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Unified Riccati equation expansion method and its application to two new classes of Benjamin–Bona–Mahony equations

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Abstract The Bäcklund transformations and the superposition formulas of the Riccati equation with constant coefficients are constructed. Two fractional type solutions of the Riccati equation are obtained from its Bäcklund transformations. The equivalence relations between fractional solutions and previous known solutions are proved. A so-called unified Riccati equation expansion method for generating infinite number of exact traveling wave solutions for nonlinear evolution equations is then developed on the basis of the fractional solutions. With the method, infinitely many exact traveling wave solutions of two new classes of Benjamin–Bona–Mahony equations are presented.

Keywords Riccati equation · Fractional solution · Traveling wave solution · Periodic solution

1 Introduction

In recent years, many effective direct methods [1–18] have been developed to solve nonlinear evolution equations (NLEEs) which describes the wave propagation in fluid, plasma, optical fibers, elastic media, and so on. In general, most of the direct methods are related to an auxiliary ordinary differential equation (ODE) and some known special solutions of the auxiliary equation.

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For example, the generalized Riccati equation mapping method (GREMM) is an effective direct method to construct the exact solitary wave solutions, the periodic solutions and the rational solutions for NLEEs. This method used the constant coefficients Riccati equation

$$F'(\xi) = c_0 + c_1 F(\xi) + c_2 F^2(\xi), \tag{1}$$

as the auxiliary equation and took its special solutions as the following [19]

$$F(\xi) = \begin{cases} -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right), & \Delta > 0, \\ -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right), & \Delta > 0, \\ -\frac{c_1}{2c_2} - \frac{1}{c_2\xi + c}, & \Delta = 0, \\ -\frac{c_1}{2c_2} + \frac{\sqrt{-\Delta}}{2c_2}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right), & \Delta < 0, \\ -\frac{c_1}{2c_2} - \frac{\sqrt{-\Delta}}{2c_2}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right), & \Delta < 0, \end{cases} \tag{2}$$

where $\Delta = c_1^2 - 4c_0c_2$ and c is an arbitrary constant. Although twenty-seven solutions of Eq. (1) were cited in many studies [20–23], these solutions are equivalent to the solutions (2) in the sense of wave translation or just their forms are different in expressions. However, no work is done to deal with the classification and prove the equivalence relations for those solutions. On the contrary, more and more repeated solutions for NLEEs which induced by the solutions of Eq. (1) appear in the recent works [24–28]. It is clear that the classification of the solutions of Eq. (1) is very important to avoid those repeated solutions and establish a systematic theory of GREMM. Therefore, it remains open problems as how to prove the equivalence relations and look for the independent solutions for Eq. (1) and how to give an essential generalization of the GREMM. Obviously, this problem cannot be solved by the usual detection method and have to seek another systematic way. The aim of this work is to give a method for solving these problems. To serve the purpose, we shall first construct new fractional type solutions of the Riccati Eq. (1) through use of the technique named Bäcklund transformation and superposition formula. And then we shall prove the equivalence relations between new fractional solutions and previous known solutions. Third, we shall propose a novel generalization of the GREMM using our obtained fractional type solutions. Finally, the presented unified Riccati equation expansion method will be performed to construct infinite number of exact traveling wave solutions of two new classes of Benjamin–Bona–Mahony (BBM) equation. These two examples will show that the previously known solutions can be recovered by our method, and the GREMM is only a special case of our method.

2 Bäcklund transformations and superposition formulas

2.1 Bäcklund transformations

Let $F_n(\xi)$ and $F_{n-1}(\xi)$ be two solutions of the Riccati Eq. (1), that is

$$F_n'(\xi) = c_0 + c_1 F_n(\xi) + c_2 F_n^2(\xi), \tag{3}$$

$$F_{n-1}'(\xi) = c_0 + c_1 F_{n-1}(\xi) + c_2 F_{n-1}^2(\xi). \tag{4}$$

From (3) and (4), we can obtain

$$\frac{dF_n}{c_0 + c_1 F_n + c_2 F_n^2} = \frac{dF_{n-1}}{c_0 + c_1 F_{n-1} + c_2 F_{n-1}^2}. \tag{5}$$

Integrating (5) with respect to ξ yields

$$\operatorname{artanh}\frac{2c_2 F_n + c_1}{\sqrt{\Delta}} - \operatorname{artanh}\frac{2c_2 F_{n-1} + c_1}{\sqrt{\Delta}} = \operatorname{artanh}(A), \quad \Delta > 0, \tag{6}$$

$$\arctan\frac{2c_2 F_n + c_1}{\sqrt{-\Delta}} - \arctan\frac{2c_2 F_{n-1} + c_1}{\sqrt{-\Delta}} = \arctan(B), \quad \Delta < 0, \tag{7}$$

where A and B are integration constants.

To solve (6) and (7) for F_n and setting $A = \alpha^{-1}$ and $B = \beta^{-1}$, we obtain the Bäcklund transformations of the Riccati Eq. (1) as

$$F_n(\xi) = \frac{\left(\alpha\sqrt{\Delta} - c_1\right) F_{n-1}(\xi) - 2c_0}{2c_2 F_{n-1}(\xi) + \alpha\sqrt{\Delta} + c_1}, \quad \Delta > 0, \tag{8}$$

$$F_n(\xi) = \frac{\left(\beta\sqrt{-\Delta} + c_1\right) F_{n-1}(\xi) + 2c_0}{\beta\sqrt{-\Delta} - c_1 - 2c_2 F_{n-1}(\xi)}, \quad \Delta < 0. \tag{9}$$

2.2 Superposition formulas

2.2.1 The case $\Delta > 0$

Take four solutions $F_0 = F_0(\xi)$, $F_1 = F_1(\xi;\alpha_1)$, $F_2 = F_2(\xi;\alpha_2)$, $F_3 = F_{12}(\xi;\alpha_1,\alpha_2) = F_{21}(\xi;\alpha_2,\alpha_1)$ of the Riccati Eq. (1) and insert them into the Bäcklund transformation (8), we have

$$F_1 = \frac{\left(\alpha_1\sqrt{\Delta} - c_1\right) F_0 - 2c_0}{2c_2 F_0 + \alpha_1\sqrt{\Delta} + c_1}, \tag{10}$$

$$F_2 = \frac{\left(\alpha_2\sqrt{\Delta} - c_1\right) F_0 - 2c_0}{2c_2 F_0 + \alpha_2\sqrt{\Delta} + c_1}, \tag{11}$$

$$F_3 = \frac{\left(\alpha_2\sqrt{\Delta} - c_1\right) F_1 - 2c_0}{2c_2 F_1 + \alpha_2\sqrt{\Delta} + c_1}, \tag{12}$$

$$F_3 = \frac{\left(\alpha_1\sqrt{\Delta} - c_1\right) F_2 - 2c_0}{2c_2 F_2 + \alpha_1\sqrt{\Delta} + c_1}, \tag{13}$$

or equivalently

$$\left(\alpha_1\sqrt{\Delta} + c_1\right) F_1 + 2c_2 F_0 F_1 = \left(\alpha_1\sqrt{\Delta} - c_1\right) F_0 - 2c_0, \tag{14}$$

$$\left(\alpha_2\sqrt{\Delta} + c_1\right) F_2 + 2c_2 F_0 F_2 = \left(\alpha_2\sqrt{\Delta} - c_1\right) F_0 - 2c_0, \tag{15}$$

$$\left(\alpha_2\sqrt{\Delta} + c_1\right) F_3 + 2c_2 F_1 F_3 = \left(\alpha_2\sqrt{\Delta} - c_1\right) F_1 - 2c_0, \tag{16}$$

$$\left(\alpha_1\sqrt{\Delta} + c_1\right) F_3 + 2c_2 F_2 F_3 = \left(\alpha_1\sqrt{\Delta} - c_1\right) F_2 - 2c_0. \tag{17}$$

Multiplying (14) by F_2, (15) by F_1 and subtracting yields

$$\sqrt{\Delta}\left(\alpha_1-\alpha_2\right)F_1F_2=\sqrt{\Delta}\left(\alpha_1F_2-\alpha_2F_1\right)+c_1\left(F_1-F_2\right)F_0+2c_0\left(F_1-F_2\right). \quad (18)$$

Following the same procedure, multiplying (16) by F_2, (17) by F_1 and subtracting leads

$$\sqrt{\Delta}\left(\alpha_2F_2-\alpha_1F_1\right)F_3=\sqrt{\Delta}\left(\alpha_2-\alpha_1\right)F_1F_2+2c_0\left(F_1-F_2\right). \quad (19)$$

Solving F_3 from (18) and (19) yields the superposition formula

$$F_3=\frac{\alpha_1F_2-\alpha_2F_1}{\alpha_1F_1-\alpha_2F_2}F_0-\frac{c_1}{\sqrt{\Delta}}\left(F_2-F_1\right)F_0, \quad \Delta>0. \quad (20)$$

2.2.2 The case $\Delta<0$

In this case, taking the four solutions of the Riccati Eq. (1) as $F_0=F_0(\xi)$, $F_1=F_1(\xi;\beta_1)$, $F_2=F_2(\xi;\beta_2)$, $F_3=F_{12}(\xi;\beta_1,\beta_2)=F_{21}(\xi;\beta_2,\beta_1)$ and insert them into the Bäcklund transformation (9), we obtain

$$F_1=\frac{\left(\beta_1\sqrt{-\Delta}+c_1\right)F_0+2c_0}{\beta_1\sqrt{-\Delta}-c_1-2c_2F_0}, \quad (21)$$

$$F_2=\frac{\left(\beta_2\sqrt{-\Delta}+c_1\right)F_0+2c_0}{\beta_2\sqrt{-\Delta}-c_1-2c_2F_0}, \quad (22)$$

$$F_3=\frac{\left(\beta_2\sqrt{-\Delta}+c_1\right)F_1+2c_0}{\beta_2\sqrt{-\Delta}-c_1-2c_2F_1}, \quad (23)$$

$$F_3=\frac{\left(\beta_1\sqrt{-\Delta}+c_1\right)F_2+2c_0}{\beta_1\sqrt{-\Delta}-c_1-2c_2F_2}, \quad (24)$$

or equivalently

$$\left(\beta_1\sqrt{-\Delta}-c_1\right)F_1-2c_2F_0F_1=\left(\beta_1\sqrt{-\Delta}+c_1\right)F_0+2c_0, \quad (25)$$

$$\left(\beta_2\sqrt{-\Delta}-c_1\right)F_2-2c_2F_0F_2=\left(\beta_2\sqrt{-\Delta}+c_1\right)F_0+2c_0, \quad (26)$$

$$\left(\beta_2\sqrt{-\Delta}-c_1\right)F_3-2c_2F_1F_3=\left(\beta_2\sqrt{-\Delta}+c_1\right)F_1+2c_0, \quad (27)$$

$$\left(\beta_1\sqrt{-\Delta}-c_1\right)F_3-2c_2F_2F_3=\left(\beta_1\sqrt{-\Delta}+c_1\right)F_2+2c_0. \quad (28)$$

Proceeding as in the case of $\Delta>0$, we obtain the following two expressions

$$\sqrt{-\Delta}\left(\beta_1-\beta_2\right)F_1F_2=\sqrt{-\Delta}\left(\beta_1F_2-\beta_2F_1\right)F_0+c_1\left(F_2-F_1\right)F_0+2c_0\left(F_2-F_1\right), \quad (29)$$

$$\sqrt{-\Delta}\left(\beta_2F_2-\beta_1F_1\right)F_3=\sqrt{-\Delta}\left(\beta_2-\beta_1\right)F_1F_2+2c_0\left(F_2-F_1\right). \quad (30)$$

To solve this pair of equations for F_3 gives the superposition formula, that is

$$F_3=\frac{\beta_1F_2-\beta_2F_1}{\beta_1F_1-\beta_2F_2}F_0-\frac{c_1}{\sqrt{-\Delta}}\left(F_2-F_1\right)F_0, \quad \Delta<0. \quad (31)$$

3 Fractional solutions

Choosing the initial solution as the first solution in (2) and taking it into the Bäcklund transformation (8), we get

$$F(\xi)=\frac{\sqrt{\Delta}-\alpha c_1-\left(\alpha\sqrt{\Delta}-c_1\right)\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{2c_2\left(\alpha-\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right)}, = -\frac{c_1}{2c_2}-\frac{\sqrt{\Delta}\left(\alpha\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)-1\right)}{2c_2\left(\alpha-\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right)}. \quad (32)$$

If we set $r_1=\alpha, r_2=-1$, then the expression (32) can be rewritten as:

$$F_+(\xi)=-\frac{c_1}{2c_2}-\frac{\sqrt{\Delta}\left(r_1\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)+r_2\right)}{2c_2\left(r_1+r_2\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right)}, \quad (33)$$

where $\Delta>0$ and which is non-trivial and non-degenerate if and only if

$$r_2\neq\pm r_1, r_1^2+r_2^2\neq 0. \quad (34)$$

Under condition (34), we can check that (33) is just the solution of the Riccati Eq. (1). Therefore, (33) expresses a two parametric fractional solutions of the Riccati Eq. (1). We also can see that when $r_2 = 0$ and $r_1 = 0$ the solution (33) is reduced to the first solution and the second solution in (2), respectively.
If we choose the third solution in (2) as the initial solution and take it into the Bäcklund transformation (9), then we obtain

$$F(\xi) = \frac{\sqrt{-\Delta} - \beta c_1 + \left(\beta\sqrt{-\Delta} + c_1\right)\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{2c_2\left(\beta - \tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right)},$$
$$= -\frac{c_1}{2c_2} + \frac{\sqrt{-\Delta}\left(\beta\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + 1\right)}{2c_2\left(\beta - \tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right)}. \quad (35)$$

By setting $r_3 = \beta, r_4 = -1$ we can obtain another two parametric fractional solutions of the Riccati Eq. (1)

$$F_-(\xi) = -\frac{c_1}{2c_2} + \frac{\sqrt{-\Delta}\left(r_3\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - r_4\right)}{2c_2\left(r_3 + r_4\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right)}, \quad (36)$$

where $\Delta < 0$ and which is non-trivial and non-degenerate if and only if

$$r_3^2 + r_4^2 \neq 0. \quad (37)$$

It is also easy to see that when $r_4 = 0$ and $r_3 = 0$, the solution (36) is reduced to the third solution and the fourth solution in (2), respectively.

4 Equivalence relations for solutions

Let us now turn to consider the equivalence relations between our new fractional type solutions and the twenty-seven previously known solutions of the Riccati Eq. (1). For the sake of simplicity, we shall now set the first, second, fourth and fifth solutions in (2) as $F_1(\xi)$, $F_2(\xi)$, $F_4(\xi)$ and $F_5(\xi)$. Following the notation in [24], the twenty-seven previously known solutions will be put as $G_i(\xi)\,(i = 1, 2, \ldots, 27)$ in which the coefficients w, u, v, η will be replaced by c_0, c_1, c_2, ξ. Under this assumption, we can see that

$$G_1(\xi) = -\frac{1}{2c_2}\left(c_1 + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right),$$
$$= F_1(\xi) = F_+(\xi)|_{r_2=0},$$
$$G_2(\xi) = -\frac{1}{2c_2}\left(c_1 + \sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right),$$
$$= F_2(\xi) = F_+(\xi)|_{r_1=0},$$
$$G_{13}(\xi) = \frac{1}{2c_2}\left(-c_1 + \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right),$$
$$= F_4(\xi) = F_-(\xi)|_{r_4=0},$$
$$G_{14}(\xi) = -\frac{1}{2c_2}\left(c_1 + \sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right),$$
$$= F_5(\xi) = F_-(\xi)|_{r_3=0},$$

Two waves are called equivalence in the sense of wave translation if they have same waveform but their phase difference is equal to a constant. In our work, this equivalence relation will be expressed by the notation "$\approxeq$". Based on this definition and the following easily proved identities

$$\begin{aligned}
&\tanh\eta + i\,\mathrm{sech}\eta = \tanh\left(\tfrac{\eta}{2} + \tfrac{\pi i}{4}\right),\\
&\tanh\eta - i\,\mathrm{sech}\eta = \coth\left(\tfrac{\eta}{2} + \tfrac{\pi i}{4}\right),\\
&\tanh\eta + \coth\eta = 2\coth(2\eta),\\
&\coth\eta + \mathrm{csch}\eta = \coth\left(\tfrac{\eta}{2}\right),\\
&\coth\eta - \mathrm{csch}\eta = \tanh\left(\tfrac{\eta}{2}\right)
\end{aligned} \quad (38)$$

and

$$\begin{aligned}
&\tan\eta + \sec\eta = \tan\left(\tfrac{\eta}{2} + \tfrac{\pi}{4}\right),\\
&\tan\eta - \sec\eta = -\cot\left(\tfrac{\eta}{2} + \tfrac{\pi}{4}\right),\\
&\tan\eta - \cot\eta = -2\cot(2\eta),\\
&\cot\eta + \csc\eta = \cot\left(\tfrac{\eta}{2}\right),\\
&\cot\eta - \csc\eta = -\tan\left(\tfrac{\eta}{2}\right),
\end{aligned} \quad (39)$$

we can derive that

$$G_3(\xi) = -\frac{1}{2c_2}\left(c_1 + \sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right) \pm i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right)\right),$$
$$= \begin{cases} -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\left(\tanh\left(\sqrt{\Delta}\xi\right) + i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right), \\ -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\left(\tanh\left(\sqrt{\Delta}\xi\right) - i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right), \end{cases}$$
$$= \begin{cases} -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi + \frac{\pi i}{4}\right) \approxeq F_+(\xi)|_{r_2=0}, \\ -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\coth\left(\frac{\sqrt{\Delta}}{2}\xi + \frac{\pi i}{4}\right) \approxeq F_+(\xi)|_{r_1=0}. \end{cases}$$

$$G_4(\xi) = -\frac{1}{2c_2}\left(c_1 + \sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\xi\right) \pm \operatorname{csch}\left(\sqrt{\Delta}\xi\right)\right)\right),$$
$$= \begin{cases} -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\left(\coth\left(\sqrt{\Delta}\xi\right) + \operatorname{csch}\left(\sqrt{\Delta}\xi\right)\right), \\ -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\left(\coth\left(\sqrt{\Delta}\xi\right) - \operatorname{csch}\left(\sqrt{\Delta}\xi\right)\right), \end{cases}$$
$$= \begin{cases} -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right) = F_+(\xi)|_{r_1=0}, \\ -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right) = F_+(\xi)|_{r_2=0}. \end{cases}$$

$$G_5(\xi) = -\frac{1}{4c_2}\left(2c_1 + \sqrt{\Delta}\left(\tanh\left(\frac{\sqrt{\Delta}}{4}\xi\right) + \coth\left(\frac{\sqrt{\Delta}}{4}\xi\right)\right)\right),$$
$$= -\frac{c_1}{2c_2} - \frac{\sqrt{\Delta}}{2c_2}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right) = F_+(\xi)|_{r_1=0}.$$

$$G_{15}(\xi) = \frac{1}{2c_2}\left(-c_1 + \sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right)\right),$$
$$= \begin{cases} -\frac{c_1}{2c_2} + \frac{\sqrt{-\Delta}}{2c_2}\left(\tan\left(\sqrt{-\Delta}\xi\right) + \sec\left(\sqrt{-\Delta}\xi\right)\right), \\ -\frac{c_1}{2c_2} + \frac{\sqrt{-\Delta}}{2c_2}\left(\tan\left(\sqrt{-\Delta}\xi\right) - \sec\left(\sqrt{-\Delta}\xi\right)\right), \end{cases}$$
$$= \begin{cases} -\frac{c_1}{2c_2} + \frac{\sqrt{-\Delta}}{2c_2}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi + \frac{\pi}{4}\right) \approxeq F_-(\xi)|_{r_4=0}, \\ -\frac{c_1}{2c_2} - \frac{\sqrt{-\Delta}}{2c_2}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi + \frac{\pi}{4}\right) \approxeq F_-(\xi)|_{r_3=0}. \end{cases}$$

$$G_{16}(\xi) = -\frac{1}{2c_2}\left(c_1 + \sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\xi\right) \pm \csc\left(\sqrt{-\Delta}\xi\right)\right)\right),$$
$$= \begin{cases} -\frac{c_1}{2c_2} - \frac{\sqrt{-\Delta}}{2c_2}\left(\cot\left(\sqrt{-\Delta}\xi\right) + \csc\left(\sqrt{-\Delta}\xi\right)\right), \\ -\frac{c_1}{2c_2} - \frac{\sqrt{-\Delta}}{2c_2}\left(\cot\left(\sqrt{-\Delta}\xi\right) - \csc\left(\sqrt{-\Delta}\xi\right)\right), \end{cases}$$
$$= \begin{cases} -\frac{c_1}{2c_2} - \frac{\sqrt{-\Delta}}{2}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right) = F_-(\xi)|_{r_3=0}, \\ -\frac{c_1}{2c_2} + \frac{\sqrt{-\Delta}}{2}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right) = F_-(\xi)|_{r_4=0}. \end{cases}$$

$$G_{17}(\xi) = \frac{1}{4c_2}\left(-2c_1 + \sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}}{4}\xi\right) - \cot\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\right)\right),$$
$$= -\frac{c_1}{2c_2} - \frac{\sqrt{-\Delta}}{2c_2}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right) = F_-(\xi)|_{r_3=0}.$$

It follows from the definitions of the hyperbolic functions and trigonometric functions that

$$G_8(\xi) = \frac{2c_0\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - c_1\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)},$$
$$= \frac{2c_0}{\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - c_1},$$
$$= F_+(\xi)|_{\left\{r_1=-\frac{c_1}{2c_2},\, r_2=\frac{\sqrt{\Delta}}{2c_2}\right\}}.$$

$$G_9(\xi) = \frac{-2c_0\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - \sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)},$$
$$= \frac{2c_0}{\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right) - c_1},$$
$$= F_+(\xi)|_{\left\{r_1=\frac{\sqrt{\Delta}}{2c_2},\, r_2=-\frac{c_1}{2c_2}\right\}}.$$

$$G_{20}(\xi) = \frac{-2c_0\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)},$$
$$= \frac{-2c_0}{\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_1},$$
$$= F_-(\xi)|_{\left\{r_3=\frac{c_1}{2c_2},\, r_4=\frac{\sqrt{-\Delta}}{2c_2}\right\}}.$$

$$G_{21}(\xi) = \frac{2c_0\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{-c_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + \sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)},$$
$$= \frac{2c_0}{\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - c_1},$$
$$= F_-(\xi)|_{\left\{r_3=\frac{\sqrt{-\Delta}}{2c_2},\, r_4=-\frac{c_1}{2c_2}\right\}}.$$

Using the definitions of the hyperbolic functions together with the formulas (38), (39) and the identities

$$\sinh^2\frac{x}{2} = \frac{1}{2}(\cosh x - 1), \tag{40}$$

$$\cosh^2\frac{x}{2} = \frac{1}{2}(\cosh x + 1), \tag{41}$$

$$\sin^2\frac{x}{2} = \frac{1}{2}(1 - \cos x), \tag{42}$$

$$\cos^2\frac{x}{2} = \frac{1}{2}(1 + \cos x), \tag{43}$$

we can derive the following results

$$G_{10}(\xi)=\frac{2c_0\cosh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)-c_1\cosh\left(\sqrt{\Delta}\xi\right)\pm i\sqrt{\Delta}},$$
$$=\frac{2c_0}{\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right)-c_1},$$
$$=\begin{cases}\dfrac{2c_0}{\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)+i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right)-c_1},\\ \dfrac{2c_0}{\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)-i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right)-c_1},\end{cases}$$
$$=\begin{cases}\dfrac{2c_0}{\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi+\frac{\pi i}{4}\right)-c_1},\\ \dfrac{2c_0}{\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{2}\xi+\frac{\pi i}{4}\right)-c_1},\end{cases}$$
$$\approx\begin{cases}F_+(\xi)|_{\left\{r_1=-\frac{c_1}{2c_2},r_2=\frac{\sqrt{\Delta}}{2c_2}\right\}},\\ F_+(\xi)|_{\left\{r_1=\frac{\sqrt{\Delta}}{2c_2},r_2=-\frac{c_1}{2c_2}\right\}}.\end{cases}$$

$$G_{11}(\xi)=\frac{2c_0\sinh\left(\sqrt{\Delta}\xi\right)}{-c_1\sinh\left(\sqrt{\Delta}\xi\right)+\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\pm\sqrt{\Delta}},$$
$$=\frac{2c_0}{\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\xi\right)\pm\mathrm{csch}\left(\sqrt{\Delta}\xi\right)\right)-c_1},$$
$$=\begin{cases}\dfrac{2c_0}{\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\xi\right)+\mathrm{csch}\left(\sqrt{\Delta}\xi\right)\right)-c_1},\\ \dfrac{2c_0}{\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\xi\right)-\mathrm{csch}\left(\sqrt{\Delta}\xi\right)\right)-c_1},\end{cases}$$
$$=\begin{cases}\dfrac{2c_0}{\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right)-c_1},\\ \dfrac{2c_0}{\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)-c_1},\end{cases}$$
$$=\begin{cases}F_+(\xi)|_{\left\{r_1=\frac{\sqrt{\Delta}}{2c_2},r_2=-\frac{c_1}{2c_2}\right\}},\\ F_+(\xi)|_{\left\{r_1=-\frac{c_1}{2c_2},r_2=\frac{\sqrt{\Delta}}{2c_2}\right\}}.\end{cases}$$

$$G_{12}=\frac{4c_0\sinh\left(\frac{\sqrt{\Delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\Delta}}{4}\xi\right)}{-2c_1\sinh\left(\frac{\sqrt{\Delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\Delta}}{4}\xi\right)+2\sqrt{\Delta}\cosh^2\left(\frac{\sqrt{\Delta}}{4}\xi\right)-\sqrt{\Delta}},$$
$$=\frac{2c_0\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{-c_1\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right)+\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)},$$
$$=\frac{2c_0}{\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right)-c_1}=F_+(\xi)|_{\left\{r_1=\frac{\sqrt{\Delta}}{2c_2},r_2=-\frac{c_1}{2c_2}\right\}}.$$

$$G_{22}(\xi)=\frac{-2c_0\cos\left(\sqrt{-\Delta}\xi\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right)+c_1\cos\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}},$$
$$=\frac{-2c_0}{\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)+c_1},$$
$$=\begin{cases}\dfrac{-2c_0}{\sqrt{-\Delta}\left(\tan\left(\sqrt{\Delta}\xi\right)+\sec\left(\sqrt{-\Delta}\xi\right)\right)+c_1},\\ \dfrac{-2c_0}{\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\xi\right)-\sec\left(\sqrt{-\Delta}\xi\right)\right)+c_1},\end{cases}$$
$$=\begin{cases}\dfrac{-2c_0}{\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi+\frac{\pi}{4}\right)+c_1},\\ \dfrac{2c_0}{\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi+\frac{\pi}{4}\right)-c_1},\end{cases}$$
$$\approx\begin{cases}F_-(\xi)|_{\left\{r_3=\frac{c_1}{2c_2},r_4=\frac{\sqrt{-\Delta}}{2c_2}\right\}},\\ F_-(\xi)|_{\left\{r_3=\frac{\sqrt{-\Delta}}{2c_2},r_4=-\frac{c_1}{2c_2}\right\}}.\end{cases}$$

$$G_{23}=\frac{2c_0\sin\left(\sqrt{-\Delta}\xi\right)}{-c_1\sin\left(\sqrt{-\Delta}\xi\right)+\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}},$$
$$=\frac{2c_0}{\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\xi\right)\pm\csc\left(\sqrt{-\Delta}\xi\right)\right)-c_1},$$
$$=\begin{cases}\dfrac{2c_0}{\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\xi\right)+\csc\left(\sqrt{-\Delta}\xi\right)\right)-c_1},\\ \dfrac{2c_0}{\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\xi\right)-\csc\left(\sqrt{\Delta}\xi\right)\right)-c_1},\end{cases}$$
$$=\begin{cases}\dfrac{2c_0}{\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right)-c_1},\\ \dfrac{-2c_0}{\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)+c_1},\end{cases}$$
$$=\begin{cases}F_-(\xi)|_{\left\{r_3=\frac{\sqrt{-\Delta}}{2c_2},r_4=-\frac{c_1}{2c_2}\right\}},\\ F_-(\xi)|_{\left\{r_3=\frac{c_1}{2c_2},r_4=\frac{\sqrt{-\Delta}}{2c_2}\right\}}.\end{cases}$$

$$G_{24}=\frac{4c_0\sin\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\cos\left(\frac{-\sqrt{\Delta}}{4}\xi\right)}{2\sqrt{-\Delta}\cos^2\left(\frac{\sqrt{\Delta}}{4}\xi\right)-2c_1\sin\left(\frac{-\sqrt{\Delta}}{4}\xi\right)\cos\left(\frac{\sqrt{-\Delta}}{4}\xi\right)-\sqrt{-\Delta}},$$
$$=\frac{2c_0\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{\sqrt{-\Delta}\cos\left(\frac{-\sqrt{\Delta}}{2}\xi\right)-c_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)},$$
$$=\frac{2c_0}{\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right)-c_1},$$
$$=F_-(\xi)|_{\left\{r_3=\frac{\sqrt{-\Delta}}{2c_2},r_4=-\frac{c_1}{2c_2}\right\}}.$$

From the definitions of the hyperbolic functions and the identities

$$\begin{aligned}\frac{1}{1+e^x} &= \frac{1}{2}\left(1-\tanh\frac{x}{2}\right),\\ \frac{1}{1-e^x} &= \frac{1}{2}\left(1-\coth\frac{x}{2}\right),\end{aligned} \tag{44}$$

we can prove that

$$\begin{aligned}G_{25}(\xi) &= \frac{-c_1 f_1}{c_2\left(f_1+\cosh(c_1\xi)-\sinh(c_1\xi)\right)},\\ &= \frac{-c_1 f_1}{c_2\left(f_1+e^{-c_1\xi}\right)} = \frac{-c_1}{c_2\left(1+f_1^{-1}e^{-c_1\xi}\right)},\\ &= \begin{cases}\dfrac{-c_1}{c_2\left(1+e^{-c_1\xi-\ln(f_1)}\right)}, & f_1>0,\\ \dfrac{-c_1}{c_2\left(1-e^{-c_1\xi-\ln(-f_1)}\right)}, & f_1<0,\end{cases}\\ &= \begin{cases}-\dfrac{c_1}{2c_2}\left(1+\tanh\dfrac{1}{2}(c_1\xi+\ln(f_1))\right), & f_1>0,\\ -\dfrac{c_1}{2c_2}\left(1+\coth\dfrac{1}{2}(c_1\xi+\ln(-f_1))\right), & f_1>0,\end{cases}\\ &\approx \begin{cases}F_+(\xi)|_{\{r_2=0,c_0=0\}},\\ F_+(\xi)|_{\{r_1=0,c_0=0\}}.\end{cases}\end{aligned}$$

$$\begin{aligned}G_{26}(\xi) &= \frac{-c_1\left(\cosh(c_1\xi)+\sinh(c_1\xi)\right)}{c_2\left(f_1+\cosh(c_1\xi)+\sinh(c_1\xi)\right)},\\ &= \frac{-c_1 e^{c_1\xi}}{c_2\left(f_1+e^{c_1\xi}\right)} = \frac{-c_1}{c_2\left(1+f_1e^{-c_1\xi}\right)},\\ &= \begin{cases}\dfrac{-c_1}{c_2\left(1+e^{-c_1\xi+\ln(f_1)}\right)}, & f_1>0,\\ \dfrac{-c_1}{c_2\left(1-e^{-c_1\xi+\ln(-f_1)}\right)}, & f_1<0,\end{cases}\\ &= \begin{cases}-\dfrac{c_1}{2c_2}\left(1+\tanh\dfrac{1}{2}(c_1\xi-\ln(f_1))\right), & f_1>0,\\ -\dfrac{c_1}{2c_2}\left(1+\coth\dfrac{1}{2}(c_1\xi-\ln(-f_1))\right), & f_1<0,\end{cases}\\ &\approx \begin{cases}F_+(\xi)|_{\{r_2=0,c_0=0\}},\\ F_+(\xi)|_{\{r_1=0,c_0=0\}}.\end{cases}\end{aligned}$$

Let us now put the third solution in (2) as

$$F_0(\xi) = -\frac{c_1}{2c_2} - \frac{1}{c_2\xi+c}, \quad \Delta = c_1^2-4c_0c_2 = 0, \tag{45}$$

then it may easily found that

$$F_0(\xi)|_{\{c_1=0,c=l_1\}} = \frac{-1}{c_2\xi+l_1} = G_{27}(\xi).$$

Unfortunately, we failed to derive the solutions

$$\begin{aligned}G_6 &= \tfrac{1}{2c_2}\left(-c_1+\frac{\pm\sqrt{(D^2+E^2)\Delta}-D\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{D\sinh\left(\sqrt{\Delta}\xi\right)+E}\right),\\ G_7 &= \tfrac{1}{2c_2}\left(-c_1-\frac{\pm\sqrt{(D^2+E^2)\Delta}+D\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{D\sinh\left(\sqrt{\Delta}\xi\right)+E}\right),\\ G_{18} &= \tfrac{1}{2c_2}\left(-c_1+\frac{\pm\sqrt{-(D^2-E^2)\Delta}-D\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{D\sin\left(\sqrt{-\Delta}\xi\right)+E}\right),\\ G_{19} &= \tfrac{1}{2c_2}\left(-c_1-\frac{\pm\sqrt{-(D^2-E^2)\Delta}+D\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{D\sin\left(\sqrt{-\Delta}\xi\right)+E}\right),\end{aligned}$$

from the fractional solutions $F_+(\xi)$ and $F_-(\xi)$. In summary, if $i \neq 6, 7, 18, 19$, then the solutions $G_i(\xi)$ are equivalent to the solutions $F_+(\xi)$, $F_-(\xi)$ and $F_0(\xi)$. Therefore, the solutions $G_i(\xi)$ $(i \neq 6, 7, 18, 19)$ can be replaced by $F_+(\xi)$, $F_-(\xi)$ and $F_0(\xi)$.

In addition, if we allow $c_2 = 0$ in (1), then the Riccati equation reduces to the first-order linear equation which admits the following exponential solution

$$F_e(\xi) = -\frac{c_0}{c_1} + de^{c_1\xi},\ c_2 = 0, \tag{46}$$

where d is an integration constant.

5 Method and application

According to the previous discussions and using the fractional solutions (33) and (36), below we propose a so-called unified Riccati equation expansion method. The procedures of the method can be described by following four steps:

Step 1 Suppose that the wave transformation $u(x,t) = u(\xi)$, $\xi = x - \omega t$ can change a given NLEE

$$H(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0, \tag{47}$$

into the following ODE

$$G(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0. \tag{48}$$

Step 2 Seek solutions of Eq. (47) in the form

$$u(\xi) = \sum_{i=0}^{N} a_i F^i(\xi), \tag{49}$$

with a_i $(i = 0, 1, 2, \ldots, N)$ are constants to be determined, N is a positive integer which determined by balancing the highest order derivative terms with the highest power nonlinear terms in Eq. (48) and $F(\xi)$ expresses the solutions of the Riccati Eq. (1) given by

$$F(\xi)=\begin{cases}F_+(\xi), & \Delta=c_1^2-4c_0c_2>0,\\ F_0(\xi), & \Delta=c_1^2-4c_0c_2=0,\\ F_-(\xi), & \Delta=c_1^2-4c_0c_2<0,\\ F_e(\xi), & c_2=0.\end{cases} \qquad (50)$$

where $F_+(\xi)$, $F_-(\xi)$, $F_0(\xi)$ and $F_e(\xi)$ are defined by (33), (36), (45) and (46).

Step 3 Substitute (49) along with Eq. (1) into (48) and equate the coefficients of each power of $F(\xi)$ to zero yields a set of algebraic equations for unknowns c_0, c_1, c_2, a_i $(i=0,1,\ldots,N)$ and ω.

Step 4 Solve the set of algebraic equations with the aid of a computer algebraic system and substitute the solutions obtained in this step back into (49) so as to obtain the exact traveling wave solutions for Eq. (47).

To illustrate our suggested method, we now proceed to solve a new class of BBM equation which is called the Ostrovsky–Benjamin–Bona–Mahony (OS–BBM) equation of the form

$$\left(u_t+u_x-\alpha(u^2)_x-\beta u_{xxt}\right)_x=\gamma\left(u+u^2\right), \qquad (51)$$

which describes the motion of ocean currents. Compared with the standard BBM equation, this equation is studied very little. Yang et al. [29] obtained six types of traveling wave solutions for OS–BBM equation through the use of the G'/G-expansion method. Some exact solitary wave solutions for Eq. (51) are obtained in [30,31]. Now we make the transformation

$$u(x,t)=u(\xi),\ \xi=x-\omega t, \qquad (52)$$

so that (51) is changed to the following ODE

$$(1-\omega)u''-2\alpha\left((u')^2+uu''\right)+\beta\omega u^{(4)}$$
$$-\gamma(u+u^2)=0. \qquad (53)$$

Balancing the highest order derivative term $u^{(4)}$ with the highest power nonlinear term $(u')^2+uu''$, we obtain $N=2$. Therefore, Eq. (53) admits the solution of the form

$$u(\xi)=a_0+a_1F(\xi)+a_2F^2(\xi), \qquad (54)$$

where a_i $(i=0,1,2)$ are constants to be determined. Substituting (54) along with (1) into (53) and setting the coefficients of $F(\xi)^j$ $(j=0,1,\ldots,6)$ to zero yield a set of algebraic equations for a_i, c_i $(i=0,1,2)$ and ω as follows:

$$\begin{aligned}
&-12\alpha a_0a_2c_1c_0+30\beta\omega a_2c_1^3c_0-4\alpha a_0a_1c_2c_0-\gamma a_1\\
&\quad+16\beta\omega a_1c_2^2c_0^2+120\beta\omega a_2c_1c_2c_0^2+22\beta\omega a_1c_2c_1^2c_0\\
&\quad-12\alpha a_1a_2c_0^2-6\omega a_2c_1c_0-6\alpha a_1^2c_1c_0-2\alpha a_0a_1c_1^2\\
&\quad+\beta\omega a_1c_1^4-2\omega a_1c_2c_0-2\gamma a_1a_0+6a_2c_1c_0+a_1c_1^2\\
&\quad+2a_1c_2c_0-\omega a_1c_1^2=0,\\
&-30\alpha a_2a_1c_1c_0-16\alpha a_0a_2c_2c_0+136\beta\omega a_2c_2^2c_0^2-\gamma a_1^2\\
&\quad-6\alpha a_0a_1c_2c_1+15\beta\omega a_1c_2c_1^3+232\beta\omega a_2c_2c_1^2c_0\\
&\quad+60\beta\omega a_1c_2^2c_1c_0-\gamma a_2-8\alpha a_0a_2c_1^2+16\beta\omega a_2c_1^4\\
&\quad-8\omega a_2c_2c_0-8\alpha a_1^2c_2c_0-3\omega a_1c_2c_1-2\gamma a_2a_0\\
&\quad+8a_2c_2c_0+3a_1c_2c_1-4\omega a_2c_1^2-12\alpha a_2^2c_0^2\\
&\quad-4\alpha a_1^2c_1^2+4a_2c_1^2=0,\\
&120a_2\beta c_2^4\omega-20a_2^2\alpha c_2^2=0,\\
&24a_1\beta c_2^4\omega+336a_2\beta c_1c_2^3\omega-24a_1a_2\alpha c_2^2\\
&\quad-36a_2^2\alpha c_1c_2=0,
\end{aligned}$$

$$\begin{aligned}
&60a_1\beta c_1c_2^3\omega+240a_2\beta c_0c_2^3\omega+330a_2\beta c_1^2c_2^2\omega\\
&\quad-12a_0a_2\alpha c_2^2-6a_1^2\alpha c_2^2-42a_1a_2\alpha c_1c_2-32a_2^2\alpha c_0c_2\\
&\quad-16a_2^2\alpha c_1^2-6a_2c_2^2\omega-a_2^2\gamma+6a_2c_2^2=0,\\
&8a_1\beta c_0^2c_1c_2\omega+a_1\beta c_0c_1^3\omega+16a_2\beta c_0^3c_2\omega+14a_2\beta c_0^2c_1^2\omega\\
&\quad-2a_0a_1\alpha c_0c_1-4a_0a_2\alpha c_0^2-2a_1^2\alpha c_0^2-a_1c_0c_1\omega\\
&\quad-2a_2c_0^2\omega-a_0^2\gamma+a_1c_0c_1+2a_2c_0^2-a_0\gamma=0,\\
&40a_1\beta c_0c_2^3\omega+50a_1\beta c_1^2c_2^2\omega+440a_2\beta c_0c_1c_2^2\omega\\
&\quad+130a_2\beta c_1^3c_2\omega-4a_0a_1\alpha c_2^2-20a_0a_2\alpha c_1c_2\\
&\quad-10a_1^2\alpha c_1c_2-36a_1a_2\alpha c_0c_2-18a_1a_2\alpha c_1^2\\
&\quad-28a_2^2\alpha c_0c_1-2a_1c_2^2\omega-10a_2c_1c_2\omega-2a_1a_2\gamma\\
&\quad+2a_1c_2^2+10a_2c_1c_2=0.
\end{aligned}$$

Solving this set of algebraic equations by use of Maple, we obtain

$$\begin{aligned}
&a_0=\frac{3\beta(\alpha+1)c_1^2+\beta\gamma+\alpha}{2(\beta\gamma+\alpha)},\quad a_1=\frac{6\beta(\alpha+1)c_1c_2}{\beta\gamma+\alpha},\\
&a_2=\frac{6\beta(\alpha+1)c_2^2}{\beta\gamma+\alpha},\quad c_0=\frac{\beta(\alpha+1)c_1^2+\beta\gamma+\alpha}{4\beta(\alpha+1)c_2},\\
&\omega=\frac{\alpha(\alpha+1)}{\beta\gamma+\alpha},\quad \Delta=-\frac{\beta\gamma+\alpha}{\beta(\alpha+1)},
\end{aligned} \qquad (55)$$

$$\begin{aligned}
&a_0=\frac{3\beta(\alpha+1)c_1^2-3(\beta\gamma+\alpha)}{2(\beta\gamma+\alpha)},\quad a_1=\frac{6\beta(\alpha+1)c_1c_2}{\beta\gamma+\alpha},\\
&a_2=\frac{6\beta(\alpha+1)c_2^2}{\beta\gamma+\alpha},\quad c_0=\frac{\beta(\alpha+1)c_1^2-\beta\gamma-\alpha}{4\beta(\alpha+1)c_2},\\
&\omega=\frac{\alpha(\alpha+1)}{\beta\gamma+\alpha},\quad \Delta=\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}.
\end{aligned} \qquad (56)$$

$$a_2=0,\quad c_0=\mp\frac{a_0\gamma}{2a_1\alpha\sqrt{-\frac{\gamma}{\alpha}}},\quad c_1=\pm\frac{1}{2}\sqrt{-\frac{\gamma}{\alpha}},\quad c_2=0,$$

$$\omega = \frac{4\alpha(4\alpha+1)}{\beta\gamma+4\alpha}, \quad \Delta = -\frac{\gamma}{4\alpha}, \tag{57}$$

$$a_2 = 0, \quad c_0 = \mp\frac{(a_0+1)\gamma}{2a_1\alpha\sqrt{-\frac{\gamma}{\alpha}}}, \quad c_1 = \pm\frac{1}{2}\sqrt{-\frac{\gamma}{\alpha}}, \quad c_2 = 0,$$

$$\omega = -\frac{4\alpha(2\alpha+1)}{\beta\gamma+4\alpha}, \quad \Delta = -\frac{\gamma}{4\alpha}. \tag{58}$$

Substituting (55) with $F_+(\xi)$ and $F_-(\xi)$ into (54) and using (52) gives the exact traveling wave solutions of the OS–BBM equation as follows:

$$u_1(x,t) = -\frac{1}{2}\left\{(3r_1^2 - r_2^2)\tanh^2(\eta_1) + 4r_1r_2\tanh(\eta_1) - r_1^2 + 3r_2^2\right\} / (r_1 + r_2\tanh(\eta_1))^2,$$
$$\eta_1 = \frac{1}{2}\sqrt{-\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\left(x - \frac{\alpha(\alpha+1)}{\beta\gamma+\alpha}t\right), \tag{59}$$

where $\beta(\alpha+1)(\beta\gamma+\alpha) < 0$.

$$u_2(x,t) = \frac{1}{2}\left\{(3r_3^2 + r_4^2)\tan^2(\eta_2) - 4r_3r_4\tan(\eta_2) + r_3^2 + 3r_4^2\right\} / (r_3 + r_4\tan(\eta_2))^2,$$
$$\eta_2 = \frac{1}{2}\sqrt{\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\left(x - \frac{\alpha(\alpha+1)}{\beta\gamma+\alpha}t\right), \tag{60}$$

where $\beta(\alpha+1)(\beta\gamma+\alpha) > 0$.

In particular, when choosing the free parameters $r_2 = 0$ and $r_1 = 0$, the solution (59) is reduced to the solitary wave solutions

$$u_{1a}(x,t) = -\frac{3}{2}\tanh^2\left(\frac{1}{2}\sqrt{-\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right) + \frac{1}{2},$$
$$= \frac{3}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{-\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right) - 1, \tag{61}$$

$$u_{1b}(x,t) = -\frac{3}{2}\coth^2\left(\frac{1}{2}\sqrt{-\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right) + \frac{1}{2},$$
$$= -\frac{3}{2}\mathrm{csch}^2\left(\frac{1}{2}\sqrt{-\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right) - 1, \tag{62}$$

where $\xi = x - \frac{\alpha(\alpha+1)}{\beta\gamma+\alpha}t$.

Taking the parameters $r_4 = 0$ and $r_3 = 0$ and the solution (60) is reduced to the singular trigonometric solutions

$$u_{2a}(x,t) = \frac{3}{2}\tan^2\left(\frac{1}{2}\sqrt{\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right) + \frac{1}{2}, \tag{63}$$

$$u_{2b}(x,t) = \frac{3}{2}\cot^2\left(\frac{1}{2}\sqrt{\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right) + \frac{1}{2}, \tag{64}$$

where $\xi = x - \frac{\alpha(\alpha+1)}{\beta\gamma+\alpha}t$. The solutions (61), (62) and (64) are just the solutions (17), (18) and (19) in [29]. But the solution (63) is a new because which is not obtained in [29].

Taking (56) with $F_+(\xi)$ and $F_-(\xi)$ into (54) and using (52) yields the exact traveling wave solutions of the OS–BBM equation as follows:

$$u_3(x,t) = \frac{3}{2}\frac{(r_1^2 - r_2^2)\tanh^2(\eta_3) + r_2^2 - r_1^2}{(r_1 + r_2\tanh(\eta_3))^2},$$
$$\eta_3 = \frac{1}{2}\sqrt{\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\left(x - \frac{\alpha(\alpha+1)}{\beta\gamma+\alpha}t\right), \tag{65}$$

$$u_4(x,t) = -\frac{3}{2}\frac{(r_3^2 + r_4^2)(1 + \tan^2(\eta_4))}{(r_3 + r_4\tan(\eta_4))^2},$$
$$\eta_4 = \frac{1}{2}\sqrt{-\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\left(x - \frac{\alpha(\alpha+1)}{\beta\gamma+\alpha}t\right), \tag{66}$$

where the parameters α, β and γ in (65) and (66) satisfy the condition $\beta(\alpha+1)(\beta\gamma+\alpha) > 0$ and $\beta(\alpha+1)(\beta\gamma+\alpha) < 0$, respectively.

Setting $r_2 = 0$ and $r_1 = 0$ in (65), $r_4 = 0$ and $r_3 = 0$ in (66), then they are reduced to the solitary wave solutions and the singular trigonometric solutions

$$u_{3a}(x,t) = -\frac{3}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right), \tag{67}$$

$$u_{3b}(x,t) = \frac{3}{2}\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right), \tag{68}$$

$$u_{4a}(x,t) = -\frac{3}{2}\sec^2\left(\frac{1}{2}\sqrt{-\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right), \tag{69}$$

$$u_{4b}(x,t) = -\frac{3}{2}\cot^2\left(\frac{1}{2}\sqrt{-\frac{\beta\gamma+\alpha}{\beta(\alpha+1)}}\xi\right) - \frac{3}{2}, \tag{70}$$

where $\xi = x - \frac{\alpha(\alpha+1)}{\beta\gamma+\alpha}t$. The solutions (67), (68) and (70) are just the solutions (20), (21) and (22) in [29], but (69) is a new solution which is not obtained in [29]. And (67) is just the bright soliton solution given by Alquran in [30].

As shown above, our method can recover all previously known solutions, which is obtained by using the GREMM. At the same time, we can also obtain four sets of infinite number of exact traveling wave solutions from (59), (60), (65) and (66) by choosing different values of their free parameters r_i $(i = 1, 2, 3, 4)$. Therefore, the GREMM is only a special case of our unified Riccati equation expansion method.

Taking (57) and (58) into (54) and using (52), we obtain the following exponential solutions:

$$u_5(x,t) = Ae^{\pm\frac{1}{2}\sqrt{-\frac{\gamma}{\alpha}}\left(x-\frac{4\alpha(4\alpha+1)}{\beta\gamma+4\alpha}t\right)}, \tag{71}$$

$$u_6(x,t) = Ae^{\pm\frac{1}{2}\sqrt{-\frac{\gamma}{\alpha}}\left(x+\frac{4\alpha(2\alpha+1)}{\beta\gamma+4\alpha}t\right)} - 1, \tag{72}$$

where $A = a_1 d \neq 0$ is an arbitrary constant.

Next we consider another new class of Benjamin–Bona–Mahony equation

$$u_t + au_x + bu_{xxt} + \left(pe^u + qe^{-u}\right)_x = 0, \tag{73}$$

with real constants a, b, p, q satisfying the conditions $ab \neq 0$ and $qp \neq 0$ which recently reported by Reza Abazari in [32] and obtained its exact solutions by using the G'/G-expansion method. By the transformation

$$u(x,t) = \ln v(x,t), \tag{74}$$

Eq. (73) becomes

$$(v_t + av_x + bv_{xxt})\, v^2 - b\,(v_{xx}v_t + 2v_x v_{xt})\, v + 2bv_x^2 v_t + \left(pv^2 - q\right) vv_x = 0. \tag{75}$$

Inserting wave transformation

$$v(x,t) = v(\xi), \xi = x - \omega t, \tag{76}$$

into (75), we obtain

$$\left((a-\omega)v' - b\omega v'''\right) v^2 + b\omega\left(3vv'' - 2(v')^2\right) v' + \left(pv^2 - q\right) vv' = 0. \tag{77}$$

Balancing $v^2 v'''$ term with $v^3 v'$ term gives $N = 2$, and therefore, the solution of (77) can be taken as

$$v(\xi) = a_0 + a_1 F(\xi) + a_2 F^2(\xi), \tag{78}$$

where a_i $(i = 0, 1, 2)$ are constants to be determined. Substituting (78) along with (1) into (77) and setting the coefficients of $F^j(\xi)\,(j = 0, 1, \ldots, 9)$ to zero, we obtain a set of algebraic equations for unknowns $a_0, a_1, a_2, c_0, c_1, c_2$ and ω. One can easily check that this set of algebraic equations has no nonzero solutions for $a_2 = 0$, and for $a_2 \neq 0$, without loss of generality, we can set $a_2 = 1$, and then, the algebraic equations solve that

$$a_0 = 0, \quad a_1 = 0, \quad c_0 = \pm\sqrt{\frac{-q}{2ab}}, \quad c_1 = 0, \quad c_2 = \mp\sqrt{\frac{p}{2ab}}, \quad \omega = a, \tag{79}$$

$$a_0 = 0, \quad a_1 = 0, \quad c_0 = \pm\sqrt{\frac{-q}{2ab}}, \quad c_1 = 0, \quad c_2 = \pm\sqrt{\frac{p}{2ab}}, \quad \omega = a, \tag{80}$$

Substituting (79), (80) into (78) and using the relations (74) and (76), we obtain the exact solutions of Eq. (73) as follows:

$$u_1(x,t) = \ln\left[\frac{\sqrt{-qp}}{p}\left(\frac{r_1\tanh\sqrt{\frac{\sqrt{-qp}}{2ab}}\xi + r_2}{r_1 + r_2\tanh\sqrt{\frac{\sqrt{-qp}}{2ab}}\xi}\right)^2\right], \quad \xi = x - at, ab > 0, q < 0, p > 0, \tag{81}$$

$$u_2(x,t) = \ln\left[-\frac{\sqrt{-qp}}{p}\left(\frac{r_1\tanh\sqrt{-\frac{\sqrt{-qp}}{2ab}}\xi + r_2}{r_1 + r_2\tanh\sqrt{-\frac{\sqrt{-qp}}{2ab}}\xi}\right)^2\right], \quad \xi = x - at, ab < 0, q > 0, p < 0, \tag{82}$$

$$u_3(x,t) = \ln\left[\frac{\sqrt{-qp}}{p}\left(\frac{r_3\tan\sqrt{\frac{\sqrt{-qp}}{2ab}}\xi - r_4}{r_3 + r_4\tan\sqrt{\frac{\sqrt{-qp}}{2ab}}\xi}\right)^2\right], \quad \xi = x - at, ab > 0, q < 0, p > 0, \tag{83}$$

$$u_4(x,t) = \ln\left[-\frac{\sqrt{-qp}}{p}\left(\frac{r_3\tan\sqrt{-\frac{\sqrt{-qp}}{2ab}}\xi - r_4}{r_3 + r_4\tan\sqrt{-\frac{\sqrt{-qp}}{2ab}}\xi}\right)^2\right], \quad \xi = x - at, ab < 0, q > 0, p < 0. \tag{84}$$

It should be noted that there are another eight sets of solutions for the algebraic equations, but the solutions given by them are same as the solutions (81), (82), (83) and (84), so these solutions are not listed here.

The solutions (81) and (82) with $r_2 = 0$ and with $r_1 = 0$ will give the solutions (3.9a) and (3.9b) in [32]. The solutions (83) and (84) with $r_4 = 0$ and with $r_3 = 0$ will recover the solutions (3.11a) and (3.11b) in [32], respectively. For other parameters r_i $(i = 1, 2, 3, 4)$

satisfying the conditions (34) and (37), we can obtain infinite number of new exact traveling wave solutions of Eq. (73) from (81), (82), (83) and (84).

6 Conclusion and discussion

As proved in Sect. 4, the twenty-two previously known solutions of the Riccati Eq. (1) are equivalent to our fractional solutions (33), (36), and the rational solution $G_{27}(\xi)$ is only a special case of the solution (45). Therefore, the solution (51) not only can recover all those twenty-three solutions but also can give infinitely many new independent solutions with the variation of their parameter r_i $(i = 1, 2, 3, 4)$. Because of this reason, our suggested method can effectively avoid those repeated solutions, which may be caused by those twenty-seven solutions previously used. And the computational efficiency is greatly improved by using our three solutions instead of twenty-three known solutions. At the same time, for a given NLEE, our method can catch infinite number of exact traveling wave solutions unlike the previous direct methods just capable of generating finite number of exact solutions.

More importantly, the general theoretical system of the GREMM established in this work allows one following the natural way to construct infinite number of new exact solutions of the Riccati Eq. (1) from the Bäcklund transformation and the superposition formula. We believe that the familiar theoretical system can be built for other direct methods, which is under consideration.

Both the one parametric solutions (32) and (35) have been changed to the two parametric solutions (33) and (36), respectively. The advantage of this choice is that we can get tanh −, coth −, tan- and cot-type solutions from (33) and (36) compared to the tanh- and tan-type solutions obtained from (32) and (35). So we have to construct another two one parametric fractional solutions from the Bäcklund transformation so as to obtain the coth- and cot-type solutions, and it is, of course, not worth doing.

In the special case of $c_0 = b, c_1 = 0, c_2 = 1$, we find $\Delta = -b$, and the Riccati Eq. (1) is reduced to the simple Riccati equation $F'(\xi) = b + F^2(\xi)$, which is used by Fan [3] in the extended tanh-function method. Therefore, our method also established a generalization of the extended tanh-function method. Correspondingly, the superposition formulas (20) and (31) can be reduced into one expression. And the Bäcklund transformation (8) and (9) with setting $\alpha = \frac{2}{\mu\sqrt{\Delta}}$ and $\beta = -\frac{2}{\mu\sqrt{-\Delta}}$, respectively, can be combined into the following expression:

$$F_n(\xi) = \frac{F_{n-1}(\xi) - b\mu}{1 + \mu F_{n-1}(\xi)},$$

where μ is an arbitrary constant.

As noted in Sect. 4, we failed to derive the solutions $G_6(\xi)$, $G_7(\xi)$, $G_{18}(\xi)$ and $G_{19}(\xi)$ from the fractional solutions. In other words, the relations between these four solutions and our two fractional solutions are not clear and should be further study.

Another open problem is as how to solve discrete and fractional nonlinear equations by using our suggested method, and we hope that many new results to be appear in this area.

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