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ORIGINAL PAPER

Vector multipole and vortex solitons in two-dimensional Kerr media

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Abstract We investigate a (2+1)-dimensional coupled nonlinear Schrödinger equation with spatially modulated nonlinearity and transverse modulation, and derive analytical vector multipole and vortex soliton solution. When the modulation depth q is chosen as 0 and 1, vector multipole and vortex solitons are constructed, respectively. The number of azimuthal lobes ("petals") for the multipole solitons is determined by the value of $2m$ with the topological charge m, and the number of layers in the multipole solitons is determined by the value of the soliton order number n.

Keywords Vector multipole soliton · Vector vortex soliton · (2+1)-dimensional coupled nonlinear schrödinger equation · Kerr nonlinear media

1 Introduction

In the past few decades, dynamic behaviors of optical solitons have flourished to become a research area of great importance and interest in many different contexts of nonlinear optics [1–4]. Spatial solitons, as an important nonlinear localized state, form and propagate with the balance between diffraction and self-induced nonlinear potential [5,6]. There has been a lot of interest in different type of spatial solitons, such as fundamental soliton [7,8], dromion [9], Peregrine solution [10], vortex soliton [11] and azimuthon [12].

Recently, spatial scalar solitons have been extensively studied. The sign-alternating Kerr nonlinearity in a layered medium produces stable two-dimensional (2D) solitons [13]. Wu et al. [14] investigated 2D stable vortices with the spatially modulated cubic nonlinearity and a harmonic trapping potential, respectively. Competing cubic-quintic nonlinearity in the bulk medium generates stable vortex solitons [15]. Zhong et al. [16] studied two-dimensional accessible solitons in PT-symmetric potentials.

In contrast with the spatial scalar solitons possessing one component, spatial vector solitons with two or more components mutually self-trap in a nonlinear medium. Dynamical propagation behaviors of the vector solitons are richer than those of the scalar solitons due to their multicomponent structures [17,18]. Considering multicomponent structures, one needs to consider the simultaneous propagation of optical solitons for N fields and the governing equation becomes a coupled nonlinear Schrödinger equation (NLSE). When two optical waves of different frequencies co-propagate in a medium and interact nonlinearly through the medium, or when two polarization components of a wave interact

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nonlinearly at some central frequency, the propagation equations for the two problems can be described by Manakov equation [19]. Manakov vector solitons with the equal self-phase modulation (SPM) and cross-phase modulation (XPM) can propagate in the bright-bright, bright-dark and dark-dark forms [17]. Moreover, Manakov vector solitons with the same velocities in a bound state have also been investigated [18]. However, these works discussed one-dimensional spatial vector solitons.

To our knowledge, 2D spatial vector solitons were relatively few studied. Recently, based on 2D Manakov equation, Zhong et al. discussed self-trapping of scalar and vector dipole solitary waves in Kerr media [20]. Compared with the spatial scalar solitons, spatial vector solitons have much more application in the control of optical beam diffraction, design of the logic gates, all-optical switching devices and information transformation [21,22]. Vortex solitons were created in photorefractive crystals equipped with photonic lattices [23]. As we all know, stationary solitons in 2D Kerr media are always unstable against collapse or decay, due to the critical character of the local cubic self-attractive nonlinearity in the 2D setting. However, these investigations were carried out almost completely by means of numerical methods, and analytical solutions for localized vector vortices have not been reported yet. In this paper, we report analytical 2D multipole and vortex solitons in a local Kerr medium and study the structure pattern of these solitons.

2 Exact vector multipole and vortex soliton solutions

Being motivated by the above reasons, we will devote our attention to the following coupled (2+1)-dimensional NLSE with varying coefficients

$$\mathrm{i}\frac{\partial\psi_j}{\partial z}=-\frac{1}{2}\nabla^2\psi_j+g(r)\sum_{j=1}^{N}|\psi_j|^2\psi_j+R(r)\psi_j, \tag{1}$$

which describes vector beams consisting of N mutually incoherent components co-propagating in a Kerr medium with the refractive index $n=n_0+n_1R(r)+n_2g(r)|\psi|^2$. In this equation, $\psi_j(z,r,\varphi)(j=1,2,...N)$ are the slowly varying envelopes with the propagation distance z and the polar coordinates r and φ in the transverse plane, the 2D Laplacian $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$. The Kerr nonlinearity coefficient $g(r)$, as well as the transverse modulation $R(r)$, is assumed to be a function of radial coordinate $r\equiv(x,y)$. The transverse x, y and longitudinal z coordinates are normalized to the beam width $w_0=(2k_0^2n_1)^{-1/4}$ and diffraction length $L_d=k_0w_0^2$ with the wave number $k_0=2\pi n_0/\lambda$ at the input wavelength λ. For the disappearing transverse modulation (i.e., $R(r)=0$), Eq. (1) may be considered as a 2D version of the Manakov's system, which has been studied in [20]. If ψ_j represent the macroscopic wave function of the condensate, $R(r)$ denotes the external potential, Eq. (1) is coupled Gross–Pitaevskii equation in Bose–Einstein condensates.

Spatially inhomogeneous nonlinearity (SIN) and transverse modulation have been extensively discussed [24–26]. However, analytical solutions for localized vector vortices have not been reported yet. Next, we look for the spatially localized stationary exact solution to Eq. (1) of the form

$$\psi_j(r,\varphi,z)=A(r)\Phi_j(\varphi)\exp(-\mathrm{i}\kappa z), \tag{2}$$

where κ is the propagation constant, and $A(r)$ is a real function for the localization demand $\lim_{r\to\pm\infty}A(r)=0$.

Inserting Eq. (2) into Eq. (1), one obtains

$$\frac{r^2}{A}\left\{\frac{\partial^2A}{\partial r^2}+\frac{1}{r}\frac{\partial A}{\partial r}+2[\kappa-R(r)]A-2g(r)A^3\right\}=m^2, \tag{3}$$

$$-\frac{1}{\Phi_j}\frac{\partial^2\Phi_j}{\partial\varphi^2}=m^2, \tag{4}$$

with the self-consistency condition $\sum_{j=1}^{N}|\Phi_j|^2=1$.

Equation (4) admits solution

$$\Phi_j=C_j\cos(m\varphi)+D_j\sin(m\varphi), \tag{5}$$

where m may be considered as the topological charge. Here, we consider two-component case ($N=2$), thus $C_1=1$, $D_1=\mathrm{i}p$, $C_2=0$, $C_2=\sqrt{1+p^2}$ with $p(0\le p\le 1)$. The limit values denote the multipole ($p=0$) and vortex ($p=1$) solitons, and the topological charge $m=1\sim 5$ describes dipole, quadrupole, hexapole, octopole and dodecagon solitons.

Substituting $A(r)\equiv\rho(r)U[\chi(r)]$, with $U[\chi(r)]$ satisfying

$$-\frac{\mathrm{d}^2U}{\mathrm{d}\chi^2}+G(U)=\eta U, \tag{6}$$

into Eq. (3) yields

$$\rho_{rr}+\frac{1}{r}\rho_r+\left[2\kappa-2R(r)-\frac{m^2}{r^2}\right]\rho=\frac{\eta}{r^2\rho^3}, \tag{7}$$

$$\rho r^2\chi_{rr}+\rho r\chi_r+2r^2\rho_r\chi_r=0,\quad G(U)\chi_r^2/U^3 -2g\rho^2=0, \tag{8}$$

where η and g_0 are constants.

Equation (8) admits the following solutions

$$g(r)\equiv G(U)r^{-2}\rho^{-6}(r)/(2U^3),$$
$$\chi(r)\equiv\int_0^r\rho^{-2}(s)s^{-1}\mathrm{d}s. \tag{9}$$

From the procedure above, the coupled NLSE (1) is reduced to the solvable stationary NLSE (6), thus this helps one to find exact solutions, as stationary NLSE (6) possesses rich solutions, such as Jacobian elliptic function solutions [27]. Thus, the solvability of Eq. (7) is crucial to construct exact solutions to the underlying coupled NLSE (1).

Equation (7) is not easily solved. At first, we consider the simpler case, i.e., $\eta=0$ in Eq. (7). In this case, for parabolic transverse modulation with $R(r)=\omega r^2$, we obtain exact solution

$$\rho=r^{-1}[c_1M(\kappa/2\sqrt{2\omega},m/2,\sqrt{2\omega}r^2) +c_2W(\kappa/2\sqrt{2\omega},m/2,\sqrt{2\omega}r^2)], \tag{10}$$

where functions $M(\cdot)$ and $W(\cdot)$ are Whittaker's M and W functions, respectively [28]. For disappearing transverse modulation with $R(r)=0$, one has solution

$$\rho=c_3J(m,\sqrt{2\kappa}r)+c_4Y(m,\sqrt{2\kappa}r), \tag{11}$$

where functions $J(\cdot)$ and $Y(\cdot)$ are the Bessel functions of the first and second kinds [29], respectively.

Note that the SIN strength is bounded and the integration in $R(r)$ converges [15], thus the expressions for $R(r)$ and $g(r)$ require that ρ cannot change its sign, and it must behave as $r^{-\gamma}$ with $\gamma\geq 1/3$ at $r\to 0$, and $\rho\to\infty$ (diverge) at $r\to\infty$. These restrictions require that $c_1c_2>0$, $\kappa<\kappa_0\equiv 2(1+m)\sqrt{\omega}$ in Eq. (10), and $c_3c_4>0$, $\kappa<0$ in Eq. (11).

Further, if $\eta\neq 0$, solution of $\rho(r)$ become

$$\rho=\sqrt{\frac{1}{r}(\alpha\phi_1^2+2\beta\phi_1\phi_2+\gamma\phi_2^2)}, \tag{12}$$

where $\eta=(\alpha\gamma-\beta^2)W^2$ with three constants α,β,γ and constant Wronskian $W=\phi_1\phi_{2r}-\phi_2\phi_{1r}$ with $\phi_1(r)$ and $\phi_2(r)$ being two linearly independent solutions of $\phi_{rr}+[2\kappa-2R(r)-m^2/r^2]\phi=0$.

The methodology mapping Eq. (1) into Eq. (6) provides for a systematic way to find an infinite number of the novel exact "soliton islands" in a "sea of solitary waves." Exact solutions of Eq. (1) is generated from solutions of Eq. (6). The wide choice of $G(U)$ in Eq. (6) makes Eq. (6) become some famous equations such as Schrödinger equation, NLSE, sine-Gordon equation, KdV equation and thus construct abundant solutions of Eq. (6). For example, if $G(U)$ is a linear function of U, Eq. (6) is the linear Schrödinger equation with the external potential. When the external potential is the harmonic and hyperbolic potentials, solutions of Eq. (6) have been used to construct localized modes in ref. [30]. If $G(U)=g_0U^3$, the boundary conditions $U(0)=U(\infty)=0$ leads to exact solution of Eq. (6) with $\eta=0$ as [27]

$$U(\chi)=\frac{2n\lambda}{\sqrt{-g_0}}\mathrm{sd}\left[2n\lambda\chi(r),\frac{\sqrt{2}}{2}\right], \tag{13}$$

where $g_0<0$, function $\mathrm{sd}(\cdot)\equiv\mathrm{sn}(\cdot)/\mathrm{dn}(\cdot)$ with the Jacobian elliptic sine function $\mathrm{sn}(\cdot)$ and the Jacobian elliptic of the third kind $\mathrm{dn}(\cdot)$, the soliton order number $n=1,2,3,\ldots$, and $\lambda\equiv K\left(\frac{\sqrt{2}}{2}\right)$ with the complete elliptic integral of the first kind $K(k)$ and modulus k.

If $G(U)=g_3U^3+g_5U^5$, Eq. (6) is a solvable cubic-quintic NLSE, which generates richer exact soliton solutions considering sign-changing cubic-nonlinearity coefficient for $g_3g_5<0$ [31]. Further, if $G(U)=\eta U-\sin(\eta U)$, Eq. (6) is the stationary sine-Gordon equation [31], whose solution has the form with periodical function

$$U(\chi)=2\eta^{-1}\arcsin[k\,\mathrm{sn}(\sqrt{\eta}\chi,k)], \tag{14}$$

where $2nK(k)=\sqrt{\eta}\chi$ with the positive integer n and the first-kind complete elliptic integral $K(k)$ according to the zero boundary condition at $r\to\pm\infty$. In this case, the cubic nonlinearity is defocusing sign with $g(r)>0$.

Therefore, from Eqs. (2), (5), (13) [or (14)] and (10) [or (11), or (12)], we can obtain the spatially localized stationary exact solution of Eq. (1). In this paper, we use solution (13) with (10) [or (11)].

3 Structures of vector multipole and vortex solitons

In this section, we display and discuss structures of vector multipole and vortex solitons. The multipole ($p = 0$) and vortex ($p = 1$) solitons are shown in Figs. 1, 2, 3 and 4.

In the absence of transverse modulation with $R(r) = 0$, for $p = 0, m = 1, n = 1$, vector dipole soliton is shown in the first row of Fig. 1. Two components of vector dipole soliton orthogonally arrange in Fig. 1a and b, and incoherently superpose to form a ring-like soliton in Fig. 1c. Note that the ring-like soliton is not a vortex soliton because the phase is not a 2π jump around its core in Fig. 1d. However, if $p = 1$, vector vortex soliton can be constructed. In Fig. 1e, a vortex soliton is displayed for the component $|\psi_1|^2$ because its phase exists a 2π jump around its core in Fig. 1h.

For $p = 0$, the multipole solitons with different values of m and n are exhibited in Fig. 2. The intensity of all multipole solitons equals to zero at the center. The number of azimuthal lobes ("petals") for the multipole solitons is determined by the value of $2m$, and the number of layers in the multipole solitons is determined by the value of n. According to the number of azimuthal lobes ("petals"), multipole solitons in Fig. 2a, b are called as quadrupole and hexapole solitons, respectively. For the same n, the structure expands in the radial direction with the increase in the value m [c.f. Fig. 2a–h]. Similarly, for the same m, the "petals" in the outermost layer also expands in the radial direction with the add of the value n [c.f. Figs. 1a and 2c, f; Fig. 2a, d, g; Fig. 2b, e, h].

When we consider the parabolic transverse modulation $R = \omega r^2$, we can also construct vector multipole and vortex solitons from Eqs. (2), (5), (10) and (13). Figures 3 and 4 show vector multipole and vortex solitons in the presence of the parabolic transverse modulation $R = \omega r^2$. For $p = 0$, two orthogonally arranged components of vector dipole soliton in Fig. 3a, b also

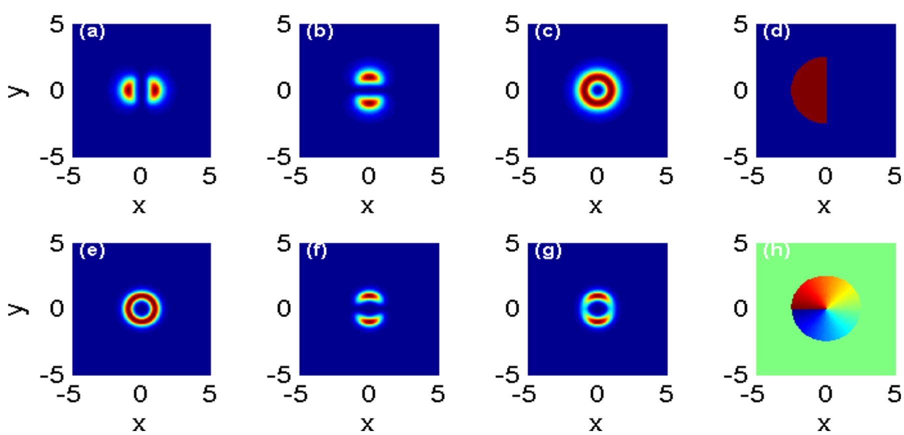

Fig. 1 (*Color online*). Dipole soliton in the first row and vortex soliton in the second row with the intensity of **a**, **e** component $|\psi_1|^2$, **b**, **f** component $|\psi_2|^2$, **c**, **g** total quantity $|\psi|^2 = |\psi_1|^2 + |\psi_2|^2$ and **d**, **h** phase in the absence of transverse modulation with $R(r) = 0$. The parameters are $c_{3,4} = -\kappa = -g_0 = 1, m = 1, n = 1$ with **a**–**d** $p = 0$ and **e**–**h** $p = 1$

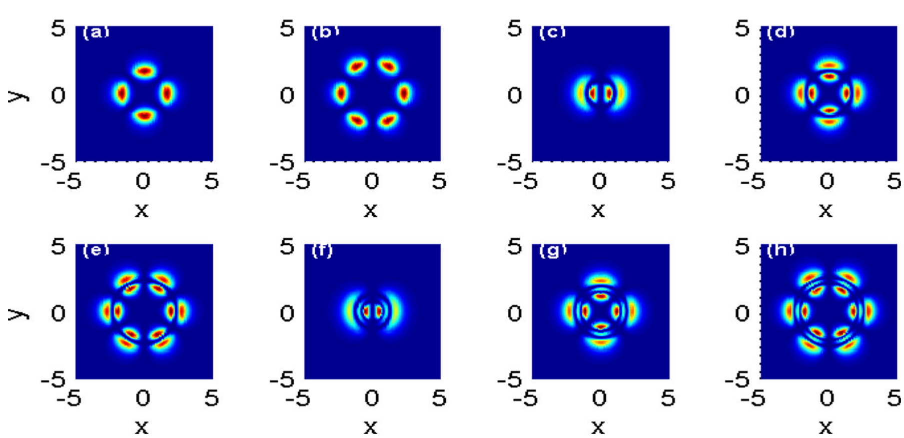

Fig. 2 (*Color online*). Multipole soliton for the intensity of component $|\psi_1|^2$ in the absence of transverse modulation with $R(r) = 0$. The parameters are $c_{3,4} = -\kappa = -g_0 = 1, p = 0$ with **a** $m = 2, n = 1$, **b** $m = 3, n = 1$, **c** $m = 1, n = 2$, **d** $m = 2, n = 2$, **e** $m = 3, n = 2$, **f** $m = 1, n = 3$, **g** $m = 2, n = 3$, **h** $m = 3, n = 3$

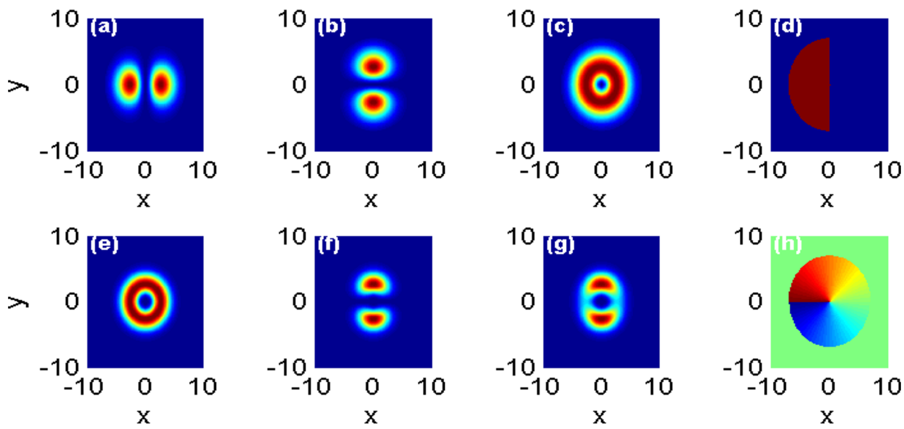

Fig. 3 (*Color online*). Dipole soliton in the first row and vortex soliton in the second row with the same quantities in Fig. 1 in the presence of the parabolic transverse modulation $R = \omega r^2$ with $\omega = 0.02$ and $c_1 = 2, c_3 = 3$. Other parameters are chosen as those in Fig. 1

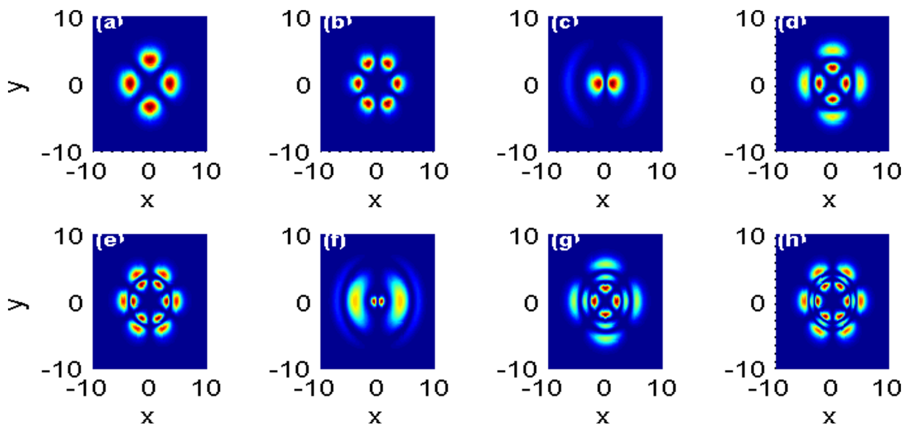

Fig. 4 (*Color online*). Multipole soliton for the intensity of component $|\psi_1|^2$ in the presence of the parabolic transverse modulation $R = \omega r^2$ with $\omega = 0.02$ and $c_1 = 2, c_3 = 3$. Other parameters are chosen as those in Fig. 2

incoherently superpose to produce a ring-like soliton in Fig. 3c. Vortex solitons in Fig. 3e possess a 2π phase jump around its core in Fig. 3h.

In the presence of the parabolic transverse modulation $R = \omega r^2$ in Fig. 4, for the same n, the structure also expands in the radial direction with the increase in the value m, and for the same m, the "petals" in the outermost layer also expands in the radial direction with the add of the value n. Comparing multipole solitons in Fig. 3 and those in Fig. 4, multipole solitons expands wider in the radial direction in the presence of the parabolic transverse modulation $R = \omega r^2$ than those in the absence of transverse modulation with $R(r) = 0$. The reason is that the effect the parabolic transverse modulation counteracts the effect of nonlinearity, and the effect of nonlinearity attenuates. Therefore, the pattern of multipole solitons in the presence of the parabolic transverse modulation possesses a wider space in the radial direction.

4 Conclusions

In conclusion, we investigate a (2+1)-dimensional coupled nonlinear Schrödinger equation with spatially modulated nonlinearity and transverse modulation, and derive analytical vector multipole and vortex soliton solution. When the modulation depth q is chosen as 0 and 1, vector multipole and vortex solitons are constructed, respectively. The number of azimuthal lobes ("petals") for the multipole solitons is determined by the value of $2m$ with the topological charge m, and the number of layers in the multipole solitons is determined by the value of the soliton order number n. Regardless of the absence or presence of transverse modulation, for the same soliton order number n, the structure of the multipole solitons expands in the radial direction with the increase in the value m, and for the same topological charge m, the "petals" in the outermost layer also expands in the radial direction with the add of the value n. The effect the parabolic transverse modulation counteracts the effect of nonlinearity; thus, the pattern of multipole solitons in the presence of the parabolic transverse modulation possesses a wider space in the radial direction.

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