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ORIGINAL PAPER

Finite-time H_∞ stability analysis of uncertain network-based control systems under random packet dropout and varying network delay

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Abstract This paper studies the finite-time H_∞ stability analysis of uncertain network-based control systems under random data packet dropout and varying network delay. The system considered as an uncertain discrete-time stochastic system benefits from Bernoulli distributed white sequence technique for modeling the packet dropout. Sufficient conditions by means of a state feedback controller are derived to suppress the inherent effects of data packet dropout and network delay simultaneously. Simulations are provided to illustrate the feasibility and applicability of the control algorithm proposed in this paper.

Keywords Network-based control system · Finite-time stabilization · Network delay · Data packet dropout · State feedback

1 Introduction

Network-based control system (NCS) as one of the most well-known time delay systems has significant attributes, such as flexibility and less wiring. However, communication channel in the closed-loop system imposes new fundamental issues that make stability analysis of the system more challenging. From the control perspective, data packet dropout and network delay are two important inevitable topics, leading to the performance degradation or even instability of the NCSs [1–9]. Consequently, it is required to take into account these inherent problems in the study of controller design in such systems.

Data packet dropout and network delay can be modeled via different probability distributions, such as Bernoulli, and Markov process [10–14]. Therefore, different control algorithms were reported to address the stability problem of NCSs, for example [15–19]. In [15], the stability conditions of NCSs under external disturbances were analyzed. In [16], a state feedback controller for NCSs considering delays in the random nature was designed. In [17], a delay-dependent stability criterion satisfying a prescribed H_∞ norm bound for NCSs with unknown bounded varying delays by means of a state feedback controller was examined. In [18], designing state feedback controllers for stabilizing NCSs in the presence of network delay was studied. In [19], H_∞ stability analysis of the NCSs under data packet loss was addressed using a state feedback control law. Nevertheless, some works were reported for continuous-time NCSs that their results may not be applicable for discrete-time NCSs [20–22]. In [23,24], the network-induced characteristics were also neglected. In [25], the stability analysis of NCSs with fixed time delay was discussed. Besides, the stability and stabilization of continuous-time NCSs were investigated by lumping the network delay and data packet dropout into one item [22,26]. However,

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adopting such a lump sum induces some difficulty to distinguish the effects of packet dropout from network delay on the stabilization of NCSs. Consequently, the problem of stability analyze of discrete-time NCSs under these phenomena is still open area. Similarly to continuous-time NCSs, they can enforce strict limitations on the overall performance of discrete-time NCSs [27–29]. For this reason, designing an appropriate controller dealing with the performance of this kind of NCSs is required. Numerous results regarding to the stabilization of NCSs were reported using the Lyapunov asymptotic stability to address the status of the system over an infinite-time period. Nevertheless, the key topic in practice is to study the system status over a prescribed time period [30]. In other words, the states of the system under saturation cannot exceed a specified bound over a constant finite-time period. In such case, the conventional Lyapunov method is not therefore applicable. To handle the temporary behavior of the dynamical systems, Finite-Time Stability (FTS) concept should be utilized [31,32]. Because of the fast convergence and appropriate performance on the robustness, the FTS technique was extended to stabilization of the time delay systems [33–40]. In particular, stability problem of NCSs using this technique has received much attention over the past decades [41–49]. It is worth pointing out that there is a significant distinction between FTS and Lyapunov concepts. That is, a finite-time stable system may not be stable in the sense of Lyapunov, and vice versa [50].

The works developed for NCSs using FTS can be categorized in different aspects, such as constant or varying network delay, linearity or nonlinearity, and certain or uncertain of the plant model [46–49]. In [46], an iterative algorithm for NCSs under data packet dropout was introduced. In order to realize finite-time boundedness for the system under study, a state feedback controller was designed without considering model uncertainty and network delay. In [47], the problem of finite-time boundedness of NCS in the presence of varying delay by designing a state feedback controller was investigated. In [48], considering a state feedback controller, finite-time boundedness for one family of NCSs over networks under packet dropout and network delay was studied. However, model uncertainty was not taken into account in the controller design. In [49], finite-time stabilization problem of the NCSs under packet dropout was discussed. To address this problem, sufficient conditions were provided using a state feedback controller without focus on model uncertainty and network delay. Compared with other related works, the key motivations to this paper come from several sources as follows.

1. The first motivation is from the model uncertainty. The main shortcoming of the previous works is that they ignore the uncertainty in the modeling of NCSs, whereas it plays a significant factor responsible for the stability and performance of the data networks,
2. The second motivation arises from the nature of time-varying delays in NCSs. The network delay is naturally varying in real time networks. Here, both actuation and measurement delays resulting from network transmissions are assumed time-varying in nature,
3. There is an open area to investigate the stability analysis of uncertain NCSs in the presence of packet dropout and network delay simultaneously, which is the third motivation of the current research work.

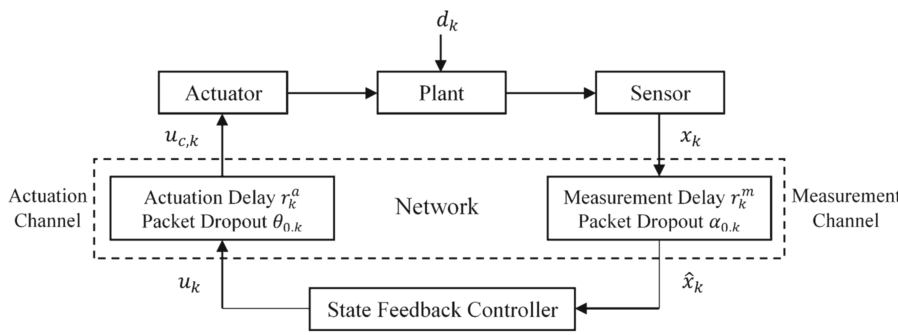
Motivated by the above discussions, this paper addresses the finite-time H_∞ stability analysis of uncertain NCSs considering uncertainty, time-varying network delay, random packet dropout, and norm-bounded disturbance. The structure of this paper is as follows. The system modeling is provided in Sect. 2. Stability analysis regarding to the uncertain NCSs is derived in Sect. 3. Simulations are provided to assess the feasibility of the control algorithm in Sect. 4. Eventually, Sect. 5 outlines the main conclusions.

Notation Throughout this paper, 0 and I are used to represent the zero and the identity matrices with compatible dimensions, $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}\left(\mathbb{R}^+\right)$ shows all real (non-negative) numbers set, $Prob$ is the probability measure, $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x, $l_2[0,\infty)$ denotes the space of square integrable vectors, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the smallest and largest eigenvalues of matrix P , respectively, and the sign $\times$ in a matrix stands for the symmetric part.

2 System modeling and prerequisites

Figure 1 shows the NCS framework. In this control framework, the controller uses the system's state data

Fig. 1 NCS framework



through the measurement channel under inherent phenomena of the network, including delay and data packet dropout in time k. Next, the control signal is transmitted to the system through the actuation channel considering network delay with together data packet dropout.

The mathematical model of the discrete-time NCS is

$$\begin{aligned} x_{k+1} &= (A+\delta A)\, x_k + (B_1+\delta B_1)\, d_k + (B_2+\delta B_2)\, u_{c,k}, \\ z_k &= (C_1+\delta C_1)\, x_k + (D_{11}+\delta D_{11})\, d_k + (D_{12} \\ &\quad + \delta D_{12})\, u_{c,k}, \\ x_k &= \varphi_k, \qquad \forall k \in [-r_M, 0]\,, \end{aligned} \tag{1}$$

where $x_k \in R^n$ and $u_{c,k} \in R^m$ are the system state and the control signal, respectively, $z_k \in R^r$ is the measured output, $A,\ B_1, B_2,\ C_1,\ D_{11}$ and D_{12} are considered to be known real constant matrices with compatible dimensions, and $\delta A,\ \delta B_1, \delta B_2,\ \delta C_1, \delta D_{11}$ and δD_{12} characterize the parameter uncertainties in the system to be in the form of

$$\begin{bmatrix} \delta A & \delta B_1 & \delta B_2 \\ \delta C_1 & \delta D_{11} & \delta D_{12} \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tilde{F}_k \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}, \tag{2}$$

where matrices $E_1,\ E_2,\ H_1,\ H_2$ and H_3 are assumed to be known real constant with compatible dimensions, and $\tilde{F}_k$ denotes an unknown real matrix which is time-varying such that $\tilde{F}_k^T \tilde{F}_k \leq I$. It is also supposed that $W_0 > 0$ exists such that

$$\tilde{F}_k^T\, W_0 \tilde{F}_k \leq W_0. \tag{3}$$

In addition, the following relationship for exogenous disturbance input $d_k \in R^q$ which belongs to $l_2[0, \infty)$ holds.

$$\sum_{k=0}^{N} d_k^T\, d_k \leq \overline{d}^2, \quad \overline{d} \geq 0, N \in \mathbb{N}. \tag{4}$$

Suppose that the state feedback control law is given by

$$u_k = K \hat{x}_k, \tag{5}$$

where K is the controller gain to be computed.
The measurement channel is expressed as

$$\hat{x}_k = \alpha_{0,k} x_{k-r_k^m}, \tag{6}$$

where r_k^m is the measurement delay which is time-varying, and $\alpha_{0,k} \in R$ as a random variable stands for the packet dropout.

$$Prob\left\{\alpha_{0,k} = 1\right\} = \mathbb{E}\left\{\alpha_{0,k}\right\} = \overline{\alpha_0}, \tag{7}$$
$$Prob\left\{\alpha_{0,k} = 0\right\} = 1 - \mathbb{E}\left\{\alpha_{0,k}\right\} = 1 - \overline{\alpha_0}. \tag{8}$$

Similarly, it is considered for the actuation channel

$$u_{c,k} = \theta_{0,k} u_{k-r_k^a}, \tag{9}$$

where r_k^a is the varying actuation delay, and $\theta_{0,k} \in R$, as random variable shows the packet dropout.
$\theta_{0,k}$ is assumed to be mutually independent of $\alpha_{0,k}$ in the form of Bernoulli distributed white sequences with the following relationships.

$$Prob\left\{\theta_{0,k} = 1\right\} = \mathbb{E}\left\{\theta_{0,k}\right\} = \overline{\theta_0}, \tag{10}$$
$$Prob\left\{\theta_{0,k} = 0\right\} = 1 - \mathbb{E}\left\{\theta_{0,k}\right\} = 1 - \overline{\theta_0}. \tag{11}$$

Combining Eqs. (5), (6) and (9), we obtain

$$u_{c,k} = \alpha_{0,k}\theta_{0,k} K x_{k-r_k^m-r_k^a} = \beta_{0,k} K x_{k-r_k}. \tag{12}$$

Lemma 1 *$\beta_{0,k}$ is a stochastic with Bernoulli distributed white sequence as*

$$Prob\left\{\beta_{0,k} = 1\right\} = \mathbb{E}\left\{\beta_{0,k}\right\} = \overline{\beta_0}, \tag{13}$$
$$Prob\left\{\beta_{0,k} = 0\right\} = 1 - \mathbb{E}\left\{\beta_{0,k}\right\} = 1 - \overline{\beta_0}. \tag{14}$$

Proof Because of the independence of variables$\alpha_{0,k}$ and $\theta_{0,k}$, we have

$$\mathbb{E}\left\{\beta_{0,k}\right\} = \mathbb{E}\left\{\alpha_{0,k}\theta_{0,k}\right\} = \overline{\alpha_0}.\overline{\theta_0} = \overline{\beta_0}, \tag{15}$$

$$\mathbb{E}\left\{\left(\beta_{0,k} - \overline{\beta_0}\right)^2\right\} = \mathbb{E}\left\{\left(\alpha_{0,k}\theta_{0,k}\right)^2\right\} - \overline{\alpha_0}^2.\ \overline{\theta_0}^2. \tag{16}$$

From there, it results

$$\begin{aligned}\mathbb{E}\left\{\left(\beta_{0,k} - \overline{\beta_0}\right)^2\right\} &= \mathbb{E}\left\{\alpha_{0,k}{}^2\right\}\mathbb{E}\left\{\theta_{0,k}{}^2\right\} - \overline{\alpha_0}^2.\overline{\theta_0}^2 \\ &= \overline{\alpha_0}.\ \overline{\theta_0} - \overline{\alpha_0}^2.\overline{\theta_0}^2.\end{aligned} \tag{17}$$

Using Eqs. (15) and (17) yields

$$\begin{aligned}&\mathbb{E}\left\{\beta_{0,k}\right\} = \overline{\beta_0}, \\ &\mathbb{E}\left\{\left(\beta_{0,k} - \overline{\beta_0}\right)^2\right\} = \overline{\beta_0}(1 - \overline{\beta_0}).\end{aligned} \tag{18}$$

Therefore, $\beta_{0,k}$ is a stochastic variable with Bernoulli distributed white sequence. Then, the resultant control input is written as

$$u_{c,k} = \beta_{0,k} K x_{k-r_k}. \tag{19}$$

It is worth mentioning that $\beta_{0,k} = 1$ reveals that the data packet is properly transmitted to the system and $\beta_{0,k} = 0$ indicates the data packet dropout. Combining Eqs. (1) and (19), the entire closed-loop system is expressed as

$$\begin{aligned}x_{k+1} &= (A + \delta A)\, x_k + (B_1 + \delta B_1)\, d_k \\ &\quad + \left(\beta_{0,k} - \overline{\beta_0}\right)(B_2 + \delta B_2)\, K x_{k-r_k} \\ &\quad + \overline{\beta_0}\,(B_2 + \delta B_2)\, K x_{k-r_k}, \\ z_k &= (C_1 + \delta C_1)\, x_k + (D_{11} + \delta D_{11})\, d_k \\ &\quad + \left(\beta_{0,k} - \overline{\beta_0}\right)(D_{12} + \delta D_{12}) K x_{k-r_k} \\ &\quad + \overline{\beta_0}\,(D_{12} + \delta D_{12})\, K x_{k-r_k}.\end{aligned} \tag{20}$$

Here, the states of the plants are available, and r_k^a and r_k^m are varying satisfying

$$0 < r_m \le r_k \le r_M. \tag{21}$$

In the following, the fundamental concepts, which are useful to derive the main results, are recalled. □

Definition 1 [51] The system (20) under $d_k = 0$ is stochastic finite-time stable with respect to ($\delta_x, \epsilon, \Gamma, N$), in which $0 < \delta_x < \epsilon$ and $\Gamma = \Gamma^T > 0$, if

$$\begin{aligned}&\mathbb{E}\left\{x_{k^*}^T\, \Gamma x_{k^*}\right\} \le \delta_x^2,\ \ \forall k^* \in [-r_M, 0] \\ &\to\ \ \mathbb{E}\left\{x_k^T \Gamma x_k\right\} < \epsilon^2,\ \ \forall k \in \{1, 2, \ldots, N\}.\end{aligned} \tag{22}$$

Definition 2 [51] The system (20) for all admissible exogenous nonzero disturbance under condition is said to be stochastic finite-time boundedness (SFTB) with respect to $\left(\delta_x, \epsilon, \Gamma, N, \overline{d}\right)$, in which $0 < \delta_x < \epsilon$ and $\Gamma = \Gamma^T > 0$, if

$$\begin{aligned}&\mathbb{E}\left\{x_{k^*}^T\, \Gamma x_{k^*}\right\} \le \delta_x^2,\ \ \forall k^* \in [-r_M, 0] \\ &\sum_{k=0}^{N} d_k^T\, d_k \le \overline{d}^2\ \to\ \ \mathbb{E}\left\{x_k^T \Gamma x_k\right\} < \epsilon^2, \\ &\forall k \in \{1, 2, \ldots, N\}.\end{aligned} \tag{23}$$

Definition 3 [51] The system (20) is stochastic finite-time stable with an H_∞ normal bound μ, (SHFTB) if the following conditions hold.

1. System (20) is SFTB.
2. Under zero initial conditions, the controlled output z_k holds

$$\mathbb{E}\left\{\sum_{k=0}^{N} z_k^T z_k\right\} < \mu^2 \mathbb{E}\left\{\sum_{k=0}^{N} d_k^T d_k\right\}. \tag{24}$$

Lemma 2 [52] *For a given symmetric matrix* $\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22}\end{bmatrix}$ *where* $X_{11} \in R^{p\times p}$, $X_{22} \in R^{q\times p}$, *and* $X_{12} \in R^{p\times q}$, *the following conditions are mutually equivalent*

1. $X < 0$,
2. $X_{11} < 0, \quad X_{22} - X_{12}^T X_{11}^{-1} X_{12} < 0$,
3. $X_{22} < 0, \quad X_{11} - X_{12} X_{22}^{-1} X_{12}^T < 0$.

3 Stability analysis

Here, first, criteria of the SFTB and SHFTB are provided in Theorems 1 and 2, respectively. Then, Theorem 3 illustrates the design procedure of the controller.

Theorem 1 *The system* (20) *is SFTB with respect to* $\left(\delta_x, \epsilon, \Gamma, N, \overline{d}\right)$ *if positive-definite matrices* P, Q_1, Q_2, Γ, T, $Z \in R^{n\times n}$, $W_1, W_2, W_3, \rho_0 \in R$, $U, S \in$

$R^{2n\times 2n}$, $L, M, F \in R^{2n\times n}$, *and scalars* $\lambda_0 > 1$, $\epsilon > 0$, *are existed such that the following relationships hold.*

1. $$\Sigma = \begin{bmatrix} \Sigma_1 & \times \\ \Sigma_2 & \Sigma_3 \end{bmatrix} \prec 0, \tag{25}$$

in which

$$\Sigma_1 = \begin{bmatrix} \Sigma_{11} & \times & \times & \times & \times & \times & \times & \times \\ 0 & -W_1 & \times & \times & \times & \times & \times & \times \\ \Sigma_{31} & 0 & \Sigma_{33} & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & -W_2 & \times & \times & \times & \times \\ M_1^T - F_1^T & 0 & M_2^T - F_2^T & 0 & -{\lambda_0}^{r_m} Q_2 & \times & \times & \times \\ -L_1^T & 0 & -L_2^T & 0 & 0 & -{\lambda_0}^{r_M} Q_1 & \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & -\rho_0 & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -W_3 \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} A & E_1 & \overline{\beta_0} B_2 K & \overline{\beta_0} E_1 & 0 & 0 & B_1 & E_1 \\ 0 & 0 & b B_2 K & b E_1 & 0 & 0 & 0 & 0 \\ (r_M - r_m)(A - I) & (r_M - r_m) E_1 & (r_M - r_m)\overline{\beta_0} B_2 K & (r_M - r_m)\overline{\beta_0} E_1 & 0 & 0 & (r_M - r_m) B_1 & (r_M - r_m) E_1 \\ 0 & 0 & (r_M - r_m) b B_2 K & (r_M - r_m) b E_1 & 0 & 0 & 0 & 0 \\ r_m (A - I) & r_m E_1 & r_m \overline{\beta_0} B_2 K & r_m \overline{\beta_0} E_1 & 0 & 0 & r_m B_1 & r_m E_1 \\ 0 & 0 & r_m b B_2 K & r_m b E_1 & 0 & 0 & 0 & 0 \\ W_1 H_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_3 K & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_3 H_2 & 0 \end{bmatrix},$$

$\Sigma_3 = diag(-P^{-1}, -bP^{-1}, -(r_M - r_m)T^{-1}, -b(r_M - r_m)T^{-1}, -r_m Z^{-1}, -br_m Z^{-1}, -W_1, -W_2^{-1}, -W_3)$,

$\Sigma_{11} = -\lambda_0 P + Q_1 + Q_2 + F_1 + F_1^T + r_m S_1 + (r_M - r_m) U_1$, $\Sigma_{31} = L_1^T - M_1^T + F_2 + r_m S_2^T + (r_M - r_m) U_2^T$,

$\Sigma_{33} = L_2 + L_2^T - M_2 - M_2^T + r_m S_3 + (r_M - r_m) U_3$,

$b = \mathbb{E}\left\{\left(\beta_0(k) - \overline{\beta_0}\right)^2\right\} = \left(1 - \overline{\beta_0}\right)\overline{\beta_0}$,

2. $$\begin{aligned} &\{\lambda_{\max}\left(\tilde{P}\right) + {\lambda_0}^{r_M - 1} r_M \lambda_{\max}(\tilde{Q}_1) \\ &+ {\lambda_0}^{r_m - 1} r_m \lambda_{\max}(\tilde{Q}_2)\}\,\delta_x^2 \\ &+ \{{\lambda_0}^{r_M - 1} \lambda_{\max}(\tilde{T}) \frac{r_M(r_M - 1) - r_m(r_m - 1)}{2} \\ &+ {\lambda_0}^{r_m - 1} \lambda_{\max}(\tilde{Z}) \frac{r_m(r_m - 1)}{2}\}\delta_y^2 \\ &+ \lambda_{\max}(\rho_0)\,\overline{d}^2 \le {\lambda_0}^{-N} \lambda_{\min}\left(\tilde{P}\right)\epsilon^2, \end{aligned} \tag{26}$$

3. $$\Phi_i \succ 0, \quad i = 1, 2, 3 \tag{27}$$

$$\Phi_1 = \begin{bmatrix} U & L \\ \times & {\lambda_0}^{r_m + 1} T \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} U & M \\ \times & {\lambda_0}^{r_m + 1} T \end{bmatrix},$$

$$\Phi_3 = \begin{bmatrix} S & F \\ \times & \lambda_0 Z \end{bmatrix}$$

$\tilde{P} = \Gamma^{-1/2} P \Gamma^{-1/2}$, $\tilde{Q}_1 = \Gamma^{-1/2} Q_1 \Gamma^{-1/2}$, $\tilde{Q}_2 = \Gamma^{-1/2} Q_2 \Gamma^{-1/2}$, $\tilde{T} = \Gamma^{-1/2} T \Gamma^{-1/2}$, $\tilde{Z} = \Gamma^{-1/2} Z \Gamma^{-1/2}$,

$$U = \begin{bmatrix} U_1 & U_2 \\ \times & U_3 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & S_2 \\ \times & S_3 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

Proof Let us define that $\tilde{y}_k = x_{k+1} - x_k$ satisfying $\tilde{y}_k^T \tilde{y}_k \le \delta_y^2$ for $k \in [-r_M, -1]$. The Lyapunov–Krasovskii-like functional is constructed as

$$V_k = V_{1,k} + V_{2,k} + V_{3,k},$$

where

$$V_{1,k} = x_k^T P x_k, \tag{28}$$

$$V_{2,k} = \sum_{i=k-r_M}^{k-1} {\lambda_0}^{k-1-i} x_i^T Q_1 x_i + \sum_{i=k-r_m}^{k-1} {\lambda_0}^{k-1-i} x_i^T Q_2 x_i, \tag{29}$$

$$V_{3,k} = \sum_{j=-r_M}^{-r_m-1} \sum_{i=k+j}^{k-1} {\lambda_0}^{k-1-i} \tilde{y}_i^T T \tilde{y}_i + \sum_{j=-r_m}^{-1} \sum_{i=k+j}^{k-1} {\lambda_0}^{k-1-i} \tilde{y}_i^T Z \tilde{y}_i. \tag{30}$$

Denote

$$\xi_{0,k} = \begin{bmatrix} x_k^T & x_{k-r_k}^T \end{bmatrix}^T,$$
$$\zeta_{0,k} = \begin{bmatrix} x_k^T & x_{k-r_k}^T & \tilde{y}_s^T \end{bmatrix}^T,$$
$$\eta_{0,k} = \begin{bmatrix} x_k^T & x_k^T H_1^T \tilde{F}_k^T & x_{k-r_k}^T & x_{k-r_k}^T K^T H_3^T \tilde{F}_k^T & x_{k-r_m}^T x_{k-r_M}^T & d_k^T & d_k^T H_2^T \tilde{F}_k^T \end{bmatrix}^T.$$

Considering $\lambda_0 > 1$, the difference of V_k is given by

$$\mathbb{E}\left\{V_{1,k+1}\right\} - \lambda_0 \mathbb{E}\left\{V_{1,k}\right\} = x_{k+1}^T P x_{k+1} - \lambda_0 x_k^T P x_k, \tag{31}$$

$$\begin{aligned}\mathbb{E}\left\{V_{2,k+1}\right\} - \lambda_0 \mathbb{E}\left\{V_{2,k}\right\} = &\sum_{i=k+1-r_M}^{k} {\lambda_0}^{k-i} x_i^T Q_1 x_i - \sum_{i=k-r_M}^{k} {\lambda_0}^{k-i} x_i^T Q_1 x_i \\ &+ \sum_{i=k+1-r_m}^{k} {\lambda_0}^{k-i} x_i^T Q_2 x_i - \sum_{i=k-r_m}^{k-1} {\lambda_0}^{k-i} x_i^T Q_2 x_i.\end{aligned} \tag{32}$$

After some calculating, we can easily obtain

$$\begin{aligned}&\mathbb{E}\left\{V_{2,k+1}\right\} - \lambda_0 \mathbb{E}\left\{V_{2,k}\right\} \\ &= x_k^T Q_1 x_k - {\lambda_0}^{r_M} x_{k-r_M}^T Q_1 x_{k-r_M} \\ &\quad + x_k^T Q_2 x_k \\ &\quad - {\lambda_0}^{r_m} x_{k-r_m}^T Q_2 x_{k-r_m}.\end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned}&\mathbb{E}\left\{V_{3,k+1}\right\} - \lambda_0 \mathbb{E}\left\{V_{3,k}\right\} \\ &= \sum_{j=-r_M}^{-r_m-1} \sum_{i=k+1+j}^{k} {\lambda_0}^{k-i} \tilde{y}_i^T T \tilde{y}_i - \sum_{j=-r_M}^{-r_m-1} \sum_{i=k+j}^{k-1} {\lambda_0}^{k-i} \tilde{y}_i^T T \tilde{y}_i \\ &\quad + \sum_{j=-r_m}^{-1} \sum_{i=k+1+j}^{k} {\lambda_0}^{k-i} \tilde{y}_i^T Z \tilde{y}_i - \sum_{j=-r_m}^{-1} \sum_{i=k+j}^{k-1} {\lambda_0}^{k-i} \tilde{y}_i^T Z \tilde{y}_i.\end{aligned} \tag{34}$$

From $\lambda_0 > 1$, it yields

$$\begin{aligned}\mathbb{E}\left\{V_{3,k+1}\right\} - \lambda_0 \mathbb{E}\left\{V_{3,k}\right\} \le\ &(r_M - r_m)\, \tilde{y}_k^T\ (k)\, T \tilde{y}_k - {\lambda_0}^{r_m+1} \sum_{j=k-r_M}^{k-r_m-1} \tilde{y}_j^T T \tilde{y}_j \\ &+ r_m \tilde{y}_k^T Z \tilde{y}_k - \lambda_0 \sum_{j=k-r_m}^{k-1} \tilde{y}_j^T Z \tilde{y}_j.\end{aligned} \tag{35}$$

$$\begin{aligned}\mathbb{E}\left\{V_{k+1}\right\} - \lambda_0 \mathbb{E}\left\{V_k\right\} \le\ &x_{k+1}^T P x_{k+1} - \lambda_0 x_k^T P x_k \\ &+ x_k^T Q_1 x_k \\ &- {\lambda_0}^{r_M} x_{k-r_M}^T Q_1 x_{k-r_M} + x_k^T Q_2 x_k \\ &- {\lambda_0}^{r_m} x_{k-r_m}^T Q_2 x_{k-r_m} \\ &+ (r_M - r_m)\, \tilde{y}_k^T T \tilde{y}_k - {\lambda_0}^{r_m+1} \sum_{j=k-r_M}^{k-r_k-1} \tilde{y}_j^T T \tilde{y}_j \\ &- {\lambda_0}^{r_m+1} \sum_{j=k-r_k}^{k-r_m-1} \tilde{y}_j^T T \tilde{y}_j + r_m \tilde{y}_k^T Z \tilde{y}_k \\ &- \lambda_0 \sum_{j=k-r_m}^{k-1} \tilde{y}_j^T Z \tilde{y}_j \\ &+ 2\xi_{0,k}^T L \left[x_{k-r_k} - x_{k-r_M} - \sum_{j=k-r_M}^{k-r_k-1} \tilde{y}_j \right] \\ &+ 2\xi_{0,k}^T M \left[x_{k-r_m} - x_{k-r_k} - \sum_{j=k-r_k}^{k-r_m-1} \tilde{y}_j \right] \\ &+ 2\xi_{0,k}^T F \left[x_k - x_{k-r_m} - \sum_{j=k-r_m}^{k-1} y_j \right].\end{aligned} \tag{36}$$

Since

$$r_m \xi_{0,k}{}^T S \xi_{0,k} - \sum_{j=k-r_m}^{k-1} \xi_{0,k}{}^T S \xi_{0,k} = 0, \quad (37)$$

$$(r_M - r_m)\, \xi_{0,k}^T U \xi_{0,k} - \sum_{j=k-r_M}^{k-r_k-1} \xi_{0,k}^T U \xi_{0,k} - \sum_{j=k-r_k}^{k-r_m-1} \xi_{0,k}^T U \xi_{0,k} = 0. \quad (38)$$

Combining Eqs. (37) and (38) into (36), and then adding and removing the expressions $x_k^T H_1^T \tilde{F}_k^T W_1 \tilde{F}_k H_1 x_k$, $d_k^T H_2^T \tilde{F}_k^T W_3 \tilde{F}_k H_2 d_k$, and $x_{k-r_k}^T K^T H_3^T \tilde{F}_k^T W_2 \tilde{F}_k H_3 K x_{k-r_k}$ into Eq. (36), it follows from Eq. (3) that

$$\begin{aligned}\mathbb{E}\{V_{k+1}\} - \lambda_0 \mathbb{E}\{V_k\} &< \eta_{0,k}^T \Omega_0 \eta_{0,k} \\ &- \sum_{j=k-r_M}^{k-r_k-1} \zeta_{0,k}^T \Phi_1 \zeta_{0,k} \\ &- \sum_{j=k-r_k}^{k-r_m-1} \zeta_{0,k}^T \Phi_2 \zeta_{0,k} \\ &- \sum_{j=k-r_m}^{k-1} \zeta_{0,k}^T \Phi_3 \zeta_{0,k}.\end{aligned} \quad (39)$$

If $\Omega < 0$, and $\Phi_i \geq 0$, then

$$\mathbb{E}\{V_{k+1} - \lambda_0 V_k\} < 0. \quad (40)$$

Since $\rho_0 > 0$, therefore

$$\mathbb{E}\{V_{k+1} - \lambda_0 V_k\} < d_k^T \rho_0 d_k. \quad (41)$$

Using Lemma 2, and applying Eqs. (39), (40) and (41), the inequalities (25) and (27) are satisfied. Hence, we get

$$\mathbb{E}(V_{k+1}) < \lambda_0 \mathbb{E}(V_k) + \lambda_{\max}(\rho_0)\, d_k^T d_k. \quad (42)$$

Applying Eqs. (41) and (42) and considering $\lambda_0 > 1$, it concludes that

$$\begin{aligned}\mathbb{E}\{V_k\} &< \lambda_0{}^k \mathbb{E}\{V_0\} + \lambda_{\max}(\rho_0) \sum_{i=0}^{k-1} \mathbb{E}\left\{\lambda_0{}^{k-i-1} d_i^T d_i\right\} \\ &\leq \lambda_0^k \mathbb{E}\{V_0\} + \lambda_{\max}(\rho_0)\, \lambda_0^k \overline{d}^2.\end{aligned} \quad (43)$$

It is straightforward to obtain that

$$\begin{aligned}\mathbb{E}\left\{V_{1,0}\right\} &= x_0^T P x_0 \leq \lambda_{\max}\left(\tilde{P}\right) \mathbb{E}\left\{x_0^T\, \Gamma x_0\right\} \\ &\leq \lambda_{\max}\left(\tilde{P}\right) \delta_x^2,\end{aligned} \quad (44)$$

$$\begin{aligned}\mathbb{E}\left\{V_{2,0}\right\} &= \sum_{i=-r_M}^{-1} \lambda_0{}^{-1-i} x_i^T Q_1 x_i + \sum_{i=-r_m}^{-1} \lambda_0{}^{-1-i} x_i^T Q_2 x_i \\ &\leq \lambda_0{}^{r_M-1} \lambda_{\max}(\tilde{Q}_1) \sum_{i=-r_M}^{-1} x_i^T \Gamma x_i \\ &+ \lambda_0{}^{r_m-1} \lambda_{\max}(\tilde{Q}_2) \sum_{i=-r_m}^{-1} x_i^T \Gamma x_i,\end{aligned} \quad (45)$$

$$\begin{aligned}\mathbb{E}\{V_3(0)\} &\leq \lambda_0^{r_M-1} \lambda_{\max}(\tilde{T}) \sum_{j=-r_M}^{-r_m-1} \sum_{i=j}^{-1} \tilde{y}_i^T \tilde{y}_i \\ &+ \lambda_0^{r_m-1} \lambda_{\max}(\tilde{Z}) \sum_{j=-r_m}^{-1} \sum_{i=j}^{-1} \tilde{y}_i^T \tilde{y}_i,\end{aligned} \quad (46)$$

$$\begin{aligned}\mathbb{E}\{V_0\} &= \mathbb{E}\left\{\sum_{i=1}^{3} V_{i,0}\right\} \leq \{\lambda_{\max}\left(\tilde{P}\right) \\ &+ \lambda_0{}^{r_M-1} r_M \lambda_{\max}(\tilde{Q}_1) \\ &+ \lambda_0{}^{r_m-1} r_m \lambda_{\max}(\tilde{Q}_2)\}\, \delta_x^2 \\ &+ \{\lambda_0{}^{r_M-1} \lambda_{\max}(\tilde{T}) \frac{r_M(r_M-1) - r_m(r_m-1)}{2} \\ &+ \lambda_0{}^{r_m-1} \lambda_{\max}(\tilde{Z}) \frac{r_m(r_m-1)}{2}\}\delta_y^2.\end{aligned} \quad (47)$$

From Eqs. (43) and (47), we obtain

$$\begin{aligned}\mathbb{E}\{V_k\} &< \{\lambda_{\max}\left(\tilde{P}\right) + \lambda_0{}^{r_M-1} r_M \lambda_{\max}(\tilde{Q}_1) \\ &+ \lambda_0{}^{r_m-1} r_m \lambda_{\max}(\tilde{Q}_2)\}\lambda_0{}^k\, \delta_x^2 \\ &+ \{\lambda_0{}^{r_M-1} \lambda_{\max}(\tilde{T}) \frac{r_M(r_M-1) - r_m(r_m-1)}{2} \\ &+ \lambda_0{}^{r_m-1} \lambda_{\max}(\tilde{Z}) \frac{r_m(r_m-1)}{2}\}\lambda_0^k \delta_y^2 \\ &+ \lambda_{\max}(\rho_0)\, \lambda_0^k \overline{d}^2, \qquad \forall k = 1, \ldots, N\end{aligned} \quad (48)$$

where $\tilde{P} = \Gamma^{-\frac{1}{2}} P\, \Gamma^{-\frac{1}{2}}$, $\tilde{Q}_1 = \Gamma^{-\frac{1}{2}} Q_1\, \Gamma^{-\frac{1}{2}}$, $\tilde{Q}_2 = \Gamma^{-\frac{1}{2}} Q_2\, \Gamma^{-\frac{1}{2}}$, $\tilde{T} = \Gamma^{-\frac{1}{2}} T\, \Gamma^{-\frac{1}{2}}$, $\tilde{Z} = \Gamma^{-\frac{1}{2}} Z\, \Gamma^{-\frac{1}{2}}$. From there,

$$\mathbb{E}\{V_k\} = \mathbb{E}\left\{\sum_{i=1}^{3} V_{i,k}\right\} \geq \mathbb{E}\{V_{1,k}\}$$
$$\geq \lambda_{\min}\left(\tilde{P}\right)\mathbb{E}\left\{x_k^T\,\Gamma x_k\right\}. \tag{49}$$

Using Eqs. (48)–(49), the following inequality can be obtained.

$$\mathbb{E}\left\{x_k^T\,\Gamma x_k\right\} \leq {\lambda_0}^k \frac{\Xi}{\lambda_{\min}\left(\tilde{P}\right)} \leq \epsilon^2, \tag{50}$$

where

$$\begin{aligned}\Xi = &\{\lambda_{\max}\left(\tilde{P}\right) + {\lambda_0}^{r_M-1} r_M \lambda_{\max}(\tilde{Q}_1) \\ &+ {\lambda_0}^{r_m-1} r_m \lambda_{\max}(\tilde{Q}_2)\}\,\delta_x^2 \\ &+ \{{\lambda_0}^{r_M-1}\lambda_{\max}(\tilde{T})\frac{r_M(r_M-1) - r_m(r_m-1)}{2} \\ &+ {\lambda_0}^{r_m-1}\lambda_{\max}(\tilde{Z})\frac{r_m(r_m-1)}{2}\}\delta_y^2 + \lambda_{\max}(\rho_0)\,\overline{d}^2.\end{aligned}$$

From Eq. (26), it yields that $\mathbb{E}\left\{x_k^T \Gamma x_k\right\} \leq \epsilon^2$, $k = 1, \ldots, N$. Therefore, the system (20) is SFTB with respect to $\left(\delta_x,\ \epsilon,\ \Gamma, N, \overline{d}\right)$. □

Remark 1 In the Lyapunov–Krasovskii-like functional used in Theorem 1, the variable ratios ${\lambda_0}^{k-1-i}$ are utilized, whereas there is no required inequality enlargement to obtain $\Delta V_k \leq (\lambda_0 - 1)V_k$. Compared with [53], V is enlarged by $\Delta V_k < (\lambda_0 - 1)x_k^T P x_k = (\lambda_0 - 1)V_{1,k} < (\lambda_0 - 1)V_k$, which indicates that the use of our method contains more information of the system states leading to less conservative stability. However, if the terms $-{\lambda_0}^{r_m+1}\sum_{j=k-r_M}^{k-r_m-1}\tilde{y}_j^T T\tilde{y}_j$ and $-\lambda_0\sum_{j=k-r_m}^{k-1}\tilde{y}_j^T Z\tilde{y}_j$ are ignored, the conservatism is unavoidable. Therefore, Eqs. (37) and (38) as well as the free-weighting matrices U, S, L, M and F are introduced in order to avoid such treatments. In the following, sufficient condition is derived for stochastic H_∞ finite-time stability of the system (20).

Theorem 2 *The system (20) is SHFTB with respect to* $\left(\delta_x, \epsilon, \Gamma, N, \overline{d}, \mu\right)$, *if positive-definite matrices* $P, Q_1, Q_2, \Gamma, T, Z \in R^{n\times n}$, $W_1, W_2, W_3 \in R$, $U, S \in R^{2n\times 2n}$ *and matrices* $L, M, F \in R^{2n\times n}$ *and scalars* $\epsilon, \mu > 0$ *and* $\lambda_0 > 1$, *are existed such that*

1. $$\Lambda = \begin{bmatrix}\Lambda_1 & \times \\ \Lambda_2 & \Lambda_3\end{bmatrix} \prec 0, \tag{51}$$

in which

$$\Lambda_1 = \begin{bmatrix}
\Sigma_{11} & \times & \times & \times & \times & \times & \times & \times \\
0 & -W_1 & \times & \times & \times & \times & \times & \times \\
\Sigma_{31} & 0 & \Sigma_{33} & \times & \times & \times & \times & \times \\
0 & 0 & 0 & -W_2 & \times & \times & \times & \times \\
M_1^T - F_1^T & 0 & M_2^T - F_2^T & 0 & -{\lambda_0}^{r_m} Q_2 & \times & \times & \times \\
-L_1^T & 0 & -L_2^T & 0 & 0 & -{\lambda_0}^{r_M} Q_1 & \times & \times \\
0 & 0 & 0 & 0 & 0 & 0 & -\mu^2{\lambda_0}^{-N} I & \times \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -W_3
\end{bmatrix},$$

$$\Lambda_2 = \begin{bmatrix}
A & E_1 & \overline{\beta}_0 B_2 K & \overline{\beta}_0 E_1 & 0 & 0 & B_1 & E_1 \\
0 & 0 & bB_2K & bE_1 & 0 & 0 & 0 & 0 \\
(r_M - r_m)(A - I) & (r_M - r_m)E_1 & (r_M - r_m)\overline{\beta}_0 B_2 K & (r_M - r_m)\overline{\beta}_0 E_1 & 0 & 0 & (r_M - r_m)B_1 & (r_M - r_m)E_1 \\
0 & 0 & (r_M - r_m)bB_2K & (r_M - r_m)bE_1 & 0 & 0 & 0 & 0 \\
r_m(A - I) & r_m E_1 & r_m\overline{\beta}_0 B_2 K & r_m\overline{\beta}_0 E_1 & 0 & 0 & r_m B_1 & r_m E_1 \\
0 & 0 & r_m bB_2K & r_m bE_1 & 0 & 0 & 0 & 0 \\
C_1 & E_2 & \overline{\beta}_0 D_{12} K & \overline{\beta}_0 E_2 & 0 & 0 & D_{11} & E_2 \\
0 & 0 & bD_{12}K & bE_2 & 0 & 0 & 0 & 0 \\
W_1 H_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H_3 K & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & W_3 H_2 & 0
\end{bmatrix},$$

$$\Lambda_3 = diag(-P^{-1}, -bP^{-1}, -(r_M - r_m)T^{-1}, -b(r_M - r_m)T^{-1}, -r_m Z^{-1}, -br_m Z^{-1},$$
$$-I, -bI, -W_1, -W_2^{-1}, -W_3),$$

2. $$\Big\{\lambda_{\max}\left(\tilde{P}\right) + {\lambda_0}^{r_M-1} r_M \lambda_{\max}(\tilde{Q}_1) + {\lambda_0}^{r_m-1} r_m \lambda_{\max}(\tilde{Q}_2)\Big\}\, \delta_x^2 + \{{\lambda_0}^{r_M-1}\lambda_{\max}(\tilde{T})\frac{r_M(r_M-1)-r_m(r_m-1)}{2} + {\lambda_0}^{r_m-1}\lambda_{\max}(\tilde{Z})\frac{r_m(r_m-1)}{2}\}\delta_y^2 + {\lambda_0}^{-N}\mu^2\overline{d}^2 \le {\lambda_0}^{-N}\lambda_{\min}\left(\tilde{P}\right)\epsilon^2, \quad (52)$$
3. $\Phi_i \succ 0, \quad i = 1, 2, 3$ (53)

where Σ_{11}, Σ_{31}, Σ_{33}, $\tilde{P}$, $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{T}$, $\tilde{Z}$, U, S, L, M, F, Φ_i and b are similar to Theorem 1.

Proof According to Theorem 1, the system (20) is SFTB. Therefore, for any admissible nonzero d_k, we have

$$\mathbb{E}\{V_{k+1}\} - \lambda_0\mathbb{E}\{V_k\} + \mathbb{E}\left\{z_k^T z_k\right\} - \mathbb{E}\left\{d_k^T \rho_0 d_k\right\} = \eta_{0,k}^T \Theta \eta_{0,k}. \quad (54)$$

Let define $\rho_0 = \mu^2{\lambda_0}^{-N} I$. It follows from Eq. (51) and Lemma 2 that $\Theta < 0$ and hence

$$\mathbb{E}\{V_{k+1}\} < \lambda_0\mathbb{E}\{V_k\} - \mathbb{E}\left\{z_k^T z_k\right\} + \mu^2{\lambda_0}^{-N}\mathbb{E}\left\{d_k^T d_k\right\}. \quad (55)$$

From Eq. (55), it is apparent that

$$\mathbb{E}\{V_k\} < {\lambda_0}^k\mathbb{E}\{V_0\} - \sum_{j=0}^{k-1}{\lambda_0}^{k-j-1}\mathbb{E}\left\{z_j^T z_j\right\} + \mu^2{\lambda_0}^{-N}\sum_{j=0}^{k-1}{\lambda_0}^{k-j-1}\mathbb{E}\left\{d_j^T d_j\right\}. \quad (56)$$

Considering zero initial conditions and using $V_k \ge 0$, we obtain

$$\sum_{j=0}^{k-1}{\lambda_0}^{k-j-1}\mathbb{E}\left\{z_j^T z_j\right\} \le \mu^2{\lambda_0}^{-N}\sum_{j=0}^{k-1}{\lambda_0}^{k-j-1}\mathbb{E}\left\{d_j^T d_j\right\}. \quad (57)$$

From Eq. (57) and using $\lambda_0 > 1$, we get

$$\sum_{j=0}^{N}\mathbb{E}\left\{z_j^T z_j\right\} \le \sum_{j=0}^{N}{\lambda_0}^{N-j}\mathbb{E}\left\{z_j^T z_j\right\} \le \mu^2{\lambda_0}^{-N}\sum_{j=0}^{N}{\lambda_0}^{N-j}\mathbb{E}\left\{d_j^T d_j\right\} \le \mu^2\sum_{j=0}^{N}\mathbb{E}\left\{d_j^T d_j\right\}, \quad (58)$$

and therefore

$$\sum_{j=0}^{N}\mathbb{E}\left\{z_j^T z_j\right\} \le \mu^2\sum_{j=0}^{N}\mathbb{E}\left\{d_j^T d_j\right\}.$$

which indicates the SHFTB of the system (20) is achieved. □

Theorem 3 *The system (20) with the controller $u_k = K\hat{x}_k$ given in (5) is SHFTB with respect to $(\delta_x, \epsilon, \Gamma, N, \overline{d}, \mu)$, if positive-definite matrices P, Q_1, Q_2, Γ, T, Z, X, $R_1, R_2 \in R^{n\times n}$, $W_1, W_2, W_3, V \in R$, $U, S \in R^{2n\times 2n}$, matrices $L, M, F \in R^{2n\times n}$, and scalars $\mu, \epsilon > 0$ existed such that*

1. $$\Lambda = \begin{bmatrix}\Lambda_1 & \times \\ \Lambda_2 & \Lambda_3\end{bmatrix} \prec 0, \quad (59)$$

in which

$$\Lambda_1 = \begin{bmatrix}
\Sigma_{11} & \times & \times & \times & \times & \times & \times & \times \\
0 & -W_1 & \times & \times & \times & \times & \times & \times \\
\Sigma_{31} & 0 & \Sigma_{33} & \times & \times & \times & \times & \times \\
0 & 0 & 0 & -W_2 & \times & \times & \times & \times \\
M_1^T - F_1^T & 0 & M_2^T - F_2^T & 0 & -{\lambda_0}^{r_m} Q_2 & \times & \times & \times \\
-L_1^T & 0 & -L_2^T & 0 & 0 & -{\lambda_0}^{r_M} Q_1 & \times & \times \\
0 & 0 & 0 & 0 & 0 & 0 & -\mu^2{\lambda_0}^{-N} I & \times \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -W_3
\end{bmatrix},$$

$$\Lambda_2 = \begin{bmatrix} A & E_1 & \overline{\beta_0} B_2 K & \overline{\beta_0} E_1 \; 0 \; 0 & B_1 & E_1 \\ 0 & 0 & b B_2 K & b E_1 \; 0 \; 0 & 0 & 0 \\ (r_M - r_m)(A - I) & (r_M - r_m) E_1 & (r_M - r_m)\overline{\beta_0} B_2 K & (r_M - r_m)\overline{\beta_0} E_1 \; 0 \; 0 & (r_M - r_m) B_1 & (r_M - r_m) E_1 \\ 0 & 0 & (r_M - r_m) b B_2 K & (r_M - r_m) b E_1 \; 0 \; 0 & 0 & 0 \\ r_m (A - I) & r_m E_1 & r_m \overline{\beta_0} B_2 K & r_m \overline{\beta_0} E_1 \; 0 \; 0 & r_m B_1 & r_m E_1 \\ 0 & 0 & r_m b B_2 K & r_m b E_1 \; 0 \; 0 & 0 & 0 \\ C_1 & E_2 & \overline{\beta_0} D_{12} K & \overline{\beta_0} E_2 \; 0 \; 0 & D_{11} & E_2 \\ 0 & 0 & b D_{12} K & b E_2 \; 0 \; 0 & 0 & 0 \\ W_1 H_1 & 0 & 0 & 0 \; 0 \; 0 & 0 & 0 \\ 0 & 0 & H_3 K & 0 \; 0 \; 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \; 0 \; 0 & W_3 H_2 & 0 \end{bmatrix},$$

$$\Lambda_3 = \mathrm{diag}(-X, -bX, -(r_M - r_m) R_1, -b(r_M - r_m) R_1, -r_m R_2, -b\, r_m R_2, -I, -bI, -W_1, -V, -W_3),$$

2. $$\{\lambda_{\max}\left(\tilde{P}\right) + {\lambda_0}^{r_M - 1} r_M \lambda_{\max}(\tilde{Q}_1) + {\lambda_0}^{r_m - 1} r_m \lambda_{\max}(\tilde{Q}_2)\}\, \delta_x^2 + \{{\lambda_0}^{r_M - 1} \lambda_{\max}(\tilde{T}) \frac{r_M(r_M - 1) - r_m(r_m - 1)}{2} + {\lambda_0}^{r_m - 1} \lambda_{\max}(\tilde{Z}) \frac{r_m(r_m - 1)}{2}\} \delta_y^2 + {\lambda_0}^{-N} \mu^2 \overline{d}^2 \le {\lambda_0}^{-N} \lambda_{\min}\left(\tilde{P}\right) \epsilon^2, \tag{60}$$
3. $$\Phi_i \succ 0, \quad i = 1, 2, 3 \tag{61}$$

where $\tilde{P}$, $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{T}$, $\tilde{Z}$, U, S, L, M, F, Φ_i and b are similar to Theorem 1.

Proof According to Lemma 2 and Theorem 2 and also denoting $P^{-1} = X$, $T^{-1} = R_1$, $Z^{-1} = R_2$, and $W_2^{-1} = V$, we can prove easily the results mentioned above on the stochastic H_∞ finite-time stability. □

Remark 2 Equation (59) is a nonlinear and cannot be solved through the standard LMI Toolbox. One way to handle this problem is to use the cone complementarity linearization algorithm (CCLM) as follows. The non-convex problem (59) is considered as a minimization problem with LMI constraints as follows.

$Minimize\ Trace\ (XP + R_1 T + R_2 Z + V W_2)$
subject to Eqs. (59)–(61) and

$$\begin{bmatrix} X & I \\ I & P \end{bmatrix} \ge 0, \quad \begin{bmatrix} R_1 & I \\ I & T \end{bmatrix} \ge 0, \quad \begin{bmatrix} R_2 & I \\ I & Z \end{bmatrix} \ge 0, \quad \begin{bmatrix} V & I \\ I & W_2 \end{bmatrix} \ge 0. \tag{62}$$

For solving above optimization problem, an algorithm is provided, which can be itemized as follows.

Step 1 Initialize the maximum number of iteration $\bar{N}$, iteration accuracy ε_0 and constant values δ_x, Γ, N, $\overline{d}$ and λ_0.

Step 2 Select an initial value of ϵ.

Step 3 Compute feasible points P_0, X_0, T_0, $R_{1,0}$, Z_0, $R_{2,0}$, $W_{2,0}$, V_0 satisfying (59)–(62), whereas $P_k = P_0$, $X_k = X_0, T_k = T_0, Z_k = Z_0, W_{2,k} = W_{2,0}$, $V_k = V_0$, $R_{1,k} = R_{1,0}$, $R_{2,k} = R_{2,0}$. In the case of they are none, exit. Set $k = 0$.

Step 4 Find P_{k+1}, X_{k+1}, T_{k+1}, $R_{1,k+1}$, Z_{k+1}, $R_{2,k+1}$, $W_{2,k+1}$, V_{k+1}. Then, the LMI problem is solved as follows.

$Minimize\ Trace\ \left(\vartheta + \mu^2\right)$ subject to Eqs. (59)–(62)

where

$$\vartheta = X_{k+1} P_k + X_k P_{k+1} + R_{1,k+1} T_k + R_{1,k} T_{k+1} + R_{2,k+1} Z_k + R_{2,k}\,(k)\, Z_{k+1} + V_{k+1} W_{2,k} + V_k W_{2,k+1}.$$

Step 5 If $|Trace\vartheta - (6n + 2)| < \varepsilon_0$ holds, then exit. Else, set $k \leftarrow k + 1$ and return to Step 2.

Step 6 In the case of $k > \overline{N}$, go to stop.

Step 7 If the problem is unfeasible, then it is required to increase ϵ. Else, decreasing ϵ till getting the its minimum value.

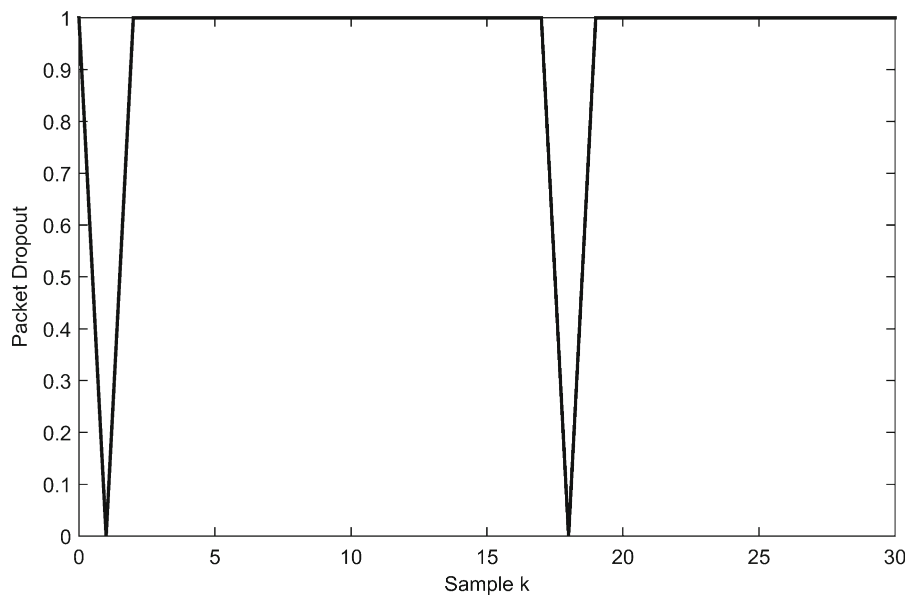


Fig. 2 Packet dropout probability with $\overline{\beta_0} = 0.9$

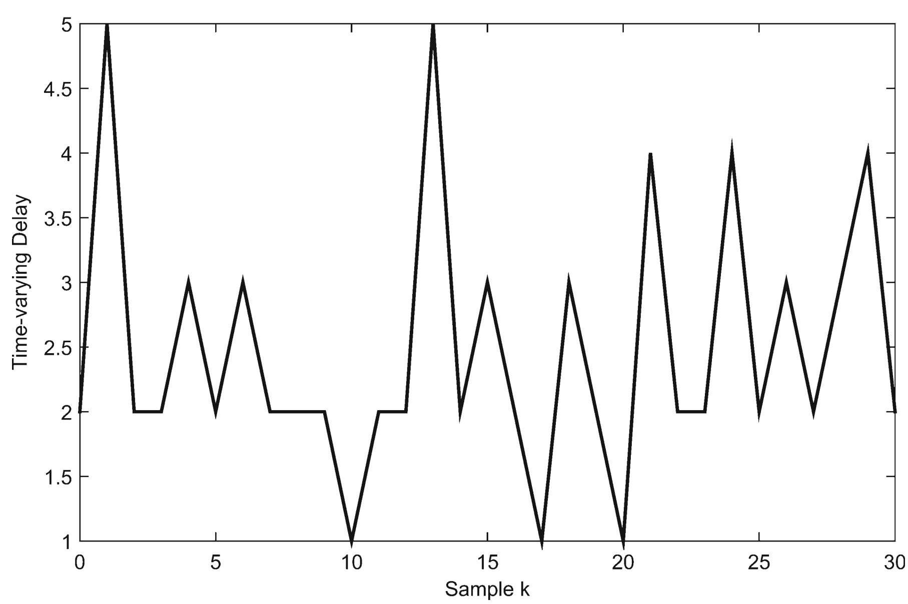


Fig. 3 Network delay with $1 \leq r_k \leq 5$

4 Application to uninterruptible power supply

Here, an uninterruptible power supply (UPS) as a practical example is adopted for elevation of the control algorithm. The network-based control problem for this kind of UPS is studied for keeping the AC voltage of the output at the desirable situation. When the signal of control is transmitted via the communication network, the inherent phenomena, that is random packet dropout and varying network delay, can degrade the system performance or even cause instability of the system. Based on this, the goal is to cope with these inherent phenomena by designing a state feedback controller (5) to achieve finite-time stabilization of the uncertain NCS with the H_∞ prescribed disturbance level. Here, the following uncertain model of the UPS is adopted from

Fig. 4 State trajectories of the NCS with $1 \leq r_k \leq 5$ and $\overline{\beta_0} = 0.9$

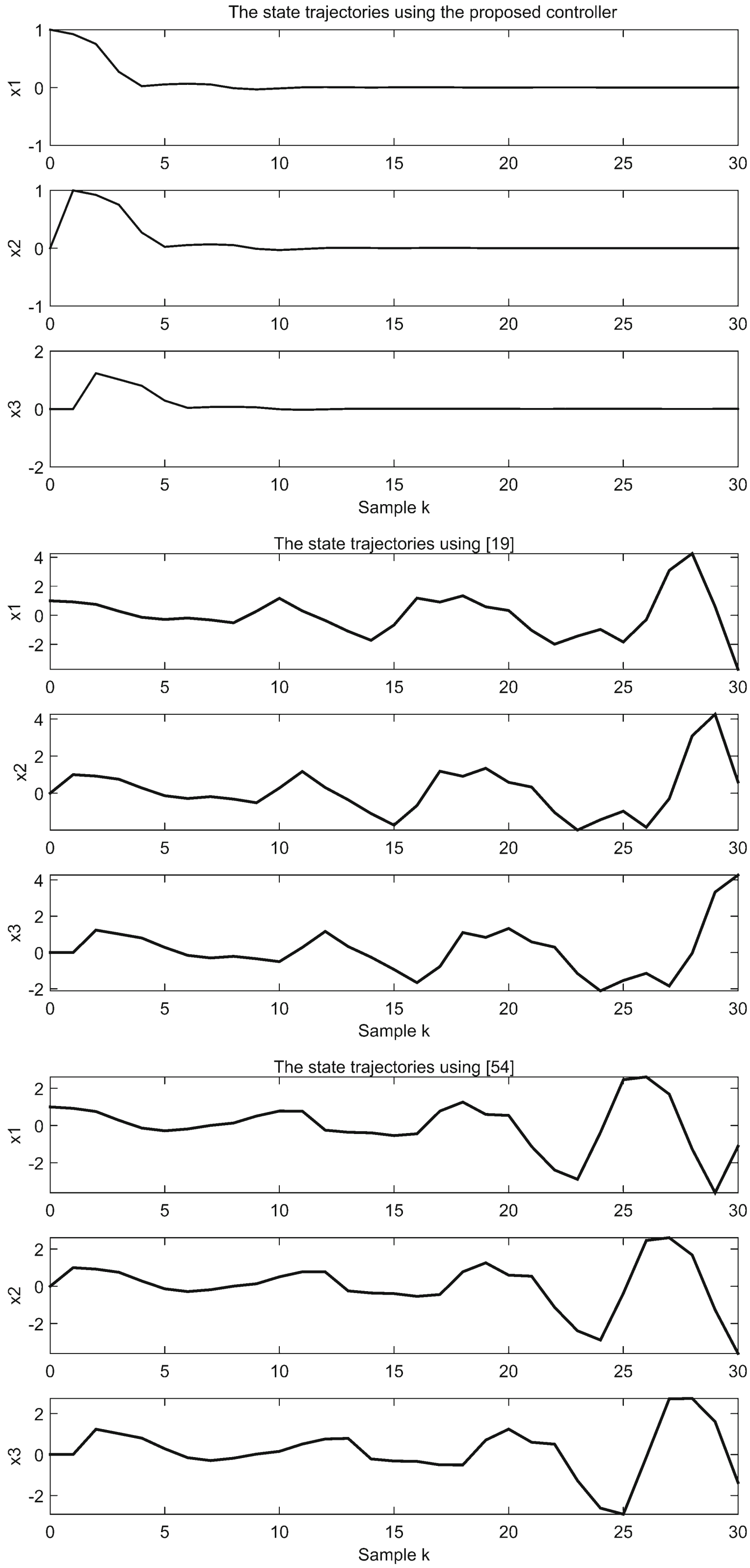
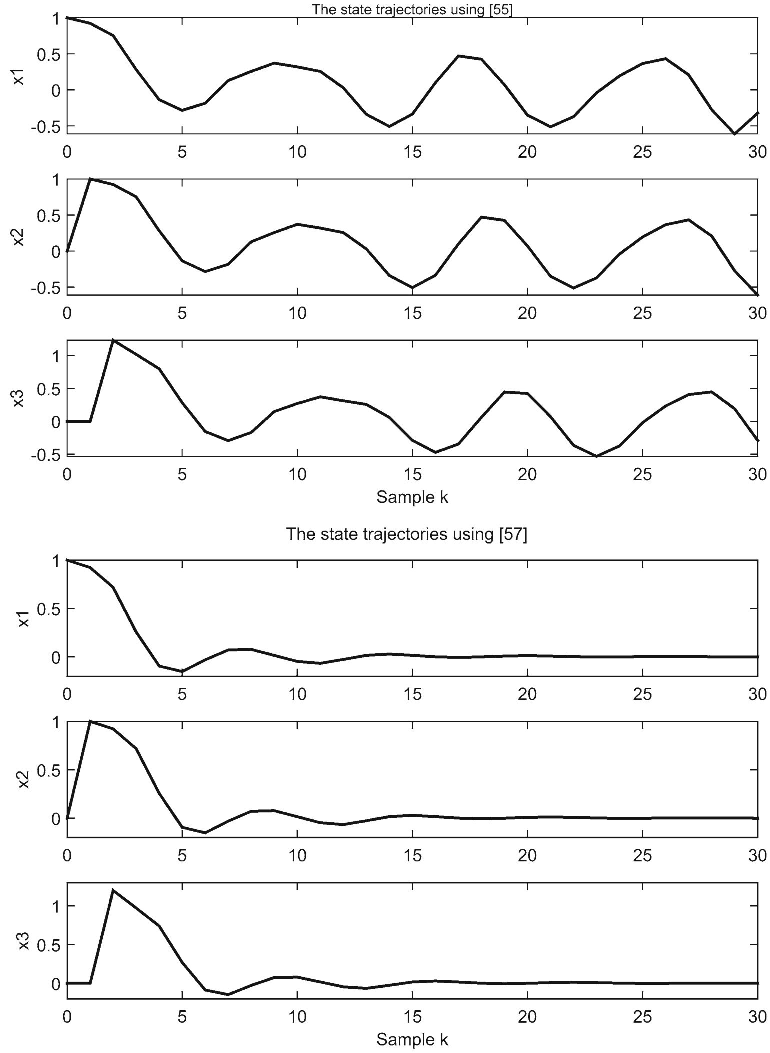


Fig. 4 continued



[19,54,55].

$$A = \begin{bmatrix} 0.9226 & -0.6330 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.5 \\ 0 \\ 0.2 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix}, \; D_{11} = 0.1, \quad D_{12} = 0.2,$$
$$E_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; E_2 = 0.2, \;\; H_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \;\; H_2 = 0.2,$$
$$H_3 = 0.1. \tag{63}$$

In the simulation, the control cost and power consumption of controller is also estimated, which is an appropriate criterion to measure the cost of controller in a theoretical way [56]. The power consumption denoted by p is computed by

$$p = |VI| = |x_1 x_3|. \tag{64}$$

The average power consumption of the controller can be estimated as

$$\overline{p}_3 = \sum_{i=1}^{N} p_{3,i}/N, \tag{65}$$

where $p_3 = p_2 - p_1$, in which p_1 and p_2 are the output power consumption from the uncontrolled and controlled NCS, respectively.

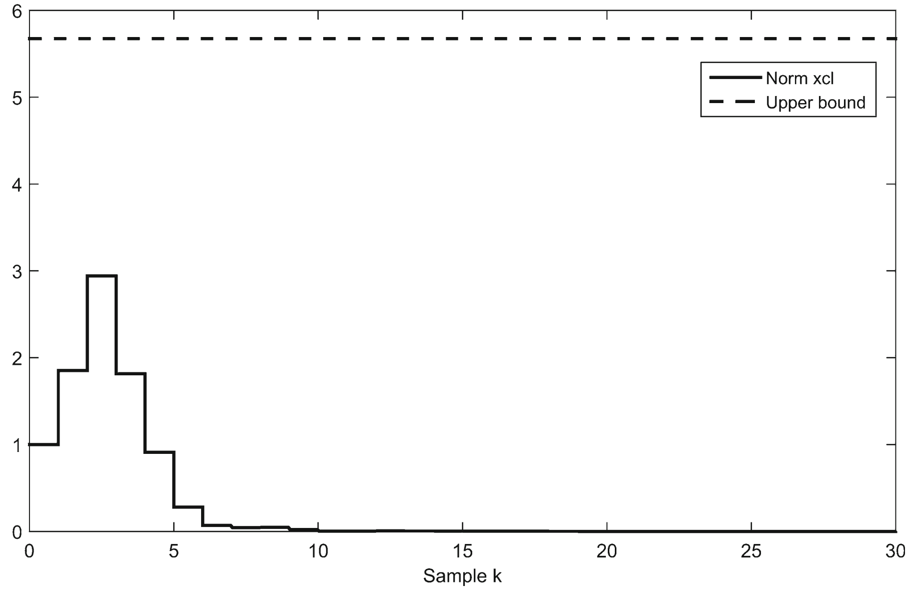


Fig. 5 Norm x_k with $1 \leq r_k \leq 5$ and $\overline{\beta_0} = 0.9$

Table 1 H_∞ norm bounded μ for different r_M

r_M	5	10	15	20
$\mu_{\min}$	0.685	0.701	0.702	0.707

Table 2 H_∞ norm bounded μ for different $\overline{\beta_0}$

$\overline{\beta_0}$	0.9	0.8	0.7	0.6
$\mu_{\min}$	0.685	0.687	0.691	0.696

In order to assess the feasibility of the designed controller, simulations are performed to compare it with other existing techniques including [19,54,55,57]. The settings are $\lambda_0 = 1.001$, $\delta_x = 1$, $\Gamma =$ I, N = 10, $\overline{d} = 1$, $1 \leq r_k \leq 5$, and $\overline{\beta_0} = 0.9$ with the initial values $x_0 = \left[1\ 0\ 0\right]^T$, $x = 0$ for $k \in [-5, -1]$, $d_k = \frac{1}{k^2}$. Solving the minimization problem given in Eqs. (59)–(62), we get $\mu_{\min} = 0.685$ and $\epsilon_{\text{opt}}^2 = 5.674$. The resulting controller is expressed as $K = [-0.0111\ \ 0.179\ \ -0.124]$. The corresponding controllers used for [19,54,55]

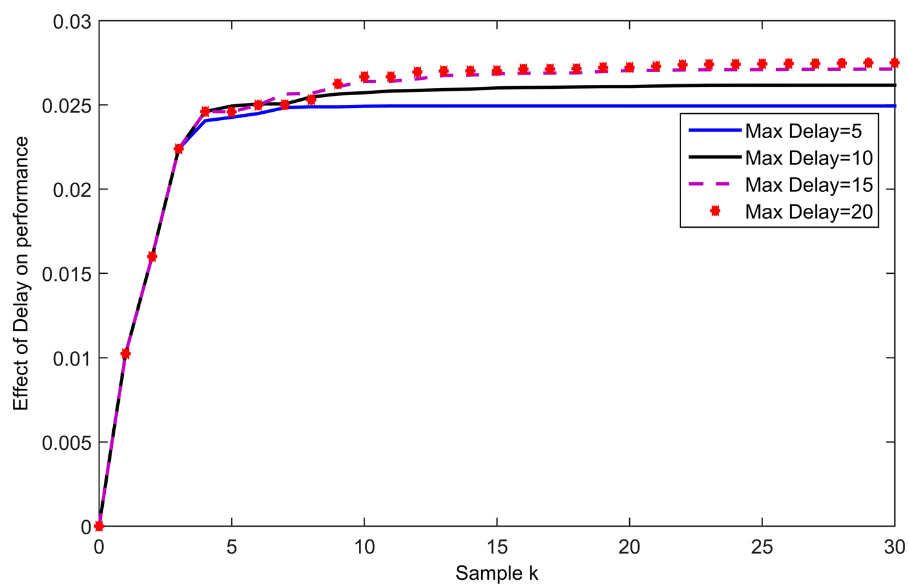


Fig. 6 Effect of upper bound of delay on H_∞ performance using the proposed controller

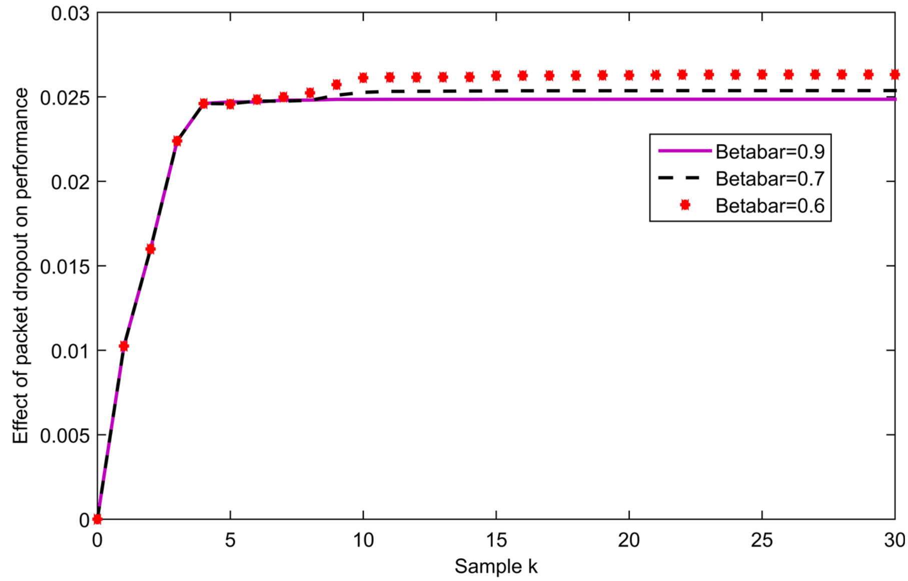


Fig. 7 Effect of packet dropout on H_∞ performance using the proposed controller

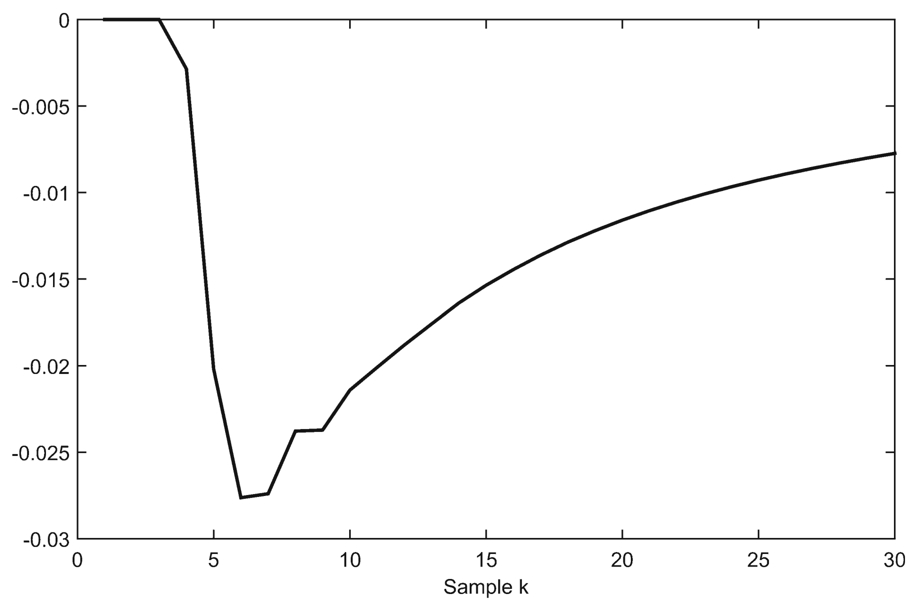


Fig. 8 Average power consumption of the proposed controller

are $\left[1.1154 - 0.6931\ 0.0007\right]$, $[-0.3291\ \ 0.2676\ -\ 0.0210]$, and $\left[-0.5960\ 0.5549\ -0.1587\right]$, respectively. The packet dropout probability and network delays are represented in Figs. 2 and 3, respectively. Figure 4 illustrates the state trajectories of the system, whereas it exhibits that all the states of the system using the proposed controller converge to zero. The results show that other methods corresponding to [19,54,55] cannot stabilize the system. Referring to Fig. 4, it can be observed that the performance of the NCS using the proposed approach is able to control the system with high convergence speed, small overshoot and high control precision. In particular, in comparison with the control method reported in [57], the proposed approach has outstanding performance in terms of settling time and accuracy. The norm x_k of the resultant system regarding to the controller proposed in this paper is also provided in Fig. 5. From Fig. 5, it can be inferred that the states trajectories stay within a given upper bound, which implies the entire system is SHFTB. Compared with the other relevant works, our results incorporate the network delays happening in both the actuation

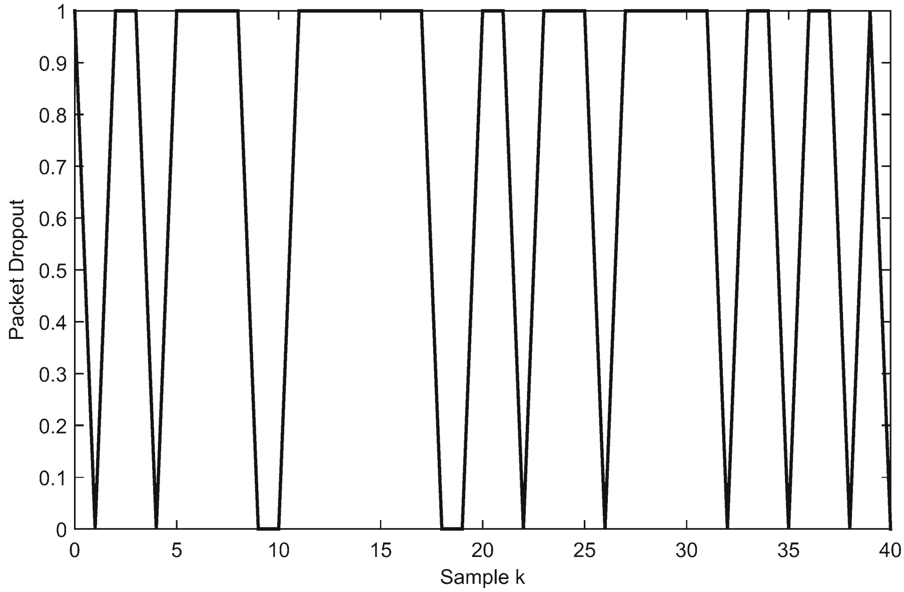


Fig. 9 Packet dropout probability with $\overline{\beta_0} = 0.8$

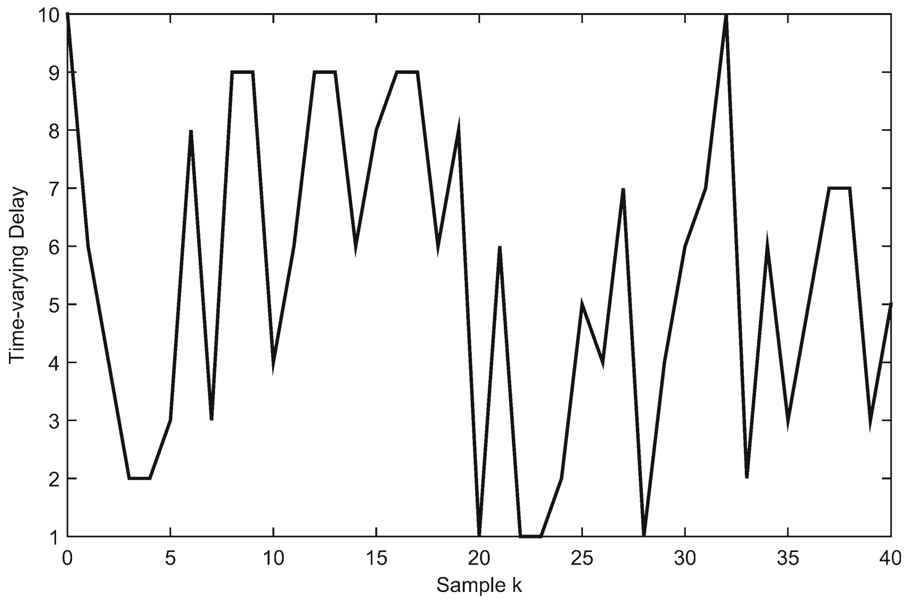


Fig. 10 Network-induced delay with$1 \leq r_k \leq 10$

and measurement channels as well as the data packet dropout. Furthermore, the proposed controller can cope with the system uncertainties appropriately.

In the sequel, the proposed controller is implemented and tested under different values of r_M and $\overline{\beta_0}$. Tables 1 and 2 list a quantitative comparison of the H_∞ performance metric of the NCS using the proposed controller. The effects of different values r_M and $\overline{\beta_0}$ on the H_∞ performance are also illustrated in Figs. 6 and 7. Results represent that increasing the upper bound of varying delay as well as decreasing the packet dropout results in decreasing the H_∞ system performance. Figure 8 shows the average power consumption of the proposed controller. Besides, in order to further assessment of the control method, we consider $1 \leq r_k \leq 10$ and $\overline{\beta_0} = 0.8$, whereas other parameters are chosen based on the values represented above. Using the proposed procedure, we have $\epsilon^2_{\text{opt}} = 6.86$. Figures 9, 10, 11 and 12 show the results. We observe that the settling time of the response is longer and minimum value of the

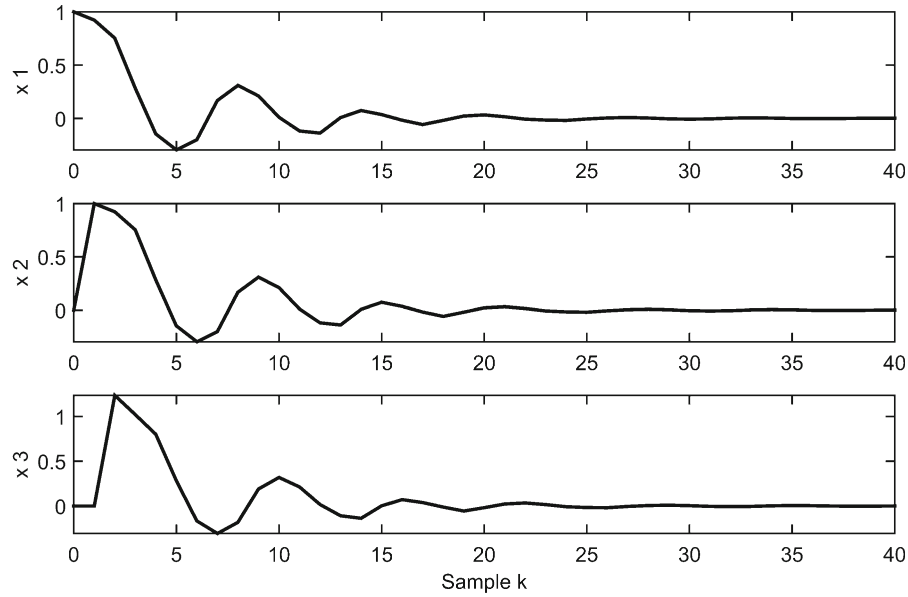


Fig. 11 State trajectories of the NCS with $1 \leq r_k \leq 10$ and $\overline{\beta_0} = 0.8$

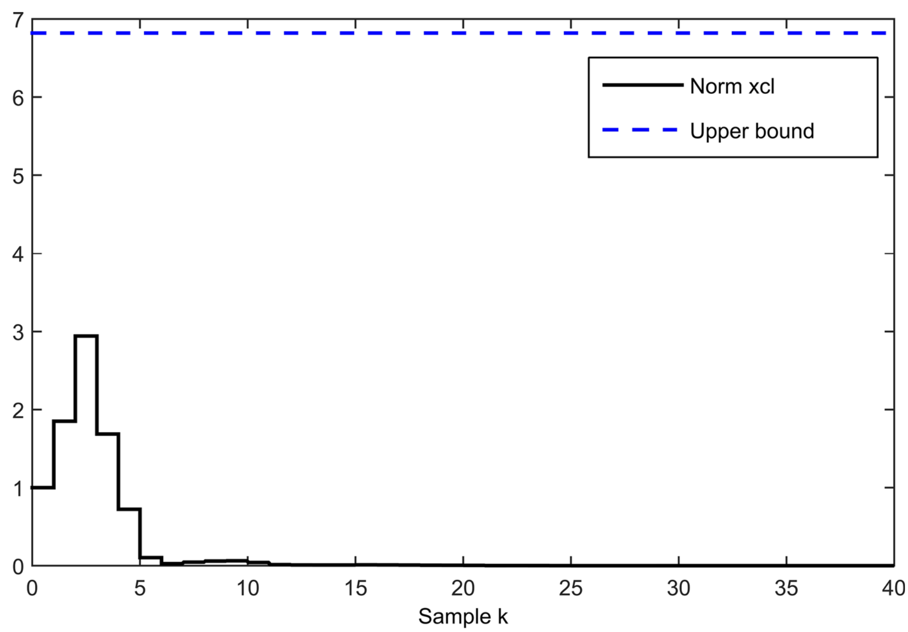


Fig. 12 Norm x_k of the system with $1 \leq r_k \leq 10$ and $\overline{\beta_0} = 0.8$

norm bounded is much larger than the previous results. In general, it can be inferred that the proposed control strategy can deal with the effects of network delay and packet dropout properly.

5 Conclusions

In this paper, we studied FTS analysis for a class of uncertain NCSs under data packet dropout and network delay simultaneously. Here, both actuation and measurement delays were varying and the data packet dropout was modeled by the independent Bernoulli distributed white sequence. Sufficient conditions by designing the state feedback controller were derived with the help of LMI approach. Results verified features of the proposed control strategy and its ability to address control challenges in the NCSs.

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