Nonlinear Dyn (2017) 90:2199–2207
DOI 10.1007/s11071-017-3795-4

ORIGINAL PAPER

Sampled-data-based lag synchronization of chaotic delayed neural networks with impulsive control

Xiaoyu Zhang · Xiaoxiao Lv · Xiaodi Li

Received: 13 April 2017 / Accepted: 30 August 2017 / Published online: 18 September 2017

Abstract In the framework of sampled-data control, this paper deals with the lag synchronization of chaotic neural networks with time delay meanwhile taking the impulsive control into account. By constructing a proper Lyapunov function and employing the impulsive control theory, some sufficient conditions for lag synchronization of the addressed chaotic neural networks are derived in terms of linear matrix inequalities (LMIs). The hybrid controller including sampled-data controller and impulsive controller is designed based on the established LMIs. A numerical example is provided to demonstrate the effectiveness and advantage of the obtained results.

Keywords Sampled-data control · Impulsive control · Lag synchronization · Time delay · Neural networks · Linear matrix inequality (LMI)

This work was supported by National Natural Science Foundation of China (11301308, 61673247) and the Research Fund for Distinguished Young Scholars and Excellent Young Scholars of Shandong Province (ZR201702100145, JQ201719). The paper has not been presented at any conference.

X. Zhang · X. Lv · X. Li (✉)
School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, People's Republic of China
e-mail: lxd@sdnu.edu.cn

X. Li
Institute of Data Science and Technology, Shandong Normal University, Jinan 250014, People's Republic of China

1 Introduction

Synchronization is a common and widespread phenomenon in many science and engineering problems [1]. Since Pecora and Carrol [2] initially proposed the master–slave concept for chaos synchronization, chaos synchronization has been investigated intensively due to their potential applications in many fields such as secure communication, finance, biology, and engineering, see [3–5]. The basic idea of the synchronization is to control the slave system such that the state trajectories of the master and salve systems are finally identical. A wide variety of synchronization phenomenon has been extensively studied such as complete synchronization [6], projective synchronization [7], phase synchronization [8], anticipated synchronization [9], and lag synchronization [10,11]. In particular, lag synchronization is characterized as the coincidence of the states of two coupled systems in which one of the systems is delayed by a given finite time. Many experimental study and computer simulations of chaotic synchronization in unidirectionally coupled external cavity semiconductor lasers have shown the presence of lag time between the drive and response laser's intensities. Moreover, it has been shown that time delays are ubiquitous in neural processing or signal transmission, for example, the release of neurotransmitters, state estimations, computation and implementation of continuous or discontinuous control forces, signal feedback and transmission [12,13]. During the past decades, a lot of efforts have been made to the study of lag synchronization of chaotic systems and neural networks with or without time delays [14–16]. Many approaches have been proposed such as adaptive control [17], fuzzy control [18], and impulsive control [19].

It is well known that the synchronization of chaotic neural networks has attracted great attention and has been extensively studied due to their potential applications in secure communication, parallel recognition, etc. [19–26]. It has been revealed that artificial neural networks can exhibit chaotic behaviors if the network's parameters, initial states, and time delays are appropriately chosen. To date, different methods have been proposed for synchronization control for delayed neural networks. In Ref. [22], authors have studied the problem of synchronization and parameters identification for chaotic neural networks with time delays by employing Lyapunov–Krasovskii functional method. In Ref. [19], the lag synchronization of neural networks with time delay has been dealt with by using impulsive control method coupled with LMIs. It is well known that impulsive control, which is an important kind of discontinuous control methods, has been used to synchronize and stabilize lots of dynamic systems [27–31]. The main idea of the impulsive control is to change the state of a system by discontinuous control input at certain time instances. Impulsive control is of distinctive advantage, from the control point of view, since only small control gains are needed at discrete instances. Until now, there are many interesting results for stabilization or synchronization control of neural networks via impulsive control [19,29–33]. In [29], authors investigated the complete synchronization of complex dynamical networks with time-varying delays via impulsive control of the distributed form. Based on dual-stage impulsive control method [32], the robust global exponential synchronization of uncertain chaotic delayed neural networks was studied in which different parametric uncertainties were addressed. Ref. [33] studied the exponential synchronization of Lurie complex neural networks with uncertain coupling strength via pinning impulsive control in which only selection of nodes is needed to control instead of the whole networks.

Since many practical problems are characterized by continuous systems, for which one usually either directly designs continuous-time controllers or extends the continuous systems to discrete systems and then designs the corresponding discrete time controllers. However, from practical applications point of view, sampled-data controller is much more favorable in many applications and admits many advantages over the continuous controller such as low-cost maintenance, reasonable structure, and high reliability. For instance, the continuous controller always occupies the communication channels all the time, but sampled-data one only needs to use the communication channels at the sampling instants. Thus, the sampled-data control theory has received much attention in recent years [34–37]. Based on sampled-data feedback control, Ref. [35] studied the exponential synchronization of neural networks with mixed delays using Lyapunov–Krasovskii functional combining with improved free-weighting matrix approach. Recently, authors in Ref. [36] have discussed the sampled-data synchronization for Markovian neural networks with generally incomplete transition rates in which the transition rate may be completely unknown or only its estimate value is known. In Ref. [37], authors proposed two novel approaches, sampling-instant-to-present-time fragmentation and free-matrix-based time-dependent discontinuous Lyapunov approach, to the problem of sampled-data synchronization of two identical chaotic Lur'e systems. Although all those references have presented some interesting sampled-data control strategies [34–37] or impulsive control strategies [19,29,32,33] for synchronization control of neural networks with various time delays, only single controller is addressed, that is, the designed controller is either sampled-data control or impulsive control. It is obvious desirable to design a hybrid controller which includes both sampled-data control and impulsive control, which is very useful in practice for the case that the effects of synchronization control are not so well via a single controller.

Inspired by the above discussion, in this paper we shall study the lag synchronization for chaotic neural networks with time delay via a class of hybrid controllers which include sampled-data controller and impulsive controller. In particular, when the sampled-data part is given in the form of the external perturbations, the lag synchronization can still be achieved via the designed impulsive controller. The rest of the paper is organized as follows. In Sect. 2, we present some notations and preliminaries. In Sect. 3, we present the control laws to ensure the lag synchronization of chaotic neural networks with time delay via hybrid control. A numerical example and its simulation are given to verify our theoretical results in Sect. 4. Concluding remarks are given in Sect. 5.

2 Preliminaries

Notations Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Z}_+$ the set of positive integers, $\mathbb{R}^n$ the n-dimensional real spaces equipped with the Euclidean norm $|\bullet|$, and $\mathbb{R}^{n\times m}$ the $n\times m$-dimensional real spaces. $\mathcal{A}>0(<0)$ denotes that the matrix $\mathcal{A}$ is a symmetric and positive (negative) definite matrix. The notations $\mathcal{A}^{-1}$ and $\mathcal{A}^T$ mean the inverse and the transpose of matrix $\mathcal{A}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are symmetric matrices, $\mathcal{A}>\mathcal{B}$ means that $\mathcal{A}-\mathcal{B}$ is positive definite matrix. I means the identity matrix with appropriate dimensions and $\Lambda=\{1,\ 2,\ldots,\ n\}$. For any interval $J\subseteq\mathbb{R}$, set $E\subseteq\mathbb{R}^k(1\le k\le n)$, $PC(J,\ E)=\{\varphi: J\to\ E$ is continuous everywhere except at finite number of instances t, at which $\varphi(t^+),\varphi(t^-)$ exist and $\varphi(t^+)=\varphi(t)\}$. $\star$ denotes the symmetric block in a symmetric matrix. Consider the following neural networks with time delay:

$$\begin{cases}\dot{x}(t)=-Cx(t)+Af_1(x(t))\\ \qquad +Bf_2(x(t-\tau))+J,\quad t>0,\\ x(s)=\phi(s),\quad s\in[-\tau,0],\end{cases}\tag{1}$$

where $\phi(\cdot)\in PC([-\tau,\ 0],\ \mathbb{R}^n)$; $x(t)=(x_1(t),\ \ldots,\ x_n(t))^T$ is the neuron state of the neural networks; $C=\mathrm{diag}(c_1,\ \ldots,\ c_n)$ is a diagonal matrix with $c_i>0,\ i=1,\ \ldots,\ n$; A and B are the connection weight and the delayed weight matrices, respectively; J is an external input; $\tau>0$ is the transmission constant delay; $f_i(x(\cdot))=(f_{i1}(x_1(\cdot)),\ \ldots,\ f_{in}(x_n(\cdot)))^T$ represents the neuron activation function satisfying

$$l_{ij}^-\le\frac{f_{ij}(u)-f_{ij}(v)}{u-v}\le l_{ij}^+,\quad i,j\in\Lambda\tag{2}$$

for any $u,\ v\in\mathbb{R},\ u\neq v$, where l_{ij}^- and l_{ij}^+ are some real constants which may be positive, zero, or negative and $f_{ij}(0)=0$ for $i,j\in\Lambda$.

To investigate the lag synchronization via control input, we consider system (1) as the drive system and the corresponding response system is given as follows:

$$\begin{cases}\dot{y}(t)=-Cy(t)+Af_1(y(t))+Bf_2(y(t-\tau))+J+u,\\ y(s)=\varphi(s),\ s\in[-\tau+\sigma,\ \sigma].\end{cases}\tag{3}$$

Lag synchronization is described by $y(t)=x(t-\sigma)$ for some constant $\sigma>0$. To derive the lag synchronization conditions, let $e(t)=y(t)-x(t-\sigma)$ be the lag synchronization error, where σ is response time delay between the drive system and response system. We design the feedback controller $u(t)$ by

$$u(t)=K_1e(t_k)+K_2e(t)\delta(t-t_{k+1}),\ t\in[t_k,\ t_{k+1}),\ k\in\mathbb{Z}_+,$$

where $K_1,\ K_2$ are control gains to be designed, $\delta(\cdot)$ is the Diral delta function, $\varphi(\cdot)\in PC([-\tau+\sigma,\ \sigma],\ \mathbb{R}^n)$. Then, system (3) can be rewritten as

$$\begin{cases}\dot{y}(t)=-Cy(t)+Af_1(y(t))+Bf_2(y(t-\tau))+J\\ \qquad +K_1e(t_k),\quad t\in[t_k,\ t_{k+1}),\\ \Delta y(t_k)=y(t_k)-y(t_k^-)=K_2e(t_k^-),\quad k\in\mathbb{Z}_+,\\ y(s)=\varphi(s),\quad s\in[-\tau+\sigma,\ \sigma].\end{cases}$$

The impulsive instants t_k satisfy $\sigma<t_1<\cdots<t_k<\cdots$ and $\lim_{k\to\infty}t_k=+\infty$. Then, the error system between systems (1) and (3) is given by

$$\begin{cases}\dot{e}(t)=-Ce(t)+Ag_1(e(t))+Bg_2(e(t-\tau))\\ \qquad +K_1e(t_k),\quad t\in[t_k,\ t_{k+1}),\\ \Delta e(t_k)=e(t_k)-e(t_k^-)=K_2e(t_k^-),\quad k\in\mathbb{Z}_+,\\ e(s)=\varphi(s)-\phi(s-\sigma),\quad s\in[-\tau+\sigma,\ \sigma],\end{cases}\tag{4}$$

where $g_1(e(t))=f_1(y(t))-f_1(x(t-\sigma))$ and $g_2(e(t-\tau))=f_2(y(t-\tau))-f_2(x(t-\tau-\sigma))$.

Definition 1 ([19]) Drive system (1) and response system (3) are lag synchronized with time lag σ if for any initial condition $x(s)=\phi(s)\in PC([-\tau,\ 0],\ \mathbb{R}^n)$, $y(s)=\varphi(s)\in PC([-\tau+\sigma,\ \sigma],\ \mathbb{R}^n)$, it holds that $|y(t)-x(t-\sigma)|\to 0$ as $t\to\infty$.

Definition 2 Let $\mathscr{F}_0$ denote the class of impulsive sequences $\{t_k, k\in\mathbb{Z}_+\}$ satisfying $\sigma<t_1<\cdots<t_k<\cdots$, $\lim_{k\to\infty}t_k=+\infty$. $\mathscr{F}_\mu\subseteq\mathscr{F}_0$ denotes the class of impulsive sequences $\{t_k, k\in\mathbb{Z}_+\}$ satisfying $t_{k+1}-t_k\le\ \mu,\ k\in\mathbb{Z}_+$.

In order to present our main results, the following theoretical result for impulsive delayed systems is needed.

Consider the following impulsive delayed systems:

$$\begin{cases} x'(t) = G(t, x_t),\ t \geq t_0, \quad t \neq t_k, \\ \Delta x|_{t=t_k} = x(t_k) - x(t_k^-) = I_k(t_k, x(t_k^-)),\ k \in \mathbb{Z}_+. \end{cases} \tag{5}$$

Some definitions for functions $G,\ I_k$ and set ν_0 can be found in [19], and here we omit them. Then, we have the following lemmas.

Lemma 1 *System (5) is globally exponentially stable (GES) over $\mathscr{F}_\mu$ with $\mu < \ln q/p$, if there exist a function $V(t,\ x(t)) \in \nu_0$ and constants $p > 0,\ q > 1,\ c_1 > 0,\ c_2 > 0,\ c_3 > 0,\ m > 0$ and $\gamma > 0$ such that*

- (I) $c_1|x(t)|^m \leq V(t,\ x(t)) \leq c_2|x(t)|^m + c_3|x(t_k)|^m$, $t \in [t_k,\ t_{k+1})$;
- (II) *For any $\sigma \geq 0$ and $\psi \in PC([-\tau,\ 0],\ \mathbb{R}^n)$, if $e^{\gamma\theta}V(t+\theta,\ \psi(\theta)) \leq qV(t,\ \psi(0)), \theta \in [-\tau,\ 0]$, $t \neq t_k$, then $D^+V(t,\ \psi(0)) \leq pV(t,\ \psi(0))$;*
- (III) *For all $(t_k, \psi) \in \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n)$, $V(t_k, \psi(0) + I_k(t_k, \psi)) \leq \frac{1}{q}V(t_k^-, \psi(0))$.*

Remark 1 *Note that condition* (I) *in Lemma 1 is more general than that in* [19]. *The proof of Lemma 1 is similar to the proofs of Lemma 2.1 in* [19] *and Lemma 1 in* [38]. *Its detailed proof is omitted here.*

Lemma 2 [39] *Given any real matrices $Y_1,\ Y_2$ and Q of appropriate dimensions and a constant $\varepsilon > 0$ such that $Q^T = Q > 0$, then it holds:*

$$Y_1^TY_2 + Y_2^TY_1 \leq \varepsilon Y_1^TQY_1 + \varepsilon^{-1}Y_2^TQ^{-1}Y_2.$$

3 Main result

Theorem 1 *Assume that there exist three constants $\alpha > 0,\ \beta > 0,\ q > 1$, four $n \times n$ matrices $P > 0,\ Q > 0,\ \Omega_1,\ \Omega_2$, two $n \times n$ diagonal matrices $Q_1 > 0,\ W > 0$, a symmetric matrix S such that the following inequalities hold:*

(i) $L_1Q_1L_1 - \beta P \leq 0$, *where* $L_1 = diag(l_{21}, \ldots, l_{2n}), l_{2j} = \max\{|l_{2j}^-|, |l_{2j}^+|\}, j \in \Lambda$;

(ii)

$$\begin{pmatrix} \Delta & \Omega_1 & PB \\ \star & -\alpha Q & 0 \\ \star & \star & -Q_1 \end{pmatrix} \leq 0,$$

$$\begin{pmatrix} -F_1W - S & F_2W + PA \\ \star & -W \end{pmatrix} \leq 0,$$

where

$$\Delta = -PC - C^TP - \alpha P + S,$$
$$F_1 = \mathrm{diag}\left(l_{11}^-l_{11}^+,\ l_{12}^-l_{12}^+,\ \ldots,\ l_{1n}^-l_{1n}^+\right),$$
$$F_2 = \mathrm{diag}\left(\frac{l_{11}^- + l_{11}^+}{2},\ \frac{l_{12}^- + l_{12}^+}{2},\ \cdots,\ \frac{l_{1n}^- + l_{1n}^+}{2}\right);$$

(iii)

$$\begin{pmatrix} -\frac{1}{q}P & \Omega_2^T \\ \star & -(P+Q) \end{pmatrix} < 0.$$

Then error system (4) is GES over the class $\mathscr{F}_\mu$ with μ satisfying

$$\mu < \frac{\ln q}{\alpha + \beta q}, \tag{6}$$

that is, drive system (1) can be exponentially lag synchronized by response system (3), where control gains are designed by

$$K_1 = P^{-1}\Omega_1, \quad K_2 = (P+Q)^{-1}\Omega_2 - I.$$

Proof Consider Lyapunov function in the form of

$$V(t) = e^T(t)Pe(t) + e^T(t_k)Qe(t_k),\ t \in [t_k,\ t_{k+1}),\ k \in \mathbb{Z}_+.$$

Since (6) holds, there exists a $\lambda > 0$ such that

$$\alpha + \beta q < \lambda < \frac{\ln q}{\mu}.$$

Then one may choose $\gamma > 0$ such that

$$\alpha + \beta q e^{\gamma\tau} = \lambda. \tag{7}$$

It follows from Lemma 1 that when

$$e^{\gamma\theta}V(t+\theta, e(t+\theta)) \leq qV(t, e(t)), \quad -\tau \leq \theta \leq 0,\ t \in [t_{k-1}, t_k),\ k \in \mathbb{Z}_+,$$

it holds that

$$e^T(t-\tau)Pe(t-\tau) \leq qe^{\gamma\tau}V(t),\ t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+. \tag{8}$$

It follows from Lemma 2 and the definition of g_2 that

$$2e^{\mathrm{T}}(t)PBg_2(e(t-\tau)) \leq e^T(t)PBQ_1^{-1}B^TPe(t)$$

$$+ g_2^T(e(t-\tau))Q_1 g_2(e(t-\tau))$$
$$\leq e^T(t)PBQ_1^{-1}B^TPe(t)$$
$$+ e^T(t-\tau)L_1Q_1L_1e(t-\tau), \quad (9)$$

which, together with (i), yields that

$$2e^T(t)PBg_2(e(t-\tau)) \leq e^T(t)PBQ_1^{-1}B^TPe(t) + \beta e^T(t-\tau)Pe(t-\tau) \leq e^T(t)PBQ_1^{-1}B^TPe(t) + \beta q e^{\gamma\tau}V(t). \quad (10)$$

It then follows from the definition of g_1 and (2) that

$$(f_{1j}(x_j(t)) - l_{1j}^- x_j(t))(f_{1j}(x_j(t)) - l_{1j}^+ x_j(t)) \leq 0,$$

for every $j \in \Lambda$. Similarly, we have

$$(g_{1j}(e_j(t)) - l_{1j}^- e_j(t))(g_{1j}(e_j(t)) - l_{1j}^+ e_j(t)) \leq 0,$$

i.e.,

$$\begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix}^T \begin{pmatrix} l_{1j}^- l_{1j}^+ E_j E_j^T & -\frac{l_{1j}^- + l_{1j}^+}{2} E_j E_j^T \\ \star & E_j E_j^T \end{pmatrix} \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix} \leq 0,$$

for every $j \in \Lambda$, where E_j means the unit column vector one element on ith unit row and zeros elsewhere. Define

$$W = \text{diag}(\omega_1, \omega_2, \ldots, \omega_n) > 0.$$

Then

$$\sum_{j=1}^n \omega_j \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix}^T \begin{pmatrix} l_{1j}^- l_{1j}^+ E_j E_j^T & -\frac{l_{1j}^- + l_{1j}^+}{2} E_j E_j^T \\ \star & E_j E_j^T \end{pmatrix} \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix} \leq 0,$$

which is equivalent to

$$\begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix}^T \begin{pmatrix} -F_1W & F_2W \\ \star & -W \end{pmatrix} \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix} \geq 0. \quad (11)$$

Then taking the derivative of V along system (4) on the continuous interval $[t_k, t_{k+1})$, $k \in \mathbb{Z}_+$ and considering (ii), (7), (10), and (11), we have

$$\begin{aligned}
&D^+V(t) \\
&= 2e^T(t)P(-Ce(t) + Ag_1(e(t)) + Bg_2(e(t-\tau)) + K_1e(t_k)) \\
&= -e^T(t)(PC + C^TP)e(t) + 2e^T(t)PAg_1(e(t)) \\
&\quad + 2e^T(t)PBg_2(e(t-\tau)) + 2e^T(t)PK_1e(t_k) \\
&\leq -e^T(t)(PC + C^TP)e(t) + 2e^T(t)PAg_1(e(t)) \\
&\quad + 2e^T(t)PBg_2(e(t-\tau)) + 2e^T(t)PK_1e(t_k) \\
&\quad + \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix}^T \begin{pmatrix} -F_1W & F_2W \\ \star & -W \end{pmatrix} \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix} \\
&\leq -e^T(t)(PC + C^TP)e(t) + e^T(t)PBQ_1^{-1}B^TPe(t) \\
&\quad + \beta q e^{\gamma\tau}V(t) + 2e^T(t)PK_1e(t_k) \\
&\quad + \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix}^T \begin{pmatrix} -F_1W & F_2W + PA \\ \star & -W \end{pmatrix} \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix} \\
&= \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix}^T \begin{pmatrix} -F_1W & F_2W + PA \\ \star & -W \end{pmatrix} \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix} \\
&\quad + \begin{pmatrix} e(t) \\ e(t_k) \end{pmatrix}^T \begin{pmatrix} -PC - C^TP + PBQ_1^{-1}B^TP & PK_1 \\ \star & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} e(t) \\ e(t_k) \end{pmatrix} + \beta q e^{\gamma\tau}V(t) \leq \begin{pmatrix} e(t) \\ e(t_k) \end{pmatrix}^{\mathrm{T}} \\
&\quad \times \begin{pmatrix} -PC - C^{\mathrm{T}}P + PBQ_1^{-1}B^{\mathrm{T}}P - \alpha P + S & PK_1 \\ \star & -\alpha Q \end{pmatrix} \\
&\quad \times \begin{pmatrix} e(t) \\ e(t_k) \end{pmatrix} + \begin{pmatrix} e(t) \\ e(t_k) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \alpha P & 0 \\ \star & \alpha Q \end{pmatrix} \begin{pmatrix} e(t) \\ e(t_k) \end{pmatrix} \\
&\quad + \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -F_1W - S & F_2W + PA \\ \star & -W \end{pmatrix} \begin{pmatrix} e(t) \\ g_1(e(t)) \end{pmatrix} \\
&\quad + \beta q e^{\gamma\tau}V(t) \\
&\leq (\alpha + \beta q e^{\gamma\tau})V(t) \\
&= \lambda V(t).
\end{aligned} \quad (12)$$

In addition, it follows from (iii) that

$$\begin{aligned}
&\begin{pmatrix} -\frac{1}{q}P & \Omega_2^T \\ \star & -(P+Q) \end{pmatrix} \leq 0 \Leftrightarrow \begin{pmatrix} I & (I+K_2)^T \\ 0 & I \end{pmatrix} \\
&\quad \times \begin{pmatrix} -\frac{1}{q}P & \Omega_2^T \\ \star & -(P+Q) \end{pmatrix} \begin{pmatrix} I & 0 \\ (I+K_2) & I \end{pmatrix} \leq 0 \\
&\Leftrightarrow \begin{pmatrix} -\frac{1}{q}P + (I+K_2)^T(P+Q)(I+K_2) & 0 \\ \star & -(P+Q) \end{pmatrix} \leq 0 \\
&\Leftrightarrow -\frac{1}{q}P + (I+K_2)^T(P+Q)(I+K_2) \leq 0.
\end{aligned}$$

Considering system (3), it holds that

$$\begin{aligned}
V(t_k) &= e^T(t_k)(P+Q)e(t_k) \\
&= e^T\left(t_k^-\right)(I+K_2)^T(P+Q)(I+K_2)e\left(t_k^-\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{q} e^T \left(t_k^-\right) P e \left(t_k^-\right) \\
&\le \frac{1}{q} e^T \left(t_k^-\right) P e \left(t_k^-\right) + \frac{1}{q} e^T (t_{k-1}) Q e(t_{k-1}) \\
&\le \frac{1}{q} V \left(t_k^-\right). \qquad (13)
\end{aligned}
$$

By (6), (12), and (13) using Lemma 1, we know that system (4) is GES, that is, systems (1) and (3) can be exponentially lag synchronized over the class $\mathscr{F}_\mu$ with μ satisfying (6). This completes the proof. □

In particular, if we take $K_2 = d \cdot I$, where d is a constant that will be estimated. Then we have the following corollary.

Corollary 1 *If there exist three constants $\alpha > 0$, $\beta > 0$, $q > 1$, three $n \times n$ matrices $P > 0$, $Q > 0$, Ω_1, two $n \times n$ diagonal matrices $Q_1 > 0$, $W > 0$, a symmetric matrix S such that conditions (i), (ii), and (6) in Theorem 1 hold. Systems (1) and (3) are exponentially lag synchronized via impulsive controller $(t_k,\ K_2)_{k\in\mathbb{Z}_+}$:*

$$
\begin{cases}
K_2 = d \cdot I, \quad d \in \left[-1 - \frac{1}{\sqrt{q}}, \quad -1 + \frac{1}{\sqrt{q}}\right], \\
\{t_k\} \in \mathscr{F}_\mu,
\end{cases} \qquad (14)
$$

where the sampled-data control gain is designed by

$K_1 = P^{-1}\Omega_1.$

Assume that $u_1(t) = K_1 e(t_k),\ t \in [t_k,\ t_{k+1}),\ k \in \mathbb{Z}_+$, is not the control input but the external bounded disturbance which may destroy the synchronization control, that is, K_1 is pre-given. Based on Corollary 1, we can design the impulsive controller.

Corollary 2 *If there exist three constants $\alpha > 0$, $\beta > 0$, $q > 1$, two $n \times n$ matrices $P > 0$, $Q > 0$, two $n \times n$ diagonal matrices $Q_1 > 0$, $W > 0$, a symmetric matrix S such that conditions (i), (ii), and (6) in Theorem 1 hold and*

$$
\begin{pmatrix}
\Delta & PK_1 & PB \\
\star & -\alpha Q & 0 \\
\star & \star & -Q_1
\end{pmatrix} \le 0,
$$

where

$$\Delta = -PC - C^T P - \alpha P + S.$$

Systems (1) and (3) are exponentially lag synchronized over the class $\mathscr{F}_\mu$ with μ satisfying (6) via the impulsive controller same as (14).

Remark 2 *In Theorem 1 and Corollary 1 some approaches are derived to design the hybrid controller including the impulsive controller and the sampled-data controller such that system (1) and system (3) can be lag synchronized. In Corollary 2, an impulsive controller is designed to achieve the lag synchronization between (1) and (3) when the sampled-data term is given from the external perturbation point of view. Compared to these impulsive control schemes in Li [19], the developed results in this paper can be applied to designing both the hybrid controller and the impulsive controller with external perturbation such that response system (3) can be lag synchronized with master system (1).*

4 Numerical simulations

In this section, we present an example and its simulations to demonstrate the effectiveness of the proposed results.

Example 1 Consider the following 2D neural network model:

$$
\begin{cases}
\dot{x}(t) = -Cx(t) + Af_1(x(t)) + Bf_2(x(t-\tau)) + J, \\
x(s) = \phi(s), \quad s \in [-\tau, 0],
\end{cases} \qquad (15)
$$

where $\tau = 1$, $J = [0,\ 0]^T$, $f_1(x) = f_2(x) = \tanh(x)$. The parameter matrices C, A, and B are given as follows

$$
C = I, \quad A = \begin{pmatrix} 2 & -0.1 \\ -5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{pmatrix}.
$$

It is well known that system (15) with initial condition $\phi(s) = [0.4,\ 0.6]^T$, $s \in [-\tau,\ 0]$ is chaotic [40], see Fig. 1. More information about the initial condition dependence of the chaotic state can be found in [41]. To design the hybrid controller, we consider $\sigma = 1$ and the response system is given in the form of

$$
\begin{cases}
\dot{y}(t) = -Cy(t) + Af_1(y(t)) + Bf_2(y(t-\tau)) + J \\
\qquad + K_1 e(t_k),\ t \in [t_k, t_{k+1}), \\
\Delta y(t_k) = y(t_k) - y\left(t_k^-\right) = K_2 e(t_k^-), \quad k \in \mathbb{Z}_+, \\
y(s) = \varphi(s), \quad s \in [-\tau + \sigma,\ \sigma],
\end{cases} \qquad (16)
$$

where the initial condition $\varphi(s) = [0,\ 0.5]^T$, $s \in [-\tau + \sigma,\ \sigma]$. Then, the error system between (15) and (16) is given by

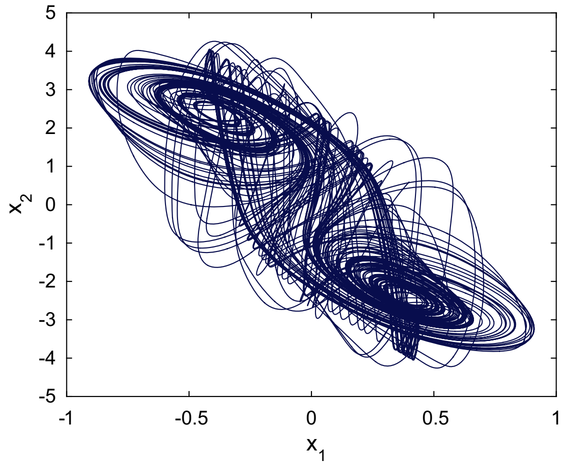

Fig. 1 The chaotic orbits of system (13)

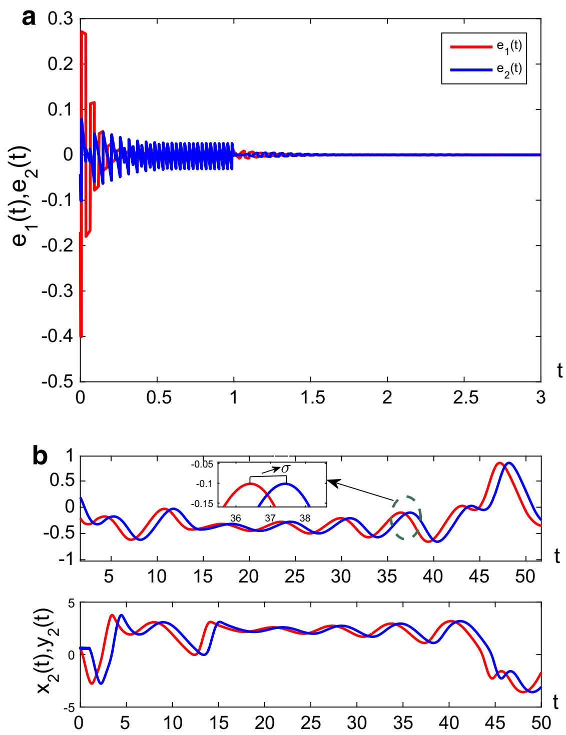

Fig. 2 Trajectories in Example 1

$$\begin{cases} \dot{e}(t) = -Ce(t) + Ag_1(e(t)) + Bg_2(e(t-\tau)) \\ \qquad +K_1 e(t_k),\ t \in [t_k,\ t_{k+1}), \\ \Delta e(t_k) = e(t_k) - e(t_k^-) = K_2 e(t_k^-), \quad k \in \mathbb{Z}_+, \\ e(s) = \varphi(s) - \phi(s-\sigma), \quad s \in [-\tau+\sigma,\ \sigma], \end{cases} \tag{17}$$

We firstly consider the hybrid control. Let $\alpha = 8$, $\beta = 10$, $q = 3$, and note that $F_1 = 0$, $F_2 = 0.5I$.

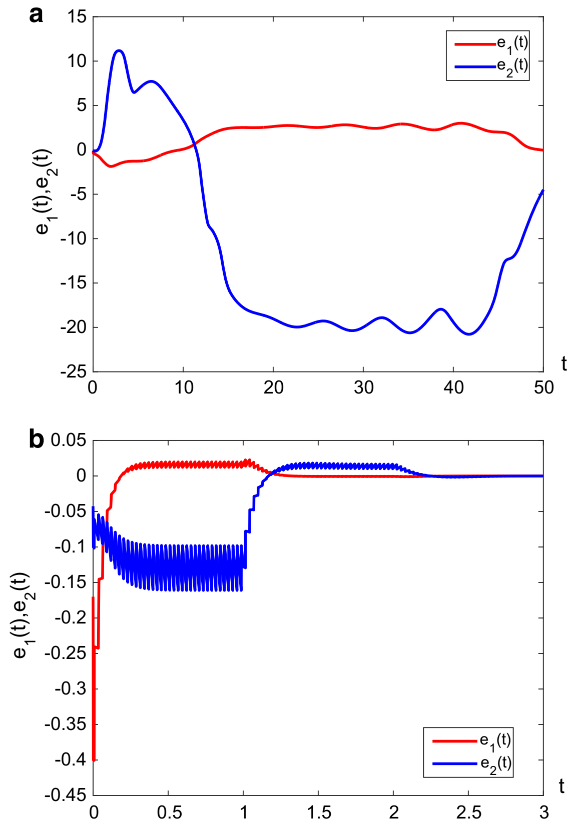

Fig. 3 Trajectories in Example 1

According to Theorem 1, via MATLAB LMI toolbox, we can obtain the following feasible solution

$$P = \begin{pmatrix} 218.5963 & 6.8960 \\ 6.8960 & 79.1988 \end{pmatrix},$$

$$Q = \begin{pmatrix} 222.2812 & -3.4839 \\ -3.4839 & 259.4708 \end{pmatrix},$$

$$\Omega_1 = \begin{pmatrix} -454.4690 & -107.8822 \\ -107.8822 & -105.1384 \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} -293.1013 & 3.0279 \\ 3.0279 & -262.6865 \end{pmatrix}.$$

Then, the gain matrices are derived as follows

$$K_1 = \begin{pmatrix} -2.0417 & -0.4529 \\ -1.1844 & -1.2881 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} -1.6649 & 0.0129 \\ 0.0156 & -1.7758 \end{pmatrix}.$$

By Theorem 1, error system (17) is GES over the class $\mathscr{F}_\mu$ with $\mu < 0.0289$, that is, drive system (15) can be exponentially lag synchronized with response system (16) when $\mu < 0.0289$. In particular, if we choose impulsive instants $t_k = \eta k,\ k \in \mathbb{Z}_+,\ \eta = 0.028$, trajectories of error variables and state variables are shown in Fig. 2.

Next we consider the impulsive control and consider $K_1 = 0.6I$. Then, error system (17) with simple-data disturbance is unstable, see Fig. 3a. According to Corollary 2, it can be deduced that the valuing range of d is [-1.577, 0.423]. If we choose $K2 = -0.4I$, then error system (17) is GES over the class $\mathscr{F}_\mu$ with $\mu < 0.0289$, that is, drive system (15) can be exponentially lag synchronized with response system (16) when $\mu < 0.0289$. In particular, if we choose impulsive instants $t_k = \eta k,\ k \in \mathbb{Z}_+,\ \eta = 0.028$, trajectories of error variables are shown in Fig. 3b.

5 Conclusion

We have studied the problem of lag synchronization of delayed chaotic neural networks based on hybrid control and established some criteria on synchronization control by employing impulsive control theory. The hybrid controller consists of the sampled-data controller and the impulsive controller. In particular, when the sampled-data are considered as the external perturbation, the lag synchronization can be completely achieved by impulsive controller according to the proposed results. Finally, a numerical example and its simulations were provided to show the effectiveness of the proposed results. In our study, we note that the sampling period is the same as the impulsive interval. Further research topics would be considered to extend the main results of this paper to the case that the sampling period is different from the impulsive interval.

References

1. Pikovsky, A., Roseblum, M., Kurths, J.: Synchronization A Universal Concept in Nonlinear Sciences. Cambridge University Press, New York (2003)
2. Pecora, L.M., Carroll, T.L.: Synchronization in chaotic systems. Phys. Rev. Lett. **64**, 821–824 (1990)
3. Zhu, Q.X., Cao, J.D.: pth moment exponential synchronization for stochastic delayed Cohen–Grossberg neural networks with Markovian switching. Nonlinear Dyn. **67**, 829–845 (2012)
4. Yang, T.: Impulsive stabilization for control and synchronization of chaotic systems: theory and application to secure communication. IEEE Trans. Circuits Syst. Part **I**(44), 976–988 (1997)
5. Shi, K.B., Liu, X.Z., Zhu, H., Zhong, S.M., Zeng, Y., Yin, C.: Novel delay-dependent master-slave synchronization criteria of chaotic Lur'e systems with time-varying-delay feedback control. Appl. Math. Comput. **282**, 351–362 (2015)
6. Li, X.D., Rakkiyappan, R., Sakthivel, N.: Non-fragile synchronization control for Markovian jumping complex dynamical networks with probabilistic time-varying coupling delay. Asian J. Control **17**, 1678–1695 (2015)
7. Chen, S., Cao, J.D.: Projective synchronization of neural networks with mixed time-varying delays and parameter mismatch. Nonlinear Dyn. **67**, 1397–1406 (2012)
8. Batista, C.A.S., Szezech, J.D., Batista, A.M., Macau, E.E.N., Viana, R.L.: Synchronization of phase oscillators with coupling mediated by a diffusing substance. Phys. A Stat. Mech. Appl. **470**, 236–248 (2017)
9. Pyragien, T., Pyragas, K.: Anticipating synchronization in a chain of chaotic oscillators with switching parameters. Phys. Lett. A **379**, 3084–3088 (2015)
10. Huang, J.J., Li, C.D., Huang, T.W., He, X.: Finite-time lag synchronization of delayed neural networks. Neurocomputing **139**, 145–149 (2014)
11. Mahmoud, G., Mahmoud, E.: Lag synchronization of hyperchaotic complex nonlinear systems. Nonlinear Dyn. **67**, 1613–1622 (2012)
12. Ma, J., Tang, J.: A review for dynamics of collective behaviors of network of neurons. Sci. China Technol. Sci. **58**, 2038–2045 (2015)
13. Song, X.L., Wang, C.N., Ma, J., et al.: Transition of electric activity of neurons induced by chemical and electric autapses. Sci. China Technol. Sci. **58**, 1007–1014 (2015)
14. Yang, Y.Q., Cao, J.D.: Exponential lag synchronization of a class of chaotic delayed neural networks with impulsive effects. Phys. A Stat. Mech. Appl. **386**, 492–502 (2007)
15. Li, C.D., Liao, X., Zhang, R.: A unified approach for impulsive lag synchronization of chaotic systems with time delay. Chaos, Solitons Fractals **23**, 1177–1184 (2005)
16. Miao, Q.Y., Tang, Y., Lu, S.J., Fang, J.N.: Lag synchronization of a class of chaotic systems with unknown parameters. Nonlinear Dyn. **57**, 107–112 (2009)
17. Xia, Y.H., Yang, Z.J., Han, M.A.: Lag synchronization of unknown chaotic delayed Yang-Yang-Type fuzzy neural networks with noise perturbation based on adaptive control and parameter identification. IEEE Trans. Neural Netw. **20**, 1165–1180 (2009)
18. Wen, S.P., Zeng, Z.G., Huang, T.W., Zhang, Y.D.: Exponential adaptive lag synchronization of memristive neural networks via fuzzy method and applications in pseudorandom number generators. IEEE Trans. Fuzzy Syst. **22**, 1704–1713 (2014)
19. Li, X.D., Fu, X.L.: Lag synchronization of chaotic delayed neural networks via impulsive control. IMA J. Math. Control Inf. **29**, 133–145 (2012)
20. Wang, C., Chu, R., Ma, J.: Controlling a chaotic resonator by means of dynamic track control. Complexity **21**, 370–378 (2015)
21. Yang, X.S., Song, Q., Liang, J.L., He, B.: Finite-time synchronization of coupled discontinuous neural networks with mixed delays and nonidentical perturbations. J. Frankl. Inst. **352**, 4382–4406 (2015)

22. Lu, J.Q., Cao, J.D.: Synchronization-based approach for parameters identification in delayed chaotic neural networks. Phys. A Stat. Mech. Appl. **382**, 672–682 (2007)
23. Syed Ali, M., Arik, S., Rani, E.M.: Passivity analysis of stochastic neural networks with leakage delay and Markovian jumping parameters. Neurocomputing **218**, 139–145 (2016)
24. Syed Ali, M., Saravanan, S., Cao, J.: Finite-time boundedness, L_2-gain analysis and control of Markovian jump switched neural networks with additive time-varying delays. Nonlinear Anal. Hybrid Syst. **23**, 27–43 (2017)
25. Syed Ali, M., Gunasekaran, N., Esther Rani, M.: Robust stability of hopfield delayed neural networks via an augmented L-K functional. Neurocomputing **234**, 198–204 (2017)
26. Rakkiyappan, R., Sivasamy, R., Li, X.D.: Synchronization of identical and nonidentical memristor-based chaotic systems via active backstepping control technique. Circuits, Syst. Signal Process. **34**, 763–778 (2015)
27. Li, X.D., Wu, J.H.: Stability of nonlinear differential systems with state-dependent delayed impulses. Automatica **64**, 63–69 (2016)
28. Li, X.D., Bohner, M., Wang, C.K.: Impulsive differential equations: periodic solutions and applications. Automatica **52**, 173–178 (2015)
29. Guan, Z.H., Liu, Z.W., Feng, G., Wang, Y.W.: Synchronization of complex dynamical networks with time-varying delays via impulsive distributed Control. IEEE Trans. Circuits Syst. I Regul. Papers **57**, 2182–2195 (2010)
30. Li, X.D., Cao, J.D.: An impulsive delay inequality involving unbounded time-varying delay and applications. IEEE Trans. Autom. Control **62**, 3618–3625 (2017)
31. Li, X.D., Song, S.J.: Stabilization of delay systems: delay-dependent impulsive control. IEEE Trans. Autom. Control **62**, 406–411 (2017)
32. Zhang, H.G., Ma, T.D., Huang, G.B., Wang, Z.L.: Robust global exponential synchronization of uncertain chaotic delayed neural networks via dual-stage impulsive control. IEEE Trans. Syst. Man Cybern. Part B (Cybernetics) **844**, 40–831 (2010)
33. Rakkiyappan, R., Velmurugan, G., George, J.N., Selvamani, R.: Exponential synchronization of Lure complex dynamical networks with uncertain inner coupling and pinning impulsive control. Appl. Math. Comput. **307**, 217–231 (2017)
34. Wu, Z.G., Ju, H.P., Su, H.Y., Chu, J.: Discontinuous Lyapunov functional approach to synchronization of time-delay neural networks using sampled-data. Nonlinear Dyn. **69**, 2021–2030 (2012)
35. Zhang, C., He, Y., Wu, M.: Exponential synchronization of neural networks with time-varying mixed delays and sampled-data. Neurocomputing **74**, 265–273 (2010)
36. Zhang, H.G., Wang, J.Y., Wang, Z.S., Liang, H.J.: Sampled-Data synchronization analysis of markovian neural networks with generally incomplete transition rates. IEEE Trans. Neural Netw. Learn. Syst. **28**, 740–752 (2017)
37. Lee, T.H., Park, J.H.: Improved criteria for sampled-data synchronization of chaotic Lur'e systems using two new approaches. Nonlinear Anal. Hybrid Syst. **24**, 132–145 (2017)
38. Lv, X.X., Li, X.D.: Finite time stability and controller design for nonlinear impulsive sampled-data systems with applications. ISA Trans. **70**, 30–36 (2017)
39. Cao, J.D., Yuan, K., Li, H.X.: Global asymptotic stability of recurrent neural networks with multiple discrete delays and distributed delays. IEEE Trans. Neural Netw. **17**, 1646–1651 (2006)
40. Lu, H.T.: Chaotic attractors in delayed neural networks. Phys. Lett. A **298**, 109–116 (2002)
41. Ma, J., Wu, F., Ren, G., et al.: A class of initials-dependent dynamical systems. Appl. Math. Comput. **298**, 65–76 (2017)