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Local discontinuous Galerkin method for distributed-order time and space-fractional convection–diffusion and Schrödinger-type equations

Tarek Aboelenen

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Abstract We develop a local discontinuous Galerkin finite element method for the distributed-order time and Riesz space-fractional convection–diffusion and Schrödinger-type equations. The stability of the presented schemes is proved and optimal order of convergence $\mathcal{O}(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2)$ for the Riesz space-fractional diffusion and Schrödinger-type equations with distributed order in time, an order of convergence of $\mathcal{O}(h^{N+\frac{1}{2}} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2)$ is provided for the Riesz space-fractional convection–diffusion equations with distributed order in time where h, θ and Δt are space step size, the distributed-order variables and the step sizes in time, respectively. Finally, the performed numerical examples confirm the optimal convergence order and illustrate the effectiveness of the method.

Keywords Fractional convection–diffusion equations with distributed order in time · Fractional Schrödinger-type equations with distributed order in time · Local discontinuous Galerkin method · Stability · Optimal convergence

1 Introduction

The distributed-order fractional differential equation (FDE), as a natural generalization of the single-order and the multi-term FDE, has become more popular in many physical and biological applications recently, for example, dielectric induction and diffusion [1], viscoelastic oscillators [2], the stress behavior of an elastic medium [3], to study properties of rheological of composite materials [4,5], distributed-order membranes in the ear [6]. This kind of equations was first introduced by Caputo [9] who used to find infinite plates and dielectric spherical shells or the eigenfunctions of the torsional models of anelastic. FDEs have recently been shown as a good tool to model the complex dynamical systems than the classical and fractional-order models. Moreover, the time-fractional anomalous diffusion equation with a constant order temporal derivative cannot be used to model the processes lacking temporal scaling, so in some sense, the diffusion equation with distributed order in time can be used to describe processes getting more anomalous in course of time, e.g., the accelerating superdiffusion and retarded subdiffusion [7,8].

In recent years, there has been much attention and effort put on developing various numerical algorithms for solving distributed-order and space-fractional equations. Diethelm and Ford [10,11] transformed the distributed-order ordinary differential equations into a multi-term FDEs by quadrature formula. Ford and Morgado [12] show the uniqueness, existence and the approximation of the solution of distributed FDEs. Podlubny et al. [13] solved the distributed-order FDE by a matrix approach method. Ye et al. [14] discussed the Riesz space-fractional diffusions equations with

T. Aboelenen (✉)
Department of Mathematics, Assiut University,
Assiut 71516, Egypt
e-mail: tarek.aboelenen@aun.edu.eg

distributed order in time by an implicit difference scheme. They prove that the implicit difference scheme is convergent and unconditionally stable. In [15], the authors have developed Jacobi collocation method for the multi-dimensional distributed-order generalized Schrödinger equations [15]. There exist many numerical methods for solving partial differential equations (PDEs) and fractional partial differential equations (FPDEs) in [16–22] and references therein. They differ in accuracy, performance and applicability. To the best of our knowledge, however, the LDG method, which is an important approach to solve FPDEs, has not been considered for the time distributed-order FPDEs. Here, we develop a LDG method to solve the time distributed-order and Riesz space-fractional convection–diffusion equation (TDO-RSFCDE)

$$\begin{aligned} &\mathcal{D}_t^{W(\alpha)}u + \varepsilon(-\Delta)^{\frac{\beta}{2}}u + \frac{\partial}{\partial x}f(u) = 0, \\ &u(x,0) = u_0(x), \end{aligned} \tag{1.1}$$

the nonlinear time distributed-order and Riesz space-fractional Schrödinger equation (TDO-RSFSE)

$$\begin{aligned} &i\mathcal{D}_t^{W(\alpha)}u - \varepsilon_1(-\Delta)^{\frac{\beta}{2}}u + \varepsilon_2 f(|u|^2)u = 0, \\ &u(x,0) = u_0(x), \end{aligned} \tag{1.2}$$

and the coupled nonlinear TDO-RSFSEs

$$\begin{aligned} &i\mathcal{D}_t^{W(\alpha)}u - \varepsilon_1(-\Delta)^{\frac{\beta}{2}}u + \varepsilon_2 f(|u|^2,|v|^2)u = 0, \\ &i\mathcal{D}_t^{W(\alpha)}v - \varepsilon_3(-\Delta)^{\frac{\beta}{2}}v + \varepsilon_4 g(|u|^2,|v|^2)v = 0, \\ &u(x,0) = u_0(x), \\ &v(x,0) = v_0(x), \end{aligned} \tag{1.3}$$

where homogeneous boundary conditions and $x \in \mathbb{R}$, $t \in (0,T]$. $\varepsilon, \varepsilon_i$, $i = 1,2,3,4$ are a real constants and $g(u)$ and $f(u)$ are nonlinear real functions, and $\mathcal{D}_t^{W(\alpha)}u(x,t)$ denotes the distributed-order fractional derivative of u in time t, given by

$$\mathcal{D}_t^{W(\alpha)}u = \int_0^1 W(\alpha)\ {}_0^C\mathcal{D}_t^{\alpha}u(x,t)\mathrm{d}\alpha, \tag{1.4}$$

where $W(\alpha)$ is the weight function, ${}_0^C\mathcal{D}_t^{\alpha}u(x,t)$ $(0 < \alpha < 1)$ is the Caputo fractional derivative of order α with respect to t. The fractional Laplacian $-(-\Delta)^{\frac{\beta}{2}}$ can be characterized using Fourier analysis as [23–25]

$$-(-\Delta)^{\frac{\beta}{2}}u(x,t) = \mathcal{F}^{-1}(|\xi|^{\beta}\hat{u}(\xi,t)),$$

where $\mathcal{F}$ is the Fourier transform.

The Discontinuous Galerkin (DG) method is famous for high accuracy properties and extreme flexibility [26–29]. There exist many applications of DG methods to solve FPDEs, for example, time-fractional wave and diffusion equations [30–33], fractional convection–diffusion equations [34,35], fractional Cahn–Hilliard equation [36] and the nonlinear Riesz space-fractional Schrödinger-type equations [35,37].

The organization of the paper is as follows: we present some preliminary definitions and a few lemmas in Sect. 2. In section 3, we present the DG scheme for the TDO-RSFCDEs. Then we prove L^2 stability as well as an error estimate in Sect. 4. In Sect. 5, we present a LDG method for the nonlinear distributed-order time and Riesz space-fractional Schrödinger-type equations. We introduce the LDG scheme for the TDO-RSFSE in Sect. 5.1. Moreover, we give L^2 stability for the nonlinear case in Sect. 5.1.1 and an error estimations for the linear equation in Sect. 5.1.2. In Sect. 5.2, we introduce the LDG scheme for the coupled nonlinear TDO-RSFSEs and give a theoretical result of L^2 stability and error estimates. Section 6 presents some numerical experiments to illustrate the efficiency of the scheme. Finally, some conclusions are given in Sect. 7.

2 Preliminaries

We first introduce the definitions of fractional derivatives and integrals [38] and review a few lemmas for our analysis.

2.1 Caputo–Liouville fractional calculus

The right-sided and left-sided Riemann–Liouville (R–L) integrals of order μ, when $0 < \mu < 1$, for the function $f(x)$ is defined , respectively, as

$$\left({}^{RL}_{-\infty}\mathcal{I}^{\mu}_{x} f\right)(x) = \frac{1}{\Gamma(\mu)}\int_{-\infty}^{x}\frac{f(s)\mathrm{d}s}{(x-s)^{1-\mu}},\quad x > -\infty, \tag{2.1}$$

and

$$\left({}^{RL}_{x}\mathcal{I}^{\mu}_{\infty} f\right)(x) = \frac{1}{\Gamma(\mu)}\int_{x}^{\infty}\frac{f(s)\mathrm{d}s}{(s-x)^{1-\mu}},\quad x < \infty. \tag{2.2}$$

For any $n \in \mathbb{N}^+$, $(n-1 < \mu < n)$, the right and left R–L fractional derivatives of function f are defined by

$$\left({}_{-\infty}^{RL}\mathcal{D}_x^{\mu} f\right)(x) = \frac{1}{\Gamma(n-\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \times \int_{-\infty}^{x} \frac{f(s)\mathrm{d}s}{(x-s)^{-n+1+\mu}}, \quad x > -\infty, \tag{2.3}$$

and

$$\left({}_{x}^{RL}\mathcal{D}_{\infty}^{\mu} f\right)(x) = \frac{1}{\Gamma(n-\mu)}\left(\frac{-\mathrm{d}}{\mathrm{d}x}\right)^n \times \int_{x}^{\infty} \frac{f(s)\mathrm{d}s}{(s-x)^{-n+1+\mu}}, \quad x < \infty. \tag{2.4}$$

Furthermore, the lift-sided and right-sided Caputo derivatives of function f are obtained as

$$\left({}_{-\infty}^{C}\mathcal{D}_x^{\mu} f\right)(x) = \frac{1}{\Gamma(n-\mu)} \times \int_{-\infty}^{x} \frac{f^{(n)}(s)\mathrm{d}s}{(x-s)^{n-1+\mu}}, \quad x > -\infty, \tag{2.5}$$

and

$$\left({}_{x}^{C}\mathcal{D}_{\infty}^{\mu} f\right)(x) = \frac{1}{\Gamma(n-\mu)} \times \int_{x}^{\infty} \frac{(-1)^n f^{(n)}(s)\mathrm{d}s}{(s-x)}^{n-1+\mu}, \quad x < \infty. \tag{2.6}$$

Proposition 1 *The R–L fractional derivatives of a function f relates to the Caputo definitions by the following equation*

$$\left({}_{-a}^{C}\mathcal{D}_x^{\mu} f\right)(x) = \left({}_{-a}^{RL}\mathcal{D}_x^{\mu} f\right)(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(1+j-\mu)}(x-a)^{j-\mu}, \tag{2.7}$$

which implies $\left({}_{-a}^{C}\mathcal{D}_x^{\mu} f\right)(x) = \left({}_{-a}^{RL}\mathcal{D}_x^{\mu} f\right)(x)$, *if* $f^{(j)}(a) = 0,\ j = 0, \ldots, n-1$.

Proposition 2 *Semigroup and linearity property*

$$\begin{aligned} {}_{-\infty}^{RL}\mathcal{I}_x^{\alpha+\mu} f(x) &= {}_{-\infty}^{RL}\mathcal{I}_x^{\alpha}\left({}_{-\infty}^{RL}\mathcal{I}_x^{\mu} f(x)\right) \\ &= {}_{-\infty}^{RL}\mathcal{I}_x^{\mu}\left({}_{-\infty}^{RL}\mathcal{I}_x^{\alpha} f(x)\right), \end{aligned} \tag{2.8}$$

$$\begin{aligned} {}_{-\infty}^{RL}\mathcal{I}_x^{\mu}(\lambda f(x) + \gamma g(x)) &= \lambda\, {}_{-\infty}^{RL}\mathcal{I}_x^{\mu} f(x) \\ &\quad + \gamma\, {}_{-\infty}^{RL}\mathcal{I}_x^{\mu} g(x). \end{aligned} \tag{2.9}$$

The Riesz fractional derivative is given by

$$\begin{aligned} \frac{\partial^{\mu}}{\partial|x|^{\mu}} f(x) &= -(-\Delta)^{\frac{\mu}{2}} f(x) \\ &= -\frac{{}_{-\infty}^{C}\mathcal{D}_x^{\mu} f(x) + {}_{x}^{C}\mathcal{D}_{\infty}^{\mu} f(x)}{2\cos\left(\frac{\pi\mu}{2}\right)}. \end{aligned} \tag{2.10}$$

If $\mu < 0$, the fractional Laplacian becomes the fractional integral operator. In this case, for any $0 < \alpha < 1$, which is formally defined as follows

$$\begin{aligned} \Delta_{-\alpha/2} f(x) &= -\frac{{}_{-\infty}^{C}\mathcal{D}_x^{-\alpha} f(x) + {}_{x}^{C}\mathcal{D}_{\infty}^{-\alpha} f(x)}{2\cos\left(\frac{\pi(2-\alpha)}{2}\right)} \\ &= \frac{{}_{-\infty}^{C}\mathcal{D}_x^{-\alpha} f(x) + {}_{x}^{C}\mathcal{D}_{\infty}^{-\alpha} f(x)}{2\cos\left(\frac{\pi\alpha}{2}\right)} \\ &= \frac{{}_{-\infty}^{RL}\mathcal{I}_x^{-\alpha} f(x) + {}_{x}^{RL}\mathcal{I}_{\infty}^{-\alpha} f(x)}{2\cos\left(\frac{\pi\alpha}{2}\right)}. \end{aligned} \tag{2.11}$$

When $1 < \mu < 2$, using (2.5), (2.6) and (2.11), we can rewrite the fractional Laplacian in the following form:

$$-(-\Delta)^{\frac{\mu}{2}} u(x) = \Delta_{\frac{(\mu-2)}{2}}\left(\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\right). \tag{2.12}$$

Definition 1 (*The right, left and symmetric fractional spaces* [39]) For any $0 < \alpha < 1$, we define the semi-norms

$$|f|_{J_R^{\alpha}(\mathbb{R})} = \left\| {}_{x}^{RL}\mathcal{D}_{x_R}^{\alpha} f \right\|_{L^2(\mathbb{R})}, \tag{2.13}$$

$$|f|_{J_L^{\alpha}(\mathbb{R})} = \left\| {}_{x_L}^{RL}\mathcal{D}_{x}^{\alpha} f \right\|_{L^2(\mathbb{R})}, \tag{2.14}$$

$$|f|_{J_S^{\alpha}(\mathbb{R})} = \left|\left({}_{x_L}^{RL}\mathcal{D}_x^{\alpha} f,\ {}_{x}^{RL}\mathcal{D}_{x_R}^{\alpha} f\right)_{L^2(\mathbb{R})}\right|^{\frac{1}{2}}, \tag{2.15}$$

and the norms

$$\|f\|_{J_R^{\alpha}(\mathbb{R})} = \left(|f|^2_{J_R^{\alpha}(\mathbb{R})} + \|f\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}, \tag{2.16}$$

$$\|f\|_{J_L^{\alpha}(\mathbb{R})} = \left(|f|^2_{J_L^{\alpha}(\mathbb{R})} + \|f\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}, \tag{2.17}$$

$$\|f\|_{J_S^{\alpha}(\mathbb{R})} = \left(|f|^2_{J_S^{\mu}(\mathbb{R})} + \|f\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}. \tag{2.18}$$

and let the three spaces $J_R^{\alpha}(\mathbb{R})$, $J_L^{\alpha}(\mathbb{R})$ and $J_S^{\alpha}(\mathbb{R})$ denote the closures of $C_0^{\infty}(\mathbb{R})$ with respect to $\|.\|_{J_R^{\alpha}(\mathbb{R})}$ and $\|.\|_{J_L^{\alpha}(\mathbb{R})}$, respectively.

Lemma 1 (see [39]) *For $\alpha > 0$, assume that $f(x)$ is a real function. Then*

$$\begin{aligned} \left({}_{-\infty}^{RL}\mathcal{I}_x^{\alpha} f,\ {}_{x}^{RL}\mathcal{I}_{\infty}^{\alpha} f\right)_{\mathbb{R}} &= \cos(\alpha\pi)|f|^2_{J_L^{-\alpha}(\mathbb{R})} \\ &= \cos(\alpha\pi)|f|^2_{J_R^{-\alpha}(\mathbb{R})}. \end{aligned} \tag{2.19}$$

Generally, we assume that the fractional problems in a bounded domain and let the domain $\Omega = [a, b]$ instead of $\mathbb{R}$.

Lemma 2 (fractional Poincaré–Friedrichs, [39]) *If* $\alpha \in \mathbb{R}$, *we have*

$$\|f\|_{L^2(\Omega)} \le C|f|_{J^{\alpha}_{L,0}(\Omega)}, \quad \forall f \in J^{\alpha}_{L,0}(\Omega). \tag{2.20}$$

and

$$\|f\|_{L^2(\Omega)} \le C|f|_{J^{\alpha}_{R,0}(\Omega)}, \quad \forall f \in J^{\alpha}_{R,0}(\Omega). \tag{2.21}$$

Lemma 3 (See [40]) *The fractional integration operator* $\mathcal{I}^s$ *is bounded in* $L^2(\Omega)$*:*

$$\|\mathcal{I}^{\alpha} f\|_{L^2(\Omega)} \le K\|f\|_{L^2(\Omega)}, \tag{2.22}$$

where $\mathcal{I}^{\alpha} = {}^{RL}_{x_L}\mathcal{I}^{\alpha}_x$ *(i.e., right-sided R–L integral of order* α*).*

Lemma 4 (See [37]) *The fractional integration operator* $\Delta_{-\alpha}$ *is bounded in* $L^2(\Omega)$*:*

$$\|\Delta_{-\alpha} f\|_{L^2(\Omega)} \le K\|f\|_{L^2(\Omega)}, \quad 0 < \alpha < 1. \tag{2.23}$$

3 LDG scheme for the TDO-RSFCDE

Let us consider the TDO-RSFCDE. To discretize the integral interval [0, 1], we choose the grid $0 = \tau_0 < \tau_1 < \cdots < \tau_S = 1$ and take $\Delta\tau_j = \tau_j - \tau_{j-1} = \frac{1}{S} = \theta$, $\alpha_j = \frac{\tau_j + \tau_{j-1}}{2} = \frac{2j-1}{2S}$, $j = 1, 2, \ldots, S$, $S \in \mathbb{N}$ and by the mid-point quadrature rule, we get

$$\mathcal{D}^{W(\alpha)}_t u(x,t) = \sum_{j=1}^{S} W(\alpha_j)\, {}^C_0\mathcal{D}^{\alpha_j}_t u(x,t)\Delta\tau_j + \mathcal{O}(\theta^2). \tag{3.1}$$

Thus the TDO-RSFCDE (1.1) is expressed as multi-term fractional equation. We approximate the time-fractional derivative (3.1) by simple quadrature formula in [41]. Let $\Delta t = T/M$ be the time mesh size, $t_n = n\Delta t$, $n = 0, 1, \ldots, M$ be mesh points and M is a positive integer.

Lemma 5 (See [41]) *Let* $z(t) \in \mathcal{C}^2[0, t_n]$*, for any* $0 < \alpha < 1$*. It holds that*

$$\left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{z'(s)\mathrm{d}s}{(t_n - s)^{\alpha}} - \frac{1}{\lambda}\left[a_0 z(t_n) - a_{n-1}z(0) - \sum_{l=1}^{n-1}(a_{n-l-1} - a_{n-l})z(t_l)\right]\right|$$
$$\le \frac{1}{\frac{2^{2-\alpha}}{2-\alpha} + \Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} - (1 + 2^{-\alpha})\right] \max_{0\le t\le t_n} |z''(t)|(\Delta t)^{2-\alpha}. \tag{3.2}$$

We suppose the notation

$${}^C_0\mathcal{D}^{\alpha}_{t_n} z \approx \delta^{\alpha}_t z^n = \frac{1}{\lambda}\left(z^n - a_{n-1}z^0 - \sum_{l=1}^{n-1}(a_{n-l-1} - a_{n-l})z^l\right). \tag{3.3}$$

From (1.4), (3.1) and (3.3), we obtain

$$\begin{aligned}\mathcal{D}^{W(\alpha)}_{t_n} u &\approx \sum_{j=1}^{S} \Delta\tau_j W(\alpha_j)\delta^{\alpha_j}_t u^n \\ &= \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} \\ &\quad \times \left(u^n - \sum_{l=1}^{n-1}(a^{\alpha_j}_{n-l-1} - a^{\alpha_j}_{n-l})u^l - a^{\alpha_j}_{n-1}u^0\right),\end{aligned} \tag{3.4}$$

where $\lambda_j = (\Delta t)^{\alpha_j}\Gamma(2-\alpha_j)$ and $a^{\alpha_j}_l = (l+1)^{1-\alpha_j} - l^{1-\alpha_j}$.

For space-fractional derivative, we rewrite it as a composite of a fractional integral and first-order derivatives and convert the TDO-RSFCDE (1.1) into a system of low order equations. However, for the first-order system, alternating fluxes are used. We suppose three variables r, p and q and set

$$p = \Delta_{(\beta-2)/2}q, \quad q = \frac{\partial}{\partial x}r, \quad r = \frac{\partial}{\partial x}u, \tag{3.5}$$

then, the TDO-RSFCDE can be rewritten as

$$\begin{aligned}&\mathcal{D}^{W(\alpha)}_t u + \frac{\partial}{\partial x}f(u) - p = 0, \\ &p = \Delta_{(\beta-2)/2}q, \quad q = \frac{\partial}{\partial x}r, \quad r = \frac{\partial}{\partial x}u.\end{aligned} \tag{3.6}$$

For any real number r, the broken Sobolev space is defined as

$$H^r(\Omega) = \{v \in L^2(\Omega) : \forall k = 1, 2, \ldots K, v|_{D^k} \in H^r(D^k)\}. \tag{3.7}$$

The $L^2(D^k)$ norm and local inner product are defined by

$$\|v\|^2_{D^k} = (v, v)_{D^k}, \quad (u, v)_{D^k} = \int_{D^k} uv\mathrm{d}x, \tag{3.8}$$

and the norm and global broken inner product

$$\|v\|^2_{L^2(\Omega)}=\sum_{k=1}^{K}(v,v)_{D^k},\quad (u,v)=\sum_{k=1}^{K}(u,v)_{D^k}. \tag{3.9}$$

We introduce some notation

$$v^{\pm}(x_i)=\lim_{x\to x_i^{\pm}}v(x),\quad \{v\}=\frac{v^{+}+v^{-}}{2},\quad [v]=v^{+}-v^{-}. \tag{3.10}$$

For simplicity, we discretize the computational domain Ω into K non-overlapping elements, $D^k=[x_{k-\frac{1}{2}},x_{k+\frac{1}{2}}]$, $k=1,\ldots,K$. Let $u_h^n,p_h^n,q_h^n,r_h^n\in V_k^N$ be the approximation of $u(.,t_n),p(.,t_n),q(.,t_n),r(.,t_n)$, respectively, where the approximation space is defined as

$$V_k^N=\{v:v_k\in\mathbb{P}(D^k),\ \forall D^k\in\Omega\}, \tag{3.11}$$

where $\mathbb{P}(D^k)$ indicate the set of degree polynomials up to N defined on the element D^k.

We define a fully discrete LDG scheme with as follows: find $u_h^n,p_h^n,q_h^n,r_h^n\in V_k^N$, such that for all test functions $v,\psi,\phi,\eta\in V_k^N$,

$$\begin{aligned}
&\left(\sum_{j=1}^{S}W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}u_h^n,v\right)_{D^k}-\varepsilon\big(p_h^n,v\big)_{D^k}\\
&-\left(f(u_h^n),\frac{\partial}{\partial x}v\right)_{D^k}+\Big((\widehat{f}(u_h^n)v^-)_{k+\frac{1}{2}}\\
&-(\widehat{f}(u_h^n)v^+)_{k-\frac{1}{2}}\Big)=0,\\
&\big(p_h^n,\psi\big)_{D^k}=\big(\Delta_{(\beta-2)/2}q_h^n,\psi\big)_{D^k},\\
&\big(q_h^n,\phi\big)_{D^k}=-\big(r_h^n,\frac{\partial\phi}{\partial x}\big)_{D^k}+\big((\widehat{r}_h^n\phi^-)_{k+\frac{1}{2}}\\
&\qquad-(\widehat{r}_h^n\phi^+)_{k-\frac{1}{2}}\big),\\
&\big(r_h^n,\eta\big)_{D^k}=-\big(u_h^n,\frac{\partial\eta}{\partial x}\big)_{D^k}+\big((\widehat{u}_h^n\eta^-)_{k+\frac{1}{2}}\\
&\qquad-(\widehat{u}_h^n\eta^+)_{k-\frac{1}{2}}\big).
\end{aligned}\tag{3.12}$$

We can choose numerical fluxes as

$$\widehat{u}_h^n=(u_h^n)^-,\quad \widehat{r}_h^n=(r_h^n)^+,\quad \widehat{f}_h=\widehat{f}((u_h^n)^-,(u_h^n)^+). \tag{3.13}$$

Note that we can also choose

$$\widehat{u}_h^n=(u_h^n)^+,\quad \widehat{r}_h^n=(r_h^n)^-,\quad \widehat{f}_h=\widehat{f}((u_h^n)^-,(u_h^n)^+). \tag{3.14}$$

4 Stability analysis and error estimates

Here, we focus on providing the stability analysis and the error estimates of the proposed scheme, for TDO-RSFCDE.

4.1 The analysis of stability for fully discrete scheme

Theorem 1 *The fully discrete LDG scheme* (3.12) *is stable, and*

$$\|u_h^n\|_{L^2(\Omega)}\le C\|u_h^0\|_{L^2(\Omega)},\quad n=1,2,\ldots,M. \tag{4.1}$$

Proof See "Appendix A". □

4.2 Error estimates

we discuss the optimal error for the LDG scheme (3.12). We want to first get the equation of error.

The exact solution of (1.1) satisfies

$$\begin{aligned}
&\left(\sum_{j=1}^{S}W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}u^n,v\right)_{D^k}-\varepsilon\big(p^n,v\big)_{D^k}\\
&+\big(\gamma(x)^n,v\big)_{D^k}-\left(f(u^n),\frac{\partial}{\partial x}v\right)_{D^k}\\
&+\Big((\widehat{f}(u^n)v^-)_{k+\frac{1}{2}}-(\widehat{f}(u^n)v^+)_{k-\frac{1}{2}}\Big)=0,\\
&\big(p^n,\psi\big)_{D^k}=\big(\Delta_{(\beta-2)/2}q^n,\psi\big)_{D^k},\\
&\big(q^n,\phi\big)_{D^k}=-\big(r^n,\frac{\partial\phi}{\partial x}\big)_{D^k}+\big((\widehat{r}^n\phi^-)_{k+\frac{1}{2}}\\
&\qquad-(\widehat{r}^n\phi^+)_{k-\frac{1}{2}}\big),\\
&\big(r^n,\eta\big)_{D^k}=-\big(u^n,\frac{\partial\eta}{\partial x}\big)_{D^k}+\big((\widehat{u}^n\eta^-)_{k+\frac{1}{2}}\\
&\qquad-(\widehat{u}^n\eta^+)_{k-\frac{1}{2}}\big).
\end{aligned}\tag{4.2}$$

Subtracting equation (3.12) from (4.2), we can obtain the error equation

$$\begin{aligned}
&\left(\sum_{j=1}^{S}W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(u^n-u_h^n),v\right)_{D^k}+\big(\gamma(x)^n,v\big)_{D^k}\\
&-\varepsilon\big(p^n-p_h^n,v\big)_{D^k}-\left(f(u^n)-f(u_h^n),\frac{\partial}{\partial x}v\right)_{D^k}\\
&+\big(p^n-p_h^n,\psi\big)_{D^k}+\big(((\widehat{f}(u^n)-\widehat{f}(u_h^n))v^-)_{k+\frac{1}{2}}\\
&-((\widehat{f}(u^n)-\widehat{f}(u_h^n))v^+)_{k-\frac{1}{2}}\big)+\big(q^n-q_h^n,\phi\big)_{D^k}\\
&-\big(\Delta_{(\beta-2)/2}(q^n-q_h^n),\psi\big)_{D^k}+\big(r^n-r_h^n,\frac{\partial\phi}{\partial x}\big)_{D^k}
\end{aligned}$$

$$- ((\widehat{r}^n - \widehat{r}_h^n)\phi^-)_{k+\frac{1}{2}} + ((\widehat{r}^n - \widehat{r}_h^n)\phi^+)_{k-\frac{1}{2}}$$
$$+ \left(r^n - r_h^n, \eta\right)_{D^k} + \left(u^n - u_h^n, \frac{\partial \eta}{\partial x}\right)_{D^k}$$
$$- ((\widehat{u}^n - \widehat{u}_h^n)\eta^-)_{k+\frac{1}{2}} + ((\widehat{u}^n - \widehat{u}_h^n)\eta^+)_{k-\frac{1}{2}} = 0, \tag{4.3}$$

where

$$|\gamma(x)^n| = |\mathcal{O}((\Delta t)^{2-\alpha_j} + \theta^2)| \le c((\Delta t)^{1+\frac{\theta}{2}} + \theta^2), \tag{4.4}$$

such that

$$1 + \frac{\theta}{2} = 2 - S\theta + \frac{\theta}{2} \le 2 - \alpha_j = 2 - j\theta + \frac{\theta}{2} \le 2 - \theta + \frac{\theta}{2} = 2 - \frac{\theta}{2}. \tag{4.5}$$

For the error estimate, we define special projections, $\mathcal{P}$ and $\mathcal{P}^{\pm}$ into V_h^k. For all the elements, D^k, $k = 1, 2, \ldots, K$ are defined to satisfy

$$(\mathcal{P}u - u, v)_{D^k} = 0, \quad \forall v \in \mathbb{P}_N^k(D^k),$$
$$(\mathcal{P}^{\pm}u - u, v)_{D^k} = 0, \quad \forall v \in \mathbb{P}_N^{k-1}(D^k), \tag{4.6}$$
$$\mathcal{P}^{\pm}u_{k+\frac{1}{2}} = u(x^{\pm}_{k+\frac{1}{2}}).$$

Denoting

$$\pi^n = \mathcal{P}^- u^n - u_h^n, \quad \pi_n^e = \mathcal{P}^- u^n - u^n,$$
$$\sigma^n = \mathcal{P}p^n - p_h^n, \quad \sigma_n^e = \mathcal{P}p^n - p^n,$$
$$\varphi^n = \mathcal{P}q^n - q_h^n, \quad \varphi_n^e = \mathcal{P}q^n - q^n, \tag{4.7}$$
$$\psi^n = \mathcal{P}^+ r^n - r_h^n, \quad \psi_n^e = \mathcal{P}^+ r^n - r^n.$$

From the standard approximation theory [42], the projections satisfy the following inequality

$$\|\pi^e\|_{L^2(\Omega)} + h\|\pi^e\|_\infty + h^{\frac{1}{2}}\|\pi\|_{\Gamma_h} \le Ch^{N+1}. \tag{4.8}$$

where $\pi^e = \mathcal{P}^{\pm}u^n - u_h^n$ or $\pi^e = \mathcal{P}u^n - u_h^n$. Γ_h denotes the set of boundary points of all elements D^k. The positive constant C, solely depending on u^n, is independent of h.

Theorem 2 (Diffusion without convection $f(u) = 0$) *The L^2 error of the scheme (3.12) satisfies:*

$$\|u(x, t_n) - u_h^n\|_{L^2(\Omega)} \le C(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2). \tag{4.9}$$

Proof See "Appendix B".

For the TDO-RSFCDE, we review a few lemmas for our analysis. □

Lemma 6 (see [43]) *For any piecewise smooth function $\pi \in L^2(\Omega)$, on each cell boundary point we define*

$$\kappa(\widehat{f}; \pi) \equiv \kappa(\widehat{f}; \pi^-, \pi^+) = \begin{cases} [w]^{-1}(f(\pi) - \widehat{f}(\pi)), & if [\pi] \ne 0; \\ \frac{1}{2}|f'(\overline{\pi})|, & if [\pi] = 0, \end{cases} \tag{4.10}$$

The nonlinear part can be rewritten as

$$\sum_{k=1}^K \mathcal{H}_k(f; u, u_h; \pi) = \sum_{k=1}^K \left(f(u) - f(u_h), \frac{\partial}{\partial x}\pi\right)_{D^k} + \sum_{k=1}^K ((f(u) - f(u_h)[\pi])_{k+\frac{1}{2}} + \sum_{k=1}^K ((f(u_h) - \widehat{f})[\pi])_{k+\frac{1}{2}}. \tag{4.11}$$

We can rewrite (4.11) as:

$$\sum_{k=1}^K \mathcal{H}_k(f; u, u_h; \pi) = \sum_{k=1}^K \left(f(u) - f(u_h), \frac{\partial}{\partial x}\pi\right)_{D^k} + \sum_{k=1}^K ((f(u) - f(\{u_h\})[\pi])_{k+\frac{1}{2}} + \sum_{k=1}^K ((f(\{u_h\}) - \widehat{f})[\pi])_{k+\frac{1}{2}}. \tag{4.12}$$

Lemma 7 (see [43]) *For $\mathcal{H}_k(f; u, u_h; \pi)$ defined above, we have the following estimate:*

$$\sum_{k=1}^K \mathcal{H}_k(f; u, u_h; v) \le -\frac{1}{4}\kappa(\widehat{f}; u_h) + (C + C_* h^{-1}\|e_u\|_\infty^2)h^{2N+1} + (C + C_*(\|v\|_\infty + h^{-1}\|e_u\|_\infty^2))\|v\|^2. \tag{4.13}$$

and

$$\|e_u\| = \|u - u_h\| \le h. \tag{4.14}$$

Theorem 3 *The L^2 error of the scheme (3.12) satisfies:*

$$\|u(x, t_n) - u_h^n\|_{L^2(\Omega)} \le C\left(h^{N+\frac{1}{2}} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2\right). \tag{4.15}$$

Proof See "Appendix C". □

5 LDG method for the nonlinear distributed-order time and Riesz space-fractional Schrödinger-type equations

5.1 LDG method for the nonlinear TDO-RSFSE

We introduce three variables e, r, s and set

$$e = \Delta_{(\beta-2)/2} r, \quad r = \frac{\partial}{\partial x} s, \quad s = \frac{\partial}{\partial x} u, \tag{5.1}$$

then, the nonlinear TDO-RSFSE can be rewritten as

$$\begin{aligned} & i\mathcal{D}_t^{W(\alpha)} u + \varepsilon_1 e + \varepsilon_2 f(|u|^2) u = 0, \\ & e = \Delta_{(\beta-2)/2} r, \quad r = \frac{\partial}{\partial x} s, \quad s = \frac{\partial}{\partial x} u. \end{aligned} \tag{5.2}$$

We set the complex function $u(x, t)$ as

$$u(x, t) = p(x, t) + iq(x, t), \tag{5.3}$$

so, the nonlinear TDO-RSFSE (5.2) can be rewritten as

$$\begin{aligned} & \mathcal{D}_t^{W(\alpha)} p + \varepsilon_1 e + \varepsilon_2 f(p^2 + q^2) q = 0, \\ & e = \Delta_{(\beta-2)/2} r, \quad r = \frac{\partial}{\partial x} s, \quad s = \frac{\partial}{\partial x} q, \\ & \mathcal{D}_t^{W(\alpha)} q - \varepsilon_1 l - \varepsilon_2 f(p^2 + q^2) p = 0, \\ & l = \Delta_{(\alpha-2)/2} w, \quad w = \frac{\partial}{\partial x} z, \quad z = \frac{\partial}{\partial x} p. \end{aligned} \tag{5.4}$$

Let $p_h^n, q_h^n, e_h^n, l_h^n, r_h^n, s_h^n, w_h^n, z_h^n \in V_k^N$ be the approximation of $p(., t_n), q(., t_n), r(., t_n), s(., t_n), e(., t_n), l(., t_n), w(., t_n), z(., t_n)$, respectively.

Let $p_h^n, q_h^n, r_h^n, s_h^n, e_h^n, l_h^n, w_h^n, z_h^n \in V_k^N$, and test functions $\vartheta_1, \rho, \phi, \varphi, \chi, \varrho, \psi, \zeta \in V_k^N$,

$$\begin{aligned} & \left(\sum_{j=1}^{S} W(\alpha_j) \Delta\tau_j \delta_t^{\alpha_j} p_h^n, \vartheta_1 \right)_{D^k} + \varepsilon_1 \left(e_h^n, \vartheta_1\right)_{D^k} \\ & \quad + \varepsilon_2 \left(\left(f((p_h^n)^2 + (q_h^n)^2)\right) q_h^n, \vartheta_1 \right)_{D^k} = 0, \\ & \left(e_h^n, \rho\right)_{D^k} = \left(\Delta_{(\beta-2)/2} r_h^n, \rho\right)_{D^k}, \\ & \left(r_h^n, \phi\right)_{D^k} = -\left(s_h^n, \phi_x\right)_{D^k} \\ & \qquad + \left((\widehat{s}_h^n \phi^-)_{k+\frac{1}{2}} - (\widehat{s}_h^n \phi^+)_{k-\frac{1}{2}} \right), \\ & \left(s_h^n, \varphi\right)_{D^k} = -\left(q_h^n, \varphi_x\right)_{D^k} \\ & \qquad + \left((\widehat{q}_h^n \varphi^-)_{k+\frac{1}{2}} - (\widehat{q}_h^n \varphi^+)_{k-\frac{1}{2}} \right), \\ & \left(\sum_{j=1}^{S} W(\alpha_j) \Delta\tau_j \delta_t^{\alpha_j} q_h^n, \chi \right)_{D^k} - \varepsilon_1 \left(l_h^n, \chi\right)_{D^k} \\ & \quad - \varepsilon_2 \left(\left(f((p_h^n)^2 + (q_h^n)^2)\right) p_h^n, \chi \right)_{D^k} = 0, \\ & \left(l_h^n, \varrho\right)_{D^k} = \left(\Delta_{(\beta-2)/2} w_h^n, \varrho\right)_{D^k}, \\ & \left(w_h^n, \psi\right)_{D^k} = -\left(z_h^n, \psi_x\right)_{D^k} \\ & \qquad + \left((\widehat{z}_h^n \psi^-)_{k+\frac{1}{2}} - (\widehat{z}_h^n \psi^+)_{k-\frac{1}{2}} \right), \\ & \left(z_h^n, \zeta\right)_{D^k} = -\left(p_h^n, \zeta_x\right)_{D^k} \\ & \qquad + \left((\widehat{p}_h^n \zeta^-)_{k+\frac{1}{2}} - (\widehat{p}_h^n \zeta^+)_{k-\frac{1}{2}} \right). \end{aligned} \tag{5.5}$$

The numerical traces $(\widehat{p}_h^n, \widehat{q}_h^n, \widehat{s}_h^n, \widehat{z}_h^n)$ are defined on interelement faces as the alternating fluxes [22,44]:

$$\widehat{p}_h^n = (p_h^n)^-, \ \widehat{s}_h^n = (s_h^n)^+, \quad \widehat{q}_h^n = (q_h^n)^-, \ \widehat{z}_h^n = (z_h^n)^+. \tag{5.6}$$

Note that we can also choose

$$\widehat{p}_h^n = (p_h^n)^+, \ \widehat{s}_h^n = (s_h^n)^-, \quad \widehat{q}_h^n = (q_h^n)^+, \ \widehat{z}_h^n = (z_h^n)^-. \tag{5.7}$$

5.1.1 The analysis of stability for fully discrete scheme

we focus on providing the stability analysis of the proposed scheme, for TDO-RSFSE

Theorem 4 (L^2 stability) *The semidiscrete scheme* (5.5) *is stable, and*

$$\begin{aligned} & \|p_h^n\|_{L^2(\Omega)}^2 + \|q_h^n\|_{L^2(\Omega)}^2 \\ & \quad \leq C \left(\|p_h^0\|_{L^2(\Omega)}^2 + \|q_h^0\|_{L^2(\Omega)}^2 \right). \end{aligned} \tag{5.8}$$

Proof We can prove by similar techniques as that in the proof of Theorem 4.1. □

5.1.2 Error estimates

Here, we focus on providing the error estimates of the proposed scheme, for TDO-RSFSE. We consider the linear TDO-RSFSE

$$i\mathcal{D}_t^{W(\alpha)} u - \varepsilon_1 (-\Delta)^{\frac{\alpha}{2}} u + \varepsilon_2 u = 0. \tag{5.9}$$

The exact solution of (5.9) satisfies

$$\begin{aligned} & \left(\sum_{j=1}^{S} W(\alpha_j) \Delta\tau_j \delta_t^{\alpha_j} p^n, \vartheta_1 \right)_{D^k} + \varepsilon_1 \left(e^n, \vartheta_1\right)_{D^k} \\ & \quad + \varepsilon_2 \left(q^n, \vartheta_1\right)_{D^k} + \left(\gamma(p)^n, \vartheta_1\right)_{D^k} = 0, \\ & \left(e^n, \rho\right)_{D^k} = \left(\Delta_{(\beta-2)/2} r^n, \rho\right)_{D^k}, \\ & \left(r^n, \phi\right)_{D^k} = -\left(s^n, \phi_x\right)_{D^k} + \left((\widehat{s}^n \phi^-)_{k+\frac{1}{2}} \right. \\ & \qquad \left. - (\widehat{s}^n \phi^+)_{k-\frac{1}{2}} \right), \\ & \left(s^n, \varphi\right)_{D^k} = -\left(q^n, \varphi_x\right)_{D^k} + \left((\widehat{q}^n \varphi^-)_{k+\frac{1}{2}} \right. \\ & \qquad \left. - (\widehat{q}^n \varphi^+)_{k-\frac{1}{2}} \right), \end{aligned}$$

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} q^n, \chi\right)_{D^k} - \varepsilon_1\big(l^n, \chi\big)_{D^k}$$
$$- \varepsilon_2\big(p^n, \chi\big)_{D^k} - \big(\gamma(p)^n, \chi\big)_{D^k} = 0 = 0,$$
$$\big(l^n, \varrho\big)_{D^k} = \big(\Delta_{(\beta-2)/2} w^n, \varrho\big)_{D^k},$$
$$\big(w^n, \psi\big)_{D^k} = -\big(z^n, \psi_x\big)_{D^k} + \big((\widehat{z}^n \psi^-)_{k+\frac{1}{2}} - (\widehat{z}^n \psi^+)_{k-\frac{1}{2}}\big),$$
$$\big(z^n, \zeta\big)_{D^k} = -\big(p^n, \zeta_x\big)_{D^k} + \big((\widehat{p}^n \psi^-)_{k+\frac{1}{2}} - (\widehat{p}^n \zeta^+)_{k-\frac{1}{2}}\big). \tag{5.10}$$

Subtracting (5.5) from (5.10), we can obtain the error equation

$$\begin{aligned}
&\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} (p^n - p_h^n), \vartheta_1\right)_{D^k} + \big(s^n - s_h^n, \phi_x\big)_{D^k} \\
&- \big(\Delta_{(\beta-2)/2}(r^n - r_h^n), \rho\big)_{D^k} + \big(q^n - q_h^n, \varphi_x\big)_{D^k} \\
&+ \Big(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} (q^n - q_h^n), \chi\Big)_{D^k} - \big(\gamma(q)^n, \chi\big)_{D^k} \\
&- \big(\Delta_{(\beta-2)/2}(w^n - w_h^n), \varrho\big)_{D^k} + \big(z^n - z_h^n, \psi_x\big)_{D^k} \\
&+ \big(\gamma(p)^n, \vartheta_1\big)_{D^k} + \varepsilon_2\big(q^n - q_h^n, \vartheta_1\big)_{D^k} \\
&- \varepsilon_2\big(p^n - p_h^n, \chi\big)_{D^k} + \big(r^n - r_h^n, \phi\big)_{D^k} \\
&+ \big(s^n - s_h^n, \varphi\big)_{D^k} + \big(e^n - e_h^n, \rho\big)_{D^k} \\
&+ \big(l^n - l_h^n, \varrho\big)_{D^k} + \big(w^n - w_h^n, \psi\big)_{D^k} \\
&+ \big(z^n - z_h^n, \zeta\big)_{D^k} - \varepsilon_1\big(l^n - l_h^n, \chi\big)_{D^k} \\
&+ \varepsilon_1\big(e^n - e_h^n, \vartheta_1\big)_{D^k} + \big(p^n - p_h^n, \zeta_x\big)_{D^k} \\
&+ \big(z^n - z_h^n, \zeta\big)_{D^k} - \big((\widehat{s}_h^n - \widehat{s}^n)\phi^+\big)_{k-\frac{1}{2}}\big) \\
&- \big(((\widehat{q}_h^n - \widehat{q}^n)\varphi^-)_{k+\frac{1}{2}} - ((\widehat{q}_h^n - \widehat{q}^n)\varphi^+)_{k-\frac{1}{2}}\big) \\
&- \big(((\widehat{z}_h^n - \widehat{z}^n)\psi^-)_{k+\frac{1}{2}} - ((\widehat{z}_h^n - \widehat{z}^n)\psi^+)_{k-\frac{1}{2}}\big) \\
&- \big(((\widehat{p}_h^n - \widehat{p}^n)\zeta^-)_{k+\frac{1}{2}} - ((\widehat{p}_h^n - \widehat{p}^n)\zeta^+)_{k-\frac{1}{2}}\big) = 0.
\end{aligned} \tag{5.11}$$

Denoting

$$\begin{aligned}
&\pi^n = \mathcal{P}^- p^n - p_h^n, \quad \pi_n^e = \mathcal{P}^- p^n - p^n, \\
&\epsilon^n = \mathcal{P} r^n - r_h^n, \quad \epsilon_n^e = \mathcal{P} r^n - r^n, \\
&\phi^n = \mathcal{P} e^n - e_h^n, \quad \phi_n^e = \mathcal{P} e^n - e^n, \\
&\tau^n = \mathcal{P}^+ s^n - s_h^n, \quad \tau_n^e = \mathcal{P}^+ s^n - s^n, \\
&\sigma^n = \mathcal{P}^- q^n - q_h^n, \quad \sigma_n^e = \mathcal{P}^- q^n - q^n, \\
&\varpi^n = \mathcal{P} l^n - l_h^n, \quad \varpi_n^e = \mathcal{P} l^n - l^n, \\
&\varphi^n = \mathcal{P} w^n - w_h^n, \quad \varphi_n^e = \mathcal{P} w^n - w^n, \\
&\vartheta^n = \mathcal{P}^+ z^n - z_h^n, \quad \vartheta_n^e = \mathcal{P}^+ z^n - z^n.
\end{aligned} \tag{5.12}$$

Lemma 8

$$\begin{aligned}
&\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} \pi^n, \pi^n\right) + \big(\Delta_{(\beta-2)/2}\epsilon^n, \epsilon^n\big) \\
&\quad + \left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} \sigma^n, \sigma^n\right) + \big(\phi^n, \phi^n\big) \\
&\quad + \big(\Delta_{(\beta-2)/2}\varphi^n, \varphi^n\big) + \big(\varpi^n, \varpi^n\big) \\
&= Q_1 + Q_2 + Q_3 + Q_4,
\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}
Q_1 &= -\big(\epsilon^n, \pi^n\big) + \big(\varphi^n, \sigma^n\big) + \big(\Delta_{(\beta-2)/2}\epsilon^n, \phi^n\big) \\
&\quad + \big(\varpi^n, \varphi^n\big) - \varepsilon_1\big(\phi^n, \pi^n\big) + \varepsilon_1\big(\varpi^n, \sigma^n\big) \\
&\quad + \big(\Delta_{(\beta-2)/2}\varphi^n, \varpi^n\big) + \big(\phi^n, \epsilon^n\big), \\
Q_2 &= \big(\tau_n^e, \pi_x^n\big) - \big(\sigma_n^e, \vartheta_x^n\big) \\
&\quad - \big(\vartheta_n^e, \sigma_x^n\big) + \big(\pi_n^e, \tau_x^n\big) + \big(\vartheta_n^e, \tau^n\big) - \big(\tau_n^e, \vartheta^n\big), \\
Q_3 &= \left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} \pi_n^e, \pi^n\right) + \big(\varpi^e, \varpi - \varphi\big) \\
&\quad + \left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} \sigma_n^e, \sigma^n\right) - \big(\varphi_n^e, \sigma^n\big) \\
&\quad + \big(\phi_n^e, \phi^n - \epsilon^n\big) + \varepsilon_2\big(\sigma_n^e, \pi^n\big) - \varepsilon_2\big(\pi_n^e, \sigma^n\big) \\
&\quad + \big(\epsilon_n^e, \pi^n\big) - \big(\Delta_{(\beta-2)/2}\epsilon_n^e, \phi^n - \epsilon^n\big) \\
&\quad + \varepsilon_1\big(\phi_n^e, \pi^n\big) - \big(\gamma(q)^n, \sigma^n\big) + \big(\gamma(p)^n, \pi^n\big) \\
&\quad - \big(\Delta_{(\beta-2)/2}\varphi_n^e, \varpi^n - \varphi^n\big) - \varepsilon_1\big(\varpi_n^e, \sigma^n\big), \\
Q_4 &= -\sum_{k=1}^{K}\big(((\tau_n^e)^+(\pi^n)^-)_{k+\frac{1}{2}} - ((\tau_n^e)^+(\pi^n)^+)_{k-\frac{1}{2}}\big) \\
&\quad + \sum_{k=1}^{K}\big(((\sigma_n^e)^-(\vartheta^n)^-)_{k+\frac{1}{2}} - ((\sigma_n^e)^-(\vartheta^n)^+)_{k-\frac{1}{2}}\big) \\
&\quad + \sum_{k=1}^{K}\big(((\vartheta_n^e)^+(\sigma^n)^-)_{k+\frac{1}{2}} - ((\vartheta_n^e)^+(\sigma^n)^+)_{k-\frac{1}{2}}\big) \\
&\quad - \sum_{k=1}^{K}\big(((\pi_n^e)^-(\tau^n)^-)_{k+\frac{1}{2}} - ((\pi_n^e)^-(\tau^n)^+)_{k-\frac{1}{2}}\big).
\end{aligned} \tag{5.14}$$

Proof See “Appendix D”. □

Theorem 5 *The L^2 error of the scheme (5.5) satisfies:*

$$\|u(x, t_n) - u_h^n\|_{L^2(\Omega)} \le C\left(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2\right). \tag{5.15}$$

Proof We estimate the term Q_i, $i = 1, \ldots, 4$ in Lemma 8. So we employ Young’s inequality, Lemma 4 and the approximation results (4.8) and similar techniques as that in the proof of Theorem 2, we obtain the result. □

5.2 LDG method for the coupled nonlinear TDO-RSFSEs

We present the LDG method for the coupled nonlinear TDO-RSFSEs

$$\begin{aligned} &i\mathcal{D}_t^{W(\alpha)}u_1 - \varepsilon_1(-\Delta)^{\frac{\beta}{2}}u_1 + \varepsilon_2 f(|u_1|^2,|u_2|^2)u_1 = 0,\\ &i\mathcal{D}_t^{W(\alpha)}u_2 - \varepsilon_3(-\Delta)^{\frac{\beta}{2}}u_2 + \varepsilon_4 g(|u_1|^2,|u_2|^2)u_2 = 0. \end{aligned} \tag{5.16}$$

We rewrite (5.16) as a first-order system:

$$\begin{aligned} &i\mathcal{D}_t^{W(\alpha)}u_1 + \varepsilon_1 e + \varepsilon_2 f(|u_1|^2,|u_2|^2)u_1 = 0,\\ &e = \Delta_{(\beta-2)/2}r, \quad r = \frac{\partial}{\partial x}s, \quad s = \frac{\partial}{\partial x}u_1,\\ &i\mathcal{D}_t^{W(\alpha)}u_2 + \varepsilon_3 l + \varepsilon_4 g(|u_1|^2,|u_2|^2)u_2 = 0,\\ &l = \Delta_{(\beta-2)/2}w, \quad w = \frac{\partial}{\partial x}z, \quad z = \frac{\partial}{\partial x}u_2. \end{aligned} \tag{5.17}$$

Setting $u_1(x,t) = p(x,t) + iq(x,t)$ and $u_2(x,t) = \upsilon(x,t) + i\vartheta(x,t)$ in system (5.16), we obtain the following coupled system

$$\begin{aligned} &\mathcal{D}_t^{W(\alpha)}p + \varepsilon_1 Q + \varepsilon_2 f(|u_1|^2,|u_2|^2)q = 0,\\ &Q = \Delta_{(\beta-2)/2}r, \quad r = \frac{\partial}{\partial x}s, \quad s = \frac{\partial}{\partial x}q,\\ &\mathcal{D}_t^{W(\alpha)}q - \varepsilon_1 H - \varepsilon_2 f(|u_1|^2,|u_2|^2)p = 0,\\ &H = \Delta_{(\beta-2)/2}w, \quad w = \frac{\partial}{\partial x}z, \quad z = \frac{\partial}{\partial x}p,\\ &\mathcal{D}_t^{W(\alpha)}\upsilon + \varepsilon_3 L + \varepsilon_4 g(|u_1|^2,|u_2|^2)\vartheta = 0,\\ &L = \Delta_{(\beta-2)/2}\rho, \quad \rho = \frac{\partial}{\partial x}\varpi, \quad \varpi = \frac{\partial}{\partial x}\vartheta,\\ &\mathcal{D}_t^{W(\alpha)}\vartheta - \varepsilon_3 E - \varepsilon_4 g(|u_1|^2,|u_2|^2)\upsilon = 0,\\ &E = \Delta_{(\beta-2)/2}\xi, \quad \xi = \frac{\partial}{\partial x}\varrho, \quad \varrho = \frac{\partial}{\partial x}\upsilon. \end{aligned} \tag{5.18}$$

Let $p_h^n, q_h^n, Q^n, r_h^n, s_h^n, H_h^n, w_h^n, z_h^n, \upsilon_h^n, \vartheta_h^n, L_h^n, \rho_h^n, \varpi_h^n, E_h^n, \xi_h^n, \varrho_h^n \in V_k^N$, and test functions $\vartheta_1, \beta_1, \phi, \varphi, \chi, \beta_2, \psi, \zeta, \gamma, \beta_3, \delta, \varsigma, o, \beta_4, \omega, \kappa \in V_k^N$,

$$\begin{aligned}
&\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}p_h^n, \vartheta_1\right)_{D^k} + \varepsilon_1\big(Q_h^n, \vartheta_1\big)_{D^k}\\
&\quad + \varepsilon_2\big(f(|u_{1h}^n|^2,|u_{2h}^n|^2)q_h^n, \vartheta_1\big)_{D^k} = 0,\\
&\big(Q_h^n, \beta_1\big)_{D^k} = \big(\Delta_{(\beta-2)/2}r_h^n, \beta_1\big)_{D^k},\\
&\big(r_h^n, \phi\big)_{D^k} = -\big(s_h^n, \phi_x\big)_{D^k} + \big((\widehat{s}_h^n\phi^-)_{k+\frac12}\\
&\quad - (\widehat{s}_h^n\phi^+)_{k-\frac12}\big),\\
&\big(s_h^n, \varphi\big)_{D^k} = -\big(q_h^n, \varphi_x\big)_{D^k} + \big((\widehat{q}_h^n\varphi^-)_{k+\frac12}\\
&\quad - (\widehat{r}_h^n\varphi^+)_{k-\frac12}\big),\\
&\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}q_h^n, \chi\right)_{D^k} - \varepsilon_1\big(H_h^n, \chi\big)_{D^k}\\
&\quad - \varepsilon_2\big(f(|u_{1h}^n|^2,|u_{2h}^n|^2)p_h^n, \chi\big)_{D^k} = 0,\\
&\big(H_h^n, \beta_2\big)_{D^k} = \big(\Delta_{(\beta-2)/2}w_h^n, \beta_2\big)_{D^k},\\
&\big(w_h^n, \psi\big)_{D^k} = -\big(z_h^n, \psi_x\big)_{D^k} + \big((\widehat{z}_h^n\psi^-)_{k+\frac12}\\
&\quad - (\widehat{z}_h^n\psi^+)_{k-\frac12}\big),\\
&\big(z_h^n, \zeta\big)_{D^k} = -\big(p_h^n, \zeta_x\big)_{D^k} + \big((\widehat{p}_h^n\zeta^-)_{k+\frac12}\\
&\quad - (\widehat{p}_h^n\zeta^+)_{k-\frac12}\big),\\
&\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}\upsilon_h^n, \gamma\right)_{D^k} + \varepsilon_3\big(L_h^n, \gamma\big)_{D^k}\\
&\quad + \varepsilon_4\big(g(|u_{1h}^n|^2,|u_{2h}^n|^2)\vartheta_h^n, \gamma\big)_{D^k} = 0,\\
&\big(L_h^n, \beta_3\big)_{D^k} = \big(\Delta_{(\beta-2)/2}\rho_h^n, \beta_3\big)_{D^k},\\
&\big(\rho_h^n, \delta\big)_{D^k} = -\big(\varpi_h^n, \delta_x\big)_{D^k} + \big((\widehat{\varpi}_h^n\delta^-)_{k+\frac12}\\
&\quad - (\widehat{\varpi}_h^n\delta^+)_{k-\frac12}\big),\\
&\big(\varpi_h^n, \varsigma\big)_{D^k} = -\big(\vartheta_h^n, \varsigma_x\big)_{D^k} + \big((\widehat{\vartheta}_h^n\varsigma^-)_{k+\frac12}\\
&\quad - (\widehat{\vartheta}_h^n\varsigma^+)_{k-\frac12}\big),\\
&\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}\vartheta_h^n, o\right)_{D^k} - \varepsilon_3\big(E_h^n, o\big)_{D^k}\\
&\quad - \varepsilon_4\big(g(|u_{1h}^n|^2,|u_{2h}^n|^2)\upsilon_h^n, o\big)_{D^k} = 0,\\
&\big(E_h^n, \beta_4\big)_{D^k} = \big(\Delta_{(\beta-2)/2}\xi_h^n, \beta_4\big)_{D^k},\\
&\big(\xi_h^n, \omega\big)_{D^k} = -\big(\varrho_h^n, \omega_x\big)_{D^k} + \big((\widehat{\varrho}_h^n\omega^-)_{k+\frac12}\\
&\quad - (\widehat{\varrho}_h^n\omega^+)_{k-\frac12}\big),\\
&\big(\varrho_h^n, \kappa\big)_{D^k} = -\big(\upsilon_h^n, \kappa_x\big)_{D^k} + \big((\widehat{\upsilon}_h^n\kappa^-)_{k+\frac12}\\
&\quad - (\widehat{\upsilon}_h^n\kappa^+)_{k-\frac12}\big).
\end{aligned} \tag{5.19}$$

The numerical traces $(\widehat{p}_h^n, \widehat{q}_h^n, \widehat{s}_h^n, \widehat{z}_h^n, \widehat{\upsilon}_h^n, \widehat{\vartheta}_h^n, \widehat{\varpi}_h^n, \widehat{\varrho}_h^n)$ are defined as

$$\begin{aligned} &\widehat{p}_h^n = (p_h^n)^-, \quad \widehat{s}_h^n = (s_h^n)^+\\ &\widehat{q}_h^n = (q_h^n)^-, \quad \widehat{z}_h^n = (z_h^n)^+,\\ &\widehat{\upsilon}_h^n = (\upsilon_h^n)^-, \quad \widehat{\varpi}_h^n = (\varpi_h^n)^+,\\ &\widehat{\varrho}_h^n = (\varrho_h^n)^+, \quad \widehat{\vartheta}_h^n = (\vartheta_h^n)^-. \end{aligned} \tag{5.20}$$

Table 1 The convergence order and L^2 errors for Example 1

K	5	10		15		20	
β	Error	Error	Order	Error	Order	Error	Order
$N = 1$							
1.2	5.97e−02	8.6e−03	2.8	3.4e−03	2.29	1.8e−03	2.21
1.4	2.84e−02	5.8e−03	2.28	2.5e−03	2.08	1.3e−03	2.27
1.8	1.91e−02	4.5e−03	2.09	1.9e−03	2.13	9.9e−04	2.27
$N = 2$							
1.2	3.52e−02	4.3e−03	3.03	1.2e−03	3.15	4.8e−04	3.19
1.4	1.57e−02	2.1e−03	2.9	5.9e−04	3.13	2.6e−04	2.85
1.8	1.45e−02	1.8e−03	3.01	5.5e−04	2.92	2.2e−04	3.19

Theorem 6 (L^2 stability) *Let $u_{1h}^n, u_{2h}^n \in V_k^N$ be the approximation of $u_1(x, t_n), u_2(x, t_n)$, then the solution to the scheme (5.19) and (5.20) satisfies the L^2 stability*

$$\|u_{1h}^n\|_{L^2(\Omega)}^2 + \|u_{2h}^n\|_{L^2(\Omega)}^2 \leq C(\|u_{1h}^0\|_{L^2(\Omega)}^2 + \|u_{2h}^0\|_{L^2(\Omega)}^2).$$

Theorem 7 *Let $u_1(x, t_n)$ and $u_2(x, t_n)$ be the exact solutions of the linear coupled TDO-RSFSEs (5.16), and let u_{1h}^n and u_{2h}^n be the numerical solutions of the scheme (5.19). Then the L^2 error satisfies:*

$$\|u_1(x, t_n) - u_{1h}^n\|_{L^2(\Omega)} + \|u_2(x, t_n) - u_{2h}^n\|_{L^2(\Omega)} \leq C\left(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2\right). \quad (5.21)$$

Theorems 7 and 6 can be proven by similar techniques as that in the proof of Theorems 4 and 5. We will thus not give the details here.

6 Numerical examples

In the following, we present some numerical experiments to show the accuracy and the performance of the present LDG method for the space-fractional convection–diffusion and Schrödinger-type equations with distributed order in time.

Example 1 Consider the following Riesz space-fractional diffusion equation with distributed-order in time

$$\mathcal{D}_t^{W(\alpha)} u(x,t) + \varepsilon(-\Delta)^{\frac{\beta}{2}} u(x,t) = g(x,t), \quad x \in [-1,1], \quad t \in (0, 0.5], \quad u(x,0) = 0, \quad (6.1)$$

and the forcing term

$$g(x,t) = \left((x^2-1)^4 \mathcal{D}_t^{W(\alpha)} t^2 + \varepsilon t^2 (-\Delta)^{\frac{\beta}{2}} (x^2-1)^4\right), \quad (6.2)$$

then the exact solution is given by $u(x,t) = t^2(x^2-1)^4$ with $\varepsilon = \frac{\Gamma(8-\beta)}{\Gamma(8)}$.

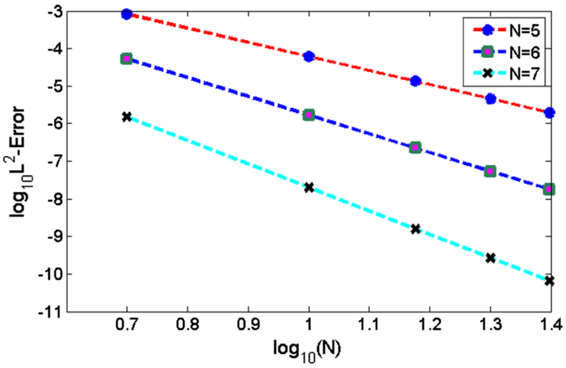

Fig. 1 Convergence tests of Example (1) with $\beta = 1.2$

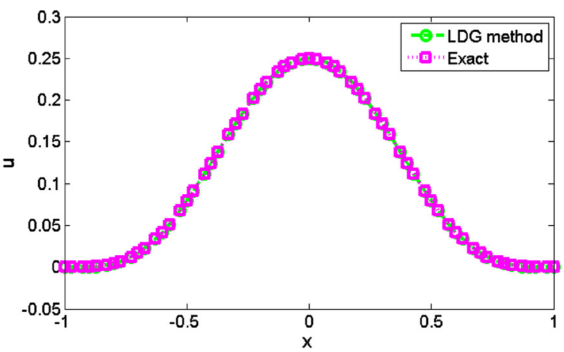

Fig. 2 The numerical and exact solutions of Example (1) with $T = 0.5$, $\beta = 1.2$, $N = 3$ and $K = 20$

The spatial convergence orders and L^2 errors are listed in Table 1. The numerical of spatial convergence orders are shown in Fig. 1 with $N = 5, 6, 7$ and $K = 5, 10, 15, 20, 25$. From these results, we show that the spatial convergence order is $\mathcal{O}(h^{N+1})$. Figure 2 exhibits a comparison between the exact and numerical solution that show the efficiency of the simulation.

Table 2 The convergence order and L^2 errors for Example 2

K	10	20		30		40	
β	Error	Error	Order	Error	Order	Error	Order
$N=1$							
1.2	7.8e−03	1.9e−03	2.04	8.5e−04	1.98	4.6e−04	2.13
1.4	4.9e−03	1.1e−03	2.16	4.6e−04	2.15	2.5e−04	2.12
1.8	1.9e−03	5.1e−04	1.9	2.2e−04	2.07	1.2e−04	2.11
$N=2$							
1.2	3.4e−03	4.2e−04	3.02	1.3e−04	2.89	5.2e−05	3.19
1.4	1.3e−03	1.8e−04	2.85	5.6e−05	2.88	2.5e−05	2.8
1.8	8.2e−04	1.1e−04	2.9	3.1e−05	3.12	1.3e−08	3.02

Example 2 We consider the following Riesz space-fractional Burgers' equation with distributed-order in time

$$\mathcal{D}_t^{W(\alpha)}u(x,t)+\varepsilon(-\Delta)^{\frac{\beta}{2}}u(x,t)+\frac{\partial}{\partial x}\left(\frac{u^2(x,t)}{2}\right)=g(x,t),\\ x\in[-1,1],\quad t\in(0,0.5],\\ u(x,0)=0, \tag{6.3}$$

and the forcing term

$$g(x,t)=\Big((x^2-1)^4\mathcal{D}_t^{W(\alpha)}t^2+8t^4x(x^2-1)^7\\ +\varepsilon t^2(-\Delta)^{\frac{\beta}{2}}(x^2-1)^4\Big). \tag{6.4}$$

and the exact solution will be $u(x,t)=t^2(x^2-1)^4$ with $\varepsilon=\frac{\Gamma(8-\beta)}{\Gamma(8)}$. For the nonlinear term, we select a Lax–Friedrichs flux and take $\Delta t=T/500$, $\theta=1/50$. The spatial convergence orders and L^2 errors are listed in Table 2. Table 3 provides the temporal convergence orders and the L^2 errors with $\beta=1.2, 1.6$, respectively, at $T=0.5$ with $N=1$, $K=30$. The numerical integration convergence orders and L^2 errors in Table 4 with $\beta=1.2, 1.6$, respectively, at $T=0.5$. From these results, we show that the order of convergence is $\mathcal{O}(h^{N+1}+(\Delta t)^{1+\frac{\theta}{2}}+\theta^2)$ when θ is small enough.

Table 3 The temporal convergence orders and L^2 errors for u with $\beta=1.2, 1.8$ at $T=0.5$

β	$\beta=1.2$		$\beta=1.6$		
Δt	Error	Order	Δt	Error	Order
T/100	4.38e−03	–	T/100	1.6e−03	–
T/200	2.21e−03	0.99	T/200	7.82e−04	1.03
T/400	1.1e−03	1.01	T/400	4.1e−04	0.93

Table 4 The numerical integration convergence orders and L^2 errors for u with $\beta=1.2, 1.8$ at $T=0.5$

β	$\beta=1.2$		$\beta=1.6$		
θ	Error	Order	θ	Error	Order
1/10	2.14e−02	–	1/10	7.37e−03	-
1/20	4.84e−03	2.15	1/20	2.01e−03	1.88
1/40	1.3e−03	1.9	1/40	4.6e−04	2.13

Example 3 Consider the following nonlinear TDO-RSFSE

$$i\mathcal{D}_t^{W(\alpha)}u(x,t)-\varepsilon(-\Delta)^{\frac{\beta}{2}}u+|u|^2u=g(x,t),\\ x\in[-1,1],\quad t\in(0,0.5],\\ u(x,0)=0, \tag{6.5}$$

and the forcing term

$$g(x,t)=(1+i)\Big(i(x^2-1)^5\mathcal{D}_t^{W(\alpha)}t^2\\ -\varepsilon t^2(-\Delta)^{\frac{\beta}{2}}(x^2-1)^5+2t^6(x^2-1)^{15}\Big). \tag{6.6}$$

The exact solution $u(x,t)=(1+i)t^2(x^2-1)^5$ with $\varepsilon=\frac{\Gamma(10-\beta)}{\Gamma(10)}$. The spatial convergence orders and L^2 errors are listed in Table 5. The numerical spatial convergence orders are shown in Fig. 3 with $N=3,4,5$ and $K=15,20,25,30$. Table 6 provides the temporal convergence orders and the L^2 errors with $\beta=1.2, 1.6$, respectively, at $T=0.5$. The numerical integration convergence orders and L^2 errors in

Table 5 The convergence orders and L^2 errors for Example 3

K	10	20		30		40	
β	Error	Error	Order	Error	Order	Error	Order
$N=1$							
1.2	1.23e−02	4.61e−03	1.42	1.97e−03	2.1	1.1e−03	2.03
1.4	1.01e−02	2.51e−03	2.01	1.11e−03	2.01	6.31e−04	1.96
1.8	7.31e−03	1.91e−03	1.94	8.35e−04	2.04	4.71e−04	1.99
$N=2$							
1.2	8.35e−03	1.21e−03	2.79	3.55e−04	3.02	1.41e−04	3.21
1.4	6.24e−03	9.23e−04	2.76	2.79e−04	2.95	1.13e−04	3.14
1.8	2.62e−03	3.54e−04	2.89	1.13e−04	2.82	4.66e−05	3.08

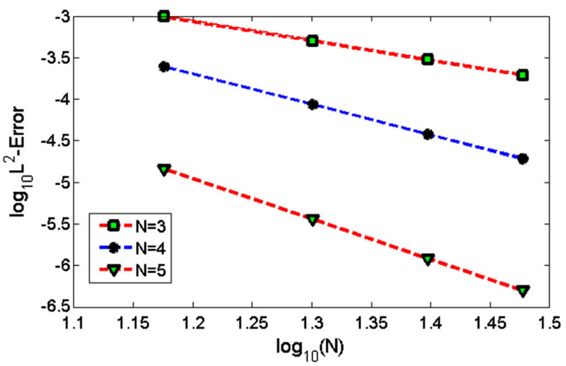

Fig. 3 Convergence tests of Example 3 with $\beta = 1.2$

Table 6 The temporal convergence orders and L^2 errors for u with $\beta = 1.2, 1.8$ at $T = 0.5$

β	$\beta = 1.2$		$\beta = 1.6$		
Δt	Error	Order	Δt	Error	Order
T/100	6.25e−03	–	T/100	5.64e−03	–
T/200	3.12e−03	1.00	T/200	2.81e−03	1.01
T/400	1.58e−03	0.98	T/400	1.25e−03	1.17

Table 7 with $\beta = 1.2, 1.6$, respectively, at $T = 0.5$. From these results it is clear that the order of convergence is $\mathcal{O}(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2)$, which matches the theoretical convergence order when θ is small enough.

Table 7 The numerical integration convergence orders and L^2 errors for u with $\beta = 1.2, 1.8$ at $T = 0.5$

β	β=1.2		β=1.6		
θ	Error	Order	θ	Error	Order
1/10	5.28e−02	–	1/10	2.25e−02	–
1/20	1.25e−02	2.08	1/20	5.4e−03	2.06
1/40	2.98e−03	2.07	1/40	1.55e−03	1.80

Example 4 We consider the coupled nonlinear TDO-RSFSEs

$$\begin{aligned} &i\mathcal{D}_t^{W(\alpha)}u_1(x,t) - \varepsilon_1(-\Delta)^{\frac{\beta}{2}}u_1(x,t) + 2(|u_1(x,t)|^2 \\ &\quad + |u_2(x,t)|^2)u_1(x,t) = g_1(x,t), \\ &\quad x \in [-1,1], \ \ t \in (0, 0.5], \\ &i\mathcal{D}_t^{W(\alpha)}u_2(x,t) - \varepsilon_2(-\Delta)^{\frac{\beta}{2}}u_2(x,t) + 4(|u_1(x,t)|^2 \\ &\quad + |u_2(x,t)|^2)u_2(x,t) = g_2(x,t), \\ &\quad x \in [-1,1], \ t \in (0, 0.5], \end{aligned} \tag{6.7}$$

Table 8 The convergence orders and L^2 errors for u_1

N	$N=1$			$N=2$			$N=3$	
K	Error	Order	K	Error	Order	K	Error	Order
10	4.23e−02	–	10	1.45e−02	–	10	7.87e−03	–
20	9.98e−03	2.08	20	1.87e−03	2.96	20	4.65e−04	4.08
40	2.54e−03	1.97	40	2.12e−04	3.14	40	2.59e−05	4.17
80	6.48e−04	1.97	80	2.68e−05	2.98	80	1.63e−06	3.99

Table 9 The convergence orders and L^2 errors for u_2

N	N = 1			N = 2			N = 3	
K	Error	Order	K	Error	Order	K	Error	Order
10	3.98e−02	–	10	1.26e−02	–	10	6.89e−03	–
20	9.19e−03	2.12	20	1.54e−03	3.03	20	3.84e−04	4.17
40	2.23e−03	2.04	40	1.82e−04	3.08	40	2.39e−05	4.01
80	5.54e−04	2.01	80	1.94e−05	3.23	80	1.47e−06	4.02

and the forcing terms

$$\begin{aligned} g_1(x,t) &= (1+i)\Big(i(x^2-1)^6\mathcal{D}_t^{W(\alpha)}t^2 \\ &\quad - \varepsilon_1 t^2(-\Delta)^{\frac{\beta}{2}}(x^2-1)^6 + 8t^6(x^2-1)^{18}\Big), \\ g_2(x,t) &= (1+i)\Big(i(x^2-1)^6\mathcal{D}_t^{W(\alpha)}t^2 \\ &\quad - \varepsilon_2(-\Delta)^{\frac{\beta}{2}}(x^2-1)^6 + 16t^6(x^2-1)^{18}\Big), \end{aligned} \tag{6.8}$$

to obtain an exact solutions $u_1(x,t) = (1+i)t^2(x^2-1)^6$ and $u_2(x,t) = (1+i)t^2(x^2-1)^6$ with $\beta = 1.3$, $\varepsilon_1 = \frac{\Gamma(13-\beta)}{2\Gamma(13)}$, $\varepsilon_2 = \frac{\Gamma(13-\beta)}{2\Gamma(13)}$. The spatial convergence orders and L^2 errors are listed in Tables 8 and 9, confirming optimal $\mathcal{O}(h^{N+1})$ order of convergence across.

7 Conclusions

In this work, we developed and analyzed a LDG method for solving the distributed-order time and Riesz space-fractional convection–diffusion and Schrödinger-type equations and have proven the stability and error estimates of these methods. The performed numerical experiments confirm the optimal convergence order. Future work will include the analysis of LDG method for two-dimensional fractional problems.

Appendix A: Proof Of Theorem 4.1

Set $(v,\psi,\phi,\eta) = (u_h^n, p_h^n - q_h^n, u_h^n, r_h^n)$ in (3.12), and define $\theta(u_h^n) = \int^{u_h^n} f(s_h^n)\mathrm{d}s_h^n$. Then the following result holds:

$$\begin{aligned} &\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} u_h^n, u_h^n\right)_{D^k} - \varepsilon\big(p_h^n, u_h^n\big)_{D^k} \\ &\quad - (\theta(u_h^n))^-_{k+\frac12} + (\theta(u_h^n))^+_{k-\frac12} + \big((\widehat{f}(u_h^n)u^-)_{k+\frac12} \\ &\quad - (\widehat{f}(u_h^n)(u_h^n)^+)_{k-\frac12}\big) - \big(p_h^n, q_h^n\big)_{D^k} + \big(p_h^n, p_h^n\big)_{D^k} \\ &\quad + \big(\Delta_{(\beta-2)/2}q_h^n, q_h^n\big)_{D^k} - \big(\Delta_{(\beta-2)/2}q_h^n, p_h^n\big)_{D^k} \\ &\quad + \big(r_h^n, r_h^n\big)_{D^k} + \big(u_h^n, \frac{\partial r_h^n}{\partial x}\big)_{D^k} + \big(q_h^n, u_h^n\big)_{D^k} \\ &\quad + \big(r_h^n, \frac{\partial u_h^n}{\partial x}\big)_{D^k} - \big((\widehat{u}_h^n(r_h^n)^-)_{k+\frac12} - (\widehat{u}_h^n(r_h^n)^+)_{k-\frac12}\big) \\ &\quad - \big((\widehat{r}_h^n(u_h^n)^-)_{k+\frac12} - (\widehat{r}_h^n(u_h^n)^+)_{k-\frac12}\big) \\ &= 0. \end{aligned} \tag{A.1}$$

Summing over k, with the numerical fluxes (3.13), we obtain

$$\begin{aligned} &\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} u_h^n, u_h^n\right) - \varepsilon\big(p_h^n, u_h^n\big) \\ &- \sum_{k=1}^{K}\big((\theta(u_h^n))^-_{k+\frac12} - (\theta(u_h^n))^+_{k-\frac12}\big) \\ &+ \sum_{k=1}^{K}\big((\widehat{f}(u_h^n)u^-)_{k+\frac12} - (\widehat{f}(u_h^n)(u_h^n)^+)_{k-\frac12}\big) \\ &- \big(p_h^n, q_h^n\big) + \big(p_h^n, p_h^n\big) \\ &+ \big(\Delta_{(\beta-2)/2}q_h^n, q_h^n\big) - \big(\Delta_{(\beta-2)/2}q_h^n, p_h^n\big) \\ &+ \big(r_h^n, r_h^n\big) + \big(q_h^n, u_h^n\big) = 0. \end{aligned} \tag{A.2}$$

Here $\widehat{f}((u_h^n)^-, (u_h^n)^+)$ is monotone flux, we have

$$\begin{aligned} &\sum_{k=1}^{K}\Big((\widehat{f}(u_h^n)u^-)_{k+\frac12} - (\widehat{f}(u_h^n)(u_h^n)^+)_{k-\frac12}\Big) \\ &\quad - \sum_{k=1}^{K}\Big((\theta(u_h^n))^-_{k+\frac12} - (\theta(u_h^n))^+_{k-\frac12}\Big) > 0. \end{aligned} \tag{A.3}$$

This implies that

$$\begin{aligned} &\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} u_h^n, u_h^n\right) + \big(r_h^n, r_h^n\big) \\ &\quad + \big(p_h^n, p_h^n\big) + \big(\Delta_{(\beta-2)/2}q_h^n, q_h^n\big) \\ &\le \big(\Delta_{(\beta-2)/2}q_h^n, p_h^n\big) - \big(q_h^n, u_h^n\big) \\ &\quad + \big(p_h^n, q_h^n\big) + \varepsilon\big(p_h^n, u_h^n\big). \end{aligned} \tag{A.4}$$

Employing Young's inequality and Lemma 4, we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} u_h^n, u_h^n\right) + \|r_h^n\|^2_{L^2(\Omega)}$$
$$+\left(\Delta_{(\beta-2)/2} q_h^n, q_h^n\right) \le c\|u_h^n\|^2_{L^2(\Omega)} + c_1\|q_h^n\|^2_{L^2(\Omega)}. \quad (A.5)$$

Recalling Lemma 4, we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j} u_h^n, u_h^n\right) + \|r_h^n\|^2_{L^2(\Omega)} \le c\|u_h^n\|^2_{L^2(\Omega)}. \quad (A.6)$$

It then follows that

$$\left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} u^n, u_h^n\right) \le \left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} \sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j}) u_h^l, u_h^n\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} a_{n-1}^{\alpha_j} u_h^0, u_h^n\right) + c\|u_h^n\|^2_{L^2(\Omega)}. \quad (A.7)$$

Using Cauchy–Schwarz inequality, we obtain

$$\|u_h^n\|^2_{L^2(\Omega)} \le c_1 \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q \sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j})\|u_h^l\|_{L^2(\Omega)} \|u_h^n\|_{L^2(\Omega)} + c_2 \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q a_{n-1}^{\alpha_j}\|u_h^0\|_{L^2(\Omega)} \|u_h^n\|_{L^2(\Omega)} + cQ\|u_h^n\|^2_{L^2(\Omega)}, \quad (A.8)$$

where $Q = \frac{1}{\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}}$ and provided c is sufficiently small such that $1 - cQ > 0$, we obtain that

$$\|u_h^n\|_{L^2(\Omega)} \le C\left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q \sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j})\|u_h^l\|_{L^2(\Omega)} + \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q a_{n-1}^{\alpha_j}\|u_h^0\|_{L^2(\Omega)}\right). \quad (A.9)$$

Clearly the theorem is true for $n = 1$. It is also true for $n = 1, 2, \ldots, m-1$. So, by (A.9), we have

$$\|u_h^m\|_{L^2(\Omega)} \le C\left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q \sum_{l=1}^{m-1}(a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j})\|u_h^l\|_{L^2(\Omega)} + \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q a_{n-1}^{\alpha_j}\|u_h^0\|_{L^2(\Omega)}\right)$$
$$\le C\left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q \sum_{l=1}^{m-1}(a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j})\, |u_h^0\|_{L^2(\Omega)} + \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q a_{n-1}^{\alpha_j}\|u_h^0\|_{L^2(\Omega)}\right)$$
$$= C\|u_h^0\|_{L^2(\Omega)}. \quad (A.10)$$

□

Appendix B: Proof Of Theorem 2

From (4.3), we can get the error equation

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j}(u^n - u_h^n), v\right)_{D^k} + \left(\gamma(x)^n, v\right)_{D^k} - \varepsilon\left(p^n - p_h^n, v\right)_{D^k} + \left(p^n - p_h^n, \psi\right)_{D^k} + \left(q^n - q_h^n, \phi\right)_{D^k} - \left(\Delta_{(\beta-2)/2}(q^n - q_h^n), \psi\right)_{D^k} + \left(r^n - r_h^n, \frac{\partial\phi}{\partial x}\right)_{D^k} - ((\widehat{r}^n - \widehat{r}_h^n)\phi^-)_{k+\frac{1}{2}} + ((\widehat{r}^n - \widehat{r}_h^n)\phi^+)_{k-\frac{1}{2}} + \left(r^n - r_h^n, \eta\right)_{D^k} + \left(u^n - u_h^n, \frac{\partial\eta}{\partial x}\right)_{D^k} - ((\widehat{u}^n - \widehat{u}_h^n)\eta^-)_{k+\frac{1}{2}} + ((\widehat{u}^n - \widehat{u}_h^n)\eta^+)_{k-\frac{1}{2}} = 0. \quad (B.1)$$

Using (4.7), the error equation (B.1) can be written

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j}(\pi^n - \pi_n^e), v\right)_{D^k} + \left(\gamma(x)^n, v\right)_{D^k} - \varepsilon\left(\sigma^n - \sigma_n^e, v\right)_{D^k} + \left(\sigma^n - \sigma_n^e, \psi\right)_{D^k} - \left(\Delta_{(\beta-2)/2}(\varphi^n - \varphi_n^e), \psi\right)_{D^k} + \left(\varphi^n - \varphi_n^e, \phi\right)_{D^k} + \left(\psi^n - \psi_n^e, \frac{\partial\phi}{\partial x}\right)_{D^k} - ((\psi^n - \psi_n^e)^-\phi^-)_{k+\frac{1}{2}} + ((\psi^n - \psi_n^e)^-\phi^+)_{k-\frac{1}{2}} + \left(\psi^n - \psi_n^e, \eta\right)_{D^k} + \left(\pi^n - \pi_n^e, \frac{\partial\eta}{\partial x}\right)_{D^k} - ((\pi^n - \pi_n^e)^+\eta^-)_{k+\frac{1}{2}} + ((\pi^n - \pi_n^e)^+\eta^+)_{k-\frac{1}{2}} = 0, \quad (B.2)$$

and taking the test functions

$$v = \pi^n, \quad \psi = \sigma^n - \varphi^n, \quad \phi = \pi^n, \quad \eta = \psi^n, \quad (B.3)$$

we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(\pi^n-\pi_n^e),\pi^n\right)+\big(\gamma(x)^n,\pi^n\big)$$
$$-\varepsilon\big(\sigma^n-\sigma_n^e,\pi^n\big)+\big(\sigma^n-\sigma_n^e,-\varphi^n+\sigma^n\big)$$
$$-\big(\Delta_{(\beta-2)/2}(\varphi^n-\varphi_n^e),-\varphi^n+\sigma^n\big)$$
$$-\big(\pi_n^e,\frac{\partial\psi^n}{\partial x}\big)+\big(\varphi^n-\varphi_n^e,\pi^n\big)$$
$$-\big(\psi_n^e,\frac{\partial\pi^n}{\partial x}\big)+\big(\psi^n-\psi_n^e,\psi^n\big)$$
$$+\sum_{k=1}^{K}\big(((\psi_n^e)^-(\pi^n)^-)_{k+\frac{1}{2}}-((\psi_n^e)^-(\pi^n)^+)_{k-\frac{1}{2}}\big)$$
$$+\sum_{k=1}^{K}\big(((\pi_n^e)^+(\psi^n)^-)_{k+\frac{1}{2}}-((\pi_n^e)^+(\psi^n)^+)_{k-\frac{1}{2}}\big)$$
$$=0, \tag{B.4}$$

by the projection properties P^+ and P^- we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(\pi^n-\pi_n^e),\pi^n\right)-\varepsilon\big(\sigma^n-\sigma_n^e,\pi^n\big)$$
$$+\big(\gamma(x)^n,\pi^n\big)+\big(\sigma^n-\sigma_n^e,-\varphi^n+\sigma^n\big)$$
$$-\big(\Delta_{(\beta-2)/2}(\varphi^n-\varphi_n^e),-\varphi^n+\sigma^n\big)$$
$$+\big(\varphi^n-\varphi_n^e,\pi^n\big)+\sum_{k=1}^{K}\big(((\psi_n^e)^-(\pi^n)^-)_{k+\frac{1}{2}}$$
$$-((\psi_n^e)^-(\pi^n)^+)_{k-\frac{1}{2}}\big)+\big(\psi^n-\psi_n^e,\psi^n\big)$$
$$+\sum_{k=1}^{K}\big(((\pi_n^e)^+(\psi^n)^-)_{k+\frac{1}{2}}-((\pi_n^e)^+(\psi^n)^+)_{k-\frac{1}{2}}\big)$$
$$=0. \tag{B.5}$$

Employing Young's inequality and Lemma 4 and the property of interpolation (4.8) and (4.4), we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}\pi^n,\pi^n\right)+\big(\sigma^n,\sigma^n\big)$$
$$+\big(\Delta_{(\beta-2)/2}\varphi^n,\varphi^n\big)+\big(\psi^n,\psi^n\big)$$
$$\le C(h^{2N+2}+(\Delta t)^{4+\theta}+\theta^4)+c_1\|\psi^n\|^2_{L^2(\Omega)}$$
$$+\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}\pi_n^e,\pi^n\right)+c_2\|\sigma^n\|^2_{L^2(\Omega)}$$
$$+c_3\|\varphi^n\|^2_{L^2(\Omega)}+c\|\pi^n\|^2_{L^2(\Omega)}, \tag{B.6}$$

by using Lemma 5, (4.7) and the interpolation property (4.8), we obtain

$$\|\delta_t^{\alpha}(\mathcal{P}^+u(x,t_n)-u(x,t_n))\|_{L^2(\Omega)}$$
$$\le C\big(h^{N+1}+(\Delta t)^{2-\alpha}\big). \tag{B.7}$$

From (3.1), (4.5) and (B.7), we obtain

$$\left\|\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(\mathcal{P}^+u(x,t_n)-u(x,t_n))\right\|_{L^2(\Omega)}$$
$$\le C\left(h^{N+1}+(\Delta t)^{1+\frac{\theta}{2}}+\theta^2\right). \tag{B.8}$$

Hence

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}\pi^n,\pi^n\right)+\big(\sigma^n,\sigma^n\big)$$
$$+\big(\Delta_{(\beta-2)/2}\varphi^n,\varphi^n\big)+\big(\psi^n,\psi^n\big)$$
$$\le C\big(h^{2N+2}+(\Delta t)^{2+\theta}+\theta^4\big)+c_2\|\sigma^n\|^2_{L^2(\Omega)}$$
$$+c_3\|\varphi^n\|^2_{L^2(\Omega)}+c\|\pi^n\|^2_{L^2(\Omega)}+c_1\|\psi^n\|^2_{L^2(\Omega)}. \tag{B.9}$$

Recalling Lemma 2 and provided $c_i,\ i=1,2$ are sufficiently small such that $c_i\le 1$, we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}\pi^n,\pi^n\right)$$
$$\le C\big(h^{2N+2}+(\Delta t)^{2+\theta}+\theta^4\big)+c\|\pi^n\|^2_{L^2(\Omega)}. \tag{B.10}$$

It then follows that

$$\left(\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}\pi^n,\pi_h^n\right)$$
$$\le\left(\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}\sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j}-a_{n-l}^{\alpha_j})\pi^l,\pi^n\right)$$
$$+\left(\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}a_{n-1}^{\alpha_j}\pi^0,\pi^n\right)+c\|\pi^n\|^2_{L^2(\Omega)}$$
$$+C\big(h^{2N+2}+(\Delta t)^{2+\theta}+\theta^4\big). \tag{B.11}$$

Employing Young's inequality, we obtain

$$\|\pi^n\|^2_{L^2(\Omega)}$$
$$\le\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Q\sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j}-a_{n-l}^{\alpha_j})\|\pi^l\|^2_{L^2(\Omega)}$$
$$+\frac{1}{4}\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Q(a_0^{\alpha_j}-a_{n-1}^{\alpha_j})\|\pi^n\|^2_{L^2(\Omega)}$$
$$+\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Qa_{n-1}^{\alpha_j}\|\pi^0\|^2_{L^2(\Omega)}+cQ\|\pi^n\|^2_{L^2(\Omega)}$$

$$+\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{4\lambda_j}Qa_{n-1}^{\alpha_j}\|\pi^n\|^2_{L^2(\Omega)} + CQ\big(h^{2N+2}+(\Delta t)^{2+\theta}+\theta^4\big). \tag{B.12}$$

Notice the facts that

$$\|\pi^0\|_{L^2(\Omega)} \le Ch^{N+1}. \tag{B.13}$$

Thus,

$$\begin{aligned}\|\pi^n\|^2_{L^2(\Omega)} &\le \sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Q\sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j}-a_{n-l}^{\alpha_j})\|\pi^l\|^2_{L^2(\Omega)} \\ &+\left(cQ+\frac{1}{4}\right)\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Q\|\pi^n\|^2_{L^2(\Omega)} \\ &+C\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Qa_{n-1}^{\alpha_j}h^{2N+2} \\ &+C\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Qa_{n-1}^{\alpha_j}\left(h^{2N+2}+(\Delta t)^{2+\theta}+\theta^4\right),\end{aligned} \tag{B.14}$$

provided c is sufficiently small such that $\frac{3}{4}-cQ>0$, we obtain that

$$\|\pi^n\|^2_{L^2(\Omega)} \le C\left(\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Q\sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j}-a_{n-l}^{\alpha_j})\|\pi^l\|^2_{L^2(\Omega)} + \sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Qa_{n-1}^{\alpha_j}\big(h^{2N+2}+(\Delta t)^{2+\theta}+\theta^4\big)\right). \tag{B.15}$$

Clearly the theorem is true for $n=0$. It is also true for $n=1,2,\ldots,m-1$. Then, by (B.15), we have

$$\begin{aligned}\|\pi^m\|^2_{L^2(\Omega)} &\le C\left(\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Q\sum_{l=1}^{m-1}(a_{n-l-1}^{\alpha_j}-a_{n-l}^{\alpha_j})\|\pi^l\|^2_{L^2(\Omega)} + \sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Qa_{n-1}^{\alpha_j}\big(h^{2N+2}+(\Delta t)^{2+\theta}+\theta^4\big)\right) \\ &\le C\left(\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Q\sum_{l=1}^{m-1}(a_{n-l-1}^{\alpha_j}-a_{n-l}^{\alpha_j})(h^{2N+2} +(\Delta t)^{2+\theta}+\theta^4) + \sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Qa_{n-1}^{\alpha_j}\big(h^{2N+2}+(\Delta t)^{2+\theta}+\theta^4\big)\right) \\ &= C\big(h^{2N+2}+(\Delta t)^{2+\theta}+\theta^4\big).\end{aligned} \tag{B.16}$$

Finally, by using triangle inequality and standard approximation theory, we get

$$\|u(x,t_m)-u_h^m\|_{L^2(\Omega)} \le C(h^{N+1}+(\Delta t)^{1+\frac{\theta}{2}}+\theta^2). \tag{B.17}$$

□

Appendix C: Proof Of Theorem 3

Using (4.7), the error equation (4.3) can be written

$$\begin{aligned}&\left(\sum_{j=1}^{S}W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(\pi^n-\pi_n^e),v\right)-\varepsilon\big(\sigma^n-\sigma_n^e,v\big) \\ &+\big(\gamma(x)^n,v\big)-\sum_{k=1}^{K}\mathcal{H}_k(f;u,u_h;v) \\ &+\big(\sigma^n-\sigma_n^e,\psi\big)-\big(\Delta_{(\beta-2)/2}(\varphi^n-\varphi_n^e),\psi\big) \\ &+\big(\varphi^n-\varphi_n^e,\phi\big)+\big(\psi^n-\psi_n^e,\frac{\partial\phi}{\partial x}\big) \\ &-\sum_{k=1}^{K}\big(((\psi^n-\psi_n^e)^-\phi^-)_{k+\frac{1}{2}}-((\psi^n-\psi_n^e)^-\phi^+)_{k-\frac{1}{2}}\big) \\ &+\big(\psi^n-\psi_n^e,\eta\big)+\big(\pi^n-\pi_n^e,\frac{\partial\eta}{\partial x}\big) \\ &-\sum_{k=1}^{K}\big(((\pi^n-\pi_n^e)^+\eta^-)_{k+\frac{1}{2}}-((\pi^n-\pi_n^e)^+\eta^+)_{k-\frac{1}{2}}\big) \\ &=0.\end{aligned} \tag{C.1}$$

Following the proof of Theorem 2, we take the test functions

$$v=\pi^n,\quad \psi=-\varphi^n+\sigma^n,\quad \phi=\pi^n,\quad \eta=\psi^n, \tag{C.2}$$

we obtain

$$\begin{aligned}&\left(\sum_{j=1}^{S}W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(\pi^n-\pi_n^e),\pi^n\right)+\big(\gamma(x)^n,\pi^n\big) \\ &-\varepsilon\big(\sigma^n-\sigma_n^e,\pi^n\big)-\sum_{k=1}^{K}\mathcal{H}_k(f;u,u_h;\pi^n) \\ &+\big(\sigma^n-\sigma_n^e,-\varphi^n+\sigma^n\big)+\big(\varphi^n-\varphi_n^e,\pi^n\big) \\ &-\big(\Delta_{(\beta-2)/2}(\varphi^n-\varphi_n^e),-\varphi^n+\sigma^n\big) \\ &+\big(\psi^n-\psi_n^e,\frac{\partial\pi^n}{\partial x}\big)-\sum_{k=1}^{K}\big(((\psi^n-\psi_n^e)^-(\pi^n)^-)_{k+\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&- ((\psi^n - \psi_n^e)^-(\pi^n)^+)_{k-\frac{1}{2}}\big) + \big(\psi^n - \psi_n^e, \psi^n\big) \\
&+ \Big(\pi^n - \pi_n^e, \frac{\partial \psi^n}{\partial x}\Big) - \sum_{k=1}^{K} \big(((\pi^n - \pi_n^e)^+(\psi^n)^-)_{k+\frac{1}{2}} \\
&- ((\pi^n - \pi_n^e)^+(\psi^n)^+)_{k-\frac{1}{2}}\big) = 0, \qquad \text{(C.3)}
\end{aligned}
$$

by the projection properties P^+ and P^-, we obtain

$$
\begin{aligned}
&\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j}(\pi^n - \pi_n^e), \pi^n\right) - \varepsilon\big(\sigma^n - \sigma_n^e, \pi^n\big) \\
&+ \big(\gamma(x)^n, \pi^n\big) - \sum_{k=1}^{K} \mathcal{H}_k(f; u, u_h; \pi^n) \\
&+ \big(\sigma^n - \sigma_n^e, -\varphi^n + \sigma^n\big) + \big(\varphi^n - \varphi_n^e, \pi^n\big) \\
&- \big(\Delta_{(\beta-2)/2}(\varphi^n - \varphi_n^e), -\varphi^n + \sigma^n\big) \\
&+ \sum_{k=1}^{K} \big(((\psi_n^e)^-(\pi^n)^-)_{k+\frac{1}{2}} - ((\psi_n^e)^-(\pi^n)^+)_{k-\frac{1}{2}}\big) \\
&+ \big(\psi^n - \psi_n^e, \psi^n\big) + \sum_{k=1}^{K} \big(((\pi_n^e)^+(\psi^n)^-)_{k+\frac{1}{2}} \\
&- ((\pi_n^e)^+(\psi^n)^+)_{k-\frac{1}{2}}\big) = 0. \qquad \text{(C.4)}
\end{aligned}
$$

Employing Young's inequality and Lemma 4 and the property of interpolation (4.8), (B.7) and (B.8), we obtain

$$
\begin{aligned}
&\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j}\pi^n, \pi^n\right) + \big(\Delta_{(\beta-2)/2}\varphi^n, \varphi^n\big) \\
&+ \big(\sigma^n, \sigma^n\big) + \big(\psi^n, \psi^n\big) \\
&\le C\big(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^4\big) + c_2\|\sigma^n\|^2_{L^2(\Omega)} \\
&+ c_3\|\varphi^n\|^2_{L^2(\Omega)} + c\|\pi^n\|^2_{L^2(\Omega)} + c_1\|\psi^n\|^2_{L^2(\Omega)} \\
&- \frac{1}{4}\kappa(f^*; u_h^n) + (C + C_*(\|\pi^n\|_\infty \\
&+ h^{-1}\|e_u\|^2_\infty))\|\pi^n\|^2 + (C + C_* h^{-1}\|e_u\|^2_\infty)h^{2N+1}. \qquad \text{(C.5)}
\end{aligned}
$$

Recalling Lemma 2 and provided $c_i,\ i = 1, 2$ are sufficiently small such that $c_i \le 1$, we obtain

$$
\begin{aligned}
&\left(\sum_{j=1}^{S} W(\alpha_j)\Delta\tau_j \delta_t^{\alpha_j}\pi^n, \pi^n\right) \\
&\le C\big(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^4\big) + c_4\|\pi^n\|^2_{L^2(\Omega)}. \qquad \text{(C.6)}
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
&\left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}\pi^n, \pi_h^n\right) \\
&\le \left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} \sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j})\pi^l, \pi^n\right) \\
&+ \left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} a_{n-1}^{\alpha_j}\pi^0, \pi^n\right) + c\|\pi^n\|^2_{L^2(\Omega)} \\
&+ C\big(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^4\big). \qquad \text{(C.7)}
\end{aligned}
$$

Employing Young's inequality, we obtain

$$
\begin{aligned}
&\|\pi^n\|^2_{L^2(\Omega_h)} \\
&\le \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q \sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j})\|\pi^l\|^2_{L^2(\Omega_h)} \\
&+ \left(cQ + \frac{1}{4}\right) \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q\|\pi^n\|^2_{L^2(\Omega_h)} \\
&+ C \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q a_{n-1}^{\alpha_j} h^{2N+2} \\
&+ C \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q a_{n-1}^{\alpha_j}\big(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^4\big), \qquad \text{(C.8)}
\end{aligned}
$$

provided c is sufficiently small such that $\frac{3}{4} - cQ > 0$, we obtain that

$$
\begin{aligned}
&\|\pi^n\|^2_{L^2(\Omega)} \\
&\le C\left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q \sum_{l=1}^{n-1}(a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j})\|\pi^l\|^2_{L^2(\Omega)} \right. \\
&\left. + \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q a_{n-1}^{\alpha_j}\big(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^4\big)\right). \qquad \text{(C.9)}
\end{aligned}
$$

Clearly the theorem is true for $n = 0$. It is also valid for $n = 1, 2, \ldots, m-1$. Then, by (C.9), we have

$$
\begin{aligned}
&\|\pi^m\|^2_{L^2(\Omega)} \\
&\le C\left(\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q \sum_{l=1}^{m-1}(a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j})\|\pi^l\|^2_{L^2(\Omega)} \right. \\
&\left. + \sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j} Q a_{n-1}^{\alpha_j}\big(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^4\big)\right)
\end{aligned}
$$

$$\leq C\left(\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Q\sum_{l=1}^{m-1}(a_{n-l-1}^{\alpha_j}-a_{n-l}^{\alpha_j})\big(h^{2N+1}+(\Delta t)^{2+\theta}+\theta^4\big)+\sum_{j=1}^{S}\frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}Qa_{n-1}^{\alpha_j}\big(h^{2N+1}+(\Delta t)^{4+\theta}+\theta^4\big)\right)$$
$$= C\big(h^{2N+1}+(\Delta t)^{2+\theta}+\theta^4\big). \quad \text{(C.10)}$$

Finally, by using triangle inequality and standard approximation theory, we can get (4.15). □

Appendix D: Proof Of Lemma 8

From the Galerkin orthogonality (5.11), we get

$$\begin{aligned}
&\left(\sum_{j=1}^{S}W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(\pi^n-\pi_n^e),\vartheta_1\right)_{D^k}+\big(\tau^n-\tau_n^e,\phi_x\big)_{D^k}\\
&-\big(\Delta_{(\beta-2)/2}(\epsilon^n-\epsilon_n^e),\rho\big)_{D^k}+\big(\vartheta^n-\vartheta_n^e,\psi_x\big)_{D^k}\\
&+\left(\sum_{j=1}^{S}W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(\sigma-\sigma^e),\chi\right)_{D^k}+\big(\sigma^n-\sigma_n^e,\varphi_x\big)_{D^k}\\
&-\big(\Delta_{(\beta-2)/2}(\varphi^n-\varphi_n^e),\varrho\big)_{D^k}+\big(\pi^n-\pi_n^e,\zeta_x\big)_{D^k}\\
&+\varepsilon_2\big(\sigma^n-\sigma_n^e,\vartheta_1\big)_{D^k}-\varepsilon_2\big(\pi^n-\pi_n^e,\chi\big)_{D^k}\\
&+\big(\epsilon^n-\epsilon_n^e,\phi\big)_{D^k}+\big(\tau^n-\tau_n^e,\varphi\big)_{D^k}+\big(\varpi^n-\varpi_n^e,\varrho\big)_{D^k}\\
&+\big(\phi^n-\phi_n^e,\rho\big)_{D^k}+\big(\varphi^n-\varphi_n^e,\psi\big)_{D^k}+\big(\vartheta^n-\vartheta_n^e,\zeta\big)_{D^k}\\
&+\varepsilon_1\big(\phi^n-\phi_n^e,\vartheta_1\big)_{D^k}-\varepsilon_1\big(\varpi^n-\varpi_n^e,\chi\big)_{D^k}\\
&-\big(\gamma(q)^n,\chi\big)+\big(\gamma(p)^n,\vartheta_1\big)\\
&-\sum_{k=1}^{K}\Big(((\tau^n-\tau_n^e)^+(\phi)^-)_{k+\frac12}-((\tau^n-\tau_n^e)^+(\phi)^+)_{k-\frac12}\Big)\\
&-\sum_{k=1}^{K}\Big(((\sigma^n-\sigma_n^e)^-(\varphi)^-)_{k+\frac12}-((\sigma^n-\sigma_n^e)^-(\varphi)^+)_{k-\frac12}\Big)\\
&-\sum_{k=1}^{K}\Big(((\vartheta^n-\vartheta_n^e)^+(\psi)^-)_{k+\frac12}-((\vartheta^n-\vartheta_n^e)^+(\psi)^+)_{k-\frac12}\Big)\\
&-\sum_{k=1}^{K}\Big(((\pi^n-\pi_n^e)^-(\zeta)^-)_{k+\frac12}-((\pi^n-\pi_n^e)^-(\zeta)^+)_{k-\frac12}\Big)\\
&=0. \qquad \text{(D.1)}
\end{aligned}$$

We take the test functions

$$\begin{aligned}
&\vartheta_1=\pi^n,\quad \rho=\phi^n-\epsilon^n,\quad \phi=\pi^n,\quad \varphi=-\vartheta^n,\\
&\chi=\sigma^n,\quad \varrho=\varpi^n-\varphi^n,\quad \psi=-\sigma^n,\quad \zeta=\tau^n, \qquad \text{(D.2)}
\end{aligned}$$

we obtain

$$\begin{aligned}
&\left(\sum_{j=1}^{S}W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(\pi^n-\pi_n^e),\pi^n\right)_{D^k}+\big(\tau^n-\tau_n^e,\pi_x^n\big)_{D^k}\\
&-\big(\Delta_{(\beta-2)/2}(\epsilon^n-\epsilon_n^e),\phi^n-\epsilon^n\big)_{D^k}+\big(\pi^n-\pi_n^e,\tau_x^n\big)_{D^k}\\
&+\left(\sum_{j=1}^{S}W(\alpha_j)\Delta\tau_j\delta_t^{\alpha_j}(\sigma^n-\sigma_n^e),\sigma^n\right)_{D^k}-\big(\sigma^n-\sigma_n^e,\vartheta_x^n\big)_{D^k}\\
&-\big(\Delta_{(\beta-2)/2}(\varphi^n-\varphi_n^e),\varpi^n-\varphi^n\big)_{D^k}-\big(\vartheta^n-\vartheta_n^e,\sigma_x^n\big)_{D^k}\\
&+\big(\epsilon^n-\epsilon_n^e,\pi\big)_{D^k}-\big(\tau^n-\tau_n^e,\vartheta\big)_{D^k}\\
&+\big(\phi^n-\phi_n^e,\phi^n-\epsilon^n\big)_{D^k}-\big(\varphi^n-\varphi_n^e,\sigma^n\big)_{D^k}\\
&+\big(\vartheta^n-\vartheta_n^e,\tau^n\big)_{D^k}+\big(\varpi^n-\varpi_n^e,\varpi^n-\varphi^n\big)_{D^k}\\
&+\varepsilon_1\big(\phi^n-\phi_n^e,\pi^n\big)_{D^k}-\varepsilon_1\big(\varpi^n-\varpi_n^e,\sigma^n\big)_{D^k}\\
&+\varepsilon_2\big(\sigma^n-\sigma_n^e,\pi\big)_{D^k}-\varepsilon_2\big(\pi_n-\pi_n^e,\sigma\big)_{D^k}\\
&-\big(\gamma(q)^n,\sigma^n\big)+\big(\gamma(p)^n,\pi^n\big)\\
&-\sum_{k=1}^{K}\Big(((\tau^n-\tau_n^e)^+(\pi^n)^-)_{k+\frac12}-((\tau^n-\tau_n^e)^+(\pi^n)^+)_{k-\frac12}\Big)\\
&+\sum_{k=1}^{K}\Big(((\sigma^n-\sigma_n^e)^-(\vartheta^n)^-)_{k+\frac12}-((\sigma^n-\sigma_n^e)^-(\vartheta^n)^+)_{k-\frac12}\Big)\\
&+\sum_{k=1}^{K}\Big(((\vartheta^n-\vartheta_n^e)^+(\sigma^n)^-)_{k+\frac12}-((\vartheta^n-\vartheta_n^e)^+(\sigma^n)^+)_{k-\frac12}\Big)\\
&-\sum_{k=1}^{K}\Big(((\pi^n-\pi_n^e)^-(\tau^n)^-)_{k+\frac12}-((\pi^n-\pi_n^e)^-(\tau^n)^+)_{k-\frac12}\Big)\\
&=0. \qquad \text{(D.3)}
\end{aligned}$$

Summing over k, simplify by integration by parts and (5.6). □

References

1. Caputo, M.: Distributed order differential equations modelling dielectric induction and diffusion. Fract. Calc. Appl. Anal. **4**, 421–442 (2001)
2. Atanackovic, T., Budincevic, M., Pilipovic, S.: On a fractional distributed-order oscillator. J. Phys. A Math. Gen. **38**, 6703 (2005)
3. Caputo, M.: Linear models of dissipation whose Q is almost frequency independentII. Geophys. J. Int. **13**, 529–539 (1967)
4. Hartley, T.T., Lorenzo, C.F.: Fractional system identification: an approach using continuous order-distributions. NASA/TM-1999-209640 (1999)
5. Lorenzo, C.F., Hartley, T.T.: Variable order and distributed order fractional operators. Nonlinear Dyn. **29**, 57–98 (2002)
6. Naghibolhosseini, M.: Estimation of outer-middle ear transmission using DPOAEs and fractional-order modeling of human middle ear Ph.D thesis, City University of New York, NY (2015)
7. Caputo, M.: Diffusion with space memory modelled with distributed order space fractional differential equations. Ann. Geophys. **46**, 223–234 (2003)
8. Sokolov, A.C.I., Klafter, J.: Distributed order fractional kinetics. Acta Phys. Pol. B **35**, 1323–1341 (2004)

9. Caputo, M.: Elasticitàe Dissipazione. Zanichelli, Bologna (1969)
10. Diethelm, K., Ford, N.J.: Numerical analysis for distributed-order differential equations. J. Comput. Appl. Math. **225**, 96–104 (2009)
11. Diethelm, K., Ford, N.J.: Numerical solution methods for distributed order differential equations. Fract. Calc. Appl. Anal. **4**, 531–542 (2001)
12. Ford, N.J., Morgado, M.L.: Distributed order equations as boundary value problems. Comput. Math. Appl. **64**, 2973–2981 (2012)
13. Podlubny, I., Skovranek, T., Jara, B.M.V., Petras, I., Verbitsky, V., Chen, Y.: Matrix approach to discrete fractional calculus III: non-equidistant grids, variable step length and distributed orders. Philos. Trans. R. Soc. A **371**, 20120153 (2013)
14. Ye, H., Liu, F., Anh, V., Turner, I.: Numerical analysis for the time distributed-order and Riesz space fractional diffusions on bounded domains. IMA J. Appl. Math. **80**, 825–838 (2013)
15. Bhrawy, A., Zaky, M.: Numerical simulation of multi-dimensional distributed-order generalized Schrödinger equations. Nonlinear Dyn. **89**, 1415–1432 (2017)
16. Shen, J., Tang, T., Wang, L.-L.: Spectral Methods: Algorithms, Analysis and Applications, vol. 41. Springer, New York (2011)
17. Zeng, S., Migórski, S.: A class of time-fractional hemivariational inequalities with application to frictional contact problem. Commun. Nonlinear Sci. Numer. Simul. **56**, 34–48 (2018)
18. Mahmoud, G.M., Ahmed, M.E., Abed-Elhameed, T.M.: On fractional-order hyperchaotic complex systems and their generalized function projective combination synchronization. Optik Int. J. Light Electron Opt. **130**, 398–406 (2017)
19. Yüzbaşı, Ş., Karaçayır, M.: An exponential galerkin method for solutions of hiv infection model of $CD4^+$ T-cells. Comput. Biol. Chem. **67**, 205–212 (2017)
20. Mahmoud, G.M., Abed-Elhameed, T.M., Ahmed, M.E.: Generalization of combination-combination synchronization of chaotic n-dimensional fractional-order dynamical systems. Nonlinear Dyn. **83**, 1885–1893 (2016)
21. Yüzbaşı, Ş., Karaçayır, M.: A galerkin-like approach to solve high-order integrodifferential equations with weakly singular kernel. Kuwait J. Sci. **43**, 106–120 (2016)
22. Cockburn, B., Shu, C.-W.: The local discontinuous Galerkin method for time-dependent convection-diffusion systems. SIAM J. Numer. Anal. **35**, 2440–2463 (1998)
23. El-Sayed, A., Gaber, M.: On the finite caputo and finite Riesz derivatives. Electron. J. Theor. Phys. **3**, 81–95 (2006)
24. Muslih, S.I., Agrawal, O.P.: Riesz fractional derivatives and fractional dimensional space. Int. J. Theor. Phys. **49**, 270–275 (2010)
25. Yang, Q., Liu, F., Turner, I.: Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. Appl. Math. Model. **34**, 200–218 (2010)
26. Bassi, F., Rebay, S.: A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier–Stokes equations. J. Comput. Phys. **131**, 267–279 (1997)
27. Cockburn, B., Shu, C.-W.: TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws. II. General framework. Math. Comput. **52**, 411–435 (1989)
28. Cockburn, B., Dawson, C.: Approximation of the velocity by coupling discontinuous Galerkin and mixed finite element methods for flow problems. Comput. Geosci. **6**, 505–522 (2002)
29. Cockburn, B., Kanschat, G., Schötzau, D.: A locally conservative LDG method for the incompressible Navier–Stokes equations. Math. Comput. **74**, 1067–1095 (2005)
30. Mustapha, K., McLean, W.: Piecewise-linear, discontinuous Galerkin method for a fractional diffusion equation. Numer. Algorithms **56**, 159–184 (2011)
31. Mustapha, K., McLean, W.: Uniform convergence for a discontinuous Galerkin, time-stepping method applied to a fractional diffusion equation. IMA J. Numer. Anal. **32**, 906–925 (2012)
32. Mustapha, K., McLean, W.: Superconvergence of a discontinuous Galerkin method for fractional diffusion and wave equations. SIAM J. Numer. Anal. **51**, 491–515 (2013)
33. Deng, W., Hesthaven, J.S.: Local discontinuous Galerkin methods for fractional diffusion equations. ESAIM Math. Model. Numer. Anal. **47**, 1845–1864 (2013)
34. Xu, Q., Hesthaven, J.S.: Discontinuous Galerkin method for fractional convection-diffusion equations. SIAM J. Numer. Anal. **52**, 405–423 (2014)
35. Aboelenen, T.: A direct discontinuous Galerkin method for fractional convection-diffusion and Schrödinger type equations. arXiv:1708.04546 (2017)
36. Aboelenen, T., El-Hawary, H.: A high-order nodal discontinuous Galerkin method for a linearized fractional Cahn–Hilliard equation. Comput. Math. Appl. **73**, 1197–1217 (2017)
37. Aboelenen, T.: A high-order nodal discontinuous Galerkin method for nonlinear fractional Schrödinger type equations. Commun. Nonlinear Sci. Numer. Simul. **54**, 428–452 (2018)
38. Miller, K., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley (1993). https://books.google.co.in/books?id=MOp_QgAACAAJ
39. Ervin, V.J., Roop, J.P.: Variational formulation for the stationary fractional advection dispersion equation. Numer. Methods Partial Differ. Equ. **22**, 558–576 (2006)
40. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations (North-Holland Mathematics Studies), vol. 204. Elsevier, New York (2006)
41. Sun, Z.-Z., Wu, X.: A fully discrete difference scheme for a diffusion-wave system. Appl. Numer. Math. **56**, 193–209 (2006)
42. Ciarlet, P.G.: Finite Element Method for Elliptic Problems. Society for Industrial and Applied Mathematics, Philadelphia (2002)
43. Zhang, Q., Shu, C.-W.: Error estimates to smooth solutions of Runge–Kutta discontinuous Galerkin methods for scalar conservation laws. SIAM J. Numer. Anal. **42**, 641–666 (2004)
44. Yan, J., Shu, C.-W.: Local discontinuous Galerkin methods for partial differential equations with higher order derivatives. J. Sci. Comput. **17**, 27–47 (2002)