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Global robust regulation control for a class of cascade nonlinear systems subject to external disturbance

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Abstract The global robust regulation problem is studied for a class of cascade nonlinear systems subject to the external disturbance. The considered system represents more general classes of nonlinear uncertain systems, for example, the much weaker integral input-to-state stable (iISS) cascaded subsystem, the unknown control coefficients, the unmeasured states appearing in the nonlinear uncertainties and the external disturbance additively in the input channel. Combined the ideas of the Nussbaum-type gain and the disturbance as a generalized state, a dynamic extended state observer (ESO) based on a Riccati differential equation is constructed to overcome these difficulties. It is shown that the global robust regulation problem is well addressed by the proposed method. In the simulation part, the fan speed control system is used as a practical example to demonstrate its efficacy.

Keywords Output feedback · Integral input-to-state stable · Nonlinear systems · Extended state observer · Additive disturbance

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1 Introduction

Nonlinear systems control has received considerable attention in the control community, and phenomenal progress has been made during the last decades. Numerous novel methodologies for nonlinear feedback control have been generated to control the engineering systems [1]. One of the influential notions is the input-to-state stability (ISS) invented by Sontag [2] in the late 1980s, which has become a central concept in nonlinear systems analysis. As an integral variant of ISS, integral input-to-state stability (iISS) is another meaningful but much weaker notion. Since it is introduced in [3,4], there has an ever increasing interest in this topic [5–9] recently. It is noted that in [10], supposing the dynamic uncertainty subject to the iISS property, a unifying framework is presented for global output feedback regulation control from ISS to iISS, which extends many known classes of output feedback form systems in several directions. Subsequently, some further results were obtained based on such class of nonlinear systems, see [11–14], etc.

However, all aforementioned output regulation controller does not consider the uncertain external disturbance except in [14]. It is known that the uncertainties that arise from external disturbance are essentially the major concern in the nonlinear control design. During the past two decades, many fruitful results have been proposed to reject the external disturbances [15,16]. Recently, the extended state observer (ESO) by Han in his pioneer works [17] is regarded as the major creativity toward the active disturbance rejection control (ADRC). As written in [18], the ESO not only possesses the state observation capability but also provides real-time estimation of generalized disturbances between the plant and the model of the considered system, such as external disturbances and modeling uncertainties caused by parameter deviations. Based on the ADRC strategy together with the ESO, the robust control was addressed for several classes of nonlinear systems with disturbances and uncertainties in [19–22]. Another fruitful tool is the disturbance observer-based control (DOBC) technique proposed in [23], which provides a promising approach to handle the system disturbances and improve robustness [24–27] when the system states are available.

In this paper, we further investigate the output regulation control problem for a class of cascade nonlinear systems subject to the external disturbance. The studied system is a perturbed version of its counterpart in [10,11]. The purpose of this paper is to construct a robust regulation controller via output feedback for a more general class of nonlinear uncertain system in the presence of external disturbance, which does not require to be square integrable or vanishing at the origin. Using the idea of ESO, we generalize the disturbance as an extended state, and then, design a novel observer whose gain is updated by a time-varying Riccati differential equation. The main contributions contained in this paper are highlighted as follows:

(1) The disturbance attenuation is addressed for a class of nonlinear systems with external disturbance in the control input channel. Different from the existing related works in [14], the external disturbance does not require to be square integrable. Consequently, the proposed control algorithm allows a larger class of disturbance signals, such as the constant signal.

(2) The system in question can accommodate some serious uncertainties such as the unknown control coefficients, which makes the state compensator design extremely difficult in the case of the unavailable system states. Technically, this hurdle is tactfully overcome by introducing a coordinate transformation and designing an ESO based a Riccati differential equation.

(3) The global set-point regulation control is solved for the fan speed control systems in the presence of external disturbances. This result improves the existing works where the external disturbance in the armature voltage is not considered.

Notations The following notations are adopted in the paper. If x is a possibly time-varying vector, then $|x(t)|$ is the Euclidean norm of x at time t, $\|x\|_p = \left[\int_0^\infty |x(\tau)|^p \mathrm{d}\tau\right]^{\frac{1}{p}}$, $p \in [1, \infty)$, $\|x\|_\infty = \sup_{0\le t} |x(t)|$, and $x \in L_p$ when $\|x\|_p$ exists, $x \in L_\infty$ when $\|x\|_\infty$ exists. A^T denotes its transpose for a matrix A. For a n-dimension vector $x = (x_1, \ldots, x_n) \in R^n$, we denote $x_{[i]} = (x_1, \ldots, x_i)$ when $i = 2, \ldots, n-1$. $\pi_1(s) = O(\pi_2(s))$ as $s \to 0^+$ means that $\pi_1(s) \le c_1\pi_2(s)$ for some constant $c_1 > 0$ and all s in a small neighborhood of zero, and $\pi_1(s) = O(\pi_2(s))$ as $s \to \infty$ means that $\pi_1(s) \le c_1\pi_2(s)$ for some constant $c_2 > 0$ and all large enough s.

2 Model description and scheme

2.1 Problem formulation

In this paper, we study the following class of cascade nonlinear systems described by

$$\begin{aligned}
&\dot{\zeta} = q(\zeta, y)\\
&\dot{x}_i = b_i x_{i+1} + \lambda_i(t)x_i + g_i(\zeta, y), i = 1, \ldots, n-1,\\
&\dot{x}_n = b_n u + \lambda_n(t)x_n + g_n(\zeta, y) + \mathrm{d}(t)\\
&y = x_1
\end{aligned} \tag{1}$$

where $(\zeta, x) \in R^r \times R^n$ are system states, $u \in R$ is control input, and $y = x_1$ is the observable output. $\lambda_i(t)(i = 1, \ldots, n)$ are known time-varying functions, and $\mathrm{d}(t)$ represents the unknown external disturbance. The signs of nonzero control coefficients $b_1, \ldots, b_{n-1}$ as well as the high-frequency gain b_n are not known a priori. The uncertain functions $q(\cdot)$ and $g_i(\cdot)(i = 1, \ldots, n)$ are supposed to be locally Lipschitz for existence and uniqueness of solutions. The state $(x_2, \ldots, x_n)$ and the state ζ of the ζ-subsystem, which is referred to as the inverse system, are not assumed to be measurable.

The objective of this paper is to design a robust controller to globally asymptotically regulate system (1) to the origin using only the output signal. For future reference, we record the following definition [28].

Definition 1 System (1) is said to be globally asymptotically regulated by the following time-varying output feedback controller

$$\dot{\hat{x}} = \nu(t, \hat{x}, y), \ u = \mu(t, \hat{x}, y) \tag{2}$$

in such a way that, for all initial conditions $(\zeta(0), x(0), \hat{x}(0))$, the solutions of the closed-loop system (1) and (2) are well defined and bounded on $[0, +\infty)$. In particular, $\zeta(t)$ and $x(t)$ converge to zero as $t \to \infty$.

The following assumptions are needed in order to achieve the stated control objective.

Assumption 1 The ζ dynamics is iISS with an iISS-Lyapunov function $U_0(\zeta)$ satisfying

$$\frac{\partial U_0}{\partial \zeta} q(\zeta, y) \le -\alpha(|\zeta|) + \gamma(|y|), \tag{3}$$

where α is a positive definite continuous function, $\gamma \in K_\infty$, and moreover

$$\gamma(s) = O(s^2) \ \text{ as } s \to 0^+. \tag{4}$$

Assumption 2 For $i = 1, \ldots, n$, there exist two unknown positive constants δ_{i1} and δ_{i2}, and two known positive semidefinite, smooth functions $\phi_{i1}(\cdot)$ and $\phi_{i2}(\cdot)$, such that

$$|g_i(\zeta, y)| \le \delta_{i1}\phi_{i1}(|\zeta|) + \delta_{i2}\phi_{i2}(|y|). \tag{5}$$

Moveover, the following additional conditions hold:

$$\phi_{i1}^2(s) = O\big(\alpha(s)\big) \ \text{ as } s \to 0^+, \tag{6}$$

and in case α is bounded,

$$\limsup_{s\to\infty} \frac{\phi_{i1}^2(s)}{\alpha(s)} < \infty, \ 1 \le i \le n. \tag{7}$$

Assumption 3 The time-varying functions $\lambda_i(t)(i = 1, \ldots, n)$ are uniformly bounded and differentiable sufficiently many times, and further assumed to have bounded derivatives. To be precisely, for each $i = 1, \ldots, n$, there exists a unknown positive number $\bar{\lambda}$, such that

$$\sup_{t\ge 0} \left\{|\lambda_i(t)|\right\} \le \bar{\lambda}. \tag{8}$$

Assumption 4 The external disturbance $\mathrm{d}(t)$ satisfies the following properties:

$$d(t) \in L_\infty, \ \dot{d}(t) \in L_\infty, \ \dot{d}(t) \in L_2. \tag{9}$$

Remark 1 Few remarks are made here.

(1) As stated in [12], the system (1) represents a larger class of nonlinear systems in output feedback form. From Assumption 1, the cascaded ζ-subsystem is iISS, which is a less restrictive condition than ISS. More generally, it could allow the presence of uncertainty in the supply rates of (α, γ) such as in [8]. As the condition C2) in [10,11], the uncertain nonlinearities $g_i(\eta, y)(i = 1, \ldots, n)$ in Assumption 2 are assumed to be vanishing or unbiased. In addition, it is also not required to satisfy any kind of polynomial bounds.

(2) Assumption 4 shows that the external disturbance $\mathrm{d}(t)$ is not required to be L_2. This is in sharp contrast to the existing closely related results, such as [14]. Here, it does only require that $\dot{d}(t) \in L_2$. This is a more relaxed version of noise signals. As a simple example, $\mathrm{d}(t) = constant$ is not square integrable, but it is clear that $\dot{d}(t) \in L_2$. It is shown that some kind of extended state observer based on a Riccati differential equation will do the job in the case of such general class of noise signals satisfying Assumption 4.

Remark 2 Considering the inverse system state ζ is not available for feedback, as in [10,14], some local small-gain type conditions in (4) (6) and (7) play key roles in dealing with the unmeasured state ζ. In fact, we have the following lemma from [10].

Lemma 1 *For $i = 1, \ldots, n$, if the local conditions (6) and (7) are available, then the following hold true*

$$\int_0^t \phi_{i1}^2(|\zeta(s)|)\mathrm{d}s \le \bar{\psi}_{i0}(|\zeta(0)|) + \int_0^t \psi_{i1}(|y(s)|)\mathrm{d}s, \tag{10}$$

where $\bar{\psi}_{i0}$ is a positive definite continuous function, and $\psi_{i1} \in K_\infty$ is quadratic near the origin.

2.2 A dynamic output feedback control scheme based on ESO

In this section, we present a step-by-step procedure to design a dynamic output feedback law to solve the global regulation problem for system (1).

To be first, we choose the new state variables ξ_i's in the form of

$$\xi_i = \frac{x_i}{b_i \cdots b_n}, \ i = 1, \ldots, n. \tag{11}$$

With this change of coordinates, system (1) is turned into

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1} + \lambda_i(t)\xi_i + \frac{1}{b_i \cdots b_n} g_i(\zeta, y), \ i = 1, \ldots, n-1, \\ \dot{\xi}_n &= u + \lambda_n(t)\xi_n + \frac{1}{b_n}\mathrm{d}(t) + \frac{1}{b_n} g_n(\zeta, y). \end{aligned} \tag{12}$$

In the following context, the external disturbance $\frac{1}{b_n}\mathrm{d}(t)$ is viewed as a generalized state. For notational consistence, we choose the following notation

$$\xi_{n+1} = \frac{1}{b_n}\mathrm{d}(t), \tag{13}$$

and its derivative is defined by $h(t)$, i.e.,

$$\dot{\xi}_{n+1} = \frac{1}{b_n}\dot{\mathrm{d}}(t) = h(t). \tag{14}$$

Then, one can get the following $(n+1)$-order augmented system

$$\begin{aligned}
\dot{\xi}_i &= \xi_{i+1} + \lambda_i(t)\xi_i + \frac{1}{b_i\cdots b_n} g_i(\zeta, y), \ i = 1, \ldots, n-1, \\
\dot{\xi}_n &= u + \lambda_n(t)\xi_n + \xi_{n+1} + \frac{1}{b_n} g_n(\zeta, y) \\
\dot{\xi}_{n+1} &= h(t).
\end{aligned} \tag{15}$$

Remark 3 It is noted that here, if the unknown disturbance $\mathrm{d}(t)$ is assumed to be constant, then its derivative $h(t)$ becomes zero. This case can usually be found in the set-point regulation or in the presence of sensor disturbances. For example, in some cases, the value of the control may not be completely known due to a desired (unknown) equilibrium point, see [29]. It can be seen that according to Assumption 4, actually a broader class of external disturbance can be allowed in (1), such as $\mathrm{d}(t) = \arctan(t)$ or $constant$, which shows the disturbance does not require to be vanishing. In this sense, the results reported in [10] with vanishing nonlinearities can be extended to the case of nonvanishing uncertainties.

Let $\xi = [\xi_1, \xi_2, \cdots, \xi_n, \xi_{n+1}]^T$, the system dynamics in (15) can be rewritten into the following compact form

$$\dot{\xi} = \Big(A + \Lambda(t)\Big)\xi + C_n u + C_{n+1} h(t) + B\cdot G(\zeta, y) \tag{16}$$

with $A = \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}$, $\Lambda(t) = \mathrm{diag}\{\lambda_1(t), \cdots, \lambda_n(t), 0\}$, $B = \mathrm{diag}\{\frac{1}{b_1\cdots b_n}, \frac{1}{b_2\cdots b_n}, \ldots, \frac{1}{b_n}, 0\}$, and

$$G(\zeta, y) = \begin{bmatrix} g_1(\zeta, y) \\ \vdots \\ g_n(\zeta, y) \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ \cdots, \ C_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \ C_{n+1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Then, for the system (16), a modified version of dynamic observer in [11] is designed as follows

$$\dot{\widehat{\xi}} = \Big(A + \Lambda(t)\Big)\widehat{\xi} + C_n u - PCC^T\,\widehat{\xi}, \tag{17}$$

where the observer gain $P(t)$ is updated by a time-varying Riccati differential equation

$$\begin{cases} \dot{P} = P\Big(A + \Lambda(t)\Big)^T + \Big(A + \Lambda(t)\Big)P - \mathrm{PCC}^T P + I \\ P(0) = P_0 > 0. \end{cases} \tag{18}$$

Furthermore, we note that the Riccati differential equation defined in (18) is solvable. In fact, we have the following lemma which ensures that the observer (17) and (18) makes sense, and its proof can be found in [30,31].

Lemma 2 *For the matrix differential equation* (18)*, if the functions $\lambda_i(t)(i = 1, \ldots, n)$ are known, continuous and uniformly bounded, the unique solution $P(t) = \big(P(t)\big)^T$ exists and there are two strictly positive real numbers $p_{\min}$ and $p_{\max}$ such that $p_{\min} I \le P(t) \le p_{\max} I, \ t \ge 0$.*

Define the observation error variables

$$\varepsilon_i = \xi_i - \widehat{\xi}_i, \ i = 1, \ldots, n+1. \tag{19}$$

From (12) *and* (17)*, it can be concluded that the ε_i's satisfy the following differential equation:*

$$\begin{aligned}
\dot{\varepsilon} &= \Big(A + \Lambda(t) - PCC^T\Big)\varepsilon \\
&\quad + \frac{PC}{b_1\cdots b_n} x_1 + C_{n+1} h(t) + B\cdot G(\zeta, y).
\end{aligned} \tag{20}$$

In order to obtain the computable gain functions of $\frac{PC}{b_1\cdots b_n} x_1$ and $B\cdot G(\zeta, y)$, we choose a scaled error variable $e = (e_1, \ldots, e_{n+1})^T$ by setting

$$\begin{aligned}
e &= \frac{1}{\delta^*}\varepsilon, \\
\delta^* &= \max\left\{ \frac{1}{|b_1\cdots b_n|}, \frac{\delta_{i1}}{|b_i\cdots b_n|}, \frac{\delta_{i2}}{|b_i\cdots b_n|}, \delta_{11}^2, \delta_{12}^2, 1 \,\middle|\, 1 \le i \le n \right\}.
\end{aligned} \tag{21}$$

Accordingly, (20) *is turned into*

$$\dot{e} = \left(A+\Lambda(t)-PCC^T\right)e+\frac{PC}{\delta^* b_1\cdots b_n}x_1 + \frac{1}{\delta^*}C_{n+1}h(t) + \frac{1}{\delta^*}B\cdot G(\zeta, y). \tag{22}$$

Take the error e as the (unmeasured) state, and the error system (22) is iISS with inputs (ζ, y) and $h(t)$. In fact, we have the following proposition, and its proof is given in "Appendix."

Proposition 1 *For the e-subsystem (22), we choose the Lyapunov function*

$$V_e = e^T P^{-1}(t)e, \tag{23}$$

then, its derivative is such that

$$\dot{V}_e \le -\frac{1}{2}e^T P^{-2}(t)e + 8\sum_{i=1}^{n}\phi_{i1}^2(|\zeta|) + y^2 + 8\sum_{i=1}^{n}\phi_{i2}^2(|y|) + 4h^2(t). \tag{24}$$

Now, we are ready to present the anti-disturbance regulation controller using the backstepping method in a recursive manner. For notational convenience, we denote $b = b_1\cdots b_n$ in what follows. The augmented system convenient for feedback design is in the form of

$$\begin{aligned}
\dot{\zeta} &= q(\zeta, y)\\
\dot{e} &= \left(A + \Lambda(t) - PCC^T\right)e + \frac{PC}{\delta^* b}x_1 + \frac{1}{\delta^*}C_{n+1}h(t) + \frac{1}{\delta^*}B\cdot G(\zeta, y)\\
\dot{x}_1 &= b\widehat{\xi}_2 + b\delta^* e_2 + \lambda_1(t)x_1 + g_1(\zeta, y)\\
\dot{\widehat{\xi}}_i &= \widehat{\xi}_{i+1} + \lambda_i(t)\widehat{\xi}_i - C_i^T PC\widehat{\xi}_1, i = 2, \ldots, n-1,\\
\dot{\widehat{\xi}}_n &= \widehat{\xi}_{n+1} + \lambda_n(t)\widehat{\xi}_n + u - C_n^T PC\widehat{\xi}_1\\
\dot{\widehat{\xi}}_{n+1} &= -C_{n+1}^T PC\widehat{\xi}_1.
\end{aligned} \tag{25}$$

Following the conventional backstepping procedure, for the $\widehat{\xi}_i$-subsystem, we take the variable of $\widehat{\xi}_{i+1}$ as the virtual control, ϑ_i as the desired control law, and $z_i = \widehat{\xi}_{i+1} - \vartheta_i (i = 1, \ldots, n-1)$ as the error.

Step 1: Choose the first Lyapunov function candidate

$$V_1 = V_e + \frac{1}{2}y^2. \tag{26}$$

In view of (24), its time derivative satisfies

$$\dot{V}_1 \le -\frac{1}{2}e^T P^{-2}e + y^2 + 8\sum_{i=1}^{n}\phi_{i2}^2(|y|) + y\left(b\vartheta_1 + bz_1 + b\delta^* e_2 + \lambda_1(t)x_1 + g_1(\zeta, y)\right) + 8\sum_{i=1}^{n}\phi_{i1}^2(|\zeta|) + 4h^2(t). \tag{27}$$

Using the completion of squares $2ab \le \frac{1}{\Delta}a^2 + \Delta b^2$, $a, b \in R$, $\Delta > 0$, it holds

$$byz_1 \le \frac{1}{2}y^2 + \frac{1}{2}b^2 z_1^2, \tag{28}$$

$$b\delta^* e_2 y \le \epsilon e^T P^{-2}e + \bar{\epsilon}\, y^2, \quad \epsilon > 0,$$
$$\bar{\epsilon} = \frac{1}{4\epsilon}b^2\delta^{*2}p_{\max}^2 > 0, \tag{29}$$

$$yg_1(\zeta, y) \le \frac{1}{2}y^2 + \delta_{11}^2\phi_{11}^2(|\zeta|) + \delta_{12}^2\phi_{12}^2(|y|). \tag{30}$$

Introduce the notation

$$\phi_1(\zeta) = 8\sum_{i=1}^{n}\phi_{i1}^2(|\zeta|) + \delta_{11}^2\phi_{11}^2(|\zeta|), \tag{31}$$

then, substitute (28), (29), (30) and (31) into (27), one can get

$$\dot{V}_1 \le -\left(\frac{1}{2} - \epsilon\right)e^T P^{-2}e + by\vartheta_1 + 8\sum_{i=1}^{n}\phi_{i2}^2(|y|) + \delta_{12}^2\phi_{12}^2(|y|) + \bar{\epsilon}\, y^2 + \lambda_1(t)y^2 + 2y^2 + \frac{1}{2}b^2 z_1^2 + \phi_1(\zeta) + 4h^2(t). \tag{32}$$

As the suggestion of the work [10], we design the virtual control law

$$\vartheta_1 = N(s)\beta(y)y, \ \dot{s} = \kappa\beta(y)y^2 \tag{33}$$

where the Nussbaum function $N(\cdot)$ is taken as $N(s) = s^2\cos(0.5\pi s)$, $\kappa > 0$ is a design constant gain, and $\beta(\cdot) \ge 1$ is a positive smooth function satisfying

$$\beta(y)y^2 \ge \max\left\{y^2, \phi_{i2}^2(|y|), \psi_{i1}(|y|), \gamma(|y|)\middle| 1 \le i \le n\right\}. \tag{34}$$

Let $r = 8n + \delta_{12}^2 + \bar{\epsilon} + \bar{\lambda} + 3$, and the following holds true due to (33) and (34):

$$8\sum_{i=1}^{n}\phi_{i2}^2(|y|) + \delta_{12}^2\phi_{12}^2(|y|) + \bar{\epsilon}\, y^2 + \lambda_1(t)y^2 + 2y^2 \le r\beta(y)y^2. \tag{35}$$

A direct substitution (35) into (32), one can get

$$\dot{V}_1 \le -(\frac{1}{2}-\epsilon)e^T P^{-2}e + bN(s)\beta(y)y^2 + r\beta(y)y^2 + \frac{1}{2}b^2 z_1^2 + \phi_1(\zeta) + 4h^2(t). \quad (36)$$

Remark 4 Thanks to the local conditions in (4) (6) and (7), the smooth positive function $\beta(\cdot)$ can be found to meet (34). For example, with the help of $\gamma(s) = O(s^2)$, there exists a nonnegative function $\hat{\gamma}(\cdot)$, such that $\gamma(s) \le \widehat{\gamma}(s)s^2$. Consequently, it only suffices to satisfy $\beta(s) \ge \widehat{\gamma}(s)$ in (34). Additionally, a requisite assumption on the bounding functions of $\phi_{i2}(|y|)$ is that ϕ_{i2}'s are vanishing, i.e., $\phi_{i2}(0) = 0 (i = 1, \ldots, n)$.

Step 2: Considering the appearance of some unknown constant gains in the following control design, we define a new unknown constant Θ in the form of

$$\Theta = \max\left\{\frac{|b|}{2}, \frac{b^2}{4}, \delta^* + \bar{\epsilon}\right\}, \quad (37)$$

and adopt $\widehat{\Theta}(\cdot)$ to supply a online estimate of Θ with the estimate error $\widetilde{\Theta}(\cdot) = \Theta - \widehat{\Theta}(\cdot)$.

Then, we choose the Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_1^2 + \frac{1}{2\Upsilon}\widetilde{\Theta}^2, \quad \Upsilon > 0. \quad (38)$$

Together with (25) (36) and $\dot{z}_1 = \dot{\widehat{\xi}}_2 - \dot{\vartheta}_1$, it can be checked that

$$\begin{aligned}\dot{V}_2 \le & -\left(\frac{1}{2}-\epsilon\right)e^T P^{-2}e \\ & + z_1\Big(\vartheta_2 + \lambda_2(t)\widehat{\xi}_2 - C_2^T PC\widehat{\xi}_1 - \frac{\partial\vartheta_1}{\partial y}\lambda_1(t)x_1 - \frac{\partial\vartheta_1}{\partial s}\dot{s}\Big) \\ & + bN(s)\beta(y)y^2 + r\beta(y)y^2 \\ & + z_1z_2 - \frac{1}{\Upsilon}\widetilde{\Theta}\dot{\widehat{\Theta}} + \phi_1(\zeta) + 4h^2(t) \\ & - z_1\frac{\partial\vartheta_1}{\partial y}\Big(b(z_1+\vartheta_1) + b\delta^* e_2 + g_1(\zeta, y)\Big). \end{aligned} \quad (39)$$

By completing the squares, the following calculations hold

$$-b\frac{\partial\vartheta_1}{\partial y}z_1^2 \le z_1^2 + \frac{b^2}{4}\Big(\frac{\partial\vartheta_1}{\partial y}\Big)^2 z_1^2, \quad -b\frac{\partial\vartheta_1}{\partial y}z_1\vartheta_1 \le \beta(y)y^2 + \frac{b^2}{4}z_1^2\beta(y)\Big(\frac{\partial\vartheta_1}{\partial y}N(s)\Big)^2, \quad (40)$$

$$-\frac{\partial\vartheta_1}{\partial y}z_1 b\delta^* e_2 \le \epsilon e^T P^{-2}e + \frac{1}{4\epsilon}b^2\delta^{*2}p_{\max}^2(\frac{\partial\vartheta_1}{\partial y}z_1)^2, \quad (41)$$

$$-\frac{\partial\vartheta_1}{\partial y}z_1 g_1(\zeta, y) \le \delta^*\Big(\frac{\partial\vartheta_1}{\partial y}\Big)^2 z_1^2 + \frac{g_1^2(\zeta, y)}{4\delta^*}, \quad \frac{g_1^2(\zeta, y)}{4\delta^*} \le \frac{1}{2}\phi_{11}^2(|\zeta|) + \frac{1}{2}\phi_{12}^2(|y|). \quad (42)$$

Take the notations $\varphi_1(t, s) = 2\big(\frac{\partial\vartheta_1}{\partial y}\big)^2 + \beta(y)\Big(\frac{\partial\vartheta_1}{\partial y}N(s)\Big)^2$, and in view of $\frac{1}{2}\phi_{12}^2(|y|) \le \frac{1}{2}\beta(y)y^2$, we have

$$\begin{aligned} & -b\frac{\partial\vartheta_1}{\partial y}z_1(z_1+\vartheta_1) - \frac{\partial\vartheta_1}{\partial y}z_1\Big(b\delta^* e_2 + g_1(\zeta, y)\Big) \\ & \le \epsilon e^T P^{-2}e + \Theta\varphi_1(y, s)z_1^2 \\ & + \frac{1}{2}\phi_{11}^2(|\zeta|) + \frac{3}{2}\beta(y)y^2. \end{aligned} \quad (43)$$

Combining (39) and (43), we have

$$\begin{aligned}\dot{V}_2 \le & \, bN(s)\beta(y)y^2 \\ & + z_1\Big(\vartheta_2 + \rho_1 z_1 + \lambda_2(t)\widehat{\xi}_2 - C_2^T PC\widehat{\xi}_1 \\ & - \frac{\partial\vartheta_1}{\partial s}\dot{s} - \frac{\partial\vartheta_1}{\partial y}\lambda_1(t)x_1 + \widehat{\Theta}\varphi_1(\cdot)z_1\Big) \\ & - \left(\frac{1}{2} - 2\epsilon\right)e^T P^{-2}e - (\rho_1 - 1)z_1^2 \\ & + \left(r + \frac{3}{2}\right)\beta(y)y^2 + \frac{1}{\Upsilon}\widetilde{\Theta}\Big(\Upsilon\varphi_1(y, s)z_1^2 - \dot{\widehat{\Theta}}\Big) \\ & + z_1z_2 + \phi_1(\zeta) + \frac{1}{2}\phi_{11}^2(|\zeta|) + 4h^2(t). \end{aligned} \quad (44)$$

Denote $\tau_1 = \Upsilon\varphi_1(y, s)z_1^2$, $\phi_2(\zeta) = \phi_1(\zeta) + \frac{1}{2}\phi_{11}^2(|\zeta|)$, $\Omega_{N,2}(t, e, z_1) = (\frac{1}{2} - 2\epsilon)e^T P^{-2}e + (\rho_1 - 1)z_1^2$, $\Omega_{P2}(t, y, \widehat{\xi}_{[2]}, s, \widehat{\Theta}) = \lambda_2(t)\widehat{\xi}_2 - C_2^T PC\widehat{\xi}_1 - \frac{\partial\vartheta_1}{\partial s}\dot{s} - \frac{\partial\vartheta_1}{\partial y}\lambda_1(t) x_1 + \widehat{\Theta}\varphi_1(y, s)z_1$, and we choose the virtual control

$$\vartheta_2 = -\rho_1 z_1 - \Omega_{P2}(t, y, \widehat{\xi}_{[2]}, s, \widehat{\Theta}), \quad \rho_1 > 0, \quad (45)$$

then, a direct substitution into (44) yields

$$\begin{aligned}\dot{V}_2 \le & -\Omega_{N,2}(t, e, z_1) + bN(s)\beta(y)y^2 \\ & + \big(r + \frac{3}{2}\big)\beta(y)y^2 + z_1z_2 + \phi_2(\zeta) \\ & + \frac{1}{\Upsilon}\widetilde{\Theta}\big(\tau_1 - \dot{\widehat{\Theta}}\big) + 4h^2(t). \end{aligned} \quad (46)$$

Step i $(3 \le i \le n)$: Assume that, in *Step* $i-1$, we have designed the virtual control $\vartheta_j(\cdot)$ and tuning function τ_{j-1}, such that, with $z_j = \widehat{\xi}_{j+1} - \vartheta_j (3 \le j \le i-1)$, the time derivative of the function

$$V_{i-1} = V_e + \frac{1}{2}y^2 + \sum_{j=1}^{i-2}\frac{1}{2}z_j^2 + \frac{1}{2\Upsilon}\widetilde{\Theta}^2 \quad (47)$$

satisfies

$$\begin{aligned}\dot{V}_{i-1} \le & -\Omega_{N,i-1}(t, e, z_{[i-2]}) + bN(s)\beta(y)y^2 \\ & + r_{i-1}\beta(y)y^2 \end{aligned}$$

$$+\frac{1}{\Upsilon}\Big(\widetilde{\Theta}+\Upsilon\sum_{j=2}^{i-2}\frac{\partial\vartheta_j}{\partial\widehat{\Theta}}z_j\Big)\big(\tau_{i-2}-\dot{\widehat{\Theta}}\big)$$
$$+z_{i-2}z_{i-1}+\phi_{i-1}(\zeta)+4h^2(t) \tag{48}$$

with $r_i=r+\frac{3}{2}(i-1)(i=1,\ldots,n)$ and

$$\Omega_{Ni}(t,e,z_{[i-1]})=\left(\frac{1}{2}-i\epsilon\right)e^TP^{-2}e+\big(\rho_1-(i-1)\big)z_1^2+\sum_{j=2}^{i-1}\rho_jz_j^2,\ \rho_j>0,\ i=3,\ldots,n. \tag{49}$$

In the sequel, one shows that a similar property of inequality (48) holds in *Step i*. To this end, we consider the function

$$V_i=V_{i-1}+\frac{1}{2}z_{i-1}^2. \tag{50}$$

Note that the variable V_i satisfies

$$\begin{aligned}\dot V_i\le{}& z_{i-1}\Big(z_i+\vartheta_i+\lambda_i(t)\widehat{\xi}_i-C_i^TPCC^T\widehat{\xi}\\&-\frac{\partial\vartheta_{i-1}}{\partial y}\big(g\widehat{\xi}_2+g\delta^*e_2+\lambda_1(t)x_1+g_1(\zeta,y)\big)\\&-\frac{\partial\vartheta_{i-1}}{\partial t}-\frac{\partial\vartheta_{i-1}}{\partial s}\dot s-\frac{\partial\vartheta_{i-1}}{\partial\widehat{\Theta}}\dot{\widehat{\Theta}}-\sum_{j=1}^{i-1}\frac{\partial\vartheta_{i-1}}{\partial\widehat{\xi}_j}\dot{\widehat{\xi}}_j\Big)\\&+bN(s)\beta(y)y^2+r_{i-1}\beta(y)y^2+\phi_{i-1}(\zeta)\\&-\Omega_{N,i-1}(t,e,z_{[i-2]})+z_{i-2}z_{i-1}\\&+\frac{1}{\Upsilon}\Big(\widetilde{\Theta}+\Upsilon\sum_{j=2}^{i-2}\frac{\partial\vartheta_j}{\partial\widehat{\Theta}}z_j\Big)\big(\tau_{i-2}-\dot{\widehat{\Theta}}\big)+4h^2(t).\end{aligned} \tag{51}$$

As in Step 2, using the completion of the squares, we have

$$\begin{aligned}-b\frac{\partial\vartheta_{i-1}}{\partial y}z_{i-1}(z_1+\vartheta_1)\le{}& z_1^2+\beta(y)y^2+\Theta\Big(\Big(\frac{\partial\vartheta_{i-1}}{\partial y}\Big)^2\\&+\beta(y)\Big(\frac{\partial\vartheta_{i-1}}{\partial y}N(s)\Big)^2\Big)z_{i-1}^2,\end{aligned} \tag{52}$$

$$\begin{aligned}-\frac{\partial\vartheta_{i-1}}{\partial y}z_{i-1}\big(b\delta^*e_2+g_1(\zeta,y)\big)\le{}&\epsilon e^TP^{-2}e\\&+\frac{1}{4\epsilon}b^2\delta^{*2}p_{\max}^2\Big(\frac{\partial\vartheta_{i-1}}{\partial y}\Big)^2z_{i-1}^2+\delta^*\Big(\frac{\partial\vartheta_{i-1}}{\partial y}\Big)^2z_{i-1}^2\\&+\frac{g_1^2(\zeta,y)}{4\delta^*},\end{aligned} \tag{53}$$

this together with (42) and $\frac{1}{2}\phi_{12}^2(|y|)\le\frac{1}{2}\beta(y)y^2$, then, the following holds

$$\begin{aligned}-\frac{\partial\vartheta_{i-1}}{\partial y}z_{i-1}\big(b\delta^*e_2+g_1(\zeta,y)\big)\le{}&\epsilon e^TP^{-2}e\\&+\Theta\Big(\frac{\partial\vartheta_{i-1}}{\partial y}\Big)^2z_{i-1}^2+\frac{1}{2}\phi_{11}^2(|\zeta|)+\frac{1}{2}\beta(y)y^2.\end{aligned} \tag{54}$$

Let

$$\varphi_i\left(t,y,\widehat{\xi}_{[i]},s,\widehat{\Theta}\right)=2\left(\frac{\partial\vartheta_i}{\partial y}\right)^2+\beta(y)\Big(\frac{\partial\vartheta_i}{\partial y}N(s)\Big)^2,\quad i=2,\ldots,n-1, \tag{55}$$

and in accordance with (52–54), we further obtain

$$\begin{aligned}&-b\frac{\partial\vartheta_{i-1}}{\partial y}z_{i-1}(z_1+\vartheta_1)-\frac{\partial\vartheta_{i-1}}{\partial y}z_{i-1}\big(b\delta^*e_2+g_1(\zeta,y)\big)\\&\le\epsilon e^TP^{-2}e+z_1^2+\frac{3}{2}\beta(y)y^2\\&\quad+\Theta\varphi_{i-1}\left(t,y,\widehat{\xi}_{[i-1]},s,\widehat{\Theta}\right)z_{i-1}^2+\frac{1}{2}\phi_{11}^2(|\zeta|).\end{aligned} \tag{56}$$

Let

$$\tau_{i-1}=\Upsilon\varphi_{i-1}(t,y,\widehat{\xi}_{[i-1]},s,\widehat{\Theta})z_{i-1}^2, \tag{57}$$

$$\phi_i(\zeta)=\phi_{i-1}(\zeta)+\frac{1}{2}\phi_{11}^2(|\zeta|), \tag{58}$$

$$\begin{aligned}&\Omega_{Pi}\left(t,y,\widehat{\xi}_{[i]},s,\widehat{\Theta}\right)\\&=z_{i-2}-C_i^TPCC^T\widehat{\xi}+\lambda_i(t)\widehat{\xi}_i-\frac{\partial\vartheta_{i-1}}{\partial s}\dot s\\&\quad-\sum_{j=1}^{i-1}\frac{\partial\vartheta_{i-1}}{\partial\widehat{\xi}_j}\dot{\widehat{\xi}}_j-\frac{\partial\vartheta_{i-1}}{\partial y}\lambda_1(t)x_1-\frac{\partial\vartheta_{i-1}}{\partial t}\\&\quad+2\widehat{\Theta}\varphi_{i-1}(t,y,\widehat{\xi}_{[i-1]},s,\widehat{\Theta})z_{i-1}\\&\quad-\frac{\partial\vartheta_{i-1}}{\partial\widehat{\Theta}}\tau_{i-2}-2\Upsilon\sum_{j=2}^{i-1}\frac{\partial\vartheta_j}{\partial\widehat{\Theta}}z_j\varphi_{i-1}(y,s)z_{i-1},\end{aligned} \tag{59}$$

substituting (57–59) into (51), and we obtain

$$\begin{aligned}\dot V_i\le{}&-\Omega_{N,i}(e,z_{[i-1]})\\&+bN(s)\beta(y)y^2+r_i\beta(y)y^2\\&+z_{i-1}\Big(\vartheta_i+\rho_{i-1}z_{i-1}+\Omega_{Pi}(t,y,\widehat{\xi}_{[i]},s,\widehat{\Theta})\Big)\end{aligned}$$

$$+z_{i-1}z_i+\phi_i(\zeta)+\frac{1}{\Upsilon}\Big(\widetilde{\Theta}+\Upsilon\sum_{j=2}^{i-1}\frac{\partial\vartheta_j}{\partial\widehat{\Theta}}z_j\Big)\times\big(\tau_{i-1}-\dot{\widehat{\Theta}}\big)+4h^2(t). \quad (60)$$

Take the virtual control input

$$\vartheta_i=-\rho_{i-1}z_{i-1}-\Omega_{Pi}(t,y,\widehat{\xi}_{[i]},s,\widehat{\Theta}), \quad (61)$$

which makes (60) further satisfy

$$\dot{V}_i\le-\Omega_{N,i}(e,z_{[i-1]})+bN(s)\beta(y)y^2+r_i\beta(y)y^2+\frac{1}{\Upsilon}\Big(\widetilde{\Theta}+\Upsilon\sum_{j=2}^{i-1}\frac{\partial\vartheta_j}{\partial\widehat{\Theta}}z_j\Big)\big(\tau_{i-1}-\dot{\widehat{\Theta}}\big)+z_{i-1}z_i+\phi_i(\zeta)+4h^2(t). \quad (62)$$

Step n: Because of the estimate $\widehat{\xi}_{n+1}$ for the external disturbance $\mathrm{d}(t)$, this step is crucial and slightly different from the *Step n* in traditional Backstepping design procedure. For notational consistence, z_n, ϑ_n and the control u can be written in the form of $z_n=0$, $u+\widehat{\xi}_{n+1}=\vartheta_n$, and $z=(z_1,\ldots,z_n)$. To obtain the real control input u, we consider the function

$$V_n=V_{n-1}+\frac{1}{2}z_{n-1}^2. \quad (63)$$

After some similar calculations such as (52–54) and (42) with $i=n$ in *Step i*, the time derivative of (63) has the following form

$$\dot{V}_n\le-\Omega_{Nn}(e,z)+z_{n-1}\Big(u+\rho_{n-1}z_{n-1}+\widehat{\xi}_{n+1}+\Omega_{Pn}(t,y,\widehat{\xi},s,\widehat{\Theta})\Big)+bN(s)\beta(y)y^2+r_n\beta(y)y^2+\frac{1}{\Upsilon}\Big(\widetilde{\Theta}+\Upsilon\sum_{j=2}^{n-1}\frac{\partial\vartheta_j}{\partial\widehat{\Theta}}z_j\Big)\big(\tau_{n-1}-\dot{\widehat{\Theta}}\big)+\phi_n(\zeta)+4h^2(t). \quad (64)$$

Take the real control input

$$u=-\widehat{\xi}_{n+1}+\vartheta_n=-\widehat{\xi}_{n+1}-\rho_{n-1}z_{n-1}-\Omega_{Pn}(t,y,\widehat{\xi},s,\widehat{\Theta}), \quad (65)$$

and the adaptive law

$$\dot{\widehat{\Theta}}=\tau_{n-1}=\sum_{i=1}^{n-1}\varphi_i(t,y,\widehat{\xi}_{[i]},s,\widehat{\Theta})z_i^2, \quad (66)$$

then, a direct substitution yields that

$$\dot{V}_n\le-\Omega_{Nn}(e,z)+bN(s)\beta(y)y^2+r_n\beta(y)y^2+\phi_n(\zeta)+4h^2(t). \quad (67)$$

Choose the positive constant $\chi=\max\left\{\frac{n-1}{2}+\delta_{11}^2,1\right\}$, and one can get

$$\phi_n(\zeta)=\frac{n-1}{2}\phi_{11}^2(|\zeta|)+\sum_{i=1}^{n}\phi_{i1}^2(|\zeta|)+\delta_{11}^2\phi_{11}^2(|\zeta|)\le\chi\sum_{i=1}^{n}\phi_{i1}^2(|\zeta|). \quad (68)$$

As a result, it holds

$$\dot{V}_n\le-\Omega_{Nn}(e,z)+bN(s)\beta(y)y^2+r_n\beta(y)y^2+\chi\sum_{i=1}^{n}\phi_{i1}^2(|\zeta|)+4h^2(t). \quad (69)$$

At this stage, the design procedure of the output feedback adaptive control has been completed.

2.3 Main results

Now we present the main theorem of this paper.

Theorem 1 *Suppose that Assumptions 1–4 hold for the controlled system (1). Then, using the observer (17) and (18), we can find a smooth adaptive control scheme, such that the global robust regulation control problem is solvable for the cascade nonlinear system (1) subject to the external disturbance. Moreover, we have the following statements:*

(a) *the solutions of the closed-loop system are globally uniformly bounded;*
(b) *the asymptotic convergence of system state is achieved, that is,*

$$\lim_{t\to\infty}\big(|\zeta(t)|+|x(t)|\big)=0; \quad (70)$$

(c) *the control input is bounded, and moreover,*

$$\lim_{t\to\infty}u(t)=0 \quad \textit{if} \quad \lim_{t\to\infty}\mathrm{d}(t)=0. \quad (71)$$

Proof It is easy to see that the right-hand sides of closed-loop system are locally Lipschitz in a domain of initial conditions and hence the system has a unique solution $\big(y(t),z(t),\widehat{\xi}(t),s(t),\widehat{\Theta}(t)\big)$ on a small interval $[0,t_f)$. Assume, without loss of generality, $[0,t_f)$ be its maximal interval of the existence and uniqueness.

We will show that $t_f = \infty$. To begin with, for the term of $\Omega_{Nn}(e, z)$ defined in (49) with $i = n$, choose the suitable constants satisfying

$$\epsilon < \frac{1}{2n}, \ \rho_1 > n-1, \ \rho_i > 0, \ i = 2, \ldots, n-1, \tag{72}$$

which guarantees

$$\Omega_{Nn}(e, z) = \left(\frac{1}{2} - n\epsilon\right) e^T P^{-2} e + \Big(\rho_1 - (n-1)\Big) z_1^2 + \sum_{j=2}^{n-1} \rho_j z_j^2 > 0. \tag{73}$$

Furthermore, it holds

$$\dot{V}_n \le bN(s)\beta(y)y^2 + r_n\beta(y)y^2 + \chi \sum_{i=1}^{n} \phi_{i1}^2(|\zeta|) + 4h^2(t). \tag{74}$$

Considering $\dot{s} = \kappa\beta(y)y^2$ in (33), (74) turns into

$$\dot{V}_n \le \frac{1}{\kappa}\Big(bN(s) + r_n\Big)\dot{s} + \chi \sum_{i=1}^{n} \phi_{i1}^2(|\zeta|) + 4h^2(t). \tag{75}$$

From (10) and (34), the following calculations hold true

$$\begin{aligned}\int_0^t \phi_{i1}^2(|\zeta(s)|)\mathrm{d}s &\le \bar{\psi}_{i0}(|\zeta(0)|) + \int_0^t \psi_{i1}(|y(s)|)\mathrm{d}s \\ &\le \bar{\psi}_{i0}(|\zeta(0)|) + n \cdot \frac{1}{\kappa}\int_0^t \dot{s}(\tau)\mathrm{d}\tau.\end{aligned} \tag{76}$$

Then, integrate both sides of (75), and one can get

$$V_n(t) \le C_0 + \frac{1}{\kappa}\int_0^t \Big(bN(s) + r_n + n\chi\Big)\dot{s}(\tau)\mathrm{d}\tau + 4\int_0^t h^2(\tau)\mathrm{d}\tau, \ \forall\, t > 0 \tag{77}$$

with constants $C_0 = V_n(0) + \chi \sum_{i=1}^{n} \bar{\psi}_{i0}(|\zeta(0)|)$. According to Assumption 4, one can get

$$h(t) = \frac{1}{b}\dot{d}(t) \in L_2, \tag{78}$$

and then, the term $4\int_0^t h^2(\tau)\mathrm{d}\tau$ in the right-hand side satisfies

$$4\int_0^t h^2(\tau)d\tau < +\infty, \ \forall\, t > 0. \tag{79}$$

Using the argument of contradiction, it can be concluded that $s(t)$, $V_n(y, \varepsilon, z, \widetilde{\Theta})$ and then $\big(y(t), e(t), z(t), \widetilde{\Theta}(t)\big)$ must be bounded over $[0, t_f)$. Also, $\widehat{\Theta}(t)$ is bounded on $[0, t_f)$. The boundedness of $y(t)$ means that $z_1(t)$ is bounded. This together with the boundedness of e(hence e_1) ensures that $\widehat{\xi}_1(t)$ is bounded. Noting the boundedness of $s(t)$ and $y(t)$ on $[0, t_f)$, we know that $\vartheta_1(\cdot)$ is bounded on $[0, t_f)$. Due to $z_1 = \widehat{\xi}_2 - \vartheta_1(\cdot)$, it follows that $\widehat{\xi}_2(t)$ is bounded. Similarly, $\vartheta_{i-1}(\cdot)$, $\widehat{\xi}_i(t)(3 \le i \le n)$ are also bounded on $[0, t_f)$. In particular, considering e_{n+1} is bounded with $e_{n+1} = \frac{1}{\delta^*}\varepsilon_{n+1} = \frac{1}{\delta^*}(z_{n+1} - \widehat{\xi}_{n+1})$, and $z_{n+1} = \frac{1}{b}d(t)$ is bounded according to Assumption 4, we can conclude that $\widehat{\xi}_{n+1} \in L_\infty$. Considering $s(t)$ is bounded and $\int_0^t \gamma\big(y(s)\big)ds < \int_0^t \dot{s}(s)ds = \big(s(t) - s(0)\big)$ for all $t > 0$, we then obtain $\int_0^t \gamma\big(y(s)\big)ds < \infty, \forall\, t > 0$. It thus follows, using Prop.6 in [3], that $\zeta(t)$ is bounded on $[0, t_f)$. So far all the closed-loop system signals are bounded on $[0, t_f)$. This shows that the finite time escape will not happen. Therefore, it is natural that t_f can be maximized to $+\infty$. This completes the property (a).

Since all signals in closed-loop are bounded over $[0, +\infty)$, from (69) and (77), one can obtain

$$\begin{aligned}&e_i \in L_2(i = 1, \ldots, n+1),\\ &z_i \in L_2(i = 1, \ldots, n-1).\end{aligned} \tag{80}$$

The property of $\dot{e}_i \in L_\infty$ and $\dot{z}_i \in L_\infty$ implies that both $e_i(t)$ and $z_i(t)$ are uniformly continuous. By Barbalat's Lemma, we have

$$\begin{aligned}&\lim_{t\to\infty} e_i(t) = 0(i = 1, \ldots, n+1),\\ &\lim_{t\to\infty} z_i(t) = 0(i = 1, \ldots, n-1).\end{aligned} \tag{81}$$

The fact of $y(t), \dot{y}(t) \in L_\infty$ implies that $\kappa\beta(y)y^2$ is uniformly continuous. Furthermore, $s(t)$ is monotonically nondecreasing over $[0, +\infty)$ because of $\dot{s}(t) \ge 0$, which together with $s(t) \in L_\infty$ results $\kappa\beta(y)y^2$ is integrable over $[0, +\infty)$. Considering the choice of $\beta(\cdot) \ge 1$ and $\kappa > 0$, one can derive that $\lim_{t\to\infty} y(t) = 0$. Hence, $z_1(t)$ and $\vartheta_1(\cdot)$ go to zero as $t \to \infty$. In view of $e_1 = \frac{1}{\delta^*}(\xi_1 - \widehat{\xi}_1)$, we have $\lim_{t\to\infty} \widehat{\xi}_1(t) = 0$. In a recursive manner, $\vartheta_i(\cdot)$ and $\widehat{\xi}_i(t)(i = 1, \ldots, n)$ asymptotically converge to zero. Particularly, with $e_i = \frac{1}{\delta^*}(\xi_i - \widehat{\xi}_i)$, it follows that $\xi_i(t) \to 0$ as $t \to \infty$, which leads to $\lim_{t\to\infty} x_i(t) = 0(i = 1, \ldots, n)$. Since $\int_0^t \gamma\big(y(s)\big)ds < \infty$, again using Prop.6 in [3], we derive that $\lim_{t\to\infty} \zeta(t) = 0$. This shows the property (b).

Next, we are ready to prove the last part (c) in Theorem 1. According to the aforementioned calculations, we have established $\widehat{\xi}_{n+1} \in L_\infty$ and $\lim_{t\to\infty} \vartheta_n(t) = 0$. Considering the control law defined by

$$u = -\widehat{\xi}_{n+1} + \vartheta_n, \tag{82}$$

it is clear that $u \in L_\infty$. Additionally, considering

$$\begin{aligned} &e_{n+1} = \xi_{n+1} - \widehat{\xi}_{n+1}, \\ &\lim_{t\to\infty} e_{n+1}(t) = 0,\ z_{n+1} = \frac{1}{b}\,\mathrm{d}(t) \in L_\infty, \end{aligned} \tag{83}$$

one can conclude

$$\begin{aligned} &\lim_{t\to\infty} \widehat{\xi}_{n+1}(t) = 0 \quad \text{if} \\ &\lim_{t\to\infty} d(t) = 0, \end{aligned} \tag{84}$$

which in turn shows that

$$\lim_{t\to\infty} u(t) = 0 \quad \text{if} \quad \lim_{t\to\infty} \mathrm{d}(t) = 0. \tag{85}$$

This ends the proof. □

3 Numerical results and discussion

In this section, the fan speed control system subject to external disturbance is used a practical example to illustrate our output feedback design methodology. It is shown that even disturbed by some external nonvanishing disturbances in the armature voltage, to realize the asymptotic regulation control of any desired constant reference signal for the fan speed, the proposed speed controller does the job.

3.1 Model analysis

From [29], the dynamics of a fan driven by a DC motor is described by

$$\begin{aligned} &J_1\dot{\upsilon} = k_1 I - \tau_L - \tau_D(\upsilon) \\ &J_2\dot{I} = u_0 - k_2\upsilon - RI + d(t) \\ &\bar{y} = \upsilon\,, \end{aligned} \tag{86}$$

where υ is the fan speed, I is the armature current, τ_L is an uncertain constant load torque, $\tau_D(\upsilon)$ is an uncertain drag torque, u_0 is the armature voltage which is considered as the input, J_1, k_1, k_2, R are known positive constants, and the inductance J_2 may be an unknown constant. The function $\mathrm{d}(t)$ represents the uncertain external disturbance. The control task is the set-point regulation control of the fan speed υ to the constant value $\upsilon_r (= \bar{y}_r)$, irrespective of the unknown τ_L, $\tau_D(\upsilon)$ and the disturbance $\mathrm{d}(t)$.

Assumption 5 For the drag torque $\tau_D(\upsilon)$ and each $\upsilon_r \in \mathcal{R}$, there exist an unknown constant $\sigma > 0$ and a known smooth function $\omega(\cdot)$ such that

$$|\tau_D(\upsilon) - \tau_D(\upsilon_r)| \le \sigma \cdot \omega(|\upsilon - \upsilon_r|), \forall\, \upsilon \in R. \tag{87}$$

In order to realize the control objective, to be first, we introduce the change of coordinates [10]

$$\begin{aligned} &\chi_1 = \upsilon, \quad \chi_2 = \frac{k_1}{J_1} I, \\ &\bar{u} = \frac{k_1}{J_1}(u_0 - k_2\upsilon - RI) + \frac{k_1}{J_1}\,\mathrm{d}(t)\,, \end{aligned} \tag{88}$$

and let $\eta(\upsilon) = -\frac{1}{J_1}(\tau_L + \tau_D(\upsilon))$, $b = \frac{1}{J_2}$, then we have

$$\begin{aligned} &\dot{\chi}_1 = \chi_2 + \eta(\bar{y}) \\ &\dot{\chi}_2 = b\bar{u} \\ &\bar{y} = \chi_1. \end{aligned} \tag{89}$$

For any $L > 0$, define the auxiliary variables

$$\dot{\Pi} = -L\,\Pi - L,\ \ \dot{\Xi} = -L\,\Xi + \bar{u}. \tag{90}$$

Let $\eta^* = \eta(\bar{y}_r)$, where $\bar{y}_r = \upsilon_r$ is some desired setting-point constant speed, and introduce the new coordinate variables (ζ, x_1, x_2)

$$\begin{aligned} &x_1 = \chi_1 - \bar{y}_r = \upsilon - \upsilon_r, \\ &\zeta = \chi_2 - Lx_1 - b\,\Xi - \Pi\eta^*, \\ &x_2 = b\,\Xi + (\Pi + 1)\eta^*. \end{aligned} \tag{91}$$

Then, in view of $x_1 = \bar{y} - \bar{y}_r$, we can newly define system output $y = x_1$ and $u = \frac{k_1}{J_1}(u_0 - k_2\upsilon - RI)$. It can be verified that

$$\begin{aligned} &\dot{\zeta} = -L\zeta - L^2x_1 - L(\eta(y + \upsilon_r) - \eta(\upsilon_r)) \\ &\dot{x}_1 = x_2 + Lx_1 + \zeta + \eta(y + \upsilon_r) - \eta(\upsilon_r) \\ &\dot{x}_2 = bu - Lx_2 + b\frac{k_1}{J_1}\mathrm{d}(t) \\ &y = x_1. \end{aligned} \tag{92}$$

Clearly, regarded ζ-subsystem as the cascaded dynamics, the system (92) falls into the form of (1) with $\lambda_1(t) = L$, $\lambda_2(t) = -L$, $g_1(\zeta, y) = \zeta + \eta(y + \upsilon_r) - \eta(\upsilon_r)$, $g_2(\zeta, y) = 0$, and the inductance coefficient J_2 (actually $1/J_2$) serving as the unknown high-frequency gain. Moreover, like the calculations in [12], the ζ dynamics is ISS (consequently iISS) with ζ as the

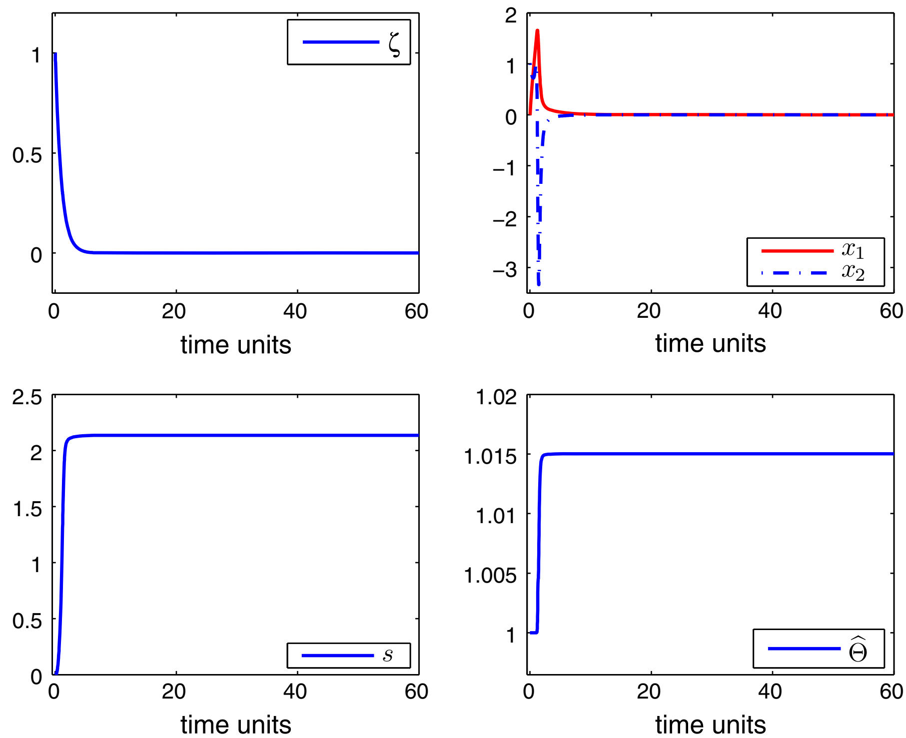

Fig. 1 The system states of the closed-loop system

state and x_1 as the input, together with an ISS-Lyapunov function $U_0(\zeta) = \zeta^2/2L$ satisfying

$$\dot{U}_0(\zeta) \leq -\frac{1}{2}\zeta^2 + L^2 x_1^2 + \frac{\sigma^2}{J_1^2}\omega^2(|x_1|). \tag{93}$$

This fulfills Assumption 1. From Assumption 5, the following calculations hold true

$$\begin{aligned} |g_1(\zeta, y)| &\leq |\zeta| + |\eta(y + \upsilon_r) - \eta(\upsilon_r)| \\ &\leq |\zeta| + \frac{\sigma}{J_1}\omega(|y|). \end{aligned} \tag{94}$$

Hence, it is evident that Assumption 2 holds with $\delta_{11} = 1$, $\delta_{12} = \sigma/J_1$, $\phi_{11}(|\zeta|) = |\zeta|$ and $\phi_{12}(|y|) = \omega(|y|)$. In simulation, we choose

$$d(t) = \theta \cdot \arctan t, \; \dot{d}(t) = \frac{\theta}{1+t^2}, \; \theta \in R. \tag{95}$$

This shows $d(t)$ and $\dot{d}(t)$ meets Assumption 4. The Assumption 3 is clear since $\lambda_1(t) = L$, $\lambda_2(t) = -L$ with the constant $L > 0$.

3.2 The fan speed regulator design

Similar to the proposed control method in Sect. 3, define

$$\begin{aligned} \xi_1 &= \frac{1}{b}x_1 = J_2 x_1, \\ \xi_2 &= \frac{1}{b}x_2 = J_2 x_2, \\ \xi_3 &= \frac{k_1}{J_1}\mathrm{d}(t), \; h(t) = \frac{k_1}{J_1}\dot{\mathrm{d}}(t), \end{aligned} \tag{96}$$

and we have the following new state equations

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + L\xi_1 + J_2\zeta + J_2\big(\eta(y + \upsilon_r) - \eta(\upsilon_r)\big) \\ \dot{\xi}_2 &= u - L\xi_2 + \xi_3 \\ \dot{\xi}_3 &= h(t) \\ y &= x_1 (= \upsilon - \upsilon_r). \end{aligned} \tag{97}$$

For simulation purposes, we set the bounding function $\omega(s) = \varpi s$ with $\varpi > 0$ in Assumption 5, and hence, according to (93) and (94), the gain function $\beta(\cdot)$ can be picked as some constant $\beta > 0$. Then, using the proposed control algorithm in Sect. 3, we design the control law

$$\begin{aligned} u = &-\rho_1 z_1 + L\widehat{\xi}_2 + p_{12}(t)\widehat{\xi}_1 - \widehat{\xi}_3 \\ &+ \frac{\partial \vartheta_1}{\partial y}Lx_1 + \frac{\partial \vartheta_1}{\partial s}\dot{s} - \widehat{\Theta}\varphi_1(y, s)z_1, \end{aligned} \tag{98}$$

and the adaption law

$$\dot{\widehat{\Theta}} = \Upsilon\varphi_1(y, s)z_1^2 \tag{99}$$

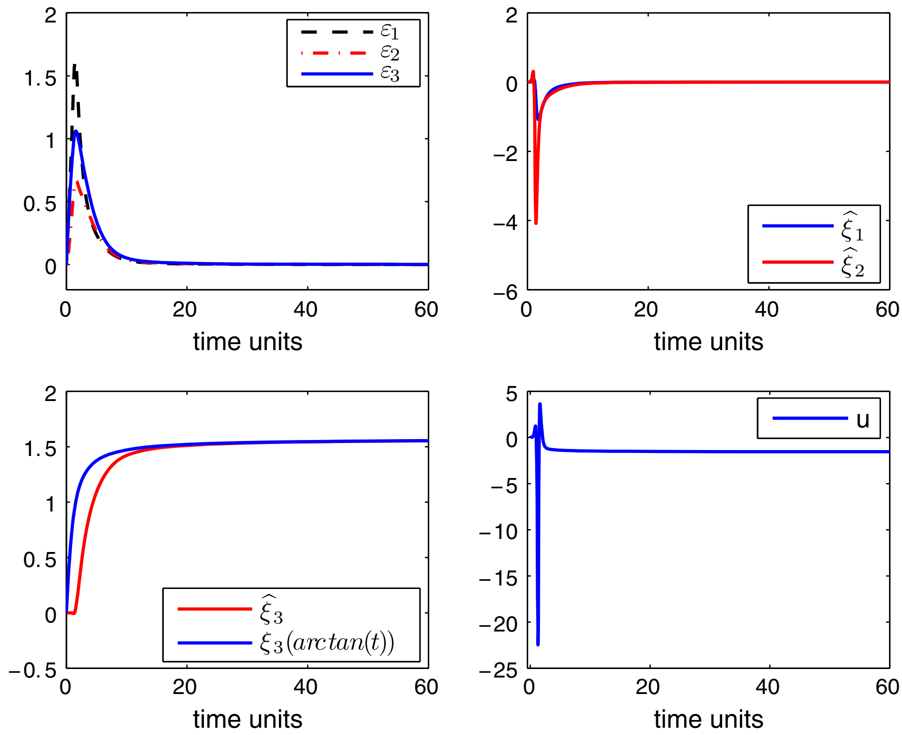

Fig. 2 The state estimates, errors and control input of the closed-loop system

with $\varphi_1(y,s) = 2\beta^2 N^2(s) + \beta^3 N^4(s)$, $\vartheta_1 = \beta N(s)y$, $\dot{s} = \kappa\beta y^2$.

3.3 Simulation results

The parameter values in (86) are set to $J_1 = J_2 = R = k_1 = k_2 = 1$, $\sigma = 1$, $L = 1$, $\rho_1 = 1$, $\Upsilon = 1$, $\kappa = 1$, $\theta = 1$, and the initial conditions: $\zeta(0) = 1$, $x(0) = (0,0)$, $s(0) = 0$, $\hat{\xi}(0) = (0,0,0.5)$, $\widehat{\Theta}(0) = 0$. The external disturbance $\mathrm{d}(t)$ is chosen as in (95). From the simulation results shown in Figs. 1, 2 and 3, it can be seen that the good set-point regulation performance can be achieved by use of the proposed methodology in the paper. In this way, the regulation of the fan speed υ to a desired value υ_r by output feedback is solved, even under the external disturbance in the armature voltage.

3.4 Discussion

It is noted that when the armature current I is not measured, the control signal

$$u_0 = \frac{J_1}{k_1}u + k_2\upsilon + RI \tag{100}$$

is not implemented. Here we construct a current observer for the unmeasured armature current I in the form of

$$J_2\dot{\hat{I}} = u_0 - k_2\upsilon - R\hat{I}. \tag{101}$$

Let the observer error $\tilde{I} = I - \hat{I}$, then the error dynamics is

$$J_2\dot{\tilde{I}} = -R\tilde{I} + \mathrm{d}(t). \tag{102}$$

Form the above analysis, it can be proved that $\mathrm{d}(t)$ tends to zero so does $\tilde{I}$.

Remark 5 In consideration of the armature voltage u_0 and the armature current I in (86), we can calculate the power of $P = u_0 I$ with the help of our proposed controller (98). It can be shown that the power consumption is dependent on the target speed signal to be tracked. The larger the fan speed, the more power it consumes. For example, the power consumption increases along with the set-point speed value from $\upsilon_r = 1$ to $\upsilon_r = 3$. Observe that the numerical results shown in Fig.3 are consistent with the analytical dis-

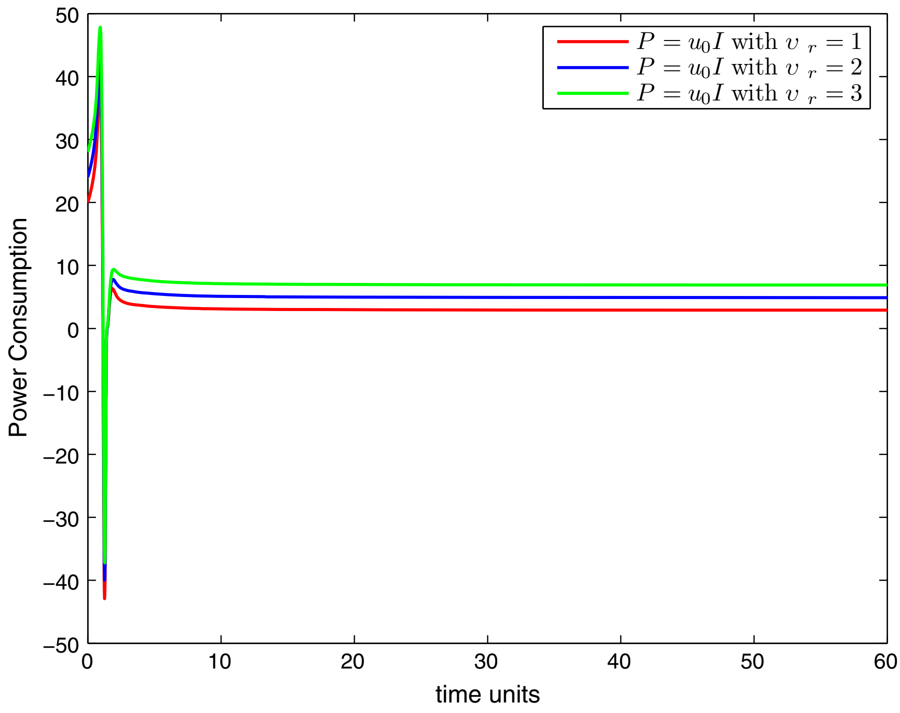

Fig. 3 The power consumption of the closed-loop system

cussion. More details on this topic can be found in [32].

4 Conclusion

In this paper, the global robust regulation problem is solved for a class of disturbed nonlinear uncertain systems in cascade. Using the ideas of the Nussbaum-type gain and the disturbance as a generalized state, we design a dynamic ESO based on a Riccati differential equation to handle the unknown control coefficients and external disturbances. Moreover, in this setting, it is shown that the additive disturbance does not require to be square integrable, which broadens the family of admissible noise signals. As a continuing research of our previous results, the proposed control scheme is verified by the fan speed control system with some external disturbance in the armature voltage. It is of interest to investigate the global robust regulation control in the presence of nonvanishing nonlinear uncertainties in the future research works.

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Appendix: Proof of Proposition 1

From $\overbrace{P^{-1}(t)}^{\cdot} = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$, by virtue of (18) and (22), we can show that

$$\dot{V}_e = -e^T P^{-2}(t)e - e_1^2 + \frac{2e_1 x_1}{\delta^* b} + \frac{2e^T P^{-1}(t)C_{n+1}h(t)}{\delta^*} + \frac{2e^T P^{-1}(t)B\cdot G(\zeta, y)}{\delta^* b}. \tag{103}$$

Given the choice of δ^*, by completing the squares, we have

$$\frac{2e_1 x_1}{\delta^* b} \le e_1^2 + y^2, \tag{104}$$

$$\frac{2e^T P^{-1}(t)B\cdot G(\zeta, y)}{\delta^* b} \le \frac{1}{4}e^T P^{-2}(t)e + 8\sum_{i=1}^{n}\left(\phi_{i1}^2(|\zeta|) + \phi_{i2}^2(|y|)\right), \tag{105}$$

$$\frac{2e^T P^{-1}(t)C_{n+1}h(t)}{\delta^*} \le \frac{1}{4}e^T P^{-2}(t)e + 4h^2(t). \tag{106}$$

A direct substitution leads to (24).

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