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ORIGINAL PAPER

Vibration control for a nonlinear three-dimensional Euler–Bernoulli beam under input magnitude and rate constraints

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Abstract In this paper, the vibration control problem for the nonlinear three-dimensional Euler–Bernoulli beam with input magnitude and rate constraints is addressed. By using the backstepping method with smooth hyperbolic tangent function, a boundary control scheme is designed to suppress vibration of the nonlinear three-dimensional Euler–Bernoulli beam and to satisfy the input magnitude and rate constraints. It is proved that the proposed control scheme can handle the vibration and input magnitude and rate constraints simultaneously. Finally, numerical simulations illustrate the effectiveness of the proposed control.

Keywords Vibration control · Three-dimensional Euler–Bernoulli beam · Input magnitude and rate constraints · Smooth hyperbolic tangent function

1 Introduction

Recently, a number of researchers are devoted to design control laws and stability analysis of partial differential equation systems and many researchers have done a lot of significant work in this field [1–7]. In [8], the vibration control design is proposed to suppress the vibration of a flexible Euler–Bernoulli beam under the boundary output constraint. In [9], the boundary stabilization of a one-dimensional tip-force destabilized shear beam equation is considered with boundary control. In [10], the finite-dimensional backstepping control and Lyapunov's direct method are applied in boundary control law design. An adaptive boundary control is developed in [11] to suppress the belt's vibration. In [12], boundary control law is designed on the original PDE dynamics to reduce the hose's vibration. A hybrid control scheme which combines the advantages of task-space and joint-space control is presented in [13]. In [14], the sliding mode control (SMC) with the backstepping approach is developed to deal with the disturbance in the boundary feedback stabilization design of heat PDE–ODE system. To suppress the vibration of the nonuniform system in [15], full state feedback boundary control is developed for a class of axially moving nonuniform system. In [16], boundary vibration suppression for an axially moving belt with high acceleration/deceleration is studied. In [17], robust adaptive control applied at the top boundary is developed for a thruster assisted position mooring system via Lyapunov's direct method.

In practice, many researchers do not take the control input constraints into account. However, constraint problem is a very important problem for the

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research because of physical limits of the controller. Researchers are supposed to pay more attention on it. In [18], state feedback boundary control with an auxiliary system is designed, which is proposed to compensate for the input constraint. The anti-windup design in [19] is proposed for the boundary control problem of a flexible manipulator. By using the backstepping method in [20], a boundary control scheme with smooth hyperbolic function is proposed based on the original PDEs to regulate the hose's vibration with input constraint. In [21], a neural network (NN) controller approximated by a radial basis function neural network (RBFNN) is designed to suppress the vibration of a flexible robotic manipulator system with input deadzone. In [22], boundary control law is designed to regulate angular position and suppress elastic vibration simultaneously. And smooth hyperbolic functions are used to satisfy input constraints. These above works are all talking about how to solve the input constraint problem. However, they do not solve input magnitude and rate constraints simultaneously. In [23], the paper studies the control problem of spacecraft under control input magnitude and rate constraints, but it aims at ordinary differential equation.

In this work, we study the vibration suppression problem for the nonlinear three-dimensional Euler–Bernoulli beam with input magnitude and rate constraints. The backstepping method with smooth hyperbolic tangent function is proposed to control the effect of the input magnitude and input rate constraints. The system stability is tested on the basis of the Lyapunov's direct method. The main contributions in this paper are summarized below: (1) Boundary control system with smooth hyperbolic tangent function by using backstepping method is designed to stabilize the nonlinear three-dimensional Euler–Bernoulli beam under the condition of input magnitude and rate constraints; (2) the system stability analysis is on the basis of the Lyapunov's direct method without simplification.

The rest of this paper is organized as follows. An Euler–Bernoulli beam with a payload in the three-dimensional space with input magnitude and rate constraints is shown in Sect. 2. In Sect. 3, a boundary control scheme is designed and analyzed. In Sect. 4, numerical simulations are demonstrated to show the effectiveness of the proposed controller. A conclusion is drawn in Sect. 5.

2 System statement

2.1 Dynamics analysis

In this paper, we study Euler–Bernoulli beam with a payload in the three-dimensional space with input magnitude and rate constraints. The effect of gravity is ignored due to the flexibility of the beam. In the engineering field, the model of three-dimensional Euler–Bernoulli beam has been widely used in many areas, such as the flexible marine risers in [18] and flexible aerial refueling hose in [20]. The system is shown in Fig. 1.

In Fig. 1, $OUVW$ is the fixed global inertial coordinate. The bending stiffness, the axial stiffness, and the tension of the beam are represented by EI, EA, and T. ρ represents the mass per meter of the link. The length of the link is represented by l. The mass of the payload at the top of the beam is represented by m. U_u, U_v, U_w and $\dot{U}_u, \dot{U}_v, \dot{U}_w$ are the inputs and the input rates generated at the top of the beam by the controllers, respectively.

Remark 1 For clarity, the following notations are used throughout this article:

$$(\dot{*}) = \frac{\partial(*)}{\partial t}, (\ddot{*}) = \frac{\partial^2(*)}{\partial t^2}, (*)' = \frac{\partial(*)}{\partial s},$$

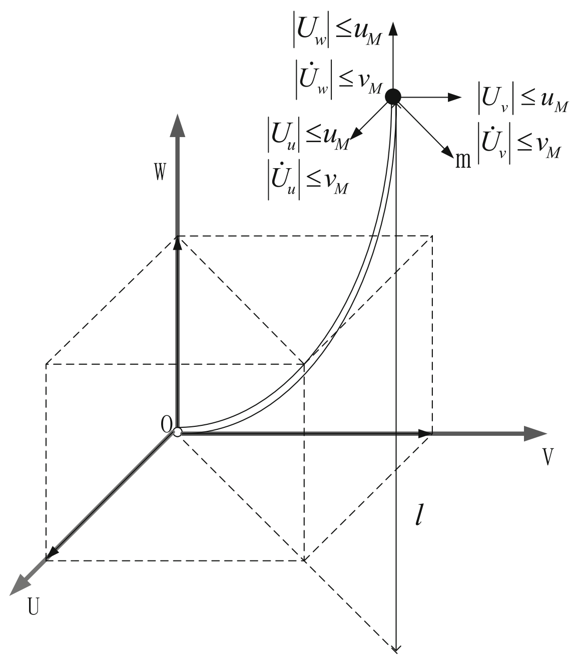


Fig. 1 A nonlinear three-dimensional Euler–Bernoulli beam with a payload

$$(*)'' = \frac{\partial^2(*)}{\partial s^2}, (*)'''' = \frac{\partial^4(*)}{\partial s^4}, (\dot{*})' = \frac{\partial^2(*)}{\partial t \partial s},$$
$$(\dot{*})'' = \frac{\partial^3(*)}{\partial t \partial s^2}, (*)_0 = *(0,t), (\dot{*})_0 = \frac{\partial(*(0,t))}{\partial t},$$
$$(\ddot{*})_0 = \frac{\partial^2(*(0,t))}{\partial t^2}, (*)'_0 = \frac{\partial(*(s,t))}{\partial s}\Big|_{s=0},$$
$$(*)''_0 = \frac{\partial^2(*(s,t))}{\partial s^2}\Big|_{s=0}, (*)'''_0 = \frac{\partial^3(*(s,t))}{\partial s^3}\Big|_{s=0},$$
$$(\dot{*})_l = \frac{\partial(*(l,t))}{\partial t}, (\ddot{*})_l = \frac{\partial^2(*(l,t))}{\partial t^2},$$
$$(*)'_l = \frac{\partial(*(s,t))}{\partial s}\Big|_{s=l}, (*)''_l = \frac{\partial^2(*(s,t))}{\partial s^2}\Big|_{s=l},$$
$$(*)'''_l = \frac{\partial^3(*(s,t))}{\partial s^3}\Big|_{s=l}$$

The model of the nonlinear three-dimensional Euler–Bernoulli beam with a payload is given by [24].

$$\begin{aligned}\rho\ddot{u} &= Tu'' + \mathrm{EA}(w''u' + u''w') \\ &\quad + \frac{3}{2}\mathrm{EA}(u')^2u'' + \frac{\mathrm{EA}}{2}[u''(v')^2 \\ &\quad + 2u'v'v''] - \mathrm{EI}u''''\end{aligned} \tag{1}$$

$$\begin{aligned}\rho\ddot{v} &= Tv'' + \mathrm{EA}(w''v' + v''w') \\ &\quad + \frac{3}{2}\mathrm{EA}(v')^2v'' + \frac{\mathrm{EA}}{2}[v''(u')^2 \\ &\quad + 2v'u'u''] - \mathrm{EI}v''''\end{aligned} \tag{2}$$

$$\rho\ddot{w} = \mathrm{EA}w'' + \mathrm{EA}u'u'' + \mathrm{EA}v'v'' \tag{3}$$

$\forall(s,t) \in [0,l] \times [0,\infty)$, and the boundary conditions are shown as follows:

$$u''_0 = v''_0 = 0 \tag{4}$$

$$u''_l = v''_l = 0 \tag{5}$$

$$\begin{aligned}U_u &= m\ddot{u}_l + Tu'_l + \frac{1}{2}\mathrm{EA}\left(u'_l\right)^3 \\ &\quad + \mathrm{EA}u'_lw'_l + \frac{1}{2}\mathrm{EA}u'_l\left(v'_l\right)^2 - \mathrm{EI}u'''_l\end{aligned} \tag{6}$$

$$\begin{aligned}U_v &= m\ddot{v}_l + Tv'_l \\ &\quad + \frac{1}{2}\mathrm{EA}\left(v'_l\right)^3 + \mathrm{EA}v'_lw'_l \\ &\quad + \frac{1}{2}\mathrm{EA}v'_l\left(u'_l\right)^2 - \mathrm{EI}v'''_l\end{aligned} \tag{7}$$

$$\begin{aligned}U_w &= m\ddot{w}_l + \mathrm{EA}w'_l \\ &\quad + \frac{1}{2}\mathrm{EA}\left(u'_l\right)^2 + \frac{1}{2}\mathrm{EA}\left(v'_l\right)^2\end{aligned} \tag{8}$$

2.2 Preliminaries

Lemma 1 [25] *Let $\varphi(s,t) \in R$ be a function defined on $s \in [0,l]$ and $t \in [0,\infty]$ that satisfies the boundary conditions $\varphi(0,t) = 0, \forall t \in [0,\infty)$, then the following inequalities hold:*

$$\int_0^l \varphi^2(s,t)ds \le l^2 \int_0^l \left[\varphi'(s,t)\right]^2 ds, \quad \forall s \in [0,l],$$

$$\varphi^2(s,t) \le l \int_0^l \left[\varphi'(s,t)\right]^2 ds, \quad \forall s \in [0,l]$$

Lemma 2 [26] *Let $\varphi_1(s,t), \varphi_2(s,t) \in R$ with $s \in [0,l]$ and $t \in [0,\infty]$, then the following inequalities hold:*

$$\begin{aligned}|\varphi_1(s,t)\varphi_2(s,t)| &= \left|\left(\frac{1}{\sqrt{\delta}}\varphi_1(s,t)\right)\left(\sqrt{\delta}\varphi_2(s,t)\right)\right| \\ &\le \frac{1}{\delta}\varphi_1^2(s,t) + \delta\varphi_2^2(s,t), \\ &\quad \forall\varphi_1(s,t), \varphi_2(s,t) \in R\end{aligned}$$

3 Control design and stability analysis

In this paper, the usual block diagram of the closed-loop control of the nonlinear three-dimensional Euler–Bernoulli beam is given as follows (Fig. 2).

3.1 Control design

The control objective is to suppress elastic vibration of the nonlinear three-dimensional Euler–Bernoulli beam in the presence of input magnitude and rate constraints. The backstepping method with smooth hyperbolic tangent function is used to design control laws U_u, U_v, U_w. The Lyapunov's direct method is adopted to analyze the closed-loop stability of the system.

In this paper, the control inputs with magnitude constraints are described as:

$$U_u(t) = g_u(u_1) = u_M \tanh\left(\frac{u_1}{u_M}\right) \tag{9}$$

$$U_v(t) = g_u(u_2) = u_M \tanh\left(\frac{u_2}{u_M}\right) \tag{10}$$

$$U_w(t) = g_u(u_3) = u_M \tanh\left(\frac{u_3}{u_M}\right) \tag{11}$$

In this paper, the control inputs with rate constraints are described as:

$$\dot{U}_u(t) = g_v(v_1) = v_M \tanh\left(\frac{v_1}{v_M}\right) \tag{12}$$

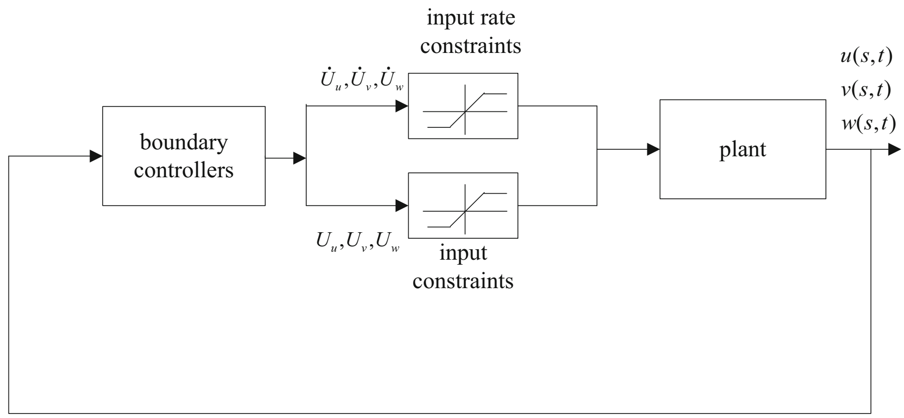


Fig. 2 the usual block diagram of the closed-loop control of the nonlinear three-dimensional Euler–Bernoulli beam system

$$\dot{U}_v(t) = g_v(v_2) = v_M \tanh\left(\frac{v_2}{v_M}\right) \tag{13}$$

$$\dot{U}_w(t) = g_v(v_3) = v_M \tanh\left(\frac{v_3}{v_M}\right) \tag{14}$$

As the usual backstepping approach, we define the variables as follows:

$$d_{11} = \dot{u}_l + \beta l u_l' \tag{15}$$
$$d_{12} = \dot{v}_l + \beta l v_l' \tag{16}$$
$$d_{13} = \dot{w}_l + \beta l w_l' \tag{17}$$
$$d_{21} = g_u(u_1) - a_1 \tag{18}$$
$$d_{22} = g_u(u_2) - a_2 \tag{19}$$
$$d_{23} = g_u(u_3) - a_3 \tag{20}$$
$$d_{31} = g_v(v_1) - a_4 \tag{21}$$
$$d_{32} = g_v(v_2) - a_5 \tag{22}$$
$$d_{33} = g_v(v_3) - a_6 \tag{23}$$

Step 1 We consider the Lyapunov function as

$$\begin{aligned} V_{b1}(t) = {} & \frac{1}{2}md_{11}^2 + \frac{1}{2}md_{12}^2 \\ & + \frac{1}{2}md_{13}^2 + V_1(t) + V_2(t) \end{aligned} \tag{24}$$

where

$$\begin{aligned} V_1(t) = {} & \frac{1}{2}\rho \int_0^l \left[(\dot{u})^2 + (\dot{v})^2 + (\dot{w})^2\right] \mathrm{d}s \\ & + \frac{1}{2}\mathrm{EA} \int_0^l \left[w' + \frac{1}{2}\left(u'\right)^2 + \frac{1}{2}\left(v'\right)^2\right]^2 \mathrm{d}s \\ & + \frac{1}{2}\mathrm{EI} \int_0^l \left[\left(u''\right)^2 + \left(v''\right)^2\right] \mathrm{d}s \\ & + \frac{1}{2}T \int_0^l \left[\left(u'\right)^2 + \left(v'\right)^2\right] \mathrm{d}s \end{aligned} \tag{25}$$

Differentiating the Lyapunov function $V_1(t)$, we can obtain

$$\begin{aligned} \dot{V}_1(t) = {} & \rho \int_0^l \dot{u}\ddot{u}\mathrm{d}s + \rho \int_0^l \dot{v}\ddot{v}\mathrm{d}s + \rho \int_0^l \dot{w}\ddot{w}\mathrm{d}s \\ & + \mathrm{EA} \int_0^l \left(w' + \frac{1}{2}\left(u'\right)^2 + \frac{1}{2}\left(v'\right)^2\right) \\ & \left(\dot{w}' + u'\dot{u}' + v'\dot{v}'\right) \mathrm{d}s \\ & + \mathrm{EI} \int_0^l \left(u''\dot{u}'' + v''\dot{v}''\right) \mathrm{d}s \\ & + T \int_0^l \left(u'\dot{u}' + v'\dot{v}'\right) \mathrm{d}s \\ = {} & A_1 + A_2 + A_3 + A_4, \end{aligned} \tag{26}$$

where

$$A_1 = \rho \int_0^l \dot{u}\ddot{u}\mathrm{d}s + \rho \int_0^l \dot{v}\ddot{v}\mathrm{d}s + \rho \int_0^l \dot{w}\ddot{w}\mathrm{d}s \tag{27}$$

$$\begin{aligned} A_2 = {} & \mathrm{EA} \int_0^l \left(w' + \frac{1}{2}\left(u'\right)^2 + \frac{1}{2}\left(v'\right)^2\right) \\ & \left(\dot{w}' + u'\dot{u}' + v'\dot{v}'\right) \mathrm{d}s \end{aligned} \tag{28}$$

$$A_3 = \mathrm{EI} \int_0^l \left(u''\dot{u}'' + v''\dot{v}''\right) \mathrm{d}s \tag{29}$$

$$A_4 = T \int_0^l \left(u'\dot{u}' + v'\dot{v}'\right) \mathrm{d}s \tag{30}$$

Substituting (1)–(3) into A_1, we can obtain

$$\begin{aligned} A_1 = {} & \int_0^l \dot{u} \left\{T u'' + \mathrm{EA}(w''u' + u''w') \right. \\ & \left. + \frac{3}{2}\mathrm{EA}(u')^2u'' + \frac{\mathrm{EA}}{2}[u''(v')^2 + 2u'v'v''] - \mathrm{EI}u'''' \right\} \mathrm{d}s \end{aligned}$$

$$+\int_0^l \dot{v}\left\{Tv''+\mathrm{EA}(w''v'+v''w')+\frac{3}{2}\mathrm{EA}(v')^2v'' +\frac{\mathrm{EA}}{2}[v''(u')^2+2v'u'u'']-\mathrm{EI}v''''\right\}\mathrm{d}s$$
$$+\int_0^l \dot{w}\left\{\mathrm{EA}w''+\mathrm{EA}u'u''+\mathrm{EA}v'v''\right\}\mathrm{d}s$$
$$=\int_0^l \left(\begin{array}{l}(Tu''\dot{u}+Tv''\dot{v})-(\mathrm{EI}u''''\dot{u}+\mathrm{EI}v''''\dot{v})\\ +\frac{3}{2}\mathrm{EA}(u')^2u''\dot{u}+\frac{3}{2}\mathrm{EA}(v')^2v''\dot{v}\\ +\mathrm{EA}(w''u'+u''w')\dot{u}+\mathrm{EA}(w''v'+v''w')\dot{v}\\ +\frac{\mathrm{EA}}{2}[u''(v')^2\\ +2u'v'v'']\dot{u}+\frac{\mathrm{EA}}{2}[v''(u')^2+2v'u'u'']\dot{v}\\ +\mathrm{EA}\dot{w}w''+\mathrm{EA}\left(u'u''+v'v''\right)\dot{w}\end{array}\right)\mathrm{d}s \tag{31}$$

Then, integrating A_2-A_4 by parts, we can obtain

$$\begin{aligned}A_2 =& \mathrm{EA}w_l'\dot{w}_l-\mathrm{EA}\int_0^l \dot{w}w''\mathrm{d}s+\frac{1}{2}\mathrm{EA}\left(u_l'\right)^3\dot{u}_l\\ &-\frac{3}{2}\mathrm{EA}\int_0^l \dot{u}(u')^2u''\mathrm{d}s+\frac{1}{2}\mathrm{EA}\left(v_l'\right)^3\dot{v}_l\\ &-\frac{3}{2}\mathrm{EA}\int_0^l \dot{v}(v')^2v''\mathrm{d}s+\mathrm{EA}w_l'u_l'\dot{u}_l\\ &-\mathrm{EA}\int_0^l (w''u'+u''w')\dot{u}\mathrm{d}s+\mathrm{EA}w_l'v_l'\dot{v}_l\\ &-\mathrm{EA}\int_0^l (w''v'+v''w')\dot{v}\mathrm{d}s+\frac{1}{2}\mathrm{EA}\left(u_l'\right)^2\dot{v}_lv_l'\\ &-\frac{1}{2}\mathrm{EA}\int_0^l \left[v''(u')^2+2u'u''v'\right]\dot{v}\mathrm{d}s\\ &+\frac{1}{2}\mathrm{EA}\left(v_l'\right)^2u_l'\dot{u}_l\\ &-\frac{1}{2}\mathrm{EA}\int_0^l \left[u''(v')^2+2v'v''u'\right]\dot{u}\mathrm{d}s\\ &+\frac{1}{2}\mathrm{EA}\left(u_l'\right)^2\dot{w}_l\\ &-\mathrm{EA}\int_0^l u''u'\dot{w}\mathrm{d}s+\frac{1}{2}\mathrm{EA}\left(v_l'\right)^2\dot{w}_l\\ &-\mathrm{EA}\int_0^l v''v'\dot{w}\mathrm{d}s\end{aligned} \tag{32}$$

$$\begin{aligned}A_3 =& -\mathrm{EI}\dot{u}_lu_l'''+\mathrm{EI}\int_0^l \dot{u}u''''\mathrm{d}s-\mathrm{EI}\dot{v}_lv_l'''\\ &+\mathrm{EI}\int_0^l \dot{v}v''''\mathrm{d}s\end{aligned} \tag{33}$$

$$\begin{aligned}A_4 =& Tu_l'\dot{u}_l+Tv_l'\dot{v}_l\\ &-T\int_0^l (u''\dot{u}+v''\dot{v})\mathrm{d}s\end{aligned} \tag{34}$$

Noticing that (6)–(8) and combining A_1-A_4, we can have

$$\begin{aligned}\dot{V}_1(t) =& \left(\frac{1}{2}\mathrm{EA}\left(u_l'\right)^3+\mathrm{EA}u_l'w_l'+\frac{1}{2}\mathrm{EA}\left(v_l'\right)^2\ u_l'\right.\\ &\left.-\mathrm{EI}u_l'''+Tu_l'\right)\dot{u}_l\\ &+\left(\frac{1}{2}\mathrm{EA}\left(v_l'\right)^3+\mathrm{EA}v_l'w_l'\right.\\ &\left.+\frac{1}{2}\mathrm{EA}\left(u_l'\right)^2v_l'-\mathrm{EI}v_l'''+Tv_l'\right)\dot{v}_l\\ &+\left(\mathrm{EA}w_l'+\frac{1}{2}\mathrm{EA}\left(u_l'\right)^2+\frac{1}{2}\mathrm{EA}\left(v_l'\right)^2\right)\dot{w}_l\\ =& (U_u-m\ddot{u}_l)\dot{u}_l+(U_v-m\ddot{v}_l)\dot{v}_l+(U_w-m\ddot{w}_l)\dot{w}_l,\end{aligned} \tag{35}$$

where

$$V_2(t)=\beta\rho\int_0^l s(\dot{u}u'+\dot{v}v'+\dot{w}w')\mathrm{d}s \tag{36}$$

The derivative of $V_2(t)$ is

$$\begin{aligned}\dot{V}_2(t) =& \beta\rho\int_0^l s(\ddot{u}u'+\dot{u}\dot{u}'+\ddot{v}v'+\dot{v}\dot{v}'+\ddot{w}w'+\dot{w}\dot{w}')\mathrm{d}s\\ =& B_1+B_2+B_3+B_4,\end{aligned} \tag{37}$$

where

$$B_1=\beta\rho\int_0^l s\ddot{u}u'\mathrm{d}s \tag{38}$$

$$B_2=\beta\rho\int_0^l s\ddot{v}v'\mathrm{d}s \tag{39}$$

$$B_3=\beta\rho\int_0^l s\ddot{w}w'\mathrm{d}s \tag{40}$$

$$B_4=\beta\rho\int_0^l s(\dot{u}\dot{u}'+\dot{v}\dot{v}'+\dot{w}\dot{w}')\mathrm{d}s \tag{41}$$

Substituting (1) into B_1, we obtain

$$\begin{aligned}B_1 =& \beta\int_0^l su'\left(Tu''+\mathrm{EA}(w''u'+u''w')\right.\\ &+\frac{3}{2}\mathrm{EA}(u')^2u''+\frac{\mathrm{EA}}{2}[u''(v')^2+2u'v'v'']\\ &\left.-\mathrm{EI}u''''\right)\mathrm{d}s\\ =& \beta T\int_0^l su'u''\mathrm{d}s+\beta\mathrm{EA}\int_0^l su'(w''u'+u''w')\mathrm{d}s\\ &+\frac{3}{2}\beta\mathrm{EA}\int_0^l su'(u')^2u''\mathrm{d}s\\ &+\frac{1}{2}\beta\mathrm{EA}\int_0^l su'[u''(v')^2+2u'v'v'']\mathrm{d}s\\ &-\beta\mathrm{EI}\int_0^l su'u''''\mathrm{d}s\end{aligned} \tag{42}$$

Then, integrating B_1 by parts, we obtain

$$\begin{aligned}B_1 = &\frac{1}{2}\beta T l\left(u_l'\right)^2-\frac{1}{2}\beta T\int_0^l (u')^2\mathrm{d}s\\&+\beta \mathrm{EA}l\left(u_l'\right)^2 w_l'\\&-\beta\mathrm{EA}\int_0^l (u')^2w'\mathrm{d}s\\&-\beta\mathrm{EA}\int_0^l su''u'w'\mathrm{d}s+\frac{3}{8}\beta\mathrm{EA}l\left(u_l'\right)^4\\&-\frac{3}{8}\beta\mathrm{EA}\int_0^l (u')^4\mathrm{d}s\\&+\frac{1}{4}\beta\mathrm{EA}l\left(u_l'v_l'\right)^2\mathrm{d}s\\&-\frac{1}{4}\beta\mathrm{EA}\int_0^l (u'v')^2\mathrm{d}s\\&+\frac{1}{2}\beta\mathrm{EA}\int_0^l s(u')^2v'v''\mathrm{d}s\\&-\beta\mathrm{EI}lu_l'u_l'''-\frac{3}{2}\beta\mathrm{EI}\int_0^l (u'')^2\mathrm{d}s\end{aligned}\tag{43}$$

Similarly, substituting (2)–(3) into $B_2 - B_3$, respectively, and then integrating $B_2 - B_3$ by parts, we can obtain

$$\begin{aligned}B_2 = &\frac{1}{2}\beta T l\left(v_l'\right)^2-\frac{1}{2}\beta T\int_0^l (v')^2\mathrm{d}s\\&+\beta \mathrm{EA}l\left(v_l'\right)^2 w_l'\\&-\beta\mathrm{EA}\int_0^l (v')^2w'\mathrm{d}s\\&-\beta\mathrm{EA}\int_0^l sv''v'w'\mathrm{d}s+\frac{3}{8}\beta\mathrm{EA}l\left(v_l'\right)^4\\&-\frac{3}{8}\beta\mathrm{EA}\int_0^l (v')^4\mathrm{d}s\\&+\frac{1}{4}\beta\mathrm{EA}l\left(u_l'v_l'\right)^2\mathrm{d}s\\&-\frac{1}{4}\beta\mathrm{EA}\int_0^l (v'u')^2\mathrm{d}s+\frac{1}{2}\beta\mathrm{EA}\int_0^l s(v')^2u'u''\mathrm{d}s\\&-\beta\mathrm{EI}lv_l'v_l'''-\frac{3}{2}\beta\mathrm{EI}\int_0^l (v'')^2\mathrm{d}s\end{aligned}\tag{44}$$

$$\begin{aligned}B_3 = &\frac{1}{2}\beta\mathrm{EA}l(w_l')^2-\frac{1}{2}\beta\mathrm{EA}\int_0^l (w')^2\mathrm{d}s\\&+\beta\int_0^l (\mathrm{EA}su'u''w'+\mathrm{EA}sv'v''w')\mathrm{d}s\end{aligned}\tag{45}$$

Then, integrating $B_4$by parts, we obtain

$$\begin{aligned}B_4 = &\frac{1}{2}\beta\rho l(\dot{u}_l)^2-\frac{1}{2}\beta\rho\int_0^l (\dot{u})^2\mathrm{d}s\\&+\frac{1}{2}\beta\rho l(\dot{v}_l)^2-\frac{1}{2}\beta\rho\int_0^l (\dot{v})^2\mathrm{d}s\\&+\frac{1}{2}\beta\rho l(\dot{w}_l)^2-\frac{1}{2}\beta\rho\int_0^l (\dot{w})^2\mathrm{d}s\end{aligned}\tag{46}$$

Noticing that

$$\begin{aligned}&\frac{1}{2}\beta\mathrm{EA}\int_0^l s(u')^2v'v''\mathrm{d}s+\frac{1}{2}\beta\mathrm{EA}\int_0^l s(v')^2u'u''\mathrm{d}s\\&=\frac{1}{4}\beta\mathrm{EA}l\left(u_l'v_l'\right)^2-\frac{1}{4}\beta\mathrm{EA}\int_0^l (u'v')^2\mathrm{d}s\end{aligned}\tag{47}$$

Combining $B_1 - B_4$ and noticing that (6)–(8), we have

$$\begin{aligned}\dot{V}_2(t) = &-\frac{1}{2}\beta\rho\int_0^l (\dot{u})^2\mathrm{d}s-\frac{1}{2}\beta\rho\int_0^l (\dot{v})^2\mathrm{d}s\\&-\frac{1}{2}\beta\rho\int_0^l (\dot{w})^2\mathrm{d}s-\frac{1}{2}\beta\mathrm{EA}\int_0^l (w')^2\mathrm{d}s\\&-\frac{3}{8}\beta\mathrm{EA}\int_0^l (u')^4\mathrm{d}s-\frac{3}{8}\beta\mathrm{EA}\int_0^l (v')^4\mathrm{d}s\\&-\beta\mathrm{EA}\int_0^l (u')^2w'\mathrm{d}s-\beta\mathrm{EA}\int_0^l (v')^2w'\mathrm{d}s\\&-\frac{3}{4}\beta\mathrm{EA}\int_0^l (u'v')^2\mathrm{d}s-\frac{3}{2}\beta\mathrm{EI}\int_0^l (u'')^2\mathrm{d}s\\&-\frac{3}{2}\beta\mathrm{EI}\int_0^l (v'')^2\mathrm{d}s-\frac{1}{2}\beta T\int_0^l (u')^2\mathrm{d}s\\&-\frac{1}{2}\beta T\int_0^l (v')^2\mathrm{d}s+\frac{1}{2}\beta\rho l(\dot{u}_l)^2\\&+\frac{1}{2}\beta\rho l(\dot{v}_l)^2+\frac{1}{2}\beta\rho l(\dot{w}_l)^2\\&+\beta l\big[(U_u-m\ddot{u}_l)u_l'+(U_v-m\ddot{v}_l)v_l'\\&+(U_w-m\ddot{w}_l)w_l'\big]\\&-\frac{1}{2}\beta T l\left(u_l'\right)^2-\frac{1}{2}\beta T l\left(v_l'\right)^2\\&-\frac{1}{2}\beta\mathrm{EA}l\left[\frac{1}{2}\left(u_l'\right)^2+\frac{1}{2}\left(v_l'\right)^2+w_l'\right]^2\end{aligned}\tag{48}$$

Noting that (6)–(8), the derivative of (24) is

$$\begin{aligned}\dot{V}_{b1}(t) = &md_{11}\dot{d}_{11}+md_{12}\dot{d}_{12}\\&+md_{13}\dot{d}_{13}+\dot{V}_1(t)+\dot{V}_2(t)\\=&d_{11}\left(m\ddot{u}_l+m\beta l\dot{u'}_l\right)+d_{12}\left(m\ddot{v}_l+m\beta l\dot{v'}_l\right)\\&+d_{13}\left(m\ddot{w}_l+m\beta l\dot{w'}_l\right)+\dot{V}_1(t)+\dot{V}_2(t)\\=&d_{11}(U_u-Tu_l'-\frac{1}{2}\mathrm{EA}\left(u_l'\right)^3-\mathrm{EA}u_l'w_l'\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{EA}u_l'\left(v_l'\right)^2+\mathrm{EI}u_l'''+m\beta l\dot{u_l'})\\
&+d_{12}(U_v-Tv_l'-\frac{1}{2}\mathrm{EA}\left(v_l'\right)^3-\mathrm{EA}v_l'w_l'\\
&-\frac{1}{2}\mathrm{EA}v_l'\left(u_l'\right)^2+\mathrm{EI}v_l'''+m\beta l\dot{v'}_l)\\
&+d_{13}(U_w-\mathrm{EA}w_l'-\frac{1}{2}\mathrm{EA}\left(u_l'\right)^2\\
&-\frac{1}{2}\mathrm{EA}\left(v_l'\right)^2+m\beta l\dot{w'}_l)+(U_u-m\ddot{u}_l)\dot{u}_l\\
&+(U_v-m\ddot{v}_l)\dot{v}_l+(U_w-m\ddot{w}_l)\dot{w}_l\\
&-\frac{1}{2}\beta\rho\int_0^l(\dot{u})^2\mathrm{d}s-\frac{1}{2}\beta\rho\int_0^l(\dot{v})^2\mathrm{d}s\\
&-\frac{1}{2}\beta\rho\int_0^l(\dot{w})^2\mathrm{d}s-\frac{1}{2}\beta\mathrm{EA}\int_0^l(w')^2\mathrm{d}s\\
&-\frac{3}{8}\beta\mathrm{EA}\int_0^l(u')^4\mathrm{d}s\\
&-\frac{3}{8}\beta\mathrm{EA}\int_0^l(v')^4\mathrm{d}s-\beta\mathrm{EA}\int_0^l(u')^2w'\mathrm{d}s\\
&-\beta\mathrm{EA}\int_0^l(v')^2w'\mathrm{d}s\\
&-\frac{3}{4}\beta\mathrm{EA}\int_0^l(u'v')^2\mathrm{d}s-\frac{3}{2}\beta\mathrm{EI}\int_0^l(u'')^2\mathrm{d}s\\
&-\frac{3}{2}\beta\mathrm{EI}\int_0^l(v'')^2\mathrm{d}s\\
&-\frac{1}{2}\beta T\int_0^l(u')^2\mathrm{d}s-\frac{1}{2}\beta T\int_0^l(v')^2\mathrm{d}s\\
&+\frac{1}{2}\beta\rho l(\dot{u}_l)^2+\frac{1}{2}\beta\rho l(\dot{v}_l)^2+\frac{1}{2}\beta\rho l(\dot{w}_l)^2\\
&+\beta l\left[(U_u-m\ddot{u}_l)u_l'+(U_v-m\ddot{v}_l)v_l'\right.\\
&\left.+(U_w-m\ddot{w}_l)w_l'\right]-\frac{1}{2}\beta Tl\left(u_l'\right)^2\\
&-\frac{1}{2}\beta Tl\left(v_l'\right)^2-\frac{1}{2}\beta\mathrm{EA}l\\
&\quad\times\left[\frac{1}{2}\left(u_l'\right)^2+\frac{1}{2}\left(v_l'\right)^2+w_l'\right]^2
\end{aligned}
\tag{49}
$$

By combining (49), we can get

$$
\begin{aligned}
\dot{V}_{b1}(t)=&\,d_{11}\left(U_u+m\beta l\dot{u'}_l\right)+d_{12}\left(U_v+m\beta l\dot{v'}_l\right)\\
&+d_{13}\left(U_w+m\beta l\dot{w'}_l\right)-\frac{1}{2}\beta\rho\int_0^l(\dot{u})^2\mathrm{d}s\\
&-\frac{1}{2}\beta\rho\int_0^l(\dot{v})^2\mathrm{d}s-\frac{1}{2}\beta\rho\int_0^l(\dot{w})^2\mathrm{d}s\\
&-\frac{1}{2}\beta\mathrm{EA}\int_0^l(w')^2\mathrm{d}s-\frac{3}{8}\beta\mathrm{EA}\int_0^l(u')^4\mathrm{d}s\\
&-\frac{3}{8}\beta\mathrm{EA}\int_0^l(v')^4\mathrm{d}s-\beta\mathrm{EA}\int_0^l(u')^2w'\mathrm{d}s\\
&-\beta\mathrm{EA}\int_0^l(v')^2w'\mathrm{d}s-\frac{3}{4}\beta\mathrm{EA}\int_0^l(u'v')^2\mathrm{d}s\\
&-\frac{3}{2}\beta\mathrm{EI}\int_0^l(u'')^2\mathrm{d}s-\frac{3}{2}\beta\mathrm{EI}\int_0^l(v'')^2\mathrm{d}s\\
&-\frac{1}{2}\beta T\int_0^l(u')^2\mathrm{d}s-\frac{1}{2}\beta T\int_0^l(v')^2\mathrm{d}s\\
&+\frac{1}{2}\beta\rho l(\dot{u}_l)^2+\frac{1}{2}\beta\rho l(\dot{v}_l)^2+\frac{1}{2}\beta\rho l(\dot{w}_l)^2\\
&-\frac{1}{2}\beta Tl\left(u_l'\right)^2-\frac{1}{2}\beta Tl\left(v_l'\right)^2\\
&-\frac{1}{2}\beta\mathrm{EA}l\left[\frac{1}{2}\left(u_l'\right)^2+\frac{1}{2}\left(v_l'\right)^2+w_l'\right]^2
\end{aligned}
\tag{50}
$$

Noting that (18)–(20), we can obtain

$$
\begin{aligned}
V_{b1}(t)=&\,d_{11}\left(d_{21}+a_1+m\beta l\dot{u'}_l\right)\\
&+d_{12}\left(d_{22}+a_2+m\beta l\dot{v'}_l\right)\\
&+d_{13}\left(d_{23}+a_3+m\beta l\dot{w'}_l\right)+D,
\end{aligned}
\tag{51}
$$

where

$$
\begin{aligned}
D=&-\frac{1}{2}\beta\rho\int_0^l(\dot{u})^2\mathrm{d}s-\frac{1}{2}\beta\rho\int_0^l(\dot{v})^2\mathrm{d}s\\
&-\frac{1}{2}\beta\rho\int_0^l(\dot{w})^2\mathrm{d}s-\frac{1}{2}\beta\mathrm{EA}\int_0^l(w')^2\mathrm{d}s\\
&-\frac{3}{8}\beta\mathrm{EA}\int_0^l(u')^4\mathrm{d}s\\
&-\frac{3}{8}\beta\mathrm{EA}\int_0^l(v')^4\mathrm{d}s\\
&-\beta\mathrm{EA}\int_0^l(u')^2w'\mathrm{d}s\\
&-\beta\mathrm{EA}\int_0^l(v')^2w'\mathrm{d}s\\
&-\frac{3}{4}\beta\mathrm{EA}\int_0^l(u'v')^2\mathrm{d}s\\
&-\frac{3}{2}\beta\mathrm{EI}\int_0^l(u'')^2\mathrm{d}s\\
&-\frac{3}{2}\beta\mathrm{EI}\int_0^l(v'')^2\mathrm{d}s\\
&-\frac{1}{2}\beta T\int_0^l(u')^2\mathrm{d}s\\
&-\frac{1}{2}\beta T\int_0^l(v')^2\mathrm{d}s\\
&+\frac{1}{2}\beta\rho l(\dot{u}_l)^2+\frac{1}{2}\beta\rho l(\dot{v}_l)^2
\end{aligned}
$$

$$+\frac{1}{2}\beta\rho l(\dot{w}_l)^2-\frac{1}{2}\beta Tl\left(u_l'\right)^2$$
$$-\frac{1}{2}\beta Tl\left(v_l'\right)^2$$
$$-\frac{1}{2}\beta \mathrm{EA}l\left[\frac{1}{2}\left(u_l'\right)^2+\frac{1}{2}\left(v_l'\right)^2+w_l'\right]^2 \tag{52}$$

Choosing the virtual control laws as

$$a_1=-m\beta l\dot{u}_l'-2c_1\dot{u}_l \tag{53}$$
$$a_2=-m\beta l\dot{v}_l'-2c_2\dot{v}_l \tag{54}$$
$$a_3=-m\beta l\dot{w}_l'-2c_3\dot{w}_l \tag{55}$$

where $c_1>0$, $c_2>0$, $c_3>0$,we have

$$\dot{V}_{b1}(t)=d_{11}d_{21}-2d_{11}c_1\dot{u}_l+d_{12}d_{22}-2d_{12}c_2\dot{v}_l+d_{13}d_{23}-2d_{13}c_3\dot{w}_l+D \tag{56}$$

Step 2 Then, we consider a Lyapunov function candidate

$$V_{b2}(t)=V_{b1}(t)+\frac{1}{2}md_{21}^2+\frac{1}{2}md_{22}^2+\frac{1}{2}md_{23}^2 \tag{57}$$

Then, we design the auxiliary system as

$$\dot{u}_1=\left(\frac{\partial g_u}{\partial u_1}\right)^{-1}g_v(v_1) \tag{58}$$
$$\dot{u}_2=\left(\frac{\partial g_u}{\partial u_2}\right)^{-1}g_v(v_2) \tag{59}$$
$$\dot{u}_3=\left(\frac{\partial g_u}{\partial u_3}\right)^{-1}g_v(v_3) \tag{60}$$

Noting that (18)–(20) and (58)–(60), the derivative of (57) is

$$\begin{aligned}\dot{V}_{b2}(t)&=\dot{V}_{b1}(t)+md_{21}\dot{d}_{21}+md_{22}\dot{d}_{22}+md_{23}\dot{d}_{23}\\&=\dot{V}_{b1}(t)+d_{21}(mg_v(v_1)-m\dot{a}_1)\\&\quad+d_{22}(mg_v(v_2)-m\dot{a}_2)\\&\quad+d_{23}(mg_v(v_3)-m\dot{a}_3)\\&=d_{11}d_{21}-2d_{11}c_1\dot{u}_l+d_{12}d_{22}\\&\quad-2d_{12}c_2\dot{v}_l+d_{13}d_{23}-2d_{13}c_3\dot{w}_l+D\\&\quad+d_{21}(mg_v(v_1)-m\dot{a}_1)\\&\quad+d_{22}(mg_v(v_2)-m\dot{a}_2)\\&\quad+d_{23}(mg_v(v_3)-m\dot{a}_3)\end{aligned} \tag{61}$$

Applying (21)–(23) into (61), we can obtain

$$\begin{aligned}\dot{V}_{b2}(t)&=\dot{V}_{b1}(t)+md_{21}\dot{d}_{21}+md_{22}\dot{d}_{22}+md_{23}\dot{d}_{23}\\&=\dot{V}_{b1}(t)+d_{21}(mg_v(v_1)-m\dot{a}_1)\\&\quad+d_{22}(mg_v(v_2)-m\dot{a}_2)\\&\quad+d_{23}(mg_v(v_3)-m\dot{a}_3)\\&=d_{11}d_{21}-2d_{11}c_1\dot{u}_l+d_{12}d_{22}\\&\quad-2d_{12}c_2\dot{v}_l+d_{13}d_{23}-2d_{13}c_3\dot{w}_l+D\\&\quad+d_{21}(md_{31}+ma_4-m\dot{a}_1)\\&\quad+d_{22}(md_{32}+ma_5-m\dot{a}_2)\\&\quad+d_{23}(md_{33}+ma_6-m\dot{a}_3)\end{aligned} \tag{62}$$

Then, we choose the virtual control laws as

$$a_4=-\frac{1}{m}d_{11}-\frac{1}{m}c_4d_{21}+\dot{a}_1 \tag{63}$$
$$a_5=-\frac{1}{m}d_{12}-\frac{1}{m}c_5d_{22}+\dot{a}_2 \tag{64}$$
$$a_6=-\frac{1}{m}d_{13}-\frac{1}{m}c_6d_{23}+\dot{a}_3 \tag{65}$$

where $c_4>0$, $c_5>0$, $c_6>0$, we can obtain

$$\begin{aligned}\dot{V}_{b2}(t)&=-2d_{11}c_1\dot{u}_l-2d_{12}c_2\dot{v}_l-2d_{13}c_3\dot{w}_l+D\\&\quad-c_4d_{21}^2-c_5d_{22}^2-c_6d_{23}^2+md_{21}d_{31}\\&\quad+md_{22}d_{32}+md_{23}d_{33}\end{aligned} \tag{66}$$

Step 3 Then, we consider a Lyapunov function candidate

$$\begin{aligned}V_b(t)&=V_{b2}(t)+\frac{1}{2}md_{31}^2\\&\quad+\frac{1}{2}md_{32}^2+\frac{1}{2}md_{33}^2\end{aligned} \tag{67}$$

Then, we design the another auxiliary system as

$$\dot{v}_1=\left(\frac{\partial g_v}{\partial v_1}\right)^{-1}U_1 \tag{68}$$
$$\dot{v}_2=\left(\frac{\partial g_v}{\partial v_2}\right)^{-1}U_2 \tag{69}$$
$$\dot{v}_3=\left(\frac{\partial g_v}{\partial v_3}\right)^{-1}U_3 \tag{70}$$

Noting that (21)–(23) and (68)–(70), the derivative of (67) is

$$\begin{aligned}\dot{V}_b(t)&=\dot{V}_{b2}(t)+md_{31}\dot{d}_{31}+md_{32}\dot{d}_{32}+md_{33}\dot{d}_{33}\\&=-2d_{11}c_1\dot{u}_l-2d_{12}c_2\dot{v}_l-2d_{13}c_3\dot{w}_l\\&\quad+D-c_4d_{21}^2-c_5d_{22}^2-c_6d_{23}^2+md_{21}d_{31}\\&\quad+md_{22}d_{32}+md_{23}d_{33}+d_{31}(mU_1-m\dot{a}_4)\\&\quad+d_{32}(mU_2-m\dot{a}_5)+d_{33}(mU_3-m\dot{a}_6)\end{aligned} \tag{71}$$

Then, we choose the virtual control laws as

$$U_1=-\frac{1}{m}c_7d_{31}-d_{21}+\dot{a}_4 \tag{72}$$
$$U_2=-\frac{1}{m}c_8d_{32}-d_{22}+\dot{a}_5 \tag{73}$$

$$U_3 = -\frac{1}{m}c_9 d_{33} - d_{23} + \dot{a}_6 \tag{74}$$

where $c_7 > 0$, $c_8 > 0$, $c_9 > 0$, we can obtain

$$\begin{aligned}\dot{V}_b(t) &= \dot{V}_{b2}(t) + md_{31}\dot{d}_{31} + md_{32}\dot{d}_{32} + md_{33}\dot{d}_{33} \\ &= -2d_{11}c_1\dot{u}_l - 2d_{12}c_2\dot{v}_l - 2d_{13}c_3\dot{w}_l \\ &\quad + D - c_4 d_{21}^2 - c_5 d_{22}^2 - c_6 d_{23}^2 \\ &\quad - c_7 d_{31}^2 - c_8 d_{32}^2 - c_9 d_{33}^2\end{aligned} \tag{75}$$

Substituting $d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23}, d_{31}, d_{32}, d_{33}$, D into (75), we can get

$$\begin{aligned}\dot{V}_b(t) &= \dot{V}_{b2}(t) + md_{31}\dot{d}_{31} + md_{32}\dot{d}_{32} + md_{33}\dot{d}_{33} \\ &= -2\left(\dot{u}_l + \beta l u_l'\right)c_1\dot{u}_l - 2\left(\dot{v}_l + \beta l v_l'\right)c_2\dot{v}_l \\ &\quad - 2\left(\dot{w}_l + \beta l w_l'\right)c_3\dot{w}_l \\ &\quad - c_4 d_{21}^2 - c_5 d_{22}^2 - c_6 d_{23}^2 - c_7 d_{31}^2 \\ &\quad - c_8 d_{32}^2 - c_9 d_{33}^2 \\ &\quad - \frac{1}{2}\beta\rho\int_0^l (\dot{u})^2 \mathrm{d}s - \frac{1}{2}\beta\rho\int_0^l (\dot{v})^2 \mathrm{d}s \\ &\quad - \frac{1}{2}\beta\rho\int_0^l (\dot{w})^2 \mathrm{d}s \\ &\quad - \frac{1}{2}\beta \mathrm{EA}\int_0^l (w')^2 \mathrm{d}s - \frac{3}{8}\beta \mathrm{EA}\int_0^l (u')^4 \mathrm{d}s \\ &\quad - \frac{3}{8}\beta \mathrm{EA}\int_0^l (v')^4 \mathrm{d}s \\ &\quad - \beta \mathrm{EA}\int_0^l (u')^2 w' \mathrm{d}s - \beta \mathrm{EA}\int_0^l (v')^2 w' \mathrm{d}s \\ &\quad - \frac{3}{4}\beta \mathrm{EA}\int_0^l (u'v')^2 \mathrm{d}s \\ &\quad - \frac{3}{2}\beta \mathrm{EI}\int_0^l (u'')^2 \mathrm{d}s - \frac{3}{2}\beta \mathrm{EI}\int_0^l (v'')^2 \mathrm{d}s \\ &\quad - \frac{1}{2}\beta T\int_0^l (u')^2 \mathrm{d}s \\ &\quad - \frac{1}{2}\beta T\int_0^l (v')^2 \mathrm{d}s + \frac{1}{2}\beta\rho l(\dot{u}_l)^2 \\ &\quad + \frac{1}{2}\beta\rho l(\dot{v}_l)^2 + \frac{1}{2}\beta\rho l(\dot{w}_l)^2 \\ &\quad - \frac{1}{2}\beta T l\left(u_l'\right)^2 - \frac{1}{2}\beta T l\left(v_l'\right)^2 \\ &\quad - \frac{1}{2}\beta \mathrm{EA} l\left[\frac{1}{2}\left(u_l'\right)^2 + \frac{1}{2}\left(v_l'\right)^2 + w_l'\right]^2\end{aligned} \tag{76}$$

Noticing that $d_{11} = \dot{u}_l + \beta l u_l'$, $d_{12} = \dot{v}_l + \beta l v_l'$, $d_{13} = \dot{w}_l + \beta l w_l'$, we have

$$\begin{aligned}\dot{V}_b(t) &= -c_1 d_{11}^2 - c_2 d_{12}^2 - c_3 d_{13}^2 - c_4 d_{21}^2 \\ &\quad - c_5 d_{22}^2 - c_6 d_{23}^2 - c_7 d_{31}^2 - c_8 d_{32}^2 \\ &\quad - c_9 d_{33}^2 - \frac{1}{2}\beta\rho\int_0^l (\dot{u})^2 \mathrm{d}s - \frac{1}{2}\beta\rho\int_0^l (\dot{v})^2 \mathrm{d}s \\ &\quad - \frac{1}{2}\beta\rho\int_0^l (\dot{w})^2 \mathrm{d}s - \frac{1}{2}\beta \mathrm{EA}\int_0^l (w')^2 \mathrm{d}s \\ &\quad - \frac{3}{8}\beta \mathrm{EA}\int_0^l (u')^4 \mathrm{d}s - \frac{3}{8}\beta \mathrm{EA}\int_0^l (v')^4 \mathrm{d}s \\ &\quad - \beta \mathrm{EA}\int_0^l (u')^2 w' \mathrm{d}s - \beta \mathrm{EA}\int_0^l (v')^2 w' \mathrm{d}s \\ &\quad - \frac{3}{4}\beta \mathrm{EA}\int_0^l (u'v')^2 \mathrm{d}s - \frac{3}{2}\beta \mathrm{EI}\int_0^l (u'')^2 \mathrm{d}s \\ &\quad - \frac{3}{2}\beta \mathrm{EI}\int_0^l (v'')^2 \mathrm{d}s - \frac{1}{2}\beta T\int_0^l (u')^2 \mathrm{d}s \\ &\quad - \frac{1}{2}\beta T\int_0^l (v')^2 \mathrm{d}s - \left(c_1 - \frac{1}{2}\beta\rho l\right)(\dot{u}_l)^2 \\ &\quad - \left(c_2 - \frac{1}{2}\beta\rho l\right)(\dot{v}_l)^2 - \left(c_3 - \frac{1}{2}\beta\rho l\right)(\dot{w}_l)^2 \\ &\quad - \left(\frac{1}{2}\beta T l - c_1\beta^2 l^2\right)\left(u_l'\right)^2 \\ &\quad - \left(\frac{1}{2}\beta T l - c_2\beta^2 l^2\right)\left(v_l'\right)^2 + c_3\beta^2 l^2\left(w_l'\right)^2 \\ &\quad - \frac{1}{2}\beta \mathrm{EA} l\left[\frac{1}{2}\left(u_l'\right)^2 + \frac{1}{2}\left(v_l'\right)^2 + w_l'\right]^2\end{aligned} \tag{77}$$

Applying the inequality $2\left[w'(s,t)\right]^2 \le \left[u'(s,t)\right]^2$, $2\left[w'(s,t)\right]^2 \le \left[v'(s,t)\right]^2$ in [27], we can get

$$\begin{aligned}\dot{V}_b(t) &\le -c_1 d_{11}^2 - c_2 d_{12}^2 - c_3 d_{13}^2 \\ &\quad - c_4 d_{21}^2 - c_5 d_{22}^2 - c_6 d_{23}^2 \\ &\quad - c_7 d_{31}^2 - c_8 d_{32}^2 - c_9 d_{33}^2 \\ &\quad - \frac{1}{2}\beta\rho\int_0^l (\dot{u})^2 \mathrm{d}s - \frac{1}{2}\beta\rho\int_0^l (\dot{v})^2 \mathrm{d}s \\ &\quad - \frac{1}{2}\beta\rho\int_0^l (\dot{w})^2 \mathrm{d}s - \frac{1}{2}\beta \mathrm{EA}\int_0^l (w')^2 \mathrm{d}s \\ &\quad - \frac{3}{8}\beta \mathrm{EA}\int_0^l (u')^4 \mathrm{d}s - \frac{3}{8}\beta \mathrm{EA}\int_0^l (v')^4 \mathrm{d}s \\ &\quad -\beta \mathrm{EA}\int_0^l (u')^2 w' \mathrm{d}s - \beta \mathrm{EA}\int_0^l (v')^2 w' \mathrm{d}s \\ &\quad - \frac{3}{4}\beta \mathrm{EA}\int_0^l (u'v')^2 \mathrm{d}s - \frac{3}{2}\beta \mathrm{EI}\int_0^l (u'')^2 \mathrm{d}s \\ &\quad - \frac{3}{2}\beta \mathrm{EI}\int_0^l (v'')^2 \mathrm{d}s \\ &\quad - \frac{1}{2}\beta T\int_0^l (u')^2 \mathrm{d}s - \frac{1}{2}\beta T\int_0^l (v')^2 \mathrm{d}s \\ &\quad - \left(c_1 - \frac{1}{2}\beta\rho l\right)(\dot{u}_l)^2\end{aligned}$$

$$-\left(c_2-\frac{1}{2}\beta\rho l\right)(\dot{v}_l)^2$$
$$-\left(c_3-\frac{1}{2}\beta\rho l\right)(\dot{w}_l)^2$$
$$-\left(\frac{1}{2}\beta T l-c_1\beta^2 l^2-\frac{1}{4}c_3\beta^2 l^2\right)\left(u_l'\right)^2$$
$$-\left(\frac{1}{2}\beta T l-c_2\beta^2 l^2-\frac{1}{4}c_3\beta^2 l^2\right)\left(v_l'\right)^2 \quad (78)$$

3.2 Stability analysis

Rewritten (25) as

$$\begin{aligned}V_1(t)=&\frac{1}{2}\rho\int_0^l\left[(\dot{u})^2+(\dot{v})^2+(\dot{w})^2\right]\mathrm{d}s\\&+\frac{1}{2}T\int_0^l\left[(u')^2+(v')^2\right]\mathrm{d}s\\&+\frac{1}{2}\mathrm{EA}\int_0^l(w')^2\mathrm{d}s\\&+\frac{1}{8}\mathrm{EA}\int_0^l(u')^4\mathrm{d}s+\frac{1}{8}\mathrm{EA}\int_0^l(v')^4\mathrm{d}s\\&+\frac{1}{2}\mathrm{EA}\int_0^l(u')^2w'\mathrm{d}s+\frac{1}{2}\mathrm{EA}\int_0^l(v')^2w'\mathrm{d}s\\&+\frac{1}{4}\mathrm{EA}\int_0^l\left(u'v'\right)^2\mathrm{d}s\\&+\frac{1}{2}\mathrm{EI}\int_0^l\left[(u'')^2+(v'')^2\right]\mathrm{d}s\end{aligned}\quad (79)$$

Applying the inequality $2\left[w'(s,t)\right]^2\le\left[u'(s,t)\right]^2$, $2\left[w'(s,t)\right]^2\le\left[v'(s,t)\right]^2$ in [27] and lemma 2, we have

$$\begin{aligned}&-\frac{1}{2\delta}\int_0^l(u')^2\mathrm{d}s-\delta\int_0^l(u')^4\mathrm{d}s\\&\le\int_0^l(u')^2w'\mathrm{d}s\\&\le\frac{1}{2\delta}\int_0^l(u')^2\mathrm{d}s+\delta\int_0^l(u')^4\mathrm{d}s\\&\quad-\frac{1}{2\delta}\int_0^l(v')^2\mathrm{d}s-\delta\int_0^l(v')^4\mathrm{d}s\\&\le\int_0^l(v')^2w'\mathrm{d}s\\&\le\frac{1}{2\delta}\int_0^l(v')^2\mathrm{d}s+\delta\int_0^l(v')^4\mathrm{d}s,\end{aligned}\quad (80)$$

where δ is a constant.

$$\begin{aligned}M(t)=\int_0^l&\Big[(\dot{u})^2+(u')^2+(\dot{v})^2+(v')^2\\&+(\dot{w})^2+(w')^2+(u')^4+(v')^4\\&+(u'v')^2+(u'')^2+(v'')^2\Big]\mathrm{d}s\end{aligned}\quad (81)$$

Let δ satisfy $T-\frac{\mathrm{EA}}{2\delta}\ge 0$ and $\frac{1}{4}-\delta\ge 0$, we can obtain

$$0\le\sigma_1 M(t)\le V_1(t)\le\sigma_2 M(t),\quad (82)$$

where

$$\begin{aligned}\sigma_1&=\frac{1}{2}\min\left[\rho,T-\frac{\mathrm{EA}}{2\delta},\frac{1}{2}\mathrm{EA},\mathrm{EA}\left(\frac{1}{4}-\delta\right),\mathrm{EI}\right]\\\sigma_2&=\frac{1}{2}\max\left[\rho,T+\frac{\mathrm{EA}}{2\delta},\mathrm{EA},\mathrm{EA}\left(\frac{1}{4}+\delta\right),\mathrm{EI}\right]\end{aligned}\quad (83)$$

Similarly, we can obtain

$$\begin{aligned}|V_2(t)|\le&\beta\rho l\int_0^l\Big[(\dot{u})^2+(u')^2\\&+(\dot{v})^2+(v')^2+(\dot{w})^2+(w')^2\Big]\mathrm{d}s\\\le&\beta\rho l M(t)\end{aligned}\quad (84)$$

Let β satisfy $\beta\rho l<\sigma_1$, we have $0<\beta\rho l<m_1$. Let $m_1=\sigma_1-\beta\rho l$, $m_2=\sigma_2+\beta\rho l$, we can get

$$0\le m_1M(t)\le V_1(t)+V_2(t)\le m_2M(t)\quad (85)$$

We conclude

$$\begin{aligned}H=&\frac{1}{2}md_{11}^2+\frac{1}{2}md_{12}^2+\frac{1}{2}md_{13}^2+\frac{1}{2}md_{21}^2\\&+\frac{1}{2}md_{22}^2+\frac{1}{2}md_{23}^2+\frac{1}{2}md_{31}^2\\&+\frac{1}{2}md_{32}^2+\frac{1}{2}md_{33}^2\end{aligned}\quad (86)$$

$$V_1(t)+V_2(t)+H=V_b(t)\quad (87)$$

Therefore, we can further obtain

$$0\le\lambda_1(M(t)+H)\le V_b(t)\le\lambda_2(M(t)+H)\quad (88)$$

where $\lambda_1=\min(m_1,1)=m_1$, $\lambda_2=\max(m_2,1)=m_2$ are two positive constants.

From (78), we can obtain

$$\begin{aligned}\dot{V}_b(t)\le&-c_1d_{11}^2-c_2d_{12}^2-c_3d_{13}^2-c_4d_{21}^2\\&-c_5d_{22}^2-c_6d_{23}^2-c_7d_{31}^2-c_8d_{32}^2\\&-c_9d_{33}^2-\frac{1}{2}\beta\rho\int_0^l(\dot{u})^2\mathrm{d}s-\frac{1}{2}\beta\rho\int_0^l(\dot{v})^2\mathrm{d}s\\&-\frac{1}{2}\beta\rho\int_0^l(\dot{w})^2\mathrm{d}s-\frac{1}{2}\beta\mathrm{EA}\int_0^l(w')^2\mathrm{d}s\\&-\left(\frac{3}{8}\beta\mathrm{EA}-\delta\beta\mathrm{EA}\right)\int_0^l(u')^4\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{3}{8}\beta\text{EA}-\delta\beta\text{EA}\right)\\
&\times\int_0^l (v')^4\mathrm{d}s-\frac{3}{4}\beta\text{EA}\int_0^l (u'v')^2\mathrm{d}s\\
&-\frac{3}{2}\beta\text{EI}\int_0^l (u'')^2\mathrm{d}s-\frac{3}{2}\beta\text{EI}\int_0^l (v'')^2\mathrm{d}s\\
&-\left(\frac{1}{2}\beta T-\frac{1}{2\delta}\beta\text{EA}\right)\int_0^l (u')^2\mathrm{d}s\\
&-\left(\frac{1}{2}\beta T-\frac{1}{2\delta}\beta\text{EA}\right)\int_0^l (v')^2\mathrm{d}s\\
&\le -\lambda_3(H+M(t))
\end{aligned} \tag{89}
$$

Therefore,

$$\dot{V}_b(t)\le -\lambda_3(H+M(t)), \tag{90}$$

where

$$
\begin{aligned}
\lambda_3=\min\Bigg\{&\frac{1}{2}\beta\rho\ ,\frac{3}{8}\beta\text{EA}-\delta\beta\text{EA},\frac{1}{2}\beta\text{EA},\\
&\frac{3}{2}\beta\text{EI},\frac{1}{2}\beta T-\frac{1}{2\delta}\beta\text{EA},\\
&\frac{2c_1}{m},\frac{2c_2}{m},\frac{2c_3}{m},\frac{2c_4}{m},\frac{2c_5}{m},\frac{2c_6}{m},\\
&\frac{2c_7}{m},\frac{2c_8}{m},\frac{2c_9}{m}\Bigg\}
\end{aligned} \tag{91}
$$

The parameters are selected to satisfy the following conditions

$$
\begin{aligned}
&c_1-\frac{1}{2}\beta\rho l\ge 0\\
&c_2-\frac{1}{2}\beta\rho l\ge 0\\
&c_3-\frac{1}{2}\beta\rho l\ge 0\\
&\frac{1}{2}\beta Tl-c_1\beta^2l^2-\frac{1}{4}c_3\beta^2l^2\ge 0\\
&\frac{1}{2}\beta Tl-c_2\beta^2l^2-\frac{1}{4}c_3\beta^2l^2\ge 0\\
&\frac{3}{8}\beta\text{EA}-\delta\beta\text{EA}\ge 0\\
&\frac{1}{2}\beta T-\frac{1}{2\delta}\beta\text{EA}\ge 0
\end{aligned} \tag{92}
$$

We further obtain

$$\dot{V}_b(t)\le -\lambda V_b(t), \tag{93}$$

where $\lambda=\frac{\lambda_3}{\lambda_2}>0$.

Then, multiplying (93) by $\mathrm{e}^{\lambda t}$ and integrating the inequality, we can get

$$V_b(t)\le V_b(0)\mathrm{e}^{-\lambda t} \tag{94}$$

Applying Lemma 1 in (81), we can obtain

$$\frac{1}{l}\left[u(s,t)\right]^2\le\int_0^l\left[u'(s,t)\right]^2\mathrm{d}s\le M(t)\le\frac{1}{\lambda_1}V_b(t) \tag{95}$$

$$\frac{1}{l}\left[v(s,t)\right]^2\le\int_0^l\left[v'(s,t)\right]^2\mathrm{d}s\le M(t)\le\frac{1}{\lambda_1}V_b(t) \tag{96}$$

$$\frac{1}{l}\left[w(s,t)\right]^2\le\int_0^l\left[w'(s,t)\right]^2\mathrm{d}s\le M(t)\le\frac{1}{\lambda_1}V_b(t) \tag{97}$$

Through clear up the above three inequalities, we can obtain

$$|u(s,t)|\le\sqrt{\frac{lV_b(0)\mathrm{e}^{-\lambda t}}{\lambda_1}} \tag{98}$$

$$|v(s,t)|\le\sqrt{\frac{lV_b(0)\mathrm{e}^{-\lambda t}}{\lambda_1}} \tag{99}$$

$$|w(s,t)|\le\sqrt{\frac{lV_b(0)\mathrm{e}^{-\lambda t}}{\lambda_1}} \tag{100}$$

Furthermore, considering (98)–(100), we can further obtain

$$
\begin{aligned}
&\lim_{t\to\infty}u(s,t)\to 0,\ \lim_{t\to\infty}v(s,t)\to 0,\\
&\lim_{t\to\infty}w(s,t)\to 0
\end{aligned} \tag{101}
$$

Moreover, considering (9)–(11), we obtain

$$|U_u(t)|=|g_u(u_1)|=u_M\left|\tanh\left(\frac{u_1}{u_M}\right)\right|\le u_M \tag{102}$$

$$|U_v(t)|=|g_u(u_2)|=u_M\left|\tanh\left(\frac{u_2}{u_M}\right)\right|\le u_M \tag{103}$$

$$|U_w(t)|=|g_u(u_3)|=u_M\left|\tanh\left(\frac{u_3}{u_M}\right)\right|\le u_M \tag{104}$$

and

$$\left|\dot{U}_u(t)\right| = \left|\frac{\partial g_u}{\partial u_1}\dot{u}_1\right| = |g_v(v_1)|$$
$$= v_M \left|\tanh\left(\frac{v_1}{v_M}\right)\right| \leq v_M \tag{105}$$
$$\left|\dot{U}_v(t)\right| = \left|\frac{\partial g_u}{\partial u_2}\dot{u}_2\right| = |g_v(v_2)|$$
$$= v_M \left|\tanh\left(\frac{v_2}{v_M}\right)\right| \leq v_M \tag{106}$$
$$\left|\dot{U}_w(t)\right| = \left|\frac{\partial g_u}{\partial u_3}\dot{u}_3\right| = |g_v(v_3)|$$
$$= v_M \left|\tanh\left(\frac{v_3}{v_M}\right)\right| \leq v_M \tag{107}$$

4 Numerical simulations

In this section, in order to demonstrate the effectiveness of the proposed model-based boundary control laws (9)–(11), we choose the finite difference method to carry out the numerical simulation. The current method to realize the simulation of PDE model is to discretize the PDE model. In the discretization process, the sampling time $\Delta t = T$ and the x axis spacing $\Delta x = \mathrm{d}x$ should satisfy the certain relationship $\Delta t \leq \frac{1}{2}\Delta x^2$ in [28]. The parameters of the Euler–Bernoulli beam in the three-dimensional space are listed in Table 1. Let $u_M > 0$, $v_M > 0$ be the given constants. The constraints on the input signals U_u, U_v, U_w are given by $|U_u| \leq u_M$, $|U_v| \leq u_M$, $|U_w| \leq u_M$. The input rate constraints are given by $\left|\dot{U}_u\right| \leq v_M$,$\left|\dot{U}_v\right| \leq v_M$, $\left|\dot{U}_w\right| \leq v_M$. The input constraints are $u_M = 8$. The input rate constraints are $v_M = 25$. The initial conditions are chosen as $u(s,0) = v(s,0) = w(s,0) = \frac{s}{2l}$, $\dot{u}(s,0) = \dot{v}(s,0) = \dot{w}(s,0) = 0$.

Table 1 Parameters of the nonlinear three-dimensional Euler–Bernoulli beam

Parameter	Description	Value
L	The length of the beam	1 m
ρ	Uniform mass per unit length of the beam	0.1 kg/m
m	The mass of the tip payload	1 kg
EI	The bending rigidity of the beam	8 Nm^2
EA	The axial stiffness of the beam	14 Nm^2
T	The tension of the beam	10 N

For analyzing and verifying the control performance, the dynamic responses of the system are simulated in the following three cases:

Case 1: Without control input.

Case 2: With the proposed control: We choose the control parameters as $\beta = 0.05, c_1 = 10, c_2 = 10, c_3 = 8.2, c_4 = 3, c_5 = 3, c_6 = 3, c_7 = 3, c_8 = 3, c_9 = 3$.

Case 3: With the control method in [24]:

$$u_u = -m\beta l\dot{u}_l' - \operatorname{sgn}\left(\dot{u}_l + \beta l u_l'\right)\bar{d}_u - 2k_1\dot{u}_l \tag{108}$$

$$u_v = -m\beta l\dot{v}_l' - \operatorname{sgn}\left(\dot{v}_l + \beta l v_l'\right)\bar{d}_v - 2k_2\dot{v}_l \tag{109}$$

$$u_w = -m\beta l\dot{w}_l' - \operatorname{sgn}\left(\dot{w}_l + \beta l w_l'\right)\bar{d}_w - 2k_3\dot{w}_l \tag{110}$$

and the control parameters are chosen as: $k_1 = 15$, $k_2 = 15$, $k_3 = 5$.

The simulation results are presented in Figs. 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, and 15. Figs. 3, 4, and 5 show the displacements of the beam in U, V, W direction for case 1, and the displacements of the beam for case 2 are shown in Figs. 6, 7, and 8. The displacements of the beam at the length of l for case 1 and case 2 are shown in Figs. 9, 10, and 11. We can clearly see that the proposed control scheme for case 2 can regulate the displacements for the nonlinear three-dimensional Euler–Bernoulli beam.

Control inputs for case 2 and case 3 are shown in Figs. 12 and 13, respectively. From Figs. 12 and 13, we can see that the input amplitude for case 2 is smaller than the input amplitude for case 3, and the input value for case 3 is larger than the input constraints. Control input rates for case 2 and case 3 are shown in Figs. 14 and 15, respectively. From Fig. 14, we can clearly see that the control input rates can be confined in $v_M = 25$.

From above analysis, we can conclude that the control system proposed in this paper can satisfy the input magnitude and rate constraints, respectively.

5 Conclusions

In this paper, boundary control systems are proposed to suppress the vibration of the nonlinear three-

Fig. 3 Displacement of the Euler–Bernoulli beam in U direction in case 1

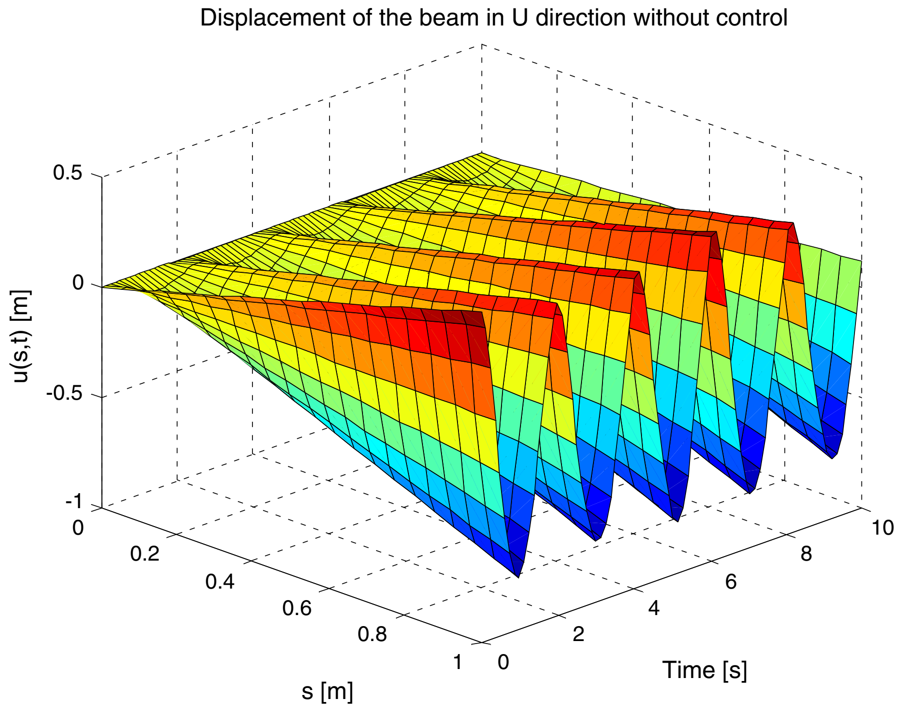
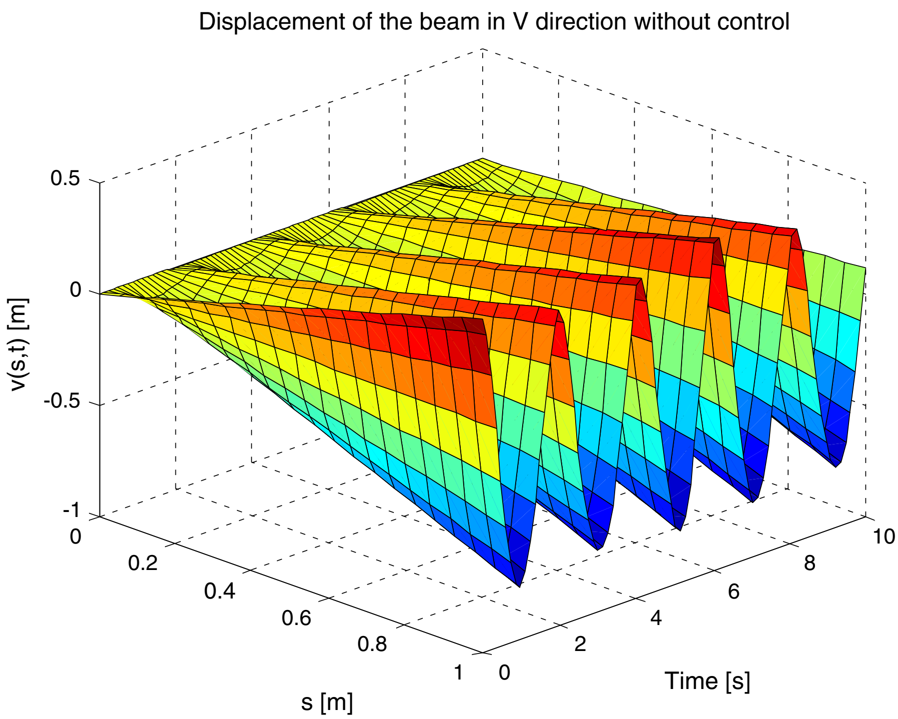


Fig. 4 Displacement of the Euler–Bernoulli beam in V direction in case 1



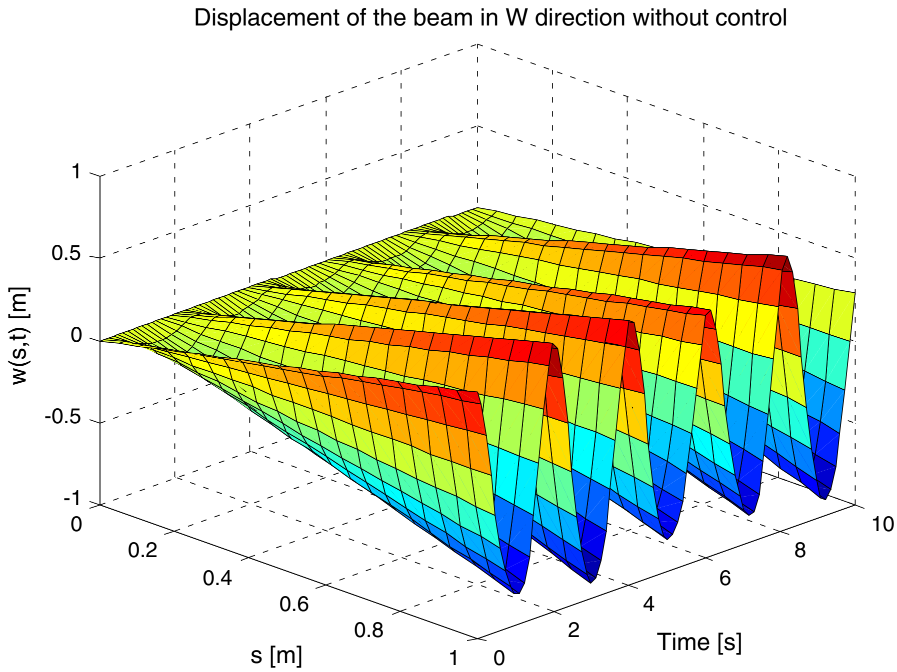


Fig. 5 Displacement of the Euler–Bernoulli beam in *W* direction in case 1

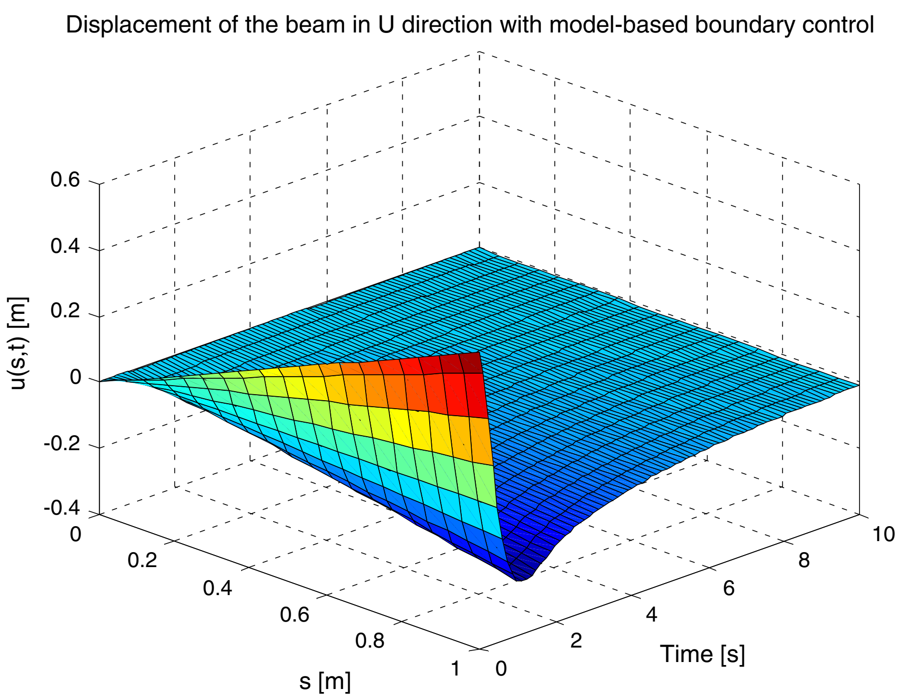


Fig. 6 Displacement of the Euler–Bernoulli beam in *U* direction in case 2

Fig. 7 Displacement of the Euler–Bernoulli beam in V direction in case 2

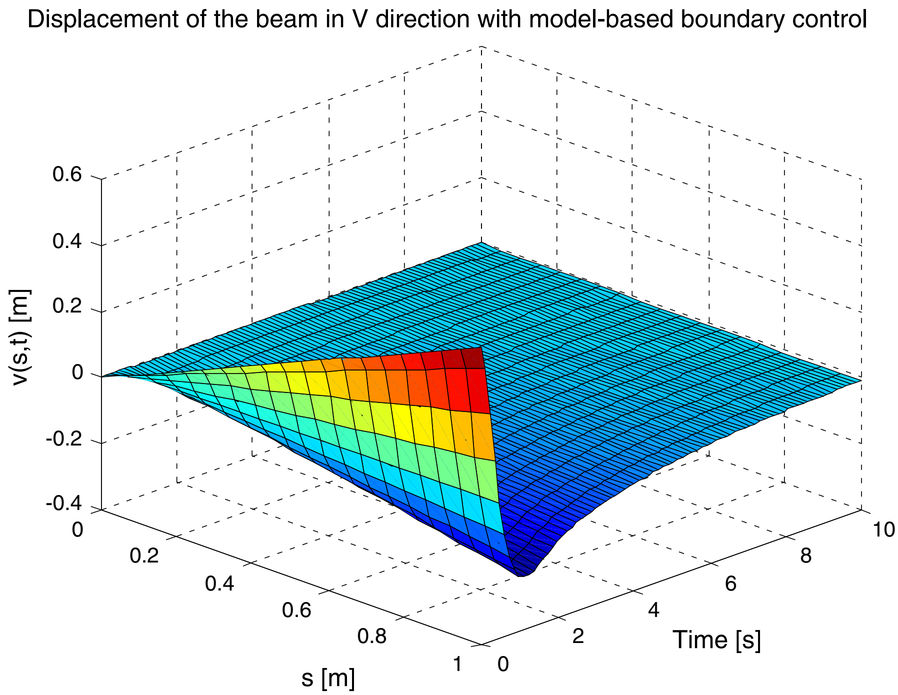
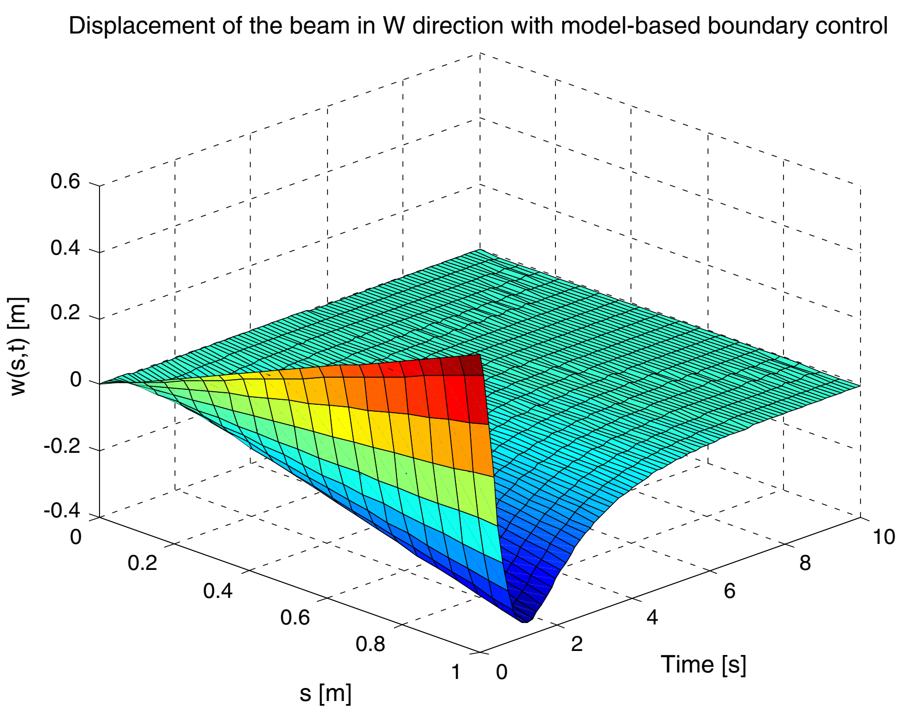


Fig. 8 Displacement of the Euler–Bernoulli beam in W direction in case 2



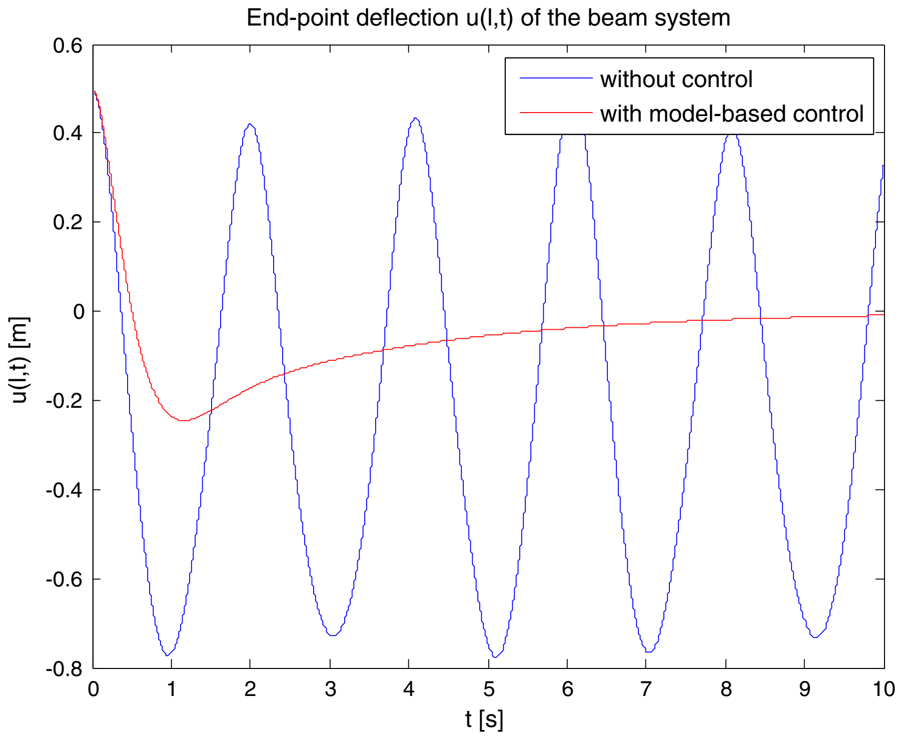


Fig. 9 Displacement of the Euler–Bernoulli beam at the length of l in U direction in case 1and case 2

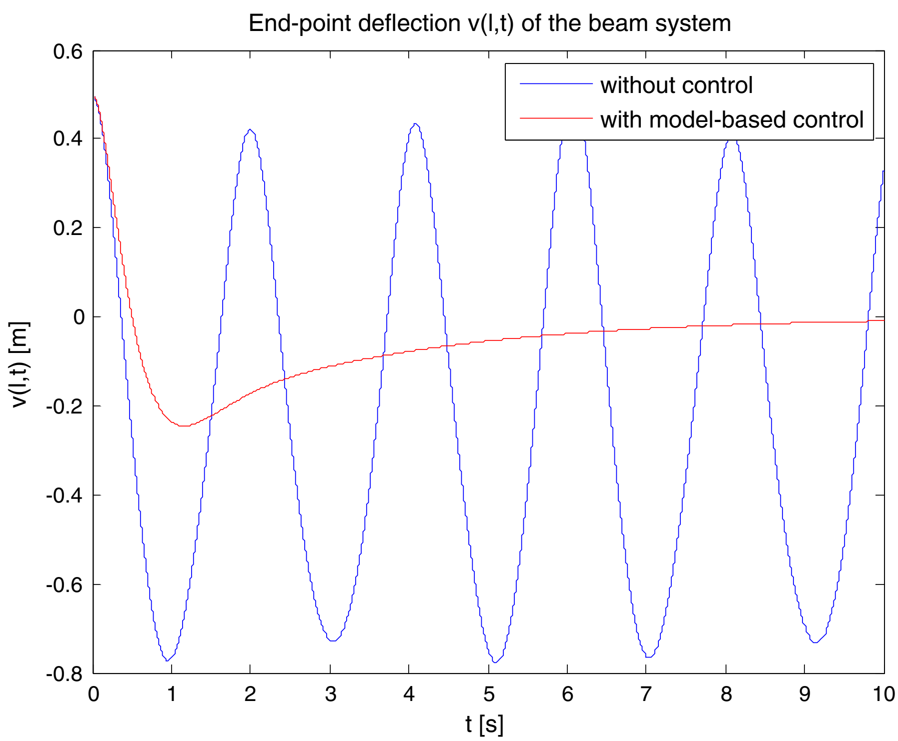


Fig. 10 Displacement of the Euler–Bernoulli beam at the length of l in V direction in case 1and case 2

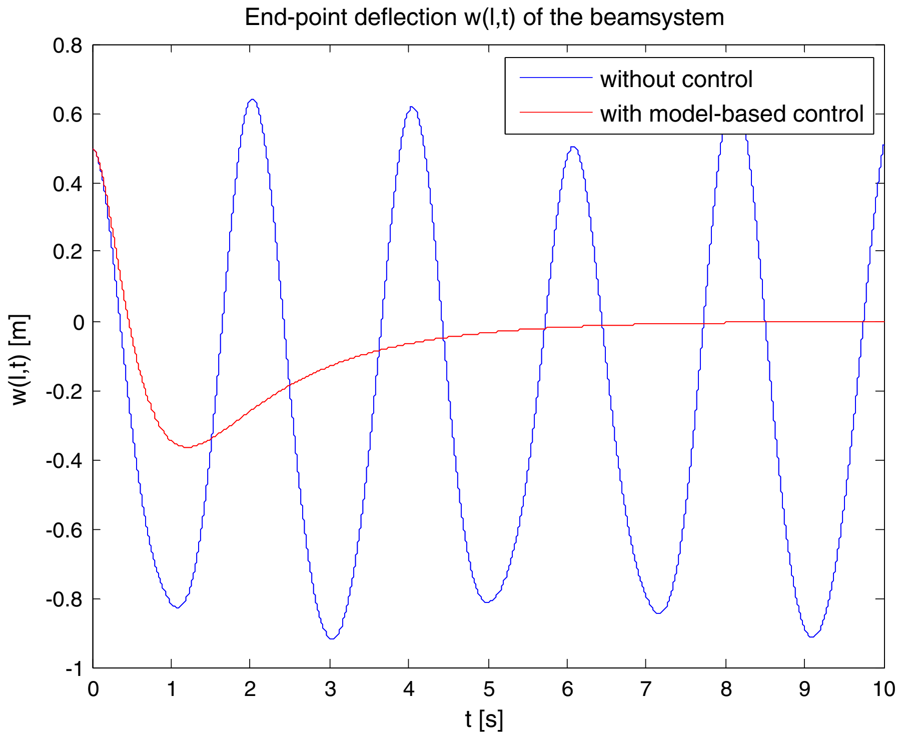


Fig. 11 Displacement of the Euler–Bernoulli beam at the length of l in W direction in case 1and case 2

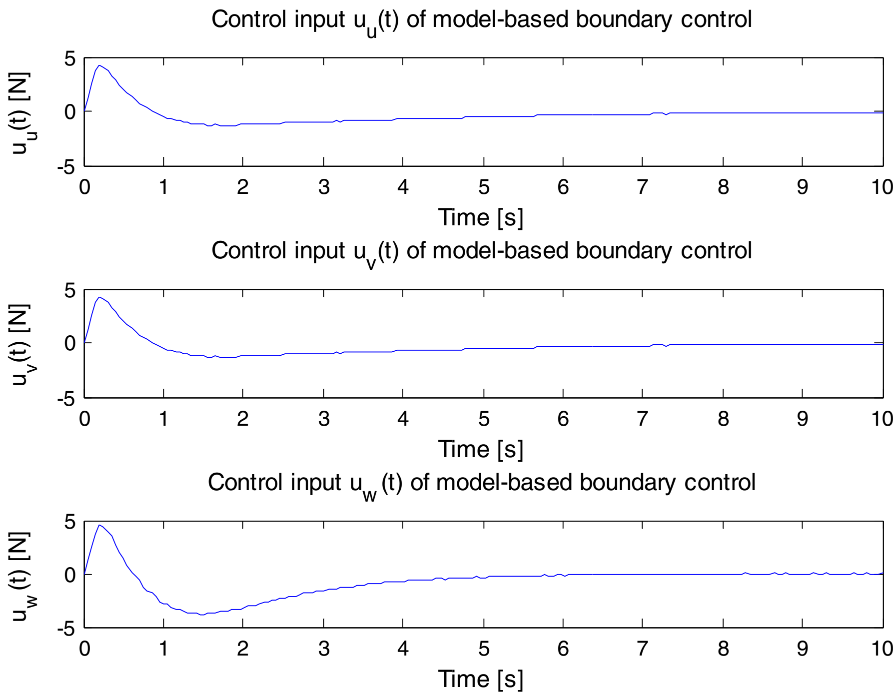


Fig. 12 Control input when $u_M = 8$ for case 2

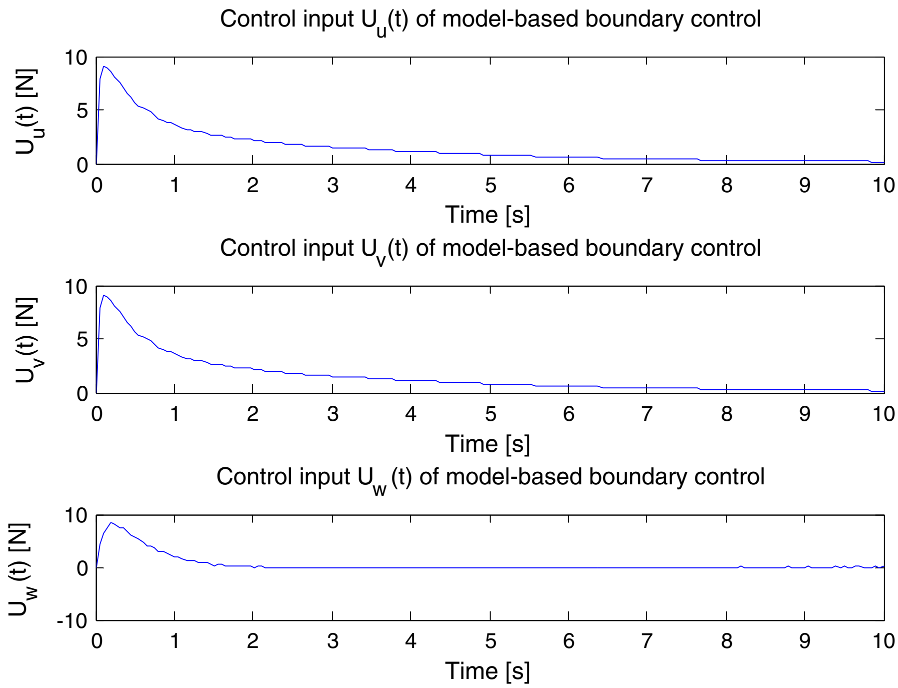


Fig. 13 Control inputs for case 3

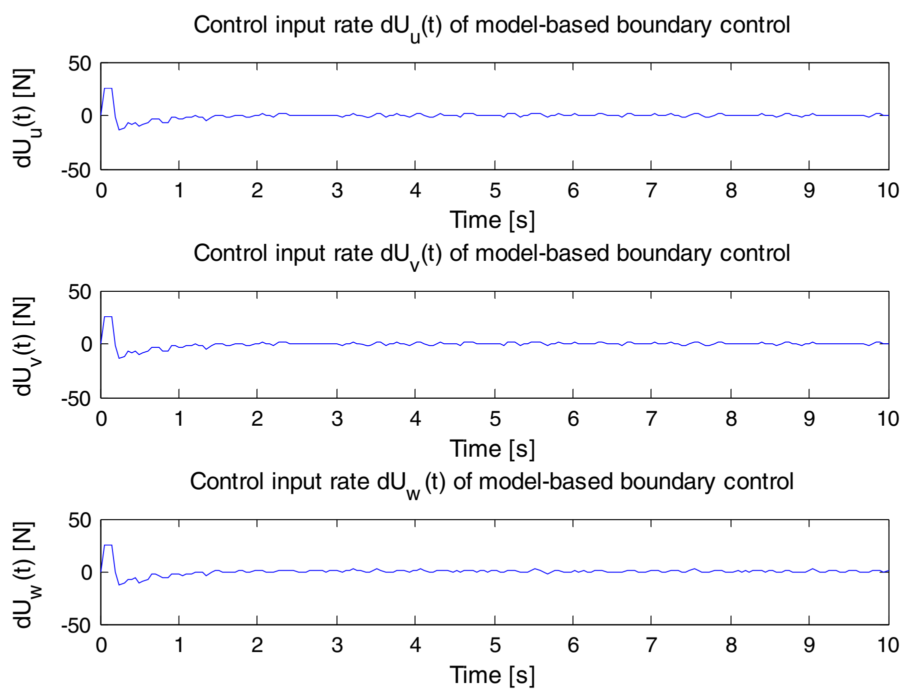


Fig. 14 Control input rate constraints when $v_M = 25$ for case 2

dimensional Euler–Bernoulli beam with input magnitude and rate constraints. To suppress the vibration of the nonlinear three-dimensional Euler–Bernoulli beam with constrained inputs and input rates, backstepping method with smooth hyperbolic tangent function is developed. In the controller design, two auxiliary systems are used to handle the impacts of the constrained input and input rates. Boundary control laws

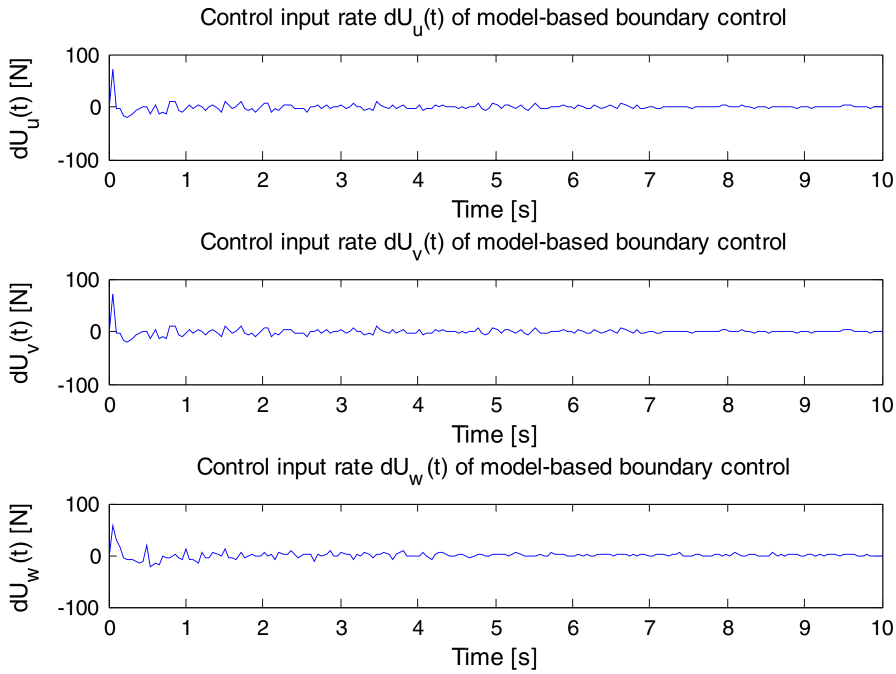


Fig. 15 Control input rates for case 3

are designed using Lyapunov's direct method. In addition, the proposed control laws have been illustrated to stabilize the beam under input magnitude and rate constraints. The closed-loop system finally has a good control performance. We can see that the control system with boundary control works well when the input magnitude and rate constraints are relatively small from simulation results.

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