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ORIGINAL PAPER

Fixed-time consensus tracking control of second-order multi-agent systems with inherent nonlinear dynamics via output feedback

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Abstract This paper is devoted to the fixed-time consensus tracking control problem for a class of second-order nonlinear multi-agent systems. Firstly, a fixed-time consensus control protocol with full state feedback is proposed, which is consisted of three parts: the higher-order components for uniformly bounded convergence for any initial states, the lower-order components for exact finite time convergence to zero and the linear components for dominating the nonlinear terms in the system dynamics, respectively. By using the Lyapunov approach and bi-limit homogeneous theory, it can be proved that under the designed control protocol the fixed-time consensus tracking control can be achieved. To overcome the lack of the velocity measurements, a fixed-time convergent state observer is designed to estimate the velocity information for each agent. Then a new observer-based fixed-time output feedback control protocol is designed, which can guarantee that the states of all agents can converge to that of the leader in a fixed time. Finally, numerical simulations are performed to demonstrate the fixed-time performance of the proposed control protocol and observer.

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1 Introduction

Consensus control of multi-agent systems has attracted considerable attention in recent years, which is not only due to its potential applications in biological systems [1], unmanned aerial vehicles [2], mobile robots [3], spacecraft formation [4] and so on, but also due to its some super advantages, such as greater efficiency, good robustness and cost reduction in contrast with individual agent [5,6]. The consensus control of multi-agent systems is to design control protocol for each agent by only using the local information interaction between neighbors such that the states of all agents converge to a common state. Many consensus control algorithms for multi-agent systems with different agent dynamics and communication topologies have been reported [7–12].

For the consensus control of multi-agent systems, the convergence rate is regarded as an important performance index to evaluate the designed consensus control protocol. In [13], it was shown that the convergence rate was related to the algebraic connectivity, i.e., the second smallest eigenvalue of the graph Laplacian matrix. Some results on how to improve the algebraic connectivity were presented in [14,15]. On the other hand, finite time control method, which not only can improve the convergence rate effectively, but also has some other advantages, such as higher precision and better robustness to the parameter uncertainties and disturbances [16–18], is more desirable than asymptotically stable control method with infinite convergence time. In [19] and [20], with the aid of the homogeneous theory method, the finite time consensus control problems for first-order and second-order integrator multi-agent systems were addressed, respectively. In [21], a finite time consensus tracking control protocol was proposed for second-order multi-agent systems by using nonsingular terminal sliding mode control method. In [22], continuous finite time consensus control protocol based on adding a power integrator method was proposed for second-order multi-agent systems.

However, the settling times of the above finite time control results are dependent on the initial states of the agents, which implies that the prescribed value of the settling time cannot be satisfied when the initial states are with large magnitudes. It is worse that the estimate of settling time is hard to provide if the initial states are not available in advance. Then it is desirable to design a consensus control protocol with finite time convergence in regardless of initial states. Based on the concept of fixed-time stability which firstly derived in [23], fixed-time control design method is proposed, whose convergence time is uniformly bounded with regardless of the initial states [24]. Due to this superior property, the fixed-time control method has been paid considerable attention. For instance, in [25,26], the fixed-time consensus control algorithms were proposed for the first-order integrator multi-agent systems. By employing a novel terminal sliding mode, the fixed-time results was generalized to the second-order multi-agent systems [27]. However, it should be pointed out that above fixed-time consensus results are mainly applicable to single-integrator and double-integrator multi-agent systems. In many practical situations, the model of the agent is governed by complicatedly inherent nonlinear dynamics. Most research results for nonlinear multi-agent systems are asymptotically stable and finite time stable [28–30]. There are only few results about the fixed-time consensus tracing control for nonlinear multi-agents systems. Thus, it is necessary to investigate the fixed-time consensus control for second-order multi-agent systems with inherent nonlinear dynamics. In addition, it is worth mentioning that most of works aforementioned above are focused on the fixed-time consensus control by using both the position and velocity information. Fewer works deal with the case that the velocity information is not available, which will arise in the case that the agents are not equipped with the velocity measurement devices or the velocity information cannot be measured precisely due to the technology constraints or the environment disturbances [30]. Thus, how to design a fixed-time consensus tracking control protocol without the velocity information for second-order multi-agent systems with inherent nonlinear dynamics is an important and challenging problem.

Motivated by the above discussion and analysis, in this paper, the problem of fixed-time consensus tracking control of second-order multi-agent systems with inherent nonlinear dynamics is investigated. Firstly, a fixed-time consensus tracking control protocol is designed by using both the position and velocity measurements for second-order nonlinear multi-agent systems. Then, based on a designed fixed-time convergent state observer, a fixed-time output feedback control protocol by only using the position measurements is proposed. Compared with finite time consensus control results in [19,20], the designed control protocols with full state feedback and output feedback both can guarantee that the tracking errors approach to zeros with fixed-time convergence, which can solve the problem brought by finite time control that the settling time is dependent on the initial states. Furthermore, the mathematical expression of the estimate of the convergence time is provided. In contrast with results in [27,32], the fixed-time consensus results are extended to the second-order nonlinear multi-agent systems, which includes the integrator multi-agent systems as a special case. Also, compared with the consensus control results by using both the position and velocity measurements for second-order nonlinear multi-agent systems [33,34], the proposed consensus protocol not only can achieve the fixed-time convergence, but also can solve the consensus tracking problem by only using the position measurements.

The rest of the paper is organized as follows. In Sect. 2, some useful preliminaries are given and the problem is formulated. In Sect. 3, the fixed-time consensus control protocols with full state feedback and output feedback are proposed, respectively. Numerical simulations results and conclusions are given in Sects. 4 and 5, respectively.

Notations Let $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n$ be the set of $n \times n$ real matrices and n-dimensional real vectors, respectively.

$\mathbb{R}_+$ is denoted as the set of positive real numbers. For a vector $x = [x_1, x_2, \ldots, x_n]^\mathrm{T}$ and $\gamma > 0$, define $\mathrm{sig}^\gamma(x) = [|x_1|^\gamma \mathrm{sign}(x_1), \ldots, |x_n|^\gamma \mathrm{sign}(x_n)]^\mathrm{T}$, where $\mathrm{sign}(\cdot)$ is the standard signum function. The function $\Gamma : \mathbb{R}_+^2 \to \mathbb{R}_+$ is defined as $\Gamma(a, b) = \frac{a}{1+a}(1 + b)$, $a, b \in \mathbb{R}_+$.

2 Background and preliminaries

2.1 Graph theory

Consider a multi-agent system with one leader and N followers. Each agent is denoted as a node. The communication relation among N followers can be described by a undirected weighted graph $G = (V, E, A)$, where $V = \{1, 2, \ldots, N\}$ is a finite and nonempty set of nodes, $E = \{e_{ij} = (i, j)\} \subseteq V \times V$ is a set of edges, and $A = [a_{ij}]$ is the adjacency matrix of the graph G, where $a_{ij} \neq 0$ if $(j, i) \in E$ while $a_{ij} = 0$ otherwise. Moreover, assume that there are no loops in the graph G, i.e., $a_{ii} = 0$. An edge (i, j) in the graph G denotes that node i and j can interact information each other. If $(j, i) \in E$, then node j is a neighbor node of node i. All neighbors of nodes i are described by $N_i = \{j|(j, i) \in E\}$. Denote $D = \mathrm{diag}\{d_1, d_2, \ldots, d_N\}$ as the degree diagonal matrix, where $d_i = \sum_{j=1}^N a_{ij}$ is the in-degree of node i. Then the Laplacian matrix of the weighted digraph G is defined as $L = D - A$. A path from node i to node s is a sequence of distinct edges in the form of $(k, k + 1) \in E, k = 1, 2, \ldots, s - 1$. The graph is said to be connected if any two nodes can be connected through a path.

Next, another graph $\bar{G}$ associated with N follower agents and one leader agent is considered. It is assumed that the leader does not receive any information from the followers, and only a part of the followers can receive information from the leader. Define a matrix $B = \mathrm{diag}\{b_1, b_2, \ldots, b_n\}$, where b_i denotes the connection weight between the ith follower and the leader. If the ith follower is connected to the leader, then $b_i > 0$; otherwise, $b_i = 0$.

2.2 Definition and Lemma

Consider the nonlinear system

$$\dot{x} = f(x),\ f(0) = 0, \quad x(0) = x_0, \quad x \in \mathbb{R}^n \tag{1}$$

where $x = [x_1, x_2, \ldots, x_n]^\mathrm{T} \in \mathbb{R}^n$ is the state vector, and $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function.

Definition 1 ([23]) A function $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ is said to homogenous in the p-limit ($p = 0$ or ∞) with associated triple (r_p, k_p, ϕ_p), where $r_p = [r_{p,1}, r_{p,2}, \ldots, r_{p,n}] \in \mathbb{R}_+^n$ is the weight vector, k_p is the degree, and $\phi_p : \mathbb{R}^n \to \mathbb{R}$ is the approximating function, if $\phi(x)$ is continuous, $\phi_p(x)$ is continuous and not identically zero, and the condition $\lim_{\varepsilon \to p} \max_{x \in C} |\frac{\phi(\Lambda_\varepsilon^{r_p}(x))}{\varepsilon^{k_p}} - \phi_p(x)| = 0$ holds for each compact set $C \subset \mathbb{R}^n \backslash 0$, where $\Lambda_\varepsilon^{r_p}(x) = [\varepsilon^{r_{p,1}} x_1, \varepsilon^{r_{p,2}} x_2, \ldots, \varepsilon^{r_{p,n}} x_n]$.

Definition 2 ([23]) A vector function $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ is said to homogeneous in the p-limit with associated triple (r_p, k_p, f_p), where $r_p = [r_{p,1}, r_{p,2}, \ldots, r_{p,n}] \in \mathbb{R}_+^n$ is the weight vector, k_p is the degree, and f_p is the approximating vector function, if $k_p + r_{p,i} > 0$ and the function $f_i(x)$ is homogeneous in the p-limit with associated with triple $(r_p, k_p + r_{p,i}, f_{p,i})$. Moreover, the system (1) is said to be homogeneous in p-limit if the function $f(x)$ is homogeneous in p-limit.

Definition 3 ([23]) A function $f(x) : \mathbb{R}^n \to \mathbb{R}$ (or a vector function $f(x) : \mathbb{R}^n \to \mathbb{R}^n$) is said to homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the ∞-limit.

Definition 4 ([31]) The equilibrium $x = 0$ of the system (1) is said to be globally finite time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of the system (1) reaches the origin at some finite time moment, i.e., $x(t, x_0) = 0, \forall t \geq T_{x_0}$, where $T : \mathbb{R}^n \to (0, \infty)$ is the settling time function.

Definition 5 ([24]) The equilibrium $x = 0$ of the system (1) is said to be fixed-time stable if it is globally finite time stable and the settling time function $T(x_0)$ is uniformly bounded for any initial states x_0, that is, $\exists\, T_{\max}$ such that $T(x_0) \leq T_{\max}, \forall\, x_0 \in \mathbb{R}^n$.

Lemma 1 ([23]) *For the nonlinear system (1), if $f(x)$ is a homogenous vector function in the bi-limit with associated triples (r_0, k_0, f_0) and $(r_\infty, k_\infty, f_\infty)$. Moreover, the original system $\dot{x} = f(x)$ and the approximating systems $\dot{x} = f_0(x), \dot{x} = f_\infty(x)$ are globally asymptotically stable; then the following results can be obtained:*

(1) *The equilibrium $x = 0$ of the system (1) is fixed-time stable if the condition $k_\infty > 0 > k_0$ is satisfied.*

(2) *The real numbers d_{V_0} and d_{V_∞} are selected such that $d_{V_0} > \max_{1\le i\le n} r_{0,i}$ and $d_{V_\infty} > \max_{1\le i\le n} r_{\infty,i}$. Then there exists a continuous and positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that for each $i = 1, 2, \ldots, n$, the function $\frac{\partial V}{\partial x_i}$ is homogeneous in the bi-limit with associated with triples $(r_0, d_{V_0} - r_{0,i}, \frac{\partial V_0}{\partial x_i})$ and $(r_0, d_{V_\infty} - r_{0,i}, \frac{\partial V_\infty}{\partial x_i})$, and the function $\frac{\partial V}{\partial x} f(x)$ is negative definite, and satisfies that*

$$\frac{\partial V}{\partial x} f(x) \le -k_v \Gamma \left(V^{\frac{d_{V_0}+k_0}{d_{V_0}}}, V^{\frac{d_{V_\infty}+k_\infty}{d_{V_\infty}}} \right) \tag{2}$$

where k_v is positive real numbers and function Γ is defined in Notations.

Lemma 2 (Barbalat's Lemma) *If a function $f(t)$ is uniformly continuous and $\lim_{t\to\infty} \int_0^t f(\tau)\mathrm{d}\tau$ exists and is finite, then $\lim_{t\to\infty} f(t) = 0$.*

Lemma 3 ([22]) *For $x \in \mathbb{R}$, $y \in \mathbb{R}$, if $c > 0, d > 0$, then $|x|^c|y|^d \le c/(c+d)|x|^{c+d} + d/(c+d)|y|^{c+d}$.*

2.3 Problem formulation

Consider the multi-agent systems with N followers labeled as 1 to N and one leader. The dynamics of the ith follower can be described as

$$\begin{aligned} &\dot{q}_i = p_i, \dot{p}_i = f(t, q_i, p_i) + u_i, \\ &y_{it} = q_i, i = 1, 2, \ldots, N \end{aligned} \tag{3}$$

where q_i and p_i are the position and velocity of the ith follower, respectively. y_{it} is the output. u_i is a control protocol to be designed, and $f(\cdot)$ is nonlinear function to describe the inherent nonlinear dynamics.

The dynamics of the leader is described as

$$\begin{aligned} &\dot{q}_d = p_d, \quad \dot{p}_d = f(t, q_d, p_d) \\ &y_d = q_d \end{aligned} \tag{4}$$

where q_d and p_d are the position and velocity of the leader, respectively.

Assumption 1 For $\forall\ x_1, x_2, y_1, y_2 \in \mathbb{R}$, there exist two nonnegative constants ρ_1 and ρ_2 such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le \rho_1|x_1 - x_2| + \rho_2|y_1 - y_2|$$

Assumption 2 For the leader-follower multi-agent systems (3) and (4), the graph G associated with the followers is connected and there exists at least one follower connected to the leader, i.e., $B \ne 0$.

Lemma 4 ([22]) *Under Assumption 2, the matrix $H = L + B$ associated with the graph $\bar{G}$ is symmetric and positive definite.*

The control objective of this paper is to design a fixed-time consensus control protocol under the constraint condition that only the output information, i.e., position measurements are available such that the trajectory of the leader can be tracked by all followers in fixed time, which can be described as: For any initial states $(q_i(0), p_i(0))$, there exists a constant T such that $q_i = q_d$, $p_i = q_d$ always hold when $t \ge T$.

3 Main results

3.1 Fixed-time consensus tracking control with full state feedback

In this section, the fixed-time consensus tracking control problem by using the relative position and velocity measurements is investigated. By considering the second-order nonlinear multi-agent systems (3) and (4), the fixed-time consensus tracking control protocol with full state feedback is proposed as

$$\begin{aligned} u_i &= u_{i1} + u_{i2} \\ u_{i1} &= -k_1 \Bigg\{ \sum_{j\in N_i} a_{ij}\mathrm{sig}^{\alpha_1}(q_i - q_j) + b_i \mathrm{sig}^{\alpha_1}(q_i - q_d) \\ &\quad + \sum_{j\in N_i} a_{ij}\mathrm{sig}^{\alpha_2}(p_i - p_j) + b_i \mathrm{sig}^{\alpha_2}(p_i - p_d) \Bigg\} \\ &\quad -k_2 \Bigg\{ \sum_{j\in N_i} a_{ij}\mathrm{sig}^{\bar{\alpha}_1}(q_i - q_j) + b_i \mathrm{sig}^{\bar{\alpha}_1}(q_i - q_d) \\ &\quad + \sum_{j\in N_i} a_{ij}\mathrm{sig}^{\bar{\alpha}_2}(p_i - p_j) + b_i \mathrm{sig}^{\bar{\alpha}_2}(p_i - p_d) \Bigg\} \\ u_{i2} &= -k_3 \Bigg\{ \sum_{j\in N_i} a_{ij}(q_i - q_j) + b_i(q_i - q_d) \\ &\quad + \sum_{j\in N_i} a_{ij}(p_i - p_j) + b_i(p_i - p_d) \Bigg\} \end{aligned} \tag{5}$$

where $k_i, i = 1, 2, 3$ are positive constants. The parameters $\alpha_i, \bar{\alpha}_i, i = 1, 2$ are selected as

$$0 < \alpha_1 < 1, \quad \alpha_2 = \frac{2\alpha_1}{\alpha_1 + 1}, \quad \bar{\alpha}_1 > 1, \quad \bar{\alpha}_2 = \frac{2\bar{\alpha}_1}{\bar{\alpha}_1 + 1} \tag{6}$$

Theorem 1 *Consider the second-order nonlinear multi-agent systems with the designed fixed-time control protocol* (5). *Suppose that Assumptions* 1 and 2 *hold, then the fixed-time consensus tracking control can be achieved if the parameters* $\alpha_i, \bar{\alpha}_i, i = 1, 2$ *satisfy the inequality* (6)*, and the control parameters* $k_i, i = 1, 2, 3$ *are selected as*

$$\begin{aligned} &k_1 > 0, \quad k_2 > 0 \\ &k_3 > \frac{k_1}{1 + \alpha_2} + \frac{k_2}{1 + \bar{\alpha}_2} + \frac{\rho_1 + 2\rho_2 + 2}{\lambda_{\min}(L + B)} \end{aligned} \tag{7}$$

where $\lambda_{\min}(L+B)$ *is the minimum eigenvalue of matrix* $L + B$. *Moreover, the upper bound of the settling time can be estimated as*

$$T_s \le \frac{2}{k_v(1 - \gamma_1)} + \frac{2}{k_v(\gamma_2 - 1)} \tag{8}$$

where $\gamma_1 = \frac{d_{V_0} + k_0}{d_{V_0}}$, $\gamma_2 = \frac{d_{V_\infty} + k_\infty}{d_{V_\infty}}$, $k_0 = -1$, $k_\infty = 1$, $d_{V_0} = \max(\frac{2}{1-\alpha_1}, \frac{1+\alpha_1}{1-\alpha_1})$, $d_{V_\infty} = \max(\frac{2}{\bar{\alpha}_1 - 1}, \frac{\bar{\alpha}_1 + 1}{\bar{\alpha}_1 - 1})$.

Proof Define the new variables $x_i = q_i - q_d$, $y_i = p_i - p_d$ as the states tracking errors. Substituting the fixed-time control protocol (5) into (3), the closed-loop system can be written as

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -k_1 \left\{ \sum_{j \in N_i} a_{ij} \text{sig}^{\alpha_1}(x_i - x_j) + b_i \text{sig}^{\alpha_1}(x_i) \right. \\ &\quad \left. + \sum_{j \in N_i} a_{ij} \text{sig}^{\alpha_2}(y_i - y_j) + b_i \text{sig}^{\alpha_2}(y_i) \right\} \\ &\quad -k_2 \left\{ \sum_{j \in N_i} a_{ij} \text{sig}^{\bar{\alpha}_1}(x_i - x_j) + b_i \text{sig}^{\bar{\alpha}_1}(x_i) \right. \\ &\quad \left. + \sum_{j \in N_i} a_{ij} \text{sig}^{\bar{\alpha}_2}(y_i - y_j) + b_i \text{sig}^{\bar{\alpha}_2}(y_i) \right\} \\ &\quad -k_3 \left\{ \sum_{j \in N_i} a_{ij}(x_i - x_j) + b_i x_i + \sum_{j \in N_i} a_{ij}(y_i \right. \\ &\quad \left. - y_j) + b_i y_i \right\} + f(t, q_i, p_i) - f(t, q_d, p_d) \end{aligned} \tag{9}$$

In what follows, it is shown that the close-loop systems (9) is fixed-time stable. The process of proof is divided into three steps. Firstly, it will be proved that the closed-loop system (9) is globally asymptotically stable. Secondly, we will show that the approximating system of (9) in the 0-limit is homogenous of negative degree k_0 and globally asymptotically stable. Finally, it will be shown that the approximating system of (9) in the ∞-limit is homogeneous with positive degree $k_\infty > 0$ and globally asymptotically stable. Then, based on Lemma 1, we can conclude that the closed-loop system (9) is fixed-time stable.

Step 1 To show that the closed-loop system is asymptotically stable, the following Lyapunov function candidate is considered as

$$\begin{aligned} W_1 =& k_1 \sum_{i=1}^N \left\{ \sum_{j=1}^N \frac{a_{ij}}{2} \int_0^{x_i - x_j} \text{sig}^{\alpha_1}(s)\text{d}s \right. \\ &\left. + b_i \int_0^{x_i} \text{sig}^{\alpha_1}(s)\text{d}s \right\} + k_2 \sum_{i=1}^N \left\{ \sum_{j=1}^N \frac{a_{ij}}{2} \right. \\ &\left. \int_0^{x_i - x_j} \text{sig}^{\bar{\alpha}_1}(s)\text{d}s + b_i \int_0^{x_i} \text{sig}^{\bar{\alpha}_1}(s)\text{d}s \right\} \\ &+ \frac{c}{2} x^{\text{T}}(L + B)x \\ W_2 =& \frac{1}{2}(x + y)^{\text{T}}(x + y) \end{aligned} \tag{10}$$

where $x = [x_1, x_2, \ldots, x_N]^{\text{T}}$, $y = [y_1, y_2, \ldots, y_N]^{\text{T}}$. c is a positive constant, which is given in the following part. Note that x_i and $\text{sig}^{\alpha_1}(x_i)$ have same sign in sense of component-wise, then we have that $\int_0^{x_i} \text{sig}^{\alpha_1}(s)ds > 0$ for any $x_i \neq 0$. Then it is concluded that W_1 is continuously differentiable and positive definite. Taking the derivative of W_1 along the trajectory of the system (9) and noting that $a_{ij} = a_{ji}$ due to the undirected graph G, we have

$$\dot{W}_1 = k_1 \sum_{i=1}^N \left\{ \sum_{j=1}^N a_{ij} y_i \text{sig}^{\alpha_1}(x_i - x_j) + b_i y_i \text{sig}^{\alpha_1}(x_i) \right\}$$

$$
\begin{aligned}
&+k_2\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}y_i\mathrm{sig}^{\bar{\alpha}_1}(x_i-x_j)+b_iy_i\mathrm{sig}^{\bar{\alpha}_1}(x_i)\right\}\\
&+cx^{\mathrm{T}}(L+B)y\\
\le\ & k_1\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}y_i\mathrm{sig}^{\alpha_1}(x_i-x_j)+b_iy_i\mathrm{sig}^{\alpha_1}(x_i)\right\}\\
&+k_2\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}y_i\mathrm{sig}^{\bar{\alpha}_1}(x_i-x_j)+b_iy_i\mathrm{sig}^{\bar{\alpha}_1}(x_i)\right\}\\
&-\frac{c}{2}x^{\mathrm{T}}(L+B)x+\frac{c}{2}(x+y)^{\mathrm{T}}(L+B)(x+y)
\end{aligned}\tag{11}
$$

Denote $u=[u_1,\ldots,u_N]^{\mathrm{T}}$, $F=[f(t,q_1,p_1)-f(t,q_d,p_d),\ldots,f(t,q_N,p_N)-f(t,q_d,p_d)]^{\mathrm{T}}$. The derivative of W_2 along the trajectory of the system (9) is

$$
\begin{aligned}
\dot{W}_2 &= (x+y)^{\mathrm{T}}(y+u+F)\\
&\le (x+y)^{\mathrm{T}}(x+y)-(x+y)^{\mathrm{T}}x+(x+y)^{\mathrm{T}}u\\
&\quad +(x+y)^{\mathrm{T}}F\\
&\le \frac{3}{2}(x+y)^{\mathrm{T}}(x+y)+\frac{1}{2}x^{\mathrm{T}}x+(x+y)^{\mathrm{T}}u\\
&\quad +(x+y)^{\mathrm{T}}F
\end{aligned}\tag{12}
$$

Substituting the fixed-time control protocol (5) into (12), we have

$$
\begin{aligned}
\dot{W}_2 \le\ & \frac{3}{2}(x+y)^{\mathrm{T}}(x+y)+\frac{1}{2}x^{\mathrm{T}}x+(x+y)^{\mathrm{T}}F\\
&-\frac{k_1}{2}(\phi+\chi)-\frac{k_2}{2}(\psi+\zeta)\\
&-k_3(x+y)^{\mathrm{T}}(L+B)(x+y)\\
&-\frac{k_1}{2}\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}(x_i-x_j)\mathrm{sig}^{\alpha_2}(y_i-y_j)\right.\\
&\left.+2b_ix_i\mathrm{sig}^{\alpha_2}(y_i)\right\}\\
&-\frac{k_2}{2}\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}(x_i-x_j)\mathrm{sig}^{\bar{\alpha}_2}(y_i-y_j)\right.\\
&\left.+2b_ix_i\mathrm{sig}^{\bar{\alpha}_2}(y_i)\right\}\\
&-k_1\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}y_i\mathrm{sig}^{\alpha_1}(x_i-x_j)+b_iy_i\mathrm{sig}^{\alpha_1}(x_i)\right\}\\
&-k_2\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}y_i\mathrm{sig}^{\bar{\alpha}_1}(x_i-x_j)+b_iy_i\mathrm{sig}^{\bar{\alpha}_1}(x_i)\right\}
\end{aligned}\tag{13}
$$

where

$$
\begin{aligned}
\phi &= \sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}|x_i-x_j|^{1+\alpha_1}+2b_i|x_i|^{1+\alpha_1}\right\}\\
\psi &= \sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}|x_i-x_j|^{1+\bar{\alpha}_1}+2b_i|x_i|^{1+\bar{\alpha}_1}\right\}\\
\chi &= \sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}|y_i-y_j|^{1+\alpha_2}+2b_i|y_i|^{1+\alpha_2}\right\}\\
\zeta &= \sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}|y_i-y_j|^{1+\bar{\alpha}_2}+2b_i|y_i|^{1+\bar{\alpha}_2}\right\}
\end{aligned}
$$

Combining (11) and (13) can yield

$$
\begin{aligned}
\dot{W} \le\ & -\frac{c}{2}x^{\mathrm{T}}(L+B)x+\frac{3}{2}(x+y)^{\mathrm{T}}(x+y)+\frac{1}{2}x^{\mathrm{T}}x\\
&+(x+y)^{\mathrm{T}}F-\left(k_3-\frac{c}{2}\right)(x+y)^{\mathrm{T}}(L+B)(x+y)\\
&-\frac{k_1}{2}(\phi+\chi)-\frac{k_2}{2}(\psi+\zeta)\\
&-\frac{k_1}{2}\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}(x_i-x_j)\mathrm{sig}^{\alpha_2}(y_i-y_j)\right.\\
&\left.+2b_ix_i\mathrm{sig}^{\alpha_2}(y_i)\right\}-\frac{k_2}{2}\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}a_{ij}\right.\\
&\left.(x_i-x_j)\mathrm{sig}^{\bar{\alpha}_2}(y_i-y_j)+2b_ix_i\mathrm{sig}^{\bar{\alpha}_2}(y_i)\right\}
\end{aligned}\tag{14}
$$

Noting that $0<\alpha_1<1, \alpha_2=\frac{2\alpha_1}{\alpha_1+1}$, we have that $\alpha_2-\alpha_1=\frac{-\alpha_1(\alpha_1-1)}{\alpha_1+1}>0$ for $0<\alpha_1<1$. Then we can obtain that $1+\alpha_1<1+\alpha_2<2$. For the last second term in (14), by using Lemma 3, we have

$$
\begin{aligned}
x_i\mathrm{sig}^{\alpha_2}(y_i) &\le \frac{1}{1+\alpha_2}|x_i|^{1+\alpha_2}+\frac{\alpha_2}{1+\alpha_2}|y_i|^{1+\alpha_2}\\
&\le \frac{1}{1+\alpha_2}(|x_i|^{1+\alpha_1}+x_i^2)\\
&\quad +\frac{\alpha_2}{1+\alpha_2}|y_i|^{1+\alpha_2}
\end{aligned}\tag{15}
$$

The above inequalities also hold for the cross-term $(x_i-x_j)\mathrm{sig}^{\alpha_2}(y_i-y_j)$.

Further, for $\bar{\alpha}_1>1, \bar{\alpha}_2=\frac{2\bar{\alpha}_1}{\bar{\alpha}_1+1}$, we obtain that $\bar{\alpha}_2-\bar{\alpha}_1=\frac{-\bar{\alpha}_1(\bar{\alpha}_1-1)}{2\bar{\alpha}_1}<0$, which implies that $2<1+\bar{\alpha}_2<1+\bar{\alpha}_1$. For the last term in (14), by using Lemma 3, we have

$$\begin{aligned} x_i \text{sig}^{\bar{\alpha}_2}(y_i) &\le \frac{1}{1+\bar{\alpha}_2}|x_i|^{1+\bar{\alpha}_2} + \frac{\bar{\alpha}_2}{1+\bar{\alpha}_2}|y_i|^{1+\bar{\alpha}_2} \\ &\le \frac{1}{1+\bar{\alpha}_2}(|x_i|^{1+\bar{\alpha}_1} + x_i^2) \\ &\quad + \frac{\bar{\alpha}_2}{1+\bar{\alpha}_2}|y_i|^{1+\bar{\alpha}_2} \end{aligned} \tag{16}$$

The above inequalities are satisfied for the cross-term $(x_i - x_j)\text{sig}^{\bar{\alpha}_2}(y_i - y_j)$.

It follows from the inequalities (15) and (16) that Eq. (14) can be simplified as

$$\begin{aligned} \dot{W} \le &- \frac{c}{2}x^{\mathrm{T}}(L+B)x + \frac{3}{2}(x+y)^{\mathrm{T}}(x+y) + \frac{1}{2}x^{\mathrm{T}}x \\ &+ (x+y)^{\mathrm{T}}F - \left(k_3 - \frac{c}{2}\right)(x+y)^{\mathrm{T}}(L+B)(x+y) \\ &- \frac{k_1\alpha_2}{2(1+\alpha_2)}\phi - \frac{k_1}{2(1+\alpha_2)}\chi - \frac{k_2\bar{\alpha}_2}{2(1+\bar{\alpha}_2)}\psi \\ &- \frac{k_2}{2(1+\bar{\alpha}_2)}\zeta + \left(\frac{k_1}{1+\alpha_2} + \frac{k_2}{1+\bar{\alpha}_2}\right)x^{\mathrm{T}}(L+B)x \end{aligned} \tag{17}$$

Under Assumption 1, we obtain that

$$\begin{aligned} (x+y)^{\mathrm{T}}F &\le \sum_{i=1}^{N}(x_i+y_i)(f(t,q_i,p_i) - f(t,q_d,p_d)) \\ &\le \sum_{i=1}^{N}|x_i+y_i|(\rho_1|x_i| + \rho_2|x_i+y_i-x_i|) \\ &\le \sum_{i=1}^{N}(\rho_1+\rho_2)|x_i||x_i+y_i| + \rho_2(x_i+y_i)^2 \\ &\le \sum_{i=1}^{N}\frac{\rho_1+\rho_2}{2}x_i^2 + \frac{\rho_1+3\rho_2}{2}(x_i+y_i)^2 \\ &= \frac{\rho_1+\rho_2}{2}x^{\mathrm{T}}x + \frac{\rho_1+3\rho_2}{2}(x+y)^{\mathrm{T}}(x+y) \end{aligned} \tag{18}$$

Substituting the inequality (18) into (17) can yield

$$\begin{aligned} \dot{W} \le &- \left(\frac{c}{2} - \frac{k_1}{1+\alpha_2} - \frac{k_2}{1+\bar{\alpha}_2}\right)x^{\mathrm{T}}(L+B)x \\ &+ \frac{(1+\rho_1+\rho_2)}{2}x^{\mathrm{T}}x \\ &- \left(k_3 - \frac{c}{2}\right)(x+y)^{\mathrm{T}}(L+B)(x+y) \\ &+ \frac{(3+\rho_1+3\rho_2)}{2}(x+y)^{\mathrm{T}}(x+y) \end{aligned} \tag{19}$$

Noting that $\lambda_{\min}(L+B)x^{\mathrm{T}}x \le x^{\mathrm{T}}(L+B)x \le \lambda_{\max}(L+B)x^{\mathrm{T}}x$, then the inequality (19) can be simplified as

$$\dot{W} \le -\tau_1 x^{\mathrm{T}}(L+B)x - \tau_2(x+y)^{\mathrm{T}}(L+B)(x+y) \tag{20}$$

where $\tau_1 = \frac{c}{2} - \frac{k_1}{1+\alpha_2} - \frac{k_2}{1+\bar{\alpha}_2} - \frac{\rho_1+\rho_2+1}{2\lambda_{\min}(L+B)}$ and $\tau_2 = k_3 - \frac{c}{2} - \frac{3+\rho_1+3\rho_2}{2\lambda_{\min}(L+B)}$. With the parameters $k_i, i = 1, 2, 3$ selected in (7), and c be chosen such that $\frac{c}{2} - \frac{k_1}{1+\alpha_2} - \frac{k_2}{1+\bar{\alpha}_2} - \frac{\rho_1+\rho_2+1}{2\lambda_{\min}(L+B)} > 0$, then we have than $\tau_1 > 0$ and $\tau_2 > 0$. Further, since matrix $L + B > 0$, then it follows from (20) that the closed-loop system (9) is globally asymptotically stable.

Step 2 We will show that the approximating system of (9) in the 0-limit is homogenous with negative degree k_0 and globally asymptotically stable. Rewrite the closed-loop system (9) as follows

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -k_1\left\{\sum_{j\in N_i}a_{ij}\text{sig}^{\alpha_1}(x_i-x_j) + b_i\text{sig}^{\alpha_1}(x_i)\right. \\ &\quad \left.+ \sum_{j\in N_i}a_{ij}\text{sig}^{\alpha_2}(y_i-y_j) + b_i\text{sig}^{\alpha_2}(y_i)\right\} \\ &\quad + \hat{g}_i(x_i,y_i) \end{aligned} \tag{21}$$

where

$$\begin{aligned} \hat{g}_i(x_i,y_i) &= \tilde{g}_i(x_i,y_i) + f(t,q_i,p_i) - f(t,q_d,p_d) \\ \tilde{g}_i(x_i,y_i) &= -k_2\left\{\sum_{j\in N_i}a_{ij}\text{sig}^{\bar{\alpha}_1}(x_i-x_j) + b_i\text{sig}^{\bar{\alpha}_1}(x_i)\right. \\ &\quad \left.+ \sum_{j\in N_i}a_{ij}\text{sig}^{\bar{\alpha}_2}(y_i-y_j) + b_i\text{sig}^{\bar{\alpha}_2}(y_i)\right\} \\ &\quad - k_3\left\{\sum_{j\in N_i}a_{ij}(x_i-x_j) + b_i x_i + \sum_{j\in N_i}a_{ij}(y_i\right. \\ &\quad \left. - y_j) + b_i y_i\right\} \end{aligned}$$

Firstly, it will be shown that the following system is the approximating system of (9) in the 0-limit and is homogeneous with negative degree k_0, which is described as

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -k_1\left\{\sum_{j\in N_i}a_{ij}\text{sig}^{\alpha_1}(x_i-x_j) + b_i\text{sig}^{\alpha_1}(x_i)\right. \\ &\quad \left.+ \sum_{j\in N_i}a_{ij}\text{sig}^{\alpha_2}(y_i-y_j) + b_i\text{sig}^{\alpha_2}(y_i)\right\} \end{aligned} \tag{22}$$

Noting that $0 < \alpha_1 < 1, \alpha_2 = \frac{2\alpha_1}{1+\alpha_1}$, it can easily be verified that the system (22) is homogeneous of degree $k_0 = -1$ with respect to (r_1, r_2), where $r_1 = \frac{2}{1-\alpha_1}$ and $r_2 = \frac{1+\alpha_1}{1-\alpha_1}$. Based on Assumption 1, it can be obtained that

$$\lim_{\varepsilon\to 0} \frac{\hat{g}(\varepsilon^{r_1}x_i, \varepsilon^{r_2}y_i)}{\varepsilon^{r_2+k_0}} \leq \lim_{\varepsilon\to 0} \frac{|\tilde{g}(\varepsilon^{r_1}x_i, \varepsilon^{r_2}y_i)| + (\rho_1|\varepsilon^{r_1}x_i| + \rho_2|\varepsilon^{r_2}y_i|)}{\varepsilon^{r_2+k_0}} \tag{23}$$

From the expression of $\tilde{g}(x_i, y_i)$, we have

$$\begin{aligned}\tilde{g}(\varepsilon^{r_1}x_i, \varepsilon^{r_2}y_i) = &-k_2\varepsilon^{\bar{\alpha}_1 r_1}\left\{\sum_{j\in N_i} a_{ij}\mathrm{sig}^{\bar{\alpha}_1}(x_i - x_j) + b_i\mathrm{sig}^{\bar{\alpha}_1}(x_i)\right\} \\ &- k_2\varepsilon^{\bar{\alpha}_2 r_2}\left\{\sum_{j\in N_i} a_{ij}\mathrm{sig}^{\bar{\alpha}_2}(y_i - y_j) + b_i\mathrm{sig}^{\bar{\alpha}_2}(y_i)\right\} \\ &- k_3\varepsilon^{r_1}\left\{\sum_{j\in N_i} a_{ij}(x_i - x_j) + b_i x_i\right\} \\ &- k_3\varepsilon^{r_2}\left\{\sum_{j\in N_i} a_{ij}(y_i - y_j) + b_i y_i\right\}\end{aligned} \tag{24}$$

Since $0 < \alpha_1, \alpha_2 < 1, \bar{\alpha}_1 > 1, \bar{\alpha}_2 > 1$, we have that $r_2 + k_0 = \alpha_1 r_1 < r_1 < \bar{\alpha}_1 r_1, r_2 + k_0 = \alpha_2 r_2 < r_2 < \bar{\alpha}_2 r_2$, which results in

$$\lim_{\varepsilon\to 0} \frac{\hat{g}(\varepsilon^{r_1}x_i, \varepsilon^{r_2}y_i)}{\varepsilon^{r_2+k_0}} = 0 \tag{25}$$

Then it is concluded that the system (22) is the approximating system of (9) in the 0-limit, which is homogeneous with negative degree.

Further, we will prove that the system (22) is globally asymptotically stable. Consider the following Lyapunov candidate function

$$W_0 = \frac{k_1}{1+\alpha_1}\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}\frac{a_{ij}}{2}|x_i - x_j|^{1+\alpha_1} + b_i|y_i|^{1+\alpha_1}\right\} + \frac{1}{2}\sum_{i=1}^{N} y_i^2 \tag{26}$$

Taking the derivative of W_0, it can be calculated that

$$\dot{W}_0 = -k_1\sum_{i=1}^{N}\left\{\sum_{j\in N_i} a_{ij}y_i\mathrm{sig}^{\alpha_2}(y_i - y_j) + b_i y_i\mathrm{sig}^{\alpha_2}(y_i)\right\} \tag{27}$$

which implies that W_0 is non-increasing. We have that $\lim_{t\to\infty}\int_0^\infty \dot{W}_0(t)\mathrm{d}t$ exists and is finite. Due to $W_0(t) \leq W_0(0)$, the states x_i, y_i are all bounded for $t \geq 0$. It follows from (22) that $\dot{y}_i$ is also bounded, from which we conclude that $\dot{W}_0$ is uniformly continuous. By using Barbalat's Lemma, we obtain that $\lim_{t\to\infty}\dot{W}_0 = 0$, which implies that $\lim_{t\to\infty}\sum_{j\in N_i} a_{ij}y_i\mathrm{sig}^{\alpha_2}(y_i - y_j) + b_i y_i \mathrm{sig}^{\alpha_2}(y_i) = \lim_{t\to\infty}\frac{1}{2}\sum_{j\in N_i} a_{ij}(y_i - y_j)\mathrm{sig}^{\alpha_2}(y_i - y_j) + b_i y_i\mathrm{sig}^{\alpha_2}(y_i) = 0$. Then we have that $\lim_{t\to\infty}\sum_{j\in N_i} a_{ij}(y_i - y_j)\mathrm{sig}^{\alpha_2}(y_i - y_j) = 0$ and $\lim_{t\to\infty} b_i y_i\mathrm{sig}^{\alpha_2}(y_i) = 0$. Based on Assumption 2, we have that $\lim_{t\to\infty} y_i = 0, i = 1, 2, \ldots, N$. Substituting $y_i = 0$ into the second equation of (22), we can get that $\lim_{t\to\infty}\sum_{j\in N_i} a_{ij}\mathrm{sig}^{\alpha_1}(x_i - x_j) + b_i\mathrm{sig}^{\alpha_1}(x_i) = 0$, which means that

$$\begin{aligned}&\lim_{t\to\infty}\sum_{i=1}^{N} x_i\left\{\sum_{j\in N_i} a_{ij}\mathrm{sig}^{\alpha_1}(x_i - x_j) + b_i\mathrm{sig}^{\alpha_1}(x_i)\right\} \\ &= \lim_{t\to\infty}\sum_{i=1}^{N}\left\{\sum_{j\in N_i}\frac{a_{ij}}{2}(x_i - x_j)\mathrm{sig}^{\alpha_1}(x_i - x_j) + b_i x_i\mathrm{sig}^{\alpha_1}(x_i)\right\} = 0\end{aligned}$$

Then we have that $\lim_{t\to\infty}\sum_{j\in N_i} a_{ij}(x_i - x_j)\mathrm{sig}^{\alpha_1}(x_i - x_j) = 0$ and $\lim_{t\to\infty} b_i x_i\mathrm{sig}^{\alpha_1}(x_i) = 0$. Under Assumption 2, we get that $x_i = 0, i = 1, 2, \ldots, N$. Then it is concluded that the equilibrium $x_i = y_i = 0$ of the system (22) is globally asymptotically stable.

Step 3 We will show that the approximating system of (9) in the ∞-limit is homogenous with positive degree k_∞ and is globally asymptotically stable. Rewrite the closed-loop system (9) as follows

$$\begin{aligned}\dot{x}_i &= y_i\\ \dot{y}_i &= -k_2\left\{\sum_{j\in N_i} a_{ij}\operatorname{sig}^{\bar{\alpha}_1}(x_i-x_j)+b_i\operatorname{sig}^{\bar{\alpha}_1}(x_i)\right.\\ &\quad\left.+\sum_{j\in N_i} a_{ij}\operatorname{sig}^{\bar{\alpha}_2}(y_i-y_j)+b_i\operatorname{sig}^{\bar{\alpha}_2}(y_i)\right\}\\ &\quad+\hat{f}_i(x_i,y_i) \end{aligned}\tag{28}$$

where

$$\begin{aligned}\hat{f}_i(x_i,y_i) &= \tilde{f}_i(x_i,y_i)+f(t,q_i,p_i)-f(t,q_d,p_d)\\ \tilde{f}_i(x_i,y_i) &= -k_1\left\{\sum_{j\in N_i} a_{ij}\operatorname{sig}^{\alpha_1}(x_i-x_j)+b_i\operatorname{sig}^{\alpha_1}(x_i)\right.\\ &\quad\left.+\sum_{j\in N_i} a_{ij}\operatorname{sig}^{\alpha_2}(y_i-y_j)+b_i\operatorname{sig}^{\alpha_2}(y_i)\right\}\\ &\quad-k_3\left\{\sum_{j\in N_i} a_{ij}(x_i-x_j)+b_i x_i\right.\\ &\quad\left.+\sum_{j\in N_i} a_{ij}(y_i-y_j)+b_i y_i\right\}\end{aligned}$$

Noting that $\bar{\alpha}_1>1, \bar{\alpha}_2=\frac{2\bar{\alpha}_1}{\bar{\alpha}_1+1}$, it is easily verified that the following system (29) is homogeneous of positive degree $k_\infty=1$ with respect to (l_1,l_2), where $l_1=\frac{2}{\bar{\alpha}_1-1}, l_2=\frac{\bar{\alpha}_1+1}{\bar{\alpha}_1-1}$, which is presented as

$$\begin{aligned}\dot{x}_i &= y_i\\ \dot{y}_i &= -k_2\left\{\sum_{j\in N_i} a_{ij}\operatorname{sig}^{\bar{\alpha}_1}(x_i-x_j)+b_i\operatorname{sig}^{\bar{\alpha}_1}(x_i)\right.\\ &\quad\left.+\sum_{j\in N_i} a_{ij}\operatorname{sig}^{\bar{\alpha}_2}(y_i-y_j)+b_i\operatorname{sig}^{\bar{\alpha}_2}(y_i)\right\}\end{aligned}\tag{29}$$

Since $0<\alpha_1,\alpha_2<1, \bar{\alpha}_1>1, \bar{\alpha}_2>1$, we have that $l_2+k_\infty=\bar{\alpha}_1 l_1>l_1>\alpha_1 l_1, l_2+k_\infty=\bar{\alpha}_2 l_2>l_2>\alpha_2 l_2$. It can be calculated that

$$\begin{aligned}&\lim_{\varepsilon\to\infty}\frac{\hat{f}(\varepsilon^{l_1}x_i,\varepsilon^{l_2}y_i)}{\varepsilon^{l_2+k_\infty}}\\ &\quad\le\lim_{\varepsilon\to\infty}\frac{|\tilde{f}(\varepsilon^{l_1}x_i,\varepsilon^{l_2}y_i)|+(\rho_1|\varepsilon^{l_1}x_i|+\rho_2|\varepsilon^{l_2}y_i|)}{\varepsilon^{l_2+k_\infty}}\\ &\quad=0\end{aligned}\tag{30}$$

Then we conclude that the system (29) is the approximating system of (9) in the ∞-limit, which is homogeneous with positive degree. To show that the system (29) is globally asymptotically stable, the Lyapunov function is chosen as

$$\begin{aligned}W_\infty &= \frac{k_1}{1+\bar{\alpha}_1}\sum_{i=1}^{N}\left\{\sum_{j=1}^{N}\frac{a_{ij}}{2}|x_i-x_j|^{1+\bar{\alpha}_1}\right.\\ &\quad\left.+b_i|y_i|^{1+\bar{\alpha}_1}\right\}+\frac{1}{2}\sum_{i=1}^{N}y_i^2\end{aligned}\tag{31}$$

Its derivative can be calculated as

$$\begin{aligned}\dot{W}_\infty &= -k_2\sum_{i=1}^{N}\left\{\sum_{j\in N_i} a_{ij}y_i\operatorname{sig}^{\bar{\alpha}_2}(y_i-y_j)\right.\\ &\quad\left.+b_i y_i\operatorname{sig}^{\bar{\alpha}_2}(y_i)\right\}\end{aligned}\tag{32}$$

Similar to the analysis in Step 2, by using Barbalat's Lemma, it is easily proved that the system (29) is globally asymptotically stable. Here the process of proof is omitted.

Thus, based on the results given in Steps 1, 2 and 3, it can be seen that all conditions required in Lemma 1 are satisfied. Then it is concluded that the fixed-time consensus tracking control problem for the nonlinear second-order multi-agent system (3) and (4) can be solved by the proposed fixed-time control protocol (5).

Further, the mathematical expression of the estimate of the settling time is given. From Lemma 1, we have that there exists a continuous, positive and proper Lyapunov candidate function V which satisfies that

$$\dot{V}\le -k_v\Gamma(V^{\gamma_1},V^{\gamma_2})\tag{33}$$

where $\gamma_1=\frac{d_{V_0}+k_0}{d_{V_0}}, \gamma_2=\frac{d_{V_\infty}+k_\infty}{d_{V_\infty}}, k_0=-1, k_\infty=1, d_{V_0}=\max(\frac{2}{1-\alpha_1},\frac{1+\alpha_1}{1-\alpha_1})>2, d_{V_\infty}=\max(\frac{2}{\bar{\alpha}_1-1},\frac{\bar{\alpha}_1+1}{\bar{\alpha}_1-1})>1$. Then we have that $0<\gamma_1<1<\gamma_2$ for $\alpha_i,\bar{\alpha}_i, i=1,2$ satisfying the inequalities (6).

For the case of $V\ge 1$, it follows that $\frac{1}{2}(1+V^{\gamma_2})<\Gamma(V^{\gamma_1},V^{\gamma_2})=\frac{V^{\gamma_1}}{1+V^{\gamma_1}}(1+V^{\gamma_2})<1+V^{\gamma_2}$ for $0<\gamma_1<1<\gamma_2$. The inequality (33) can be simplified as $\dot{V}\le-\frac{k_v}{2}V^{\gamma_2}$, which can guarantee that $V\le 1$ can be satisfied in a finite time $T_1\le\frac{2}{k_v(\gamma_2-1)}$ for any initial states satisfying that $V(0)\ge 1$. Further, for $V\le 1$, it follows from that $\frac{V^{r_1}}{2} \leq \Gamma(V^{\gamma_1}, V^{\gamma_2}) \leq V^{\gamma_1}$, then the inequality (33) can be simplified as $\dot{V} \leq -\frac{k_v}{2} V^{\gamma_1}$, which implies that $V = 0$ can be obtained in a finite time $T_2 = \frac{2}{k_v(1-\gamma_1)}$. Thus, based on the analysis above, it is concluded that the consensus tracking control can be achieved in a fixed time $T_s \leq T_1 + T_2$, which is only determined by the designed parameters with regardless of the initial and instantaneous value of the closed-loop system states. □

Remark 1 It is noted that the fixed-time consensus control protocol (5) is constructed by three parts, the lower-order components $-k_1\{\sum_{j\in N_i} a_{ij}\text{sig}^{\alpha_1}(q_i - q_j) + b_i\text{sig}^{\alpha_1}(q_i - q_d) + \sum_{j\in N_i} a_{ij}\text{sig}^{\alpha_2}(p_i - p_j) + b_i\text{sig}^{\alpha_2}(p_i - p_d)\}$ whose exponential are less than 1, the higher-order components $-k_2\{\sum_{j\in N_i} a_{ij}\text{sig}^{\bar{\alpha}_1}(q_i - q_j) + b_i\text{sig}^{\bar{\alpha}_1}(q_i - q_d) + \sum_{j\in N_i} a_{ij}\text{sig}^{\bar{\alpha}_2}(p_i - p_j) + b_i\text{sig}^{\bar{\alpha}_2}(p_i - p_d)\}$ with exponential greater than 1 and the linear components u_{i2} with exponential equal to 1. From Steps 2 and 3 in Theorem 1, it can be seen that the lower-order components and the higher-order components are the homogeneous components of the system (9) in the 0-limit and ∞-limit, respectively. Furthermore, from the analysis process of the settling time provided in Theorem 1, it is noted that the higher-order components guarantee that the states of the system (9) can converge into the small regions with uniformly bounded convergence time for any initial states and the lower-order components can drive the system states from the small regions to zero with exact finite convergence time. By combining the higher-order and lower-order components, the fixed-time convergence can be obtained. The linear components of the designed control protocol (5) are provided to dominate the nonlinear function $f_i(t, q_i, p_i)$.

3.2 Fixed-time consensus tracking control via output feedback

In many practical situations, it is difficult or impossible to obtain the velocity measurements or the velocity information can be accurately measured due to the technology constraints or environment disturbances. In these cases, the designed fixed-time control protocol with full state feedback cannot be implemented. Thus, it is highly desirable to design a fixed-time consensus control protocol by only using the position measurements.

To achieve this target, a fixed-time state observer is proposed to estimate the velocity information p_i for each agent, which is described as

$$\begin{aligned}\dot{\hat{q}}_i &= \hat{p}_i - \lambda_1\text{sig}^{\beta_1}(\hat{q}_i - q_i) - \lambda_2(\hat{q}_i - q_i)\\ &\quad - \lambda_3\text{sig}^{\bar{\beta}_1}(\hat{q}_i - q_i)\\ \dot{\hat{p}}_i &= u_i - \mu_1\text{sig}^{\beta_2}(\hat{q}_i - q_i) - \mu_2(\hat{q}_i - q_i)\\ &\quad - \mu_3\text{sig}^{\bar{\beta}_2}(\hat{q}_i - q_i) + f(t, q_i, \hat{p}_i)\end{aligned} \tag{34}$$

where $\hat{q}_i$ and $\hat{p}_i$ are the estimates of q_i and p_i, respectively. $\lambda_i, \mu_i, i = 1, 2, 3$ are positive constants. The parameters $\beta_i, \bar{\beta}_i, i = 1, 2$ are selected as

$$\begin{aligned}&\frac{1}{2} < \beta_1 < 1, \quad \beta_2 = 2\beta_1 - 1,\\ &\bar{\beta}_1 > 1, \quad \bar{\beta}_2 = 2\bar{\beta}_1 - 1\end{aligned} \tag{35}$$

Lemma 5 *Consider the proposed fixed-time state observer* (34)*. Suppose that Assumption* 1 *holds, then the states* $(\hat{q}_i, \hat{p}_i)$ *of the observer* (34) *will globally fixed-time converge to the states* (q_i, p_i) *if the parameters* $\beta_i, \bar{\beta}_i, i = 1, 2$ *satisfy the inequality* (35)*, and the following inequalities are satisfied*

$$\begin{aligned}&\mu_1 > \lambda_1, \quad \mu_3 > \lambda_3, \quad \frac{\mu_2}{\lambda_2} - \frac{3}{2}\rho_2 > 0\\ &\lambda_2 - \frac{\mu_2}{\lambda_2} - \frac{\rho_2}{2} - \frac{\lambda_1\lambda_2}{4} - \frac{\lambda_2\lambda_3}{4} > 0\end{aligned} \tag{36}$$

Proof Define the estimate errors $e_1 = \hat{q}_i - q_i$ and $e_2 = \hat{p}_i - p_i$. Based on the fixed-time state observer (34), the errors dynamic system can be obtained

$$\begin{aligned}\dot{e}_1 &= e_2 - \lambda_1\text{sig}^{\beta_1}(e_1) - \lambda_2 e_1 - \lambda_3\text{sig}^{\bar{\beta}_1}(e_1)\\ \dot{e}_2 &= -\mu_1\text{sig}^{\beta_2}(e_1) - \mu_2 e_1 - \mu_3\text{sig}^{\bar{\beta}_2}(e_1)\\ &\quad + f(t, q_i, \hat{p}_i) - f(t, q_i, p_i)\end{aligned} \tag{37}$$

As is similar in Theorem 1, the proof that the errors system (37) is fixed-time stable is divided into three steps.

Step 1 we will show that the errors system (37) is globally asymptotically stable. To this end, the Lyapunov function candidate is considered as follows

$$V=\frac{2\mu_1}{1+\beta_2}|e_1|^{1+\beta_2}+\frac{2\mu_3}{1+\bar{\beta}_2}|e_1|^{1+\bar{\beta}_2}+\frac{1}{2}(e_2-\lambda_2e_1)^2+\frac{1}{2}e_2^2 \quad (38)$$

Taking derivative of V along system (37) can yield

$$\begin{aligned}\dot{V}=&2\mu_1\text{sig}^{\beta_2}(e_1)\dot{e}_1+2\mu_3\text{sig}^{\bar{\beta}_2}(e_1)\dot{e}_1\\&+(e_2-\lambda_2e_1)(\dot{e}_2-\lambda_2\dot{e}_1)+e_2\dot{e}_2\\=&-2\mu_1\lambda_1|e_1|^{\beta_2+\beta_1}-\mu_1\lambda_2|e_1|^{\beta_2+1}\\&-2\mu_1\lambda_3|e_1|^{\beta_2+\bar{\beta}_1}\\&-2\mu_3\lambda_1|e_1|^{\bar{\beta}_2+\beta_1}-\mu_3\lambda_2|e_1|^{\bar{\beta}_2+1}\\&-2\mu_3\lambda_3|e_1|^{\bar{\beta}_2+\bar{\beta}_1}-\mu_2(2e_2-\lambda_2e_1)e_1\\&+(2e_2-\lambda_2e_1)(f(t,q_i,\hat{p}_i)\\&-f(t,q_i,p_i))-\lambda_2(e_2-\lambda_2e_1)^2\\&+\lambda_2(e_2-\lambda_2e_1)(\lambda_1\text{sig}^{\beta_1}(e_1)+\lambda_3\text{sig}^{\bar{\beta}_1}(e_1))\end{aligned} \quad (39)$$

Note that

$$\begin{aligned}&-\mu_2(2e_2-\lambda_2e_1)e_1\\&=(e_2+e_2-\lambda_2e_1)\frac{\mu_2}{\lambda_2}(e_2-\lambda_2e_1-e_2)\\&=\frac{\mu_2}{\lambda_2}(e_2-\lambda_2e_1)^2-\frac{\mu_2}{\lambda_2}e_2^2\end{aligned} \quad (40)$$

Based on Assumption 1, we can get that

$$\begin{aligned}&(2e_2-\lambda_2e_1)(f(t,q_i,\hat{p}_i)-f(t,q_i,p_i))\\&\le|e_2+e_2-\lambda_2e_1|\rho_2|e_2|\\&\le\rho_2(|e_2|+|e_2-\lambda_2e_1|)|e_2|\\&\le\rho_2|e_2|^2+\rho_2|e_2-\lambda_2e_1||e_2|\\&\le\frac{3}{2}\rho_2|e_2|^2+\frac{\rho_2}{2}(e_2-\lambda_2e_1)^2\end{aligned} \quad (41)$$

Noting that $\beta_2=2\beta_1-1$, $\bar{\beta}_2=2\bar{\beta}_1-1$, we have

$$\begin{aligned}&\lambda_2(e_2-\lambda_2e_1)(\lambda_1\text{sig}^{\beta_1}(e_1)+\lambda_3\text{sig}^{\bar{\beta}_1}(e_1))\\&\le\lambda_2\lambda_1(e_2-\lambda_2e_1)\text{sig}^{\beta_1}(e_1)\\&\quad+\lambda_2\lambda_3(e_2-\lambda_2e_1)\text{sig}^{\bar{\beta}_1}(e_1)\\&\le\left(\frac{\lambda_2\lambda_1}{4}+\frac{\lambda_2\lambda_3}{4}\right)(e_2-\lambda_2e_1)^2+\lambda_2\lambda_1\text{sig}^{2\beta_1}(e_1)\\&\quad+\lambda_2\lambda_3\text{sig}^{2\bar{\beta}_1}(e_1)\\&\le\left(\frac{\lambda_2\lambda_1}{4}+\frac{\lambda_2\lambda_3}{4}\right)(e_2-\lambda_2e_1)^2+\lambda_2\lambda_1|e_1|^{\beta_2+1}\\&\quad+\lambda_2\lambda_3|e_1|^{\bar{\beta}_2+1}\end{aligned} \quad (42)$$

Substituting (40)–(42) into (39),we have

$$\begin{aligned}\dot{V}=&-2\mu_1\lambda_1|e_1|^{\beta_2+\beta_1}-(\mu_1\lambda_2-\lambda_2\lambda_1)|e_1|^{\beta_2+1}\\&-2\mu_1\lambda_3|e_1|^{\beta_2+\bar{\beta}_1}-2\mu_3\lambda_1|e_1|^{\bar{\beta}_2+\beta_1}\\&-(\mu_3\lambda_2-\lambda_2\lambda_3)|e_1|^{\bar{\beta}_2+1}-2\mu_3\lambda_3|e_1|^{\bar{\beta}_2+\bar{\beta}_1}\\&-\tau_3e_2^2-\tau_4(e_2-\lambda_2e_1)^2\end{aligned} \quad (43)$$

where $\tau_3=\frac{\mu_2}{\lambda_2}-\frac{3}{2}\rho_2$, $\tau_4=\lambda_2-\frac{\mu_2}{\lambda_2}-\frac{\rho_2}{2}-\frac{\lambda_1\lambda_2}{4}-\frac{\lambda_2\lambda_3}{4}$. With the parameters λ_i, μ_i selected in (36),we have that $\tau_3>0$ and $\tau_4>0$. Then it follows from (43) that the errors dynamics system (37) is globally asymptotically stable.

Step 2 We will show that the approximating system of (37) in the 0-limit is homogeneous of negative degree d_0 and is globally asymptotically stable. The system (37) is written as

$$\begin{aligned}\dot{e}_1&=e_2-\lambda_1\text{sig}^{\beta_1}(e_1)+\hat{g}_1(e_1,e_2)\\\dot{e}_2&=-\mu_1\text{sig}^{\beta_2}(e_1)+\hat{g}_2(e_1,e_2)\end{aligned} \quad (44)$$

where $\hat{g}_1(e_1,e_2)=-\lambda_2e_1-\lambda_3\text{sig}^{\bar{\beta}_1}(e_1)$ and $\hat{g}_2(e_1,e_2)=-\mu_2e_1-\mu_3\text{sig}^{\bar{\beta}_2}(e_1)+f(t,q_i,\hat{p}_i)-f(t,q_i,p_i)$

Noting that $1/2<\beta_1<1$, $\beta_2=2\beta_1-1$, it can be verified that the following system (45) is homogeneous of degree $d_0=-1$ with respect to (r_1,r_2), where $r_1=\frac{1}{1-\beta_1}$, $r_2=\frac{\beta_1}{1-\beta_1}$, which is denoted as

$$\dot{e}_1=e_2-\lambda_1\text{sig}^{\beta_1}(e_1),\ \dot{e}_2=-\mu_1\text{sig}^{\beta_2}(e_1) \quad (45)$$

Due to $0<\beta_1,\beta_2<1$, $\bar{\beta}_1>1$, $\bar{\beta}_2>1$, we have that $d_0+r_1=r_2=\beta_1r_1<r_1<\bar{\beta}_1r_1$, $d_0+r_2=\beta_2r_1<r_1<\bar{\beta}_2r_1$. Under Assumption 1, we can calculate that

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{\hat{g}_1(\varepsilon^{r_1}e_1,\varepsilon^{r_2}e_2)}{\varepsilon^{r_1+d_0}}\\&\quad=\lim_{\varepsilon\to0}\frac{-\lambda_2\varepsilon^{r_1}e_1-\lambda_3\varepsilon^{\bar{\beta}_1r_1}\text{sig}^{\bar{\beta}_1}(e_1)}{\varepsilon^{r_1+d_0}}=0\\&\lim_{\varepsilon\to0}\frac{\hat{g}_2(\varepsilon^{r_1}e_1,\varepsilon^{r_2}e_2)}{\varepsilon^{r_2+d_0}}\\&\quad\le\lim_{\varepsilon\to0}\frac{|\mu_2\varepsilon^{r_1}e_1+\mu_3\varepsilon^{\bar{\beta}_2r_1}\text{sig}^{\bar{\beta}_2}(e_1)|+\rho_2|\varepsilon^{r_2}e_2|}{\varepsilon^{r_2+d_0}}\\&=0\end{aligned}$$

Then it is illustrated that the system (45) is the approximating system of (37), which is homogenous with negative degree. Further, we choose the Lyapunov function candidate $V_0 = \frac{\mu_1}{1+\beta_2}|e_1|^{1+\beta_2} + \frac{1}{2}e_2^2$ and its derivative is $\dot{V}_0 \leq -\mu_1\lambda_1|e_1|^{\beta_1+\beta_2} \leq 0$. Similar to the analysis in Step 2 in Theorem 1, by using Barbalat's Lemma, it is easily proved that the system (45) is globally asymptotically stable.

Step 3 It will be shown that the approximating system of (37) in the ∞-limit is homogenous of positive degree d_∞ and is globally asymptotically stable. The system (37) can be rewritten as

$$\begin{aligned} \dot{e}_1 &= e_2 - \lambda_3 \mathrm{sig}^{\bar{\beta}_1}(e_1) + \hat{f}_1(e_1, e_2) \\ \dot{e}_2 &= -\mu_3 \mathrm{sig}^{\bar{\beta}_2}(e_1) + \hat{f}_2(e_1, e_2) \end{aligned} \tag{46}$$

where $\hat{f}_1(e_1, e_2) = -\lambda_1 \mathrm{sig}^{\beta_1}(e_1) - \lambda_2 e_1$ and $\hat{f}_2(e_1, e_2) = -\mu_1 \mathrm{sig}^{\beta_2}(e_1) - \mu_2 e_1 + f(t, q_i, \hat{p}_i) - f(t, q_i, p_i)$

Noting that $\bar{\beta}_1 > 1$, $\bar{\beta}_2 = 2\bar{\beta}_1 - 1$, it is easily verified that the following system (47) is homogeneous of positive degree $d_\infty = 1$ with respect to (l_1, l_2), where $l_1 = 1/(\bar{\beta}_1 - 1)$, $l_2 = \bar{\beta}_1/(\bar{\beta}_1 - 1)$, which is described as

$$\dot{e}_1 = e_2 - \lambda_3 \mathrm{sig}^{\bar{\beta}_1}(e_1), \dot{e}_2 = -\mu_3 \mathrm{sig}^{\bar{\beta}_2}(e_1) \tag{47}$$

Noting that $0 < \beta_1, \beta_2 < 1$, $\bar{\beta}_1 > 1$, $\bar{\beta}_2 > 1$, we can get that $d_\infty + l_1 = l_2 = \bar{\beta}_1 l_1 > l_1 > \beta_1 l_1$, $d_\infty + l_2 = \bar{\beta}_2 l_1 > l_1 > \beta_2 l_1$. Under Assumption 1, we can obtain that

$$\begin{aligned} &\lim_{\varepsilon\to\infty} \frac{\hat{f}_1(\varepsilon^{l_1}e_1, \varepsilon^{l_2}e_2)}{\varepsilon^{l_1+d_\infty}} \\ &\quad = \lim_{\varepsilon\to\infty} \frac{-\lambda_1\varepsilon^{\beta_1 l_1}\mathrm{sig}^{\beta_1}(e_1) - \lambda_2\varepsilon^{l_1}e_1}{\varepsilon^{l_1+d_\infty}} = 0 \\ &\lim_{\varepsilon\to\infty} \frac{\hat{f}_2(\varepsilon^{l_1}e_1, \varepsilon^{l_2}e_2)}{\varepsilon^{l_2+d_\infty}} \\ &\quad \leq \lim_{\varepsilon\to\infty} \frac{|\mu_1\varepsilon^{\beta_2 l_1}\mathrm{sig}^{\beta_2}(e_1) + \mu_2\varepsilon^{l_1}e_1| + \rho_2|\varepsilon^{l_2}e_2|}{\varepsilon^{l_2+d_\infty}} \\ &\quad = 0 \end{aligned}$$

Then it is concluded that the system (47) is the approximating system of (45) in the ∞-limit, which is also homogeneous. We choose the Lyapunov function candidate $V_\infty = \frac{\mu_3}{1+\bar{\beta}_2}|e_1|^{1+\bar{\beta}_2} + \frac{1}{2}e_2^2$ and its derivative is $\dot{V}_\infty \leq -\mu_3\lambda_3|e_1|^{\bar{\beta}_1+\bar{\beta}_2} \leq 0$. With the same analysis as Step 3 in Theorem 1, it is easily proved that the system (47) is globally asymptotically stable.

Therefore, based on the results in Steps 1, 2 and 3, it follows from Lemma 1 that the proposed state observer (34) is fixed-time convergent. In addition, Similar to the analysis in Theorem 1, the convergence time of the fixed-time observer (34) can be estimated based on (8) in Theorem 1. This completes the proof. □

Remark 2 It should be pointed out that the aim of the designed fixed-time state observer (34) is to estimate each agent's velocity information by itself position measurements in a fixed time. The fixed-time observer (34) can also be employed to estimate the leader's velocity information, which is described as

$$\begin{aligned} \dot{\hat{q}}_d =& \hat{p}_d - \lambda_1 \mathrm{sig}^{\beta_1}(\hat{q}_d - q_d) - \lambda_2(\hat{q}_d - q_d) \\ & - \lambda_3 \mathrm{sig}^{\bar{\beta}_1}(\hat{q}_d - q_d) \\ \dot{\hat{p}}_d =& - \mu_1 \mathrm{sig}^{\beta_2}(\hat{q}_d - q_d) - \mu_2(\hat{q}_d - q_d) \\ & - \mu_3 \mathrm{sig}^{\bar{\beta}_2}(\hat{q}_d - q_d) + f(t, q_d, \hat{p}_d) \end{aligned} \tag{48}$$

where $\hat{q}_d$ and $\hat{p}_d$ are the estimate of q_d and p_d. Since all agents have uniform nonlinear dynamics $f(t, q_i, p_i)$, then for the agent which can receive information from the leader, i.e., $b_i \neq 0$, the velocity of the leader can be estimated by the fixed-time observer (48).

Further, based on the proposed fixed-time state observer (34), a new fixed-time output feedback control protocol is designed as

$$\begin{aligned} u_i =& u_{i1} + u_{i2} \\ u_{i1} =& -k_1 \Bigg\{ \sum_{j\in N_i} a_{ij}\mathrm{sig}^{\alpha_1}(q_i - q_j) + b_i \mathrm{sig}^{\alpha_1}(q_i - q_d) \\ & + \sum_{j\in N_i} a_{ij}\mathrm{sig}^{\alpha_2}(\hat{p}_i - \hat{p}_j) + b_i \mathrm{sig}^{\alpha_2}(\hat{p}_i - \hat{p}_d) \Bigg\} \\ & - k_2 \Bigg\{ \sum_{j\in N_i} a_{ij}\mathrm{sig}^{\bar{\alpha}_1}(q_i - q_j) + b_i \mathrm{sig}^{\bar{\alpha}_1}(q_i - q_d) \\ & + \sum_{j\in N_i} a_{ij}\mathrm{sig}^{\bar{\alpha}_2}(\hat{p}_i - \hat{p}_j) + b_i \mathrm{sig}^{\bar{\alpha}_2}(\hat{p}_i - \hat{p}_d) \Bigg\} \\ u_{i2} =& - k_3 \Bigg\{ \sum_{j\in N_i} a_{ij}(q_i - q_j) + b_i(q_i - q_d) \\ & + \sum_{j\in N_i} a_{ij}(\hat{p}_i - \hat{p}_j) + b_i(\hat{p}_i - \hat{p}_d) \Bigg\} \end{aligned} \tag{49}$$

With the developed fixed-time output feedback control protocol (49), the main results of this paper is stated in the following theorem.

Theorem 2 *Consider the second-order nonlinear multi-agent systems with the designed fixed-time output feedback control protocol* (49)*. Suppose that Assumptions* 1and 2 *hold, then the fixed-time consensus tracking control can be achieved if the parameters* $\alpha_i, \bar{\alpha}_i, i = 1, 2$ *satisfy the inequality* (6) *and the control parameters* $k_i, i = 1, 2, 3$ *are chosen by satisfying the inequality* (7)*. Furthermore, the convergence time is bounded and satisfied that* $T_3 \leq T_o + T_s$.

Proof Firstly, it follows form Lemma 5 that the designed observer (34) are fixed-time convergent, which means that $\hat{p}_i = p_i, \hat{p}_d = p_d$ are satisfied when $t \geq T_o$. Thus the proposed fixed-time output feedback control (49) is transformed into the control protocol (5) after the finite time T_o. Subsequently, based on the results in Theorem 1, it is concluded that the fixed-time consensus tracking control can be achieved with the proposed fixed-time output feedback control protocol (49). The proof is completed. □

Remark 3 It is noted that we do not prove that the states of system (3) under the designed output feedback control protocol (49) does not escape during $[0, T_o]$. Due to the nonlinear higher-order terms and lower-terms included in the control protocol (49), it is difficult to give the proof process in detail. To guarantee that the system states are bounded in the time interval $[0, T_o]$, the following control protocol is designed during the interval $[0, T_o]$

$$u_i = -k_4 \left\{ \sum_{j \in N_i} a_{ij}(q_i - q_j) + b_i(q_i - q_d) \right\} \qquad (50)$$
$$k_4 > 0, \quad i = 1, 2, \ldots, N$$

Consider the Lyapunov function candidate $V_1 = \frac{k_4}{2} x^{\mathrm{T}} (L + B)x + \frac{1}{2} y^{\mathrm{T}} y$. The derivative of V_1 with the designed control protocol (50) is $\dot{V}_1 = y^{\mathrm{T}} F \leq \frac{\rho_1}{2} x^{\mathrm{T}} x + \frac{\rho_1 + 2\rho_2}{2} y^{\mathrm{T}} y \leq \tau_m V_1$, where $\tau_m = \max(\rho_1 / \lambda_{\max}(L + B), \rho_1 + 2\rho_2)$. Then for $t \in [0, T_o]$, we have that $V_1(t) \leq V_1(0) e^{\tau_m t}$, from which we conclude that the system states will not be escaped during $[0, T_o]$.

Remark 4 In Theorem 2, if we choose $\alpha_1 = 1, \alpha_2 = \bar{\alpha}_1 = \bar{\alpha}_2 = 1$, then the fixed-time output feedback control protocol (49) will become asymptotically stable control protocol

$$u_i = -\bar{k}_1 \left\{ \sum_{j \in N_i} a_{ij}(q_i - q_j) + b_i(q_i - q_d) + \sum_{j \in N_i} a_{ij}(\hat{p}_i - \hat{p}_j) + b_i(\hat{p}_i - \hat{p}_d) \right\} \qquad (51)$$

In addition, if we select $\bar{\alpha}_1 = \bar{\alpha}_2 = 1$, then the fixed-time output feedback control protocol (49) will become the finite time control protocol

$$\begin{aligned} u_i = & -\bar{k}_2 \left\{ \sum_{j \in N_i} a_{ij} \mathrm{sig}^{\alpha_1}(q_i - q_j) + b_i \mathrm{sig}^{\alpha_1}(q_i - q_d) + \sum_{j \in N_i} a_{ij} \mathrm{sig}^{\alpha_2}(\hat{p}_i - \hat{p}_j) + b_i \mathrm{sig}^{\alpha_2}(\hat{p}_i - \hat{p}_d) \right\} \\ & - \bar{k}_3 \left\{ \sum_{j \in N_i} a_{ij}(q_i - q_j) + b_i(q_i - q_d) + \sum_{j \in N_i} a_{ij}(\hat{p}_i - \hat{p}_j) + b_i(\hat{p}_i - \hat{p}_d) \right\} \end{aligned} \qquad (52)$$

which has same expression given in [30]. In numerical simulation, we will show that the designed control protocol (49) has better convergence performance than control protocols (51) and (52).

Remark 5 It follows from Theorem 2 that the designed control protocol (49) not only can solve the consensus tracking problem for second-order nonlinear multi-agent systems, but also can achieve the fixed-time convergence, which can solve the problem brought by finite time consensus control that the convergence time may increase over the prescribed value if the magnitudes of the initial states are large. Moreover, only the position information is required to implement the control protocol (49), which can be available for the case that the velocity information is hard to obtain. In addition, the designed control protocol can be employed to solve the consensus tracking control for integrator systems, which can be regarded as a special case of the system (3) when $f(t, q_i, p_i) = 0$.

Remark 6 It is worth mentioning that the proposed fixed-time output feedback control protocol (49) is partly motivated by the works in [30]. In [30], a finite time output feedback synchronization control protocol was designed for a class of second-order nonlinear multi-agent systems. Compared with the results in [30], the proposed output feedback control protocol by using the bi-limit homogeneous theory can achieve the fixed-time convergence, and the mathematical expression of the convergence time is provided. Even though the numerical value of the convergence time is hard to compute since it is difficult to obtain the value of the parameter k_v, the expression of the convergence time can be regarded as a guideline for adjusting the control parameters such that the prescribed settling time can be satisfied.

Remark 7 Compared with asymptotic consensus control in [29,34] and finite time consensus control results in [28,30,33], although the designed consensus control protocol can achieve the fixed-time convergence performance and high control precision, the design of fixed-time control is relatively complex and the magnitude of the fixed-time control protocol may become larger than that of asymptotically stable control protocol. Then, there exists a trade-off between the control energy and convergence response performance. In addition, in this paper, we only consider the fixed-time consensus tracking control problem with fixed topology network. When the interaction topology between the agents is Markov network topology [35,36], how to design a fixed-time consensus tracking control via output feedback is a difficult and challenging problem, which will be studied in the following works.

4 Numerical results

In this section, numerical simulation examples are provided to illustrate the effective of the proposed fixed-time output feedback control protocol (49) with the designed fixed-time convergent observer (34) for second-order nonlinear multi-agent systems. Multiple nonlinear forced pendulums consisted of four followers and one leader are considered, whose communication topology is shown in Fig. 1. The dynamics of each follower agent is described by

$$\begin{aligned}\dot{q}_i =& p_i,\ \dot{p}_i \\ =& -\sin(q_i) - 0.25p_i + 1.5\cos(2.5t) + u_i \\ y_i =& q_i,\quad i = 1, 2, \ldots, N\end{aligned} \tag{53}$$

and the leader's dynamic is given as

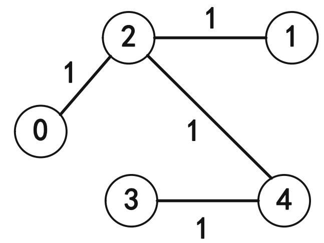

Fig. 1 Communication topology

$$\begin{aligned}&\dot{q}_d = p_d,\quad \dot{p}_d = -\sin(q_d) - 0.25p_d + 1.5\cos(2.5t) \\ &y_d = q_d\end{aligned} \tag{54}$$

It can be easily found that the nonlinear function $f(t, q_i, p_i) = -sin(q_i) - 0.25p_i + 1.5\cos(2.5t)$ satisfies Assumption 1. To demonstrate the fixed-time convergence performance of the control protocol (49), two sets of different initial states of followers and leader are selected as: Case 1: $q(0) = [1, 1, 3, 1.5]^{\mathrm{T}}$, $p(0) = [-1, 0.5, 1, 2]^{\mathrm{T}}$, $q_d(0) = -1$, $p_d(0) = 0.2$ and Case 2: $q(0) = [10, 10, 30, 15]^{\mathrm{T}}$, $p(0) = [-10, 5, 10, 20]^{\mathrm{T}}$, $q_d(0) = -10$, $p_d(0) = 2$.

The control parameters of the designed fixed-time output feedback control protocol (49) for both two different initial states cases are selected as $\alpha_1 = 1/2, \alpha_2 = 2/3, \bar{\alpha}_1 = 2, \bar{\alpha}_2 = 4/3, k_1 = 1, k_2 = 1, k_3 = 10$. It is easily verified that the selected parameters can satisfy the inequalities (6) and (7). The design parameters in the fixed-time convergent state observer (34) and (48) are set as $\beta_1 = 2/3, \beta_2 = 1/3, \bar{\beta}_1 = 3/2, \bar{\beta}_2 = 2, \lambda_1 = 1/2, \lambda_2 = 4, \lambda_3 = 1/2, \mu_1 = 1, \mu_2 = 8, \mu_3 = 1$, which can satisfy the inequalities (35) and (36). The values of initial states of the fixed-time observer (34) and (48) are $\hat{q}_i(0) = 0, \hat{p}_i(0) = 0, i = 1, 2, \ldots, N, \hat{q}_d(0) = 0, \hat{p}_d(0) = 0$.

The simulation results for both two cases under the designed fixed-time output feedback control protocol (49) are displayed in Figs. 2, 3, 4 and 5. For case 1, from Figs. 2 and 3, it can be seen that the trajectory of the leader can be tracked by the followers in finite time about 6 s. For case 2, it follows from Figs. 4 and 5 that the position and velocity tracking errors between followers and leader will approach to zeros in finite time about 8 s even though the initial positions of the followers are far away from the leader. From the simulation results of two cases, it is shown that the convergence times for different initial states are similar, which illustrates that the settling time of the design control protocol (49) is uniformly bounded for differ-

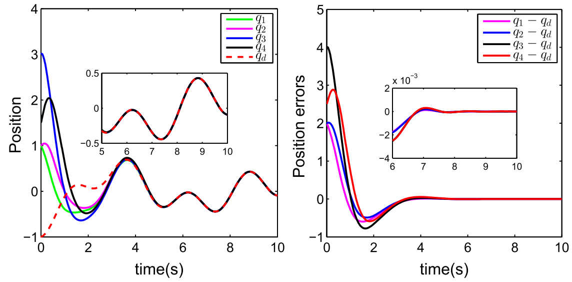

Fig. 2 Position tracking results under control protocol (49) in case 1

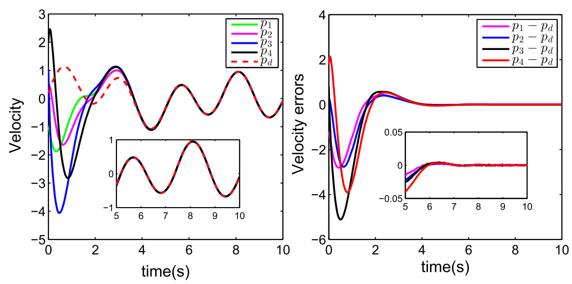

Fig. 3 Velocity tracking results under control protocol (49) in case 1

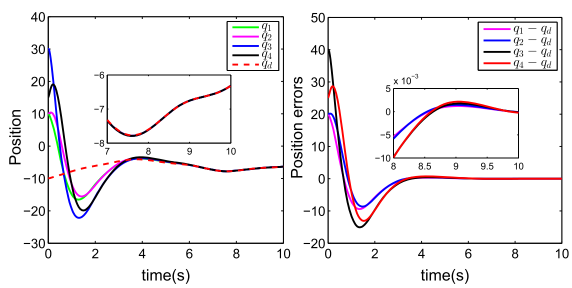

Fig. 4 Position tracking results under control protocol (49) in case 2

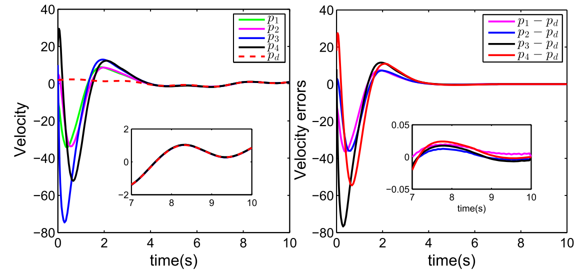

Fig. 5 Velocity tracking results under control protocol (49) in case 2

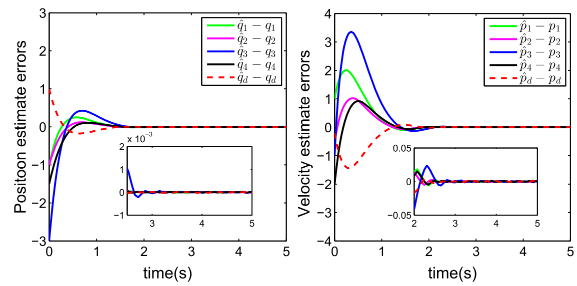

Fig. 6 Position and velocity estimate results of the observer (34) in case 1

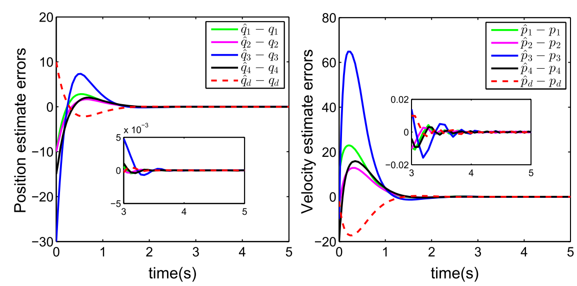

Fig. 7 Position and velocity estimate results of the observer (34) in case 2

ent initial states. In addition, the position and velocity estimate results of the designed fixed-time convergent observer (34) for each agent in both case 1 and case 2 are presented in Figs. 6 and 7, respectively. It can be seen that the position and velocity estimate errors converge to zeros about 3s in both cases 1 and 2, which verifies that the proposed observer (34) is fixed-time convergent.

Further, to highlight the convergence performance of the proposed fixed-time control scheme, the designed control protocol (FXCP) (49) is compared with asymptotically stable control protocol (ASCP) (51) and finite time control protocol (FTCP) (52). In this comparison, we consider the dynamics of system (3) as a double-integrator system to show the convergence performance of the control protocols clearly, i.e., $f(t, q_i, p_i) = 0$. For the initial states given in Case 1, the control magnitudes are limited to exceed 10 to make a fair comparison. The position response curves under control protocol (49) and (52) are shown in Figs. 8 and 9, respectively. Assume that the convergence time is defined by the time after which $|q_i - q_d| \leq 10^{-3}$ holds. Then the convergence times of control protocol (49) and (51) are presented in Table 1, from which it can be seen that the designed control protocol (49) has faster convergence performance than ASCP (51). Further, with the initial states set as $q(0) = [100, 100, 300, 150]^{\mathrm{T}}$, $p(0) = [-100, 50, 100, 200]^{\mathrm{T}}$, $q_d(0) = -100$, $p_d(0) = 20$,

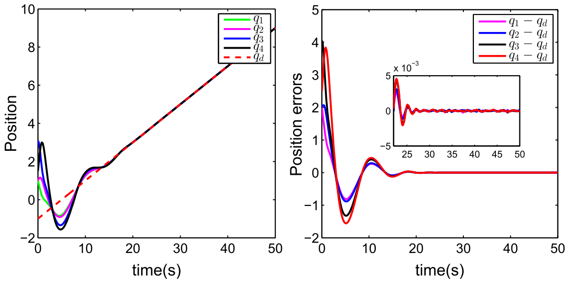

Fig. 8 Position tracking results under control protocol (49)

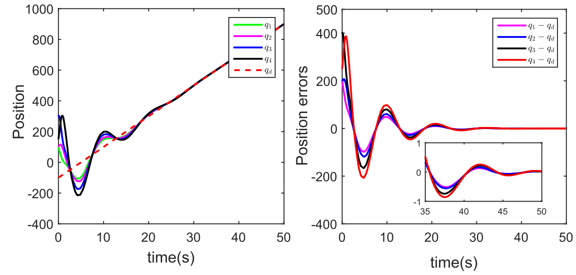

Fig. 10 Position tracking results under control protocol (49)

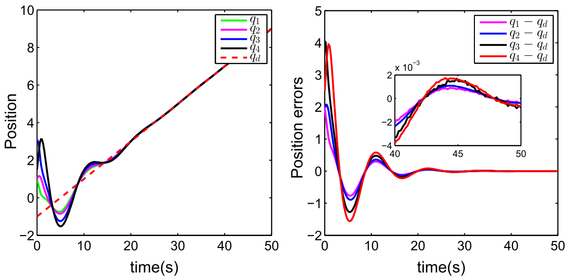

Fig. 9 Position tracking results under control protocol (51)

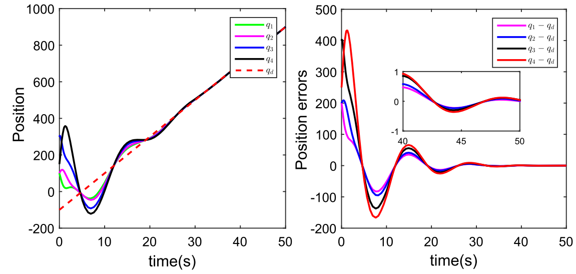

Fig. 11 Position tracking results under control protocol (52)

Table 1 Comparison of convergence time between FXCP (49) and ASCP (51)

System (3) with different control protocol	Convergence time (s)
ASCP (51)	42
FXCP (49)	23
Improvement (%)	45.24

Table 2 Comparison of convergence time between FXCP (49) and FTCP (52)

System (3) with different control protocol	Convergence time (s)
FTCP (51)	40
FXCP (49)	35
Improvement (%)	12.5

the control magnitudes are limited to exceed 1000. The position response curves under the control protocol (49) and (52) are presented in Figs. 10 and 11, respectively. For this case, the settling time is defined when $|q_i - q_d| \leq 1$ always satisfies. The convergence times of control protocol (49) and (52) are given in Table 2, from which we can see the designed control protocol (49) also has faster convergence performance than FTCP (52).

5 Conclusion

In this paper, the fixed-time consensus tracking control problem for a class of nonlinear second-order multi-agent systems without the velocity measurements has been investigated. To achieve this target, a fixed-time convergent state observer is designed, which can estimate each agents velocity information in a fixed time. Then based on the designed fixed-time observer, a fixed-time output feedback control protocol is proposed, which is constructed by the higher-order terms and lower-order terms to achieve the fixed-time convergence and the linear term for dominating the nonlinear function in the system dynamics. Rigorous proof is shown that all followers can track the leaders states in a fixed time in regardless of the initial states. The fixed-time convergence performance of the proposed output feedback control protocol has been demonstrated by finial simulation results. An extension of the fixed-time output feedback control method for nonlinear multi-agent system with switching graph is currently investigated.

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References

1. Olfati-Saber, R.: Flocking for multi-agent dynamic systems: algorithms and theory. IEEE Trans. Autom. Control **51**(3), 401–420 (2006)
2. Wang, Y., Wu, Q., Wang, Y.: Distributed cooperative control for multiple quadrotor systems via dynamic surface control. Nonlinear Dyn. **3**(75), 513–527 (2014)
3. Peng, Z., Wen, G., Yang, S., Rahmani, A.: Distributed consensus-based formation control for nonholonomic wheeled mobile robots using adaptive neural network. Nonlinear Dyn. **1**(86), 605–622 (2016)
4. Zhao, L., Jia, Y.: Decentralized adaptive attitude synchronization control for spacecraft formation using nonsingular fast terminal sliding mode. Nonlinear Dyn. **4**(78), 2779–2794 (2014)
5. Ren, W., Beard, R.W., Atkins, E.M.: Information consensus in multivehicle cooperative control. IEEE Trans. Control Syst. Mag. **27**(2), 71–82 (2007)
6. Zhang, B., Jia, Y., Matsuno, F.: Finite-time observers for multi-agent systems without velocity measurements and with input saturations. Syst. Control Lett. **68**, 86–94 (2014)
7. Wang, L., Xiao, F.: Finite-time consensus problems for networks of dynamic agents. IEEE Trans. Autom. Control **55**(4), 950–955 (2010)
8. Cheng, S., Yu, L., Zhang, D., Huo, L., Ji, J.: Consensus of second-order multi-agent systems using partial agents velocity measurements. Nonlinear Dyn. **3**(86), 1927–1935 (2016)
9. Lin, P., Jia, Y., Li, L.: Distributed robust H_∞ consensus control in directed networks of agents with time-delay. Syst. Control Lett. **57**(8), 643–653 (2008)
10. Chen, K., Wang, J., Zhang, Y., Liu, Z.: Consensus of second-order nonlinear multi-agent systems under state-controlled switching topology. Nonlinear Dyn. **4**(81), 1871–1878 (2015)
11. Meng, D., Jia, Y., Du, J.: Robust consensus tracking control for multiagent systems with initial state shifts, disturbances, and switching topologies. IEEE Trans. Neural Netw. Learn. Syst. **26**(4), 809–824 (2015)
12. Liu, K., Xie, G., Ren, W., Wang, L.: Consensus for multi-agent systems with inherent nonlinear dynamics under directed topologies. Syst. Control Lett. **62**(2), 152–162 (2013)
13. Olfati-Saber, R., Murray, R.M.: Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Autom. Control **49**(9), 1520–1533 (2004)
14. Kim, Y., Mesbahi, M.: On maximizing the second smallest eigenvalue of a state-dependent graph Laplacian. IEEE Trans. Autom. Control **51**(1), 116–120 (2006)
15. Xiao, L., Boyd, S.: Fast linear iterations for distributed averaging. Syst. Control Lett. **53**(1), 65–78 (2004)
16. Bhat, S.P., Bernstein, D.S.: Finite-time stability of continuous autonomous systems. SIAM J. Control Optim. **38**(3), 751–766 (2000)
17. Zhao, Y., Duan, Z., Wen, G., Zhang, Y.: Distributed finite-time tracking control for multi-agent systems: an observer-based approach. Syst. Control Lett. **62**(1), 22–28 (2013)
18. Zhao, L., Jia, Y., Yu, J.: Adaptive finite-time bipartite consensus for second-order multi-agent systems with antagonistic interactions. Syst. Control Lett. **102**, 22–31 (2017)
19. Wang, X., Hong, Y.: Finite-time consensus for multi-agent networks with second-order agent dynamics. In: Proceedings of the 17th IFAC World Congress, pp. 15185–15190 (2008)
20. Guan, Z.H., Sun, F.L., Wang, Y.W., Li, T.: Finite-time consensus for leader-following second-order multi-agent networks. IEEE Trans. Circuits Syst. I Regul. Pap. **59**(11), 2646–2654 (2012)
21. Zhao, L.W., Hua, C.C.: Finite-time consensus tracking of second-order multi-agent systems via nonsingular TSM. Nonlinear Dyn. **1**(75), 311–318 (2014)
22. Li, S., Du, H., Lin, X.: Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics. Automatica **47**(8), 1706–1712 (2011)
23. Andrieu, V., Praly, L., Astolfi, A.: Homogeneous approximation, recursive observer design, and output feedback. SIAM J. Control Optim. **47**(4), 1814–1850 (2008)
24. Polyakov, A.: Nonlinear feedback design for fixed-time stabilization of linear control systems. IEEE Trans. Autom. Control **57**(8), 2106–2110 (2012)
25. Parsegov, S.E., Polyakov, A.E., Shcherbakov, P.S.: Fixed-time consensus algorithm for multi-agent systems with integrator dynamics. In: Proceedings of the 4th IFAC Workshop on Distributed Estimation and Control in Networked Systems, pp. 110–115 (2013)
26. Zuo, Z., Tie, L.: A new class of finite-time nonlinear consensus protocols for multi-agent systems. Int. J. Control **87**(2), 363–370 (2014)
27. Zuo, Z.: Nonsingular fixed-time consensus tracking for second-order multi-agent networks. Automatica **54**, 305–309 (2015)
28. Cao, Y., Ren, W.: Finite-time consensus for multi-agent networks with unknown inherent nonlinear dynamics. Automatica **50**(10), 2648–2656 (2014)
29. Li, W., Chen, Z., Liu, Z.: Leader-following formation control for second-order multiagent systems with time-varying delay and nonlinear dynamics. Nonlinear Dyn. **4**(72), 803–812 (2013)
30. Du, H., He, Y., Cheng, Y.: Finite-time synchronization of a class of second-order nonlinear multi-agent systems using output feedback control. IEEE Trans. Circuits Syst. I Regul. Pap. **61**(6), 1778–1788 (2014)
31. Hong, Y., Huang, J., Xu, Y.: On an output feedback finite-time stabilization problem. IEEE Trans. Autom. Control **46**(2), 305–309 (2001)
32. Tian, B., Zuo, Z., Wang, H.: Leader-follower fixed-time consensus of multi-agent systems with high-order integrator dynamics. Int. J. Control **90**(7), 1420–1427 (2017)
33. Wen, G., Yu, Y., Peng, Z., Rahmani, A.: Distributed finite-time consensus tracking for nonlinear multi-agent systems with a time-varying reference state. Int. J. Syst. Sci. **47**(8), 1856–1867 (2016)
34. Wang, C., Wang, X., Ji, H.: Leader-following consensus for a class of second-order nonlinear multi-agent systems. Syst. Control Lett. **89**, 61–65 (2016)

35. You, K., Li, Z., Xie, L.: Consensus condition for linear multi-agent systems over randomly switching topologies. Automatica **49**(10), 3125–3132 (2013)
36. Shen, M., Park, J.H., Ye, D.: A separated approach to control of Markov jump nonlinear systems with general transition probabilities. IEEE Trans. Cybern. **46**(9), 2010–2018 (2016)