations are considered. At the same time, exact/approximate stationary solutions are taken as a benchmark to evaluate the transient PDF solutions.

The EPC method with the polynomial order being $n = 6$ is utilized in the following examples. Due to the fact that exact transient PDF solutions are not available in these two examples, MCS is conducted to present an adequate estimation on transient PDF solutions with a sample size 10^7. Since the PDF values in the tail regions play an important role in the reliability and failure analysis, the results are shown in the form of logarithmic PDFs.

More importantly, it should be noticed that the computational time of the EPC solution procedure taken is greatly less than the one of the MCS method, which further indicates the efficiency of the EPC method. For example, the time spent for the PDF results at time instant $t = 5$ s is only serval seconds for the EPC method. While it is more than ten minutes (about 16 min) for the MCS method. As evolution time continues, computational time required by the MCS method grows exponentially and is up to serval hours at time instant $t = 20$ s. But for the developed EPC solution procedure, it is spent only three minutes and takes obviously advantages.

For each example, initial PDF with a Gaussian distribution is assumed as follows

$$p(x_1(0), x_2(0); 0) = \frac{1}{2\pi\sigma_1(0)\sigma_2(0)} \exp \times \left(-\frac{1}{2} \left[\frac{(x_1^2(0) - \mu_1^2(0))^2}{\sigma_{x_1}^2(0)} + \frac{(x_2^2(0) - \mu_2^2(0))^2}{\sigma_{x_2}^2(0)} \right] \right), \tag{27}$$

where initial parameters at the beginning are set as $\mu_1(0) = 0, \mu_2(0) = 0, \sigma_1(0) = 0.05, \sigma_2(0) = 0.05$. The time step is set as 0.02 to trace the temporal evolution of PDF solutions.

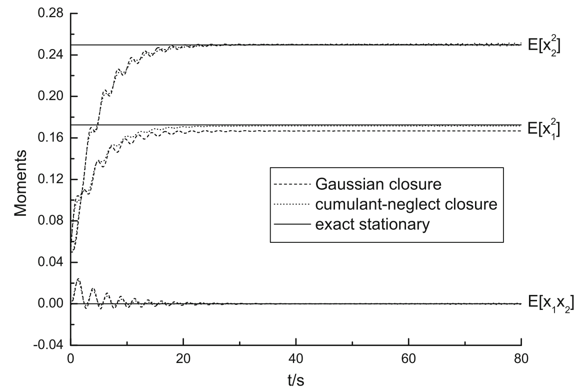

Fig. 1 Time evolution of statistical moments for Duffing oscillator under external excitation in Example 1

4.1 Example 1: Nonlinear oscillator under external excitation

A Duffing oscillator, which can be used to model a simply supported beam with large deformations under transversal loading [3], is considered first

$$\ddot{X} + 2\zeta\omega\dot{X} + X + \varepsilon X^3 = W(t), \tag{28}$$

where parameters $\zeta = 0.1, \omega = 1, \varepsilon = 1$, $W(t)$ is a zero-mean stationary white noise with correlation function $E[W(t)W(t+\tau)] = 0.1\delta(\tau)$.

Firstly, time evolution of the second-order moments of the responses from the initial condition set as $(\sigma_1^2 = \sigma_2^2 = 0.05, \sigma_{12} = 0)$ to the exact stationary state is shown in Fig. 1. Notice that the results obtained by cumulant-neglect closure method with closure scheme $n = 4$ are in good agreement with those exact ones at the stationary state. Specially, approximate velocity variance coincides with the exact ones. The relative error is 0.7% for the displacement variance. The results for the stationary variance of displacement with Gaussian closure method deviate from the exact ones with relative error 3.4%, which shows the non-Gaussian behavior of displacement. It is further observed that moment $E[x_1x_2]$ vanishes as the stationary state attains, which illustrates that the correlation between displacement and velocity disappears and independence appears at the stationary state. Besides, it is seen that the system approaches to the stationary state at time instant $t = 20$ s, when statistical moments become about constant.

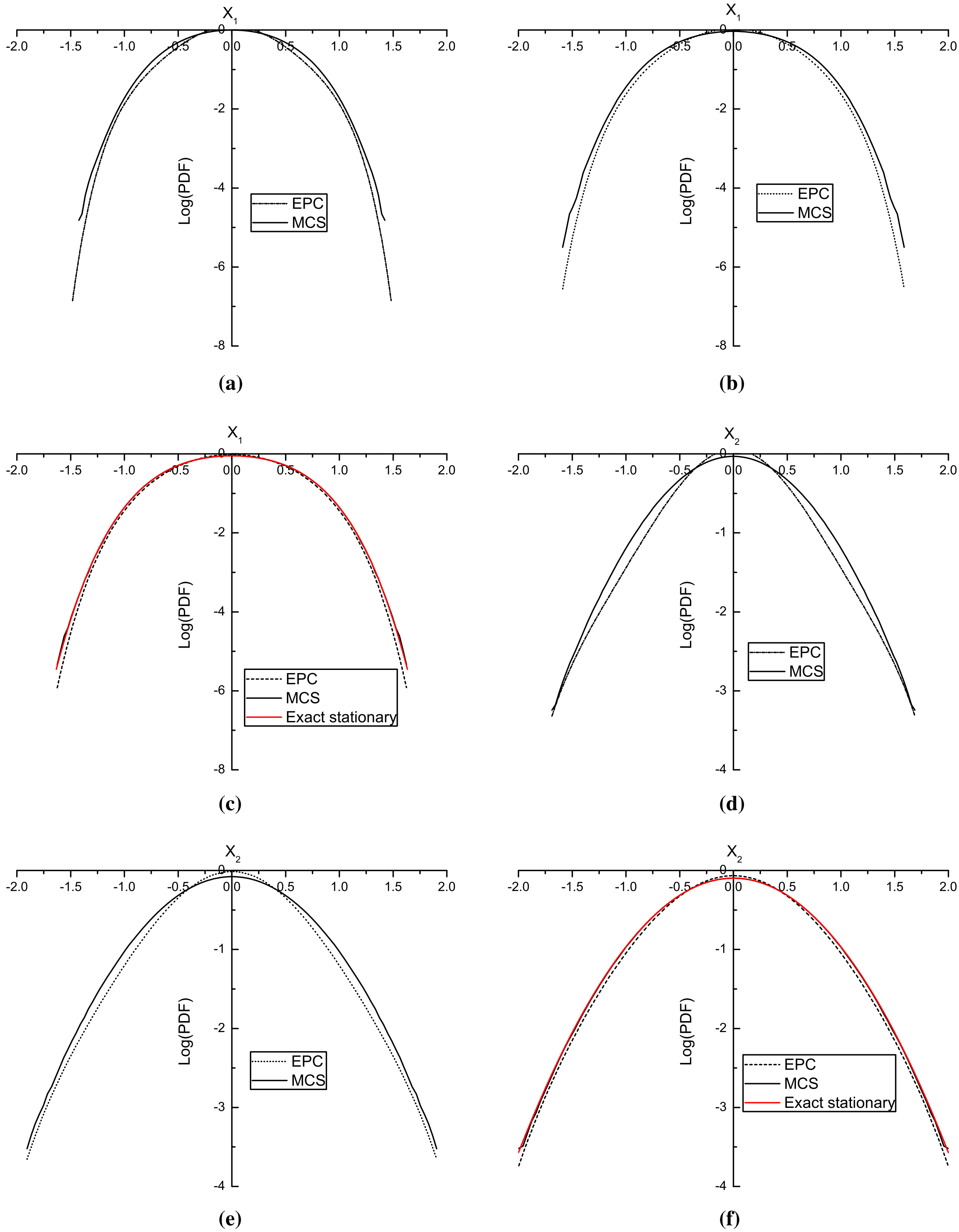

Fig. 2 Log(PDF) of displacement at different time instants: **a** $t = 5$ s, **b** $t = 10$ s, **c** $t = 20$ s. The Log(PDF) of velocity at different time instants: **d** $t = 5$ s, **e** $t = 10$ s, **f** $t = 20$ s

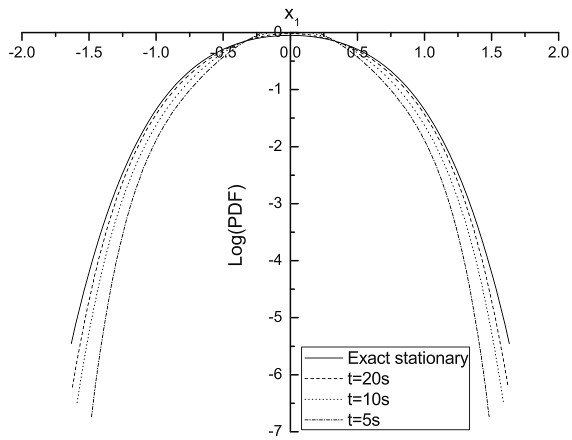

Fig. 3 Time evolution of Log(PDF) of displacement for Duffing oscillator under external excitation in Example 1

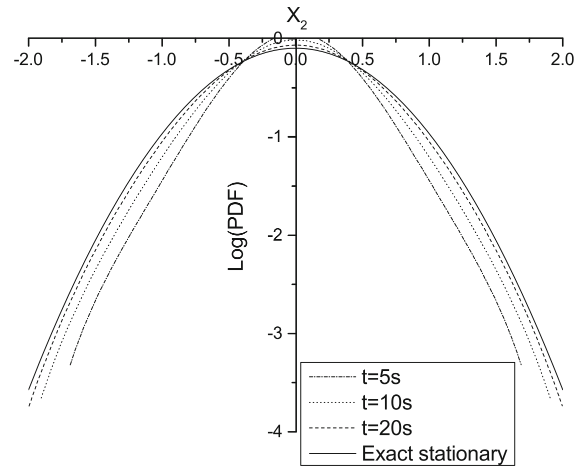

Fig. 4 Time evolution of Log(PDF) of velocity for Duffing oscillator under external excitation in Example 1

Based on the obtained moments of system responses, the EPC method is used for the transient PDF solutions. The results at different time instants are compared with simulated ones and shown in Fig. 2. In order to neglect the effect of initial PDF values, the first time instant has to be long enough and here $t = 5$ s is adopted. It is seen that the PDF values with EPC method agree well with the ones from MCS. As time evolves to $t = 20$ s, the obtained transient PDF solutions coincide with the stationary ones shown in Fig. 2c, f. The stationary joint PDF of $X = X_1$ and $\dot{X} = X_2$ for this system is given by

$$p(x_1, x_2) = C \exp\left(-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{\varepsilon}{4}x_1^4\right), \qquad (29)$$

where C is the normalization constant.

Additionally, it is seen that the PDF solutions become more spread and gradually close to the stationary state as time evolves, which are shown in Figs. 3 and 4.

4.2 Example 2: nonlinear oscillator under both external and parametric excitations

A simply supported uniform column with large deformations is subjected to combined excitations of axial compressive force and a transverse concentrated force at the middle of the column. The transverse motion of the column is governed by an ordinary differential equation as follows [3]:

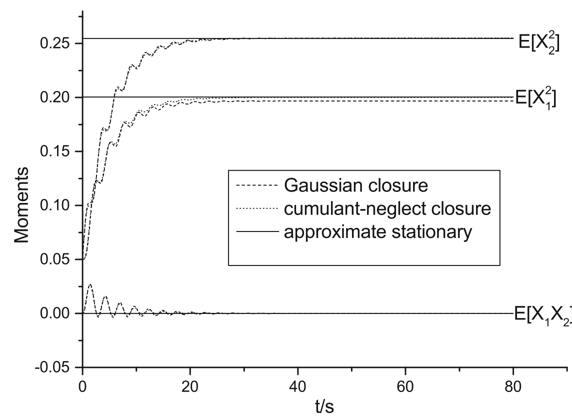

Fig. 5 Time evolution of statistical moments for nonlinear oscillator under external and parametric excitations in Example 2

$$\ddot{X} + \alpha\dot{X} + X + \varepsilon X^3 = cXW_1(t) + W_2(t), \qquad (30)$$

where W_1 and W_2 are independent zero-mean stationary Gaussian white noises with correlation functions $E[W_i(t)W_i(t+\tau)] = S_{ii}\delta(\tau)$. Other parameters are assumed $\alpha = 0.2$, $\varepsilon = 0.5$ and $c = 0.1$.

Statistical moments varied with time are computed by Gaussian closure method, cumulant-neglect closure with $n = 4$. The results are present in Fig. 5. Meanwhile, approximate stationary solutions are also provided, which are acquired with EPC method for the FPK stationary solutions.

$$E[x_i^2] = \int_{\infty}^{\infty} x_i^2 \tilde{p}(x_i)\mathrm{d}x_i \qquad (31)$$

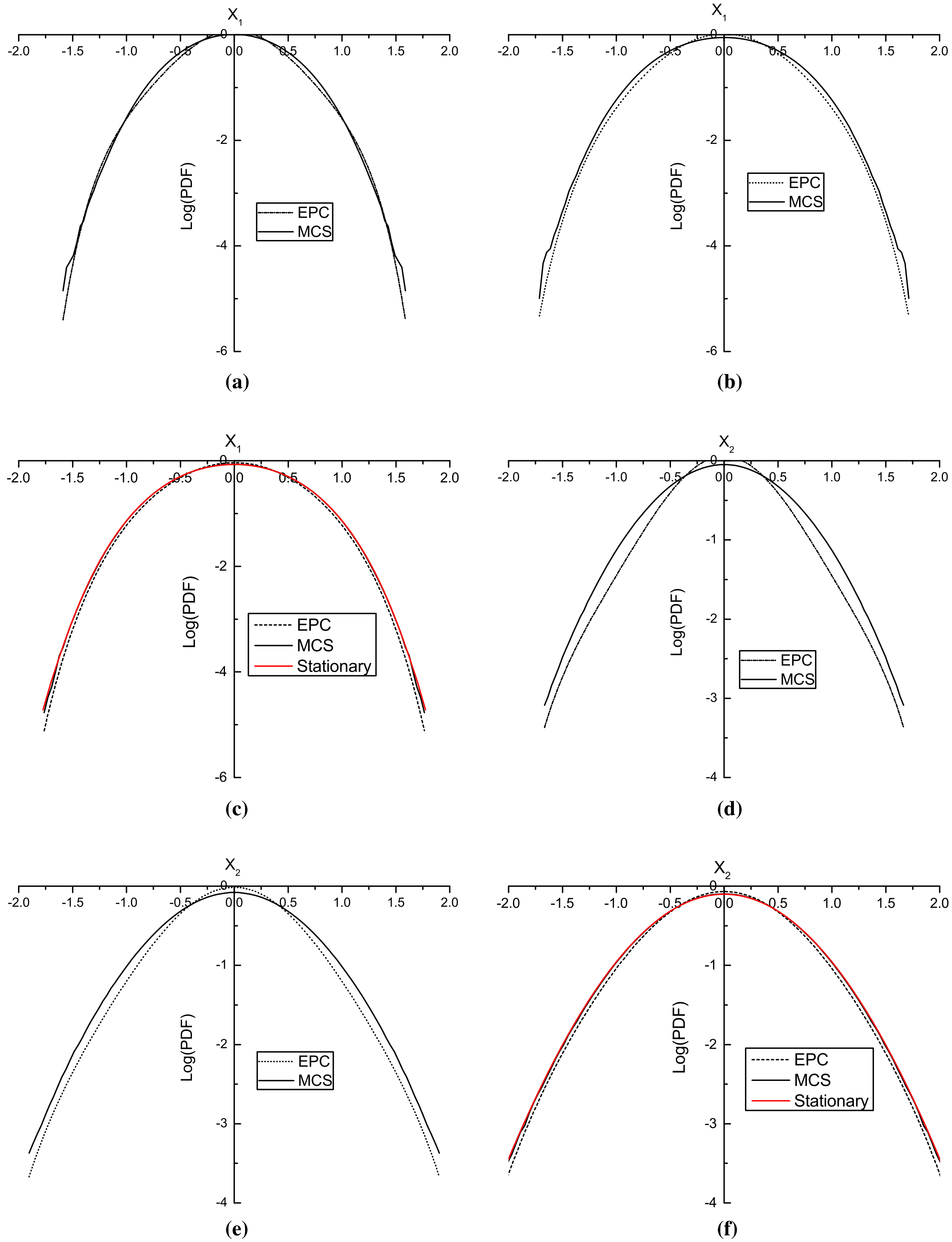

Fig. 6 Log(PDF) of displacement at different time instants: **a** $t = 5\,\text{s}$, **b** $t = 10\,\text{s}$, **c** $t = 20\,\text{s}$. The Log(PDF) of velocity at different time instants: **d** $t = 5\,\text{s}$, **e** $t = 10\,\text{s}$, **f** $t = 20\,\text{s}$

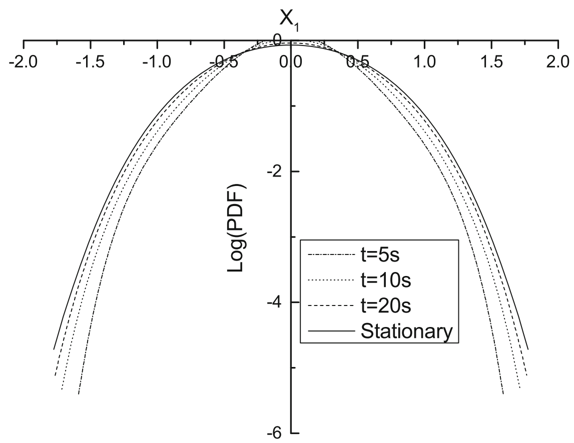

Fig. 7 Time evolution of Log(PDF) of displacement for nonlinear oscillator under external and parametric excitations in Example 2

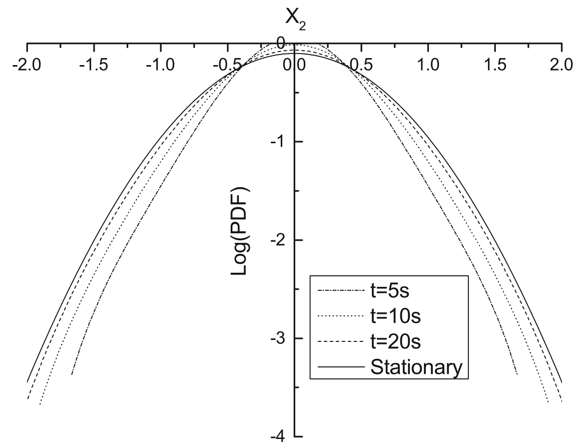

Fig. 8 Time evolution of Log(PDF) of velocity for nonlinear oscillator under external and parametric excitations in Example 2

It is observed that as time evolves, moments gradually become constant and arrive at the stationary state at about time $t = 20$ s. As in the first example, stationary moments obtained with cumulant-neglect closure method match well to the approximate stationary ones. For the stationary displacement variance, the relative error is only 0.2% and is 3% for Gaussian closure method. For other moments $E[x_2^2]$ and $E[x_1 x_2]$, it is seen that the results at the stationary state from these two methods overlap together with the approximate ones.

With above acquired moments, the EPC method and MCS method are conducted for the transient PDF solutions. The results are shown in Fig. 6a–f. At the transient state, it is found that the EPC approximate solutions close to the simulated ones, especially at the tail regions. As time evolves, transient PDF solutions gradually approach to the approximate stationary ones, which are marked to evaluate the obtained transient solutions. The solutions from the EPC method agree well to the approximate stationary ones at time instant $t = 20$ s.

Besides, it is observed that transient PDF solutions are more centralized in the initial and then spread as time increases, which are present in Figs. 7 and 8.

4.3 Example 3: Bouc–Wen hysteretic system under stationary white noise

Structural systems under dynamic loadings usually exhibit hysteretic behavior. Here Bouc–Wen hysteretic oscillator is analyzed

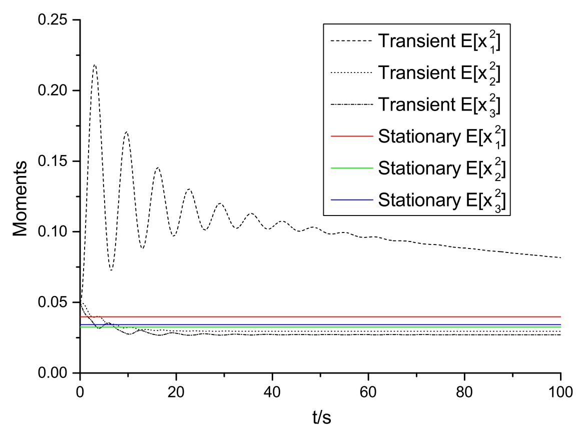

Fig. 9 Time evolution of statistical moments for Bouc–Wen hysteretic system in Example 3

$$\ddot{X} + 2\xi\omega\dot{X} + \alpha\omega^2 X + (1-\alpha)\omega^2 Z = W(t)$$
$$\dot{Z} = -\gamma|\dot{X}|Z - \beta\dot{X}|Z| + A\dot{X}, \tag{32}$$

where system parameters are set as $\xi = 0.05, \omega = 1.0, \alpha = 1/21$. For hysteretic restoring force, hardening system is assumed with $\gamma = 0.5, \beta = -0.5$, and A is set as $A = 1.0$. $W(t)$ is stationary Gaussian white noise with $E[W(t)W(t+\tau)] = 0.01\delta(\tau)$.

Statistical moments varied with time are computed by Gaussian closure method. The results are present in Fig. 9. At the same time, approximate stationary moments are also provided, which are obtained with EPC method for the FPK stationary solutions. It is observed that as time moves on, moments gradually become constant and arrive at the stationary state at about time $t = 20$ s, except for the displacement,

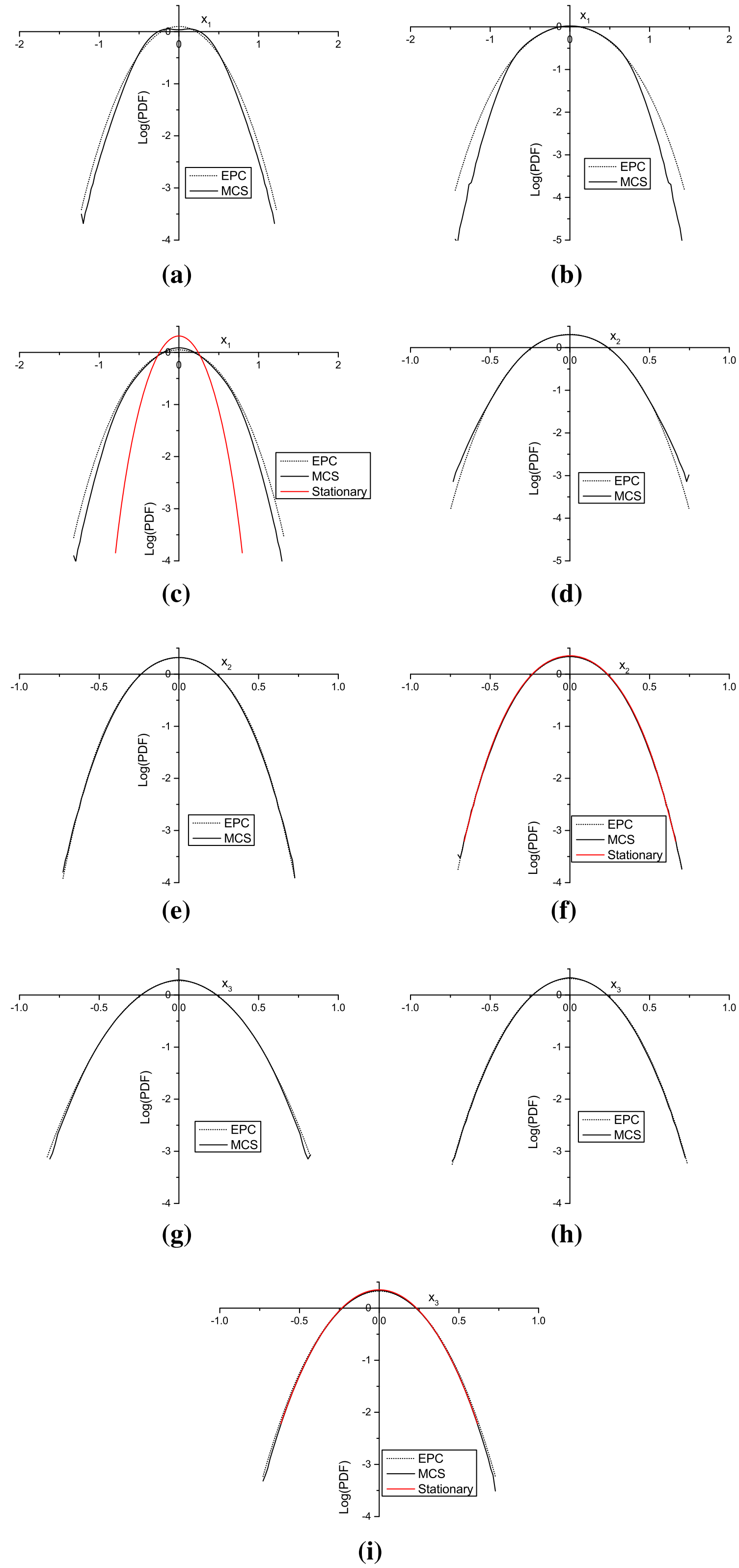

Fig. 10 Log(PDF) of displacement at different time instants: **a** $t = 5$ s, **b** $t = 10$ s, **c** $t = 20$ s. The Log(PDF) of velocity at different time instants: **d** $t = 5$ s, **e** $t = 10$ s, **f** $t = 20$ s. The Log(PDF) of hysteretic force at different time instants: **g** $t = 5$ s, **h** $t = 10$ s, **i** $t = 20$ s

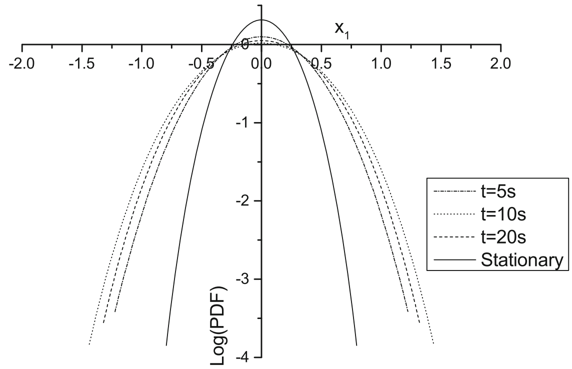

Fig. 11 Time evolution of Log(PDF) of displacement for Bouc–Wen hysteretic system in Example 3

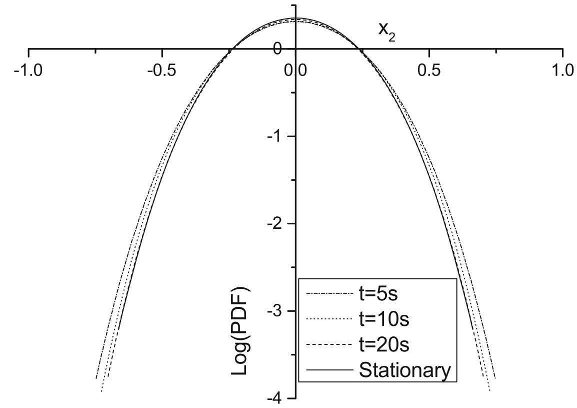

Fig. 12 Time evolution of Log(PDF) of velocity for Bouc–Wen hysteretic system in Example 3

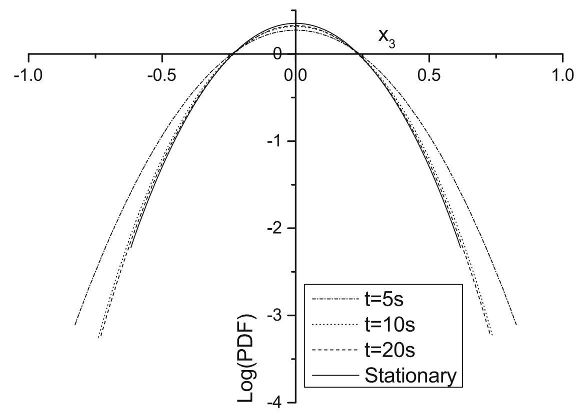

Fig. 13 Time evolution of Log(PDF) of hysteretic force for Bouc–Wen hysteretic system in Example 3

which is still far away from stationary ones even at time instant $t = 100$ s. On the other hand, obvious differences between Gaussian stationary moments and EPC stationary ones can be observed for the displacement, which illustrate the non-Gaussian properties at the transient state.

With above moments results, the EPC method and MCS method are performed for the transient PDF solutions. Approximate transient PDF results obtained with EPC method are compared with those obtained with MCS method as shown in Fig. 10. It is observed that approximations match well to the simulated ones at different time instants. As time moves to $t = 20$ s, it is seen that Log(PDF) of displacement is far from the stationary one, while Log(PDF) of velocity and Log(PDF) of hysteretic restoring force are close to the stationary ones. In order to illustrate this fact, it is seen from Fig. 9 that the time instant is up to about $t = 100$ s when the variance of displacement is still far from a nonzero constant at the stationary state, while the variances of velocity and hysteretic force reach the stationary state after much faster steps. It means that at time instant $t = 20$ s, the displacement is still at transient state. There is yet a long way to reach the stationary state. As a result, the Log(PDF) of displacement greatly different from the stationary one. But for the velocity and hysteretic force, the Log(PDFs) of velocity and of hysteretic force approach to the stationary ones since they already arrive at the stationary state at time $t = 20$ s.

Besides, it is observed that transient PDF solutions are more spread than the stationary ones, which are present in Figs. 11, 12, 13. It supports that systems are more likely to fail at the nonstationary state. Therefore, not only stationary state is taken into account, but also transient state has to be paid attention on.

The developed EPC solution is also tried on oscillators exhibiting bistabilities and limit cycles. Numerical results show that it has the ability to treat such kinds of oscillators. Due to the limitation of paper length, comparison results are not shown here.

5 Conclusions

In this paper, the potential of approximate EPC method to solving nonstationary PDF solutions of the FPK equation has been demonstrated. Specifically, this method expresses the PDF approximation in terms of polynomial functions with time-dependent coefficients. Relying on moment equations and on weighted residual method, the FPK equation is then reduced to a series of nonlinear first-order ordinary differential equations, which can be solved by the numerical method. Three examples of nonlinear oscillators under additive or/and parametric excitations to stationary white noise excitations have been analyzed. Results comparisons between different methods indicate that the developed EPC method is of fair accuracy and efficiency in solving transient solutions to stationary white noise excitations. Notably, the PDF results at the tail regions in the logarithmic form, which are important for reliability and failure analysis, agree very well with the simulated ones, and the computational time is only a small fraction of the one spent by the MCS method.

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