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Some classification of non-commutative Integrable Systems

Daryoush Talati · Abdul-Majid Wazwaz

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Abstract In this work, we develop some non-commutative (NC) second-order integrable coupled systems of weight 1 which possess a third-order symmetry. Two-component second-order NC Burgers-type systems, with non-diagonal constant matrix of leading order terms, are classified for higher symmetries. We obtain new symmetry integrable systems with their master symmetries.

Keywords Integrable coupled systems · Third-order symmetry · Two-component second-order non-commutative Burgers

1 Introduction

Integrable systems of equations, which possess sufficiently large number of conservation laws and give rise to multiple soliton solutions, play a major role in theoretical physics and in propagation of waves. An evolution equation is defined to be integrable in symmetry sense if it admits infinitely many symmetries. A partial differential equation is completely integrable if and only if it possesses infinitely many generalized symmetries. The existence of a sufficiently large number of conservation laws or symmetries guarantees complete integrability for this equation. The connection between conservation laws and symmetries from the geometric point of view was investigated in [1–17].

Integrable systems are nonlinear differential equations which can be solved analytically. Exactly solvable models and integrable evolution equations in nonlinear science play an essential role in many branches of science and engineering applications. The useful findings in integrable systems of equations have stimulated much research activities. We refer to the useful papers [1–12,14] and the some of the references therein, where useful researches were invested to make further progress in this field. The extension of integrable systems to non-commutative (NC) spaces is one of hot topics in the recent study of integrable systems [3,5,8,9]. NC extension in gauge theories corresponds to the presence of background magnetic fields and leads to the discovery of many new physical objects and successful applications to string theories. Non-commutative generalizations of the classical nonlinear evolution equations in (1+1) dimensions were classified according to the symmetry-based integrability in [15–21].

In this article, two-component non-commutative versions of Burgers equation are observed to have higher symmetry in a certain weighting scheme of symmetries. As a result, we present various NC equations

D. Talati
Sama Technical and Vocational Training College,
Islamic Azad University, Urmia, Iran
e-mail: Talati@eng.ankara.edu.tr;
Daryoush.Talati@gmail.com

A.-M. Wazwaz (✉)
Department of Mathematics, Saint Xavier University,
Chicago, IL 60655, USA
e-mail: Wazwaz@sxu.edu

which possess the higher symmetry. We mainly discuss the applications of master symmetry. The non-commutative (NC) extension of Burgers equations is quite interesting in order to explore their properties which they possess on ordinary spaces. NC spaces are characterized by the non-commutativity of the spatial coordinates; if x^μ are the space coordinates, then the non-commutativity is defined by $[x^\mu, x^\nu]_* = i\theta^\mu$ where parameter $\theta^{\mu\nu}$ is antisymmetric tensor and Lorentz invariant and $[x^\mu, x^\nu]_*$ is commutator under the star product. NC field theories on flat spaces are given by the replacement of ordinary products with the Moyal products and realized as deformed theories from the commutative ones.

Systematic classification of two-component integrable systems of second-order differential equations is considered by Olver and Sokolov [8] and the references therein. One of their extensive results was the fact that all the integrable cases of the class considered were those systems that can be written by generalized contact transformations in the form where the coefficient matrix of leading order terms $J^{\pm 1}$ is the first of the three constant matrices

$$J^{\pm 1} = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, J^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, J^3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (1)$$

No two-component NC system with non-diagonal Jordan form J^3, to our belief, has been discovered so far. Here we classify (1, 1)-homogeneous systems which are left unclassified in terms of the matrix of leading J^2 and J^3 admitting a certain type of higher symmetry. We give master symmetries of the systems determined to have only higher symmetries.

2 The class

The class of non-commutative two-component (1,1)-homogeneous systems with undetermined constant coefficients p_i and q_i has the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} p_1u_2 + p_2v_2 + p_3uu_1 + p_4u_1v + p_5u_1u + p_6vu_1 + p_7uv_1 + p_8v_1u \\ + vp_9v_1 + p_{10}v_1v + p_{11}u^3 + p_{12}v^3 + p_{13}u^2v + p_{14}uvu + p_{15}vu^2 \\ + p_{16}uv^2 + p_{17}vuv + p_{17}v^2u \\ \\ q_1u_2 + q_2v_2 + q_{3u}u_1 + q_4u_1v + q_5u_1u + q_6vu_1 + q_7uv_1 + q_8v_1u \\ + q_9vv_1 + q_{10}v_1v + q_{11}u^3 + q_{12}v^3 + q_{13}u^2v + q_{14}uvu + q_{15}vu^2 \\ + q_{16}uv^2 + q_{17}vuv + q_{17}v^2u. \end{pmatrix} \quad (2)$$

The class of non-commutative two-component (1,1)-homogeneous systems with undetermined constant coefficients k_i and l_i is

$$\begin{pmatrix} u \\ \\ v \end{pmatrix}_t = \begin{pmatrix} \begin{array}{c} k_1u_3 + k_2v_3 + k_3uu_2 + k_4u_2u + k_5u_2v + k_6vu_2 + k_7uv_2 + k_8v_2u \\ + k_9vv_2 + k_{10}v_2v + k_{11}u_1^2 + k_{12}v_1^2 + k_{13}u_1v_1 + k_{14}v_1u_1 + k_{15}u^2u_1 \\ + k_{16}u_1u^2 + k_{17}uu_1u + k_{18}uu_1v + k_{19}vu_1u + k_{20}u_1vu + k_{21}u_1uv \\ + k_{22}vuu_1 + k_{23}uvu_1 + k_{24}v^2u_1 + k_{25}u_1v^2 + k_{26}vu_1v + k_{27}uv_1v + k_{28}u^2v_1 \\ + k_{29}v_1u^2 + k_{30}uv_1u + k_{31}vv_1u + k_{32}v_1vu + k_{33}v_1uv + k_{34}vuv_1 + k_{35}uvv_1 \\ + k_{36}v^2v_1 + k_{37}v_1v^2 + k_{38}vv_1v + k_{39}u^4 + k_{40}u^3v + k_{41}vu^3 + k_{42}uvu^2 \\ + k_{43}u^2vu + k_{44}v^3u + k_{45}uv^3 + k_{46}vuv^2 + k_{47}v^2uv + k_{48}u^2v^2 + k_{49}v^2u^2 \\ + k_{50}uv^2u + k_{51}vu^2v + k_{52}vuvu + k_{53}uvuv + k_{54}v^4 \\ \\ l_1u_3 + l_2v_3 + l_3uu_2 + l_4u_2u + l_5u_2v + l_6vu_2 + l_7uv_2 + l_8v_2u + l_9vv_2 \\ + l_{10}v_2v + l_{11}u_1^2 + l_{12}v_1^2 + l_{13}u_1v_1 + l_{14}v_1u_1 + l_{15}u^2u_1 + l_{16}u_1u^2 \\ + l_{17}uu_1u + l_{18}uu_1v + l_{19}vu_1u + l_{20}u_1vu + l_{21}u_1uv + l_{22}vuu_1 + l_{23}uvu_1 \\ + l_{24}v^2u_1 + l_{25}u_1v^2 + l_{26}vu_1v + l_{27}uv_1v + l_{28}u^2v_1 + l_{29}v_1u^2 + l_{30}uv_1u \\ + l_{31}vv_1u + l_{32}v_1vu + l_{33}v_1uv + l_{34}vuv_1 + l_{35}uvv_1 + l_{36}v^2v_1 \\ + l_{37}v_1v^2 + l_{38}vv_1v + l_{39}u^4 + l_{40}u^3v + l_{41}vu^3 + l_{42}uvu^2 + l_{43}u^2vu \\ + l_{44}v^3u + l_{45}uv^3 + l_{46}vuv^2 + l_{47}v^2uv + l_{48}u^2v^2 + l_{49}v^2u^2 \\ + l_{50}uv^2u + l_{51}vu^2v + l_{52}vuvu + l_{53}uvuv + l_{54}v^4 \end{array} \end{pmatrix} \tag{3}$$

Imposing compatibility condition among the classes of systems (2) and their proposed class of symmetries (3), we obtain a system of algebraic equations among the undetermined constants. Each solution of the system of constraints on the constants $p_iq_ik_i$ and l_i determines a system and its higher symmetry. Among the systems obtained to possess a higher symmetry, the uncoupled ones and the triangular ones, i.e., those that reduce to successive scalar equations, are discarded as they are trivial. The remaining systems are investigated for master symmetry. Solutions of the algebraic conditions imposed on the undetermined coefficients by compatibility condition among classes (2) and (3) with the choice $p_1 = p_2 = q_2 = 1$ and $q_1 = 0$ give rise only to the non-trivial system given in the following section.

Theorem 1 *Any (1,1)-homogeneous system of form (2) with a non-degenerate constant coefficient matrix of leading order terms having one- dimensional eigenspace, possessing a (1,1)-homogeneous symmetry of form (3), is equivalent, by a linear change of variables u and v and rescaling of x and t, to one of the following eight equations:*

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xx} + v_{xx} - 2u_xu - 2v_xu + uv_x - vu_x - uvu + vu^2 \\ \\ v_{xx} - 2v_xu - 2v_xv + 2uv_x + vv_x + 2u^2v - uv^2 \\ - 2uvu - v^2u + 2vuv \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xx} + v_{xx} + u_xv - 3v_xu + v_xv + 2uu_x + 2uv_x - 2vu_x \\ + u^2v + uv^2 - 3uvu + 2vu^2 - vuv \\ \\ v_{xx} - 2v_xu + v_xv + 2uv_x + uv^2 - 2uvu - v^2u + 2vu^2 \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xx} + v_{xx} - u_xv + v_xu - 2uu_x - 2uv_x + u^2v - uvu \\ \\ v_{xx} + v_xv - 2uv_x - uv^2 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 4u_{xx} + 4v_{xx} + 4u_xu - 2uv_x + 4v_xu + 2vu_x - uvu + vu^2 \\ \\ 4v_{xx} + 4v_xu - v^2u - 2vv_x \end{pmatrix} \tag{7}$$

All of the systems given in the above theorem have symmetry algebra of higher-order systems. All above systems reduce to scalar second-order Burgers equation by $v = 0$ reduction.

$$u_t = u_{xx} + u_x u$$

and

$$u_t = u_{xx} + u u_x$$

2.1 System (4)

System (4) has the master symmetry

$$\mathcal{M}^1 = \begin{pmatrix} x u_t + 2v_x - u^2 + uv - 2vu \\ x v_t - v_x - \frac{1}{2} v^2 \end{pmatrix} \quad (8)$$

So one can generate symmetries of arbitrarily high orders by symmetry relation successively. Therefore, system (4), obtained by acting $\mathcal{M}^1$ on x-translation symmetry, is symmetry integrable.

2.2 System (5)

System (5) is a symmetry integrable system having the master symmetry

$$\mathcal{M}^2 = \begin{pmatrix} x u_t + v_x + u^2 + uv + \frac{1}{2} v^2 - 2vu \\ x v_t - v_x + \frac{1}{2} v^2 \end{pmatrix} \quad (9)$$

Acted on x-translation symmetry, $\mathcal{M}^2$ gives system (5).

2.3 System (6)

Symmetry integrable system (6) is obtained by acting $\mathcal{M}^3$ to x-translation symmetry where the master symmetry is

$$\mathcal{M}^3 = \begin{pmatrix} x u_t + u_x - u^2 + 2\theta uv - 2\theta v_x + (-2\theta - 1) vu \\ x v_t + (-2\theta - 1) v^2 \end{pmatrix} \quad (10)$$

here θ is arbitrary. So system (6) is divided into two systems:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -u_x v + v_{xx} + v_x u - 2uv_x + u^2 v - uvu \\ v_x v - uv^2 \end{pmatrix} \quad (11)$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xx} - 2uu_x \\ v_{xx} - 2uv_x \end{pmatrix} \quad (12)$$

2.4 System (7)

Symmetry integrable system (7) is obtained by acting $\mathcal{M}^4$ to x-translation symmetry where the master symmetry is related to $Mastersymmetry$:

$$\mathcal{M}^4 = \begin{pmatrix} 2\lambda x u_t - 4\mu u_x + \lambda u^2 + (\lambda + 2\theta) uv - 4\theta v_x - 2\theta vu \\ 2\lambda x v_t - 4\alpha v_x + (\lambda + 2\theta) v^2 + (\lambda + 2\mu - 2\alpha) vu \end{pmatrix} \quad (13)$$

Here λ is arbitrary. So system (7) is divided into two systems:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 4v_{xx} - 2uv_x + 4v_x u + 2vu_x - uvu + vu^2 \\ -v^2 u - 2vv_x \end{pmatrix} \quad (14)$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xx} + u_x u \\ v_{xx} + v_x u \end{pmatrix} \quad (15)$$

We may rewrite Eq. (14) with transformation $u \longrightarrow 2w_x \bar{w}$ and $v \longrightarrow z\bar{w}$ in the following forms

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 2z_x \\ 0 \end{pmatrix} \quad (16)$$

which is trivial.

3 Some new classes with no member

Not every scalar integrable system has a non-commutative counterpart. A key question is how many of the known commutative examples extend to non-commutative integrable equations. In recent years, studies on fifth-order systems of two-component nonlinear evolution equations have received considerable attention [10,11,13,15]. Olver and Sokolove proved [8] that in KdV weighting, there are no purely matrix analogs of the fifth-order Sawada–Kotera and Kaup Kupershmidt equations when the right-hand side of the evolution equation only involves the field variable u and its x derivatives.

The second interesting weighting is that associated with both 1-homogeneous Kupershmidt equation, So here we aim to complete the solving of classification problem for non-commutative scalar fifth- order equations.

The class of non-commutative scalar 1-homogeneous systems with undetermined constant coefficients α_i has the form

$$\begin{aligned} u_t = {} & u_5 + \alpha_2 u u_4 + \alpha_3 u_4 u + \alpha_4 u_1 u_3 + \alpha_5 u_3 u_1 \\ & + \alpha_6 u_2^2 + \alpha_7 u^2 u_3 + \alpha_8 u_3 u^2 \\ & + \alpha_9 u u_3 u + \alpha_{10} u u_1 u_2 + \alpha_{11} u u_2 u_1 + \alpha_{12} u_1 u u_2 \\ & + \alpha_{13} u_1 u_2 u + \alpha_{14} u_2 u_1 u \\ & + \alpha_{15} u_2 u u_1 + \alpha_{16} u_1^3 + \alpha_{17} u^3 u_2 \\ & + \alpha_{18} u_2 u^3 + \alpha_{19} u u_2 u^2 \\ & + \alpha_{20} u^2 u_2 u + \alpha_{21} u^2 u_1^2 \\ & + \alpha_{22} u_1^2 u^2 + \alpha_{23} u_1 u u_1 u + \alpha_{24} u u_1 u u_1 \\ & + \alpha_{25} u u_1^2 u + \alpha_{26} u_1 u^2 u_1 + \alpha_{27} u^4 u_1 \\ & + \alpha_{28} u^3 u_1 u + \alpha_{29} u^2 u_1 u^2 + \alpha_{30} u_1 u^4 \\ & + \alpha_{31} u u_1 u^3 + \alpha_{32} u^6 \end{aligned} \tag{17}$$

With arbitrary α.The class of non-commutative 1-homogeneous of weight 7 systems contains about 130 terms in each component. Therefore, we omit writing this class explicitly. Solutions of the algebraic conditions imposed on the undetermined coefficients by compatibility condition among classes 17 and related higher- order symmetry give rise no any non-trivial system given in the following Theorem .

Theorem 2 *For the scalar 1-homogeneous systems, every non-degenerate polynomial equation of weight 5 over a non-commutative associative algebra having a weight 7 symmetry is either linear or equivalent to the fifth-order NC mkdv or NC Burgers equation.*

We have done also a classification of 2-homogeneous non-diagonal second-order system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_2 + v_2 + \beta_1 u^2 + \beta_2 u v + \beta_3 v u + \beta_4 v^2 \\ v_2 + \beta_5 u^2 + \beta_6 u v + \beta_7 v u + \beta_8 v^2 \end{pmatrix}. \tag{18}$$

With arbitrary β.

Solutions of the algebraic conditions imposed on the undetermined coefficients by compatibility condition among classes 18 and related higher-order symmetry give rise no any non-trivial system given in the following Theorem .

Theorem 3 *No non-trivial (2,2)-homogeneous system of form* (18) *with a non-degenerate constant coefficient matrix of leading order terms having one-dimensional eigenspace, possessing a (2,2)-homogeneous third-order symmetry, is existed.*

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