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ORIGINAL PAPER

The nonlinear dynamics based on the nonstandard Hamiltonians

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Abstract The new action of a nonstandard $S = \int_a^b e^{[p\dot{q}-H(p,q,t)]}dt$ with a nonstandard Hamiltonian is introduced. The nonstandard Hamiltonian equations are obtained by using the standard variational method. Some new dynamical properties that the nonlinear systems hold are obtained. It is demonstrated that several constrained Hamiltonian systems have been identified to possess some interesting properties, and some additional features are discussed in details.

Keywords Nonstandard Hamiltonian · Nonstandard Hamiltonian equation · Nonlinear dynamics

1 Introduciton

It is well known that the differential equations of motion in physics, mechanics and engineering can be derived from the Hamiltonian principle $\delta S = \delta\int_{t_1}^{t_2} L(q,\dot{q},t)dt = 0$, which can be rewritten in the form $\delta S = \delta\int_{t_1}^{t_2}[p\dot{q} - H(q,p,t)]dt = 0$; L and H are the standard Lagrangian and Hamiltonian, and q is the generalized coordinate with $\dot{q} = dq/dt$, and p is the generalized momentum corresponding to generalized coordinate q. But for the description of nonlinear evolution equations [1], dissipative dynamical systems with nonconstant coefficients [2–6], Friedmann–Robertson–Walker model [7,8], the problem of quantization of a classical theory [9], etc., introducing the nonstandard Hamiltonian and Lagrangian will make the descriptions more easier. The nonstandard Hamiltonian (NSH) differs from the standard Hamiltonians that are expressed as the sum of the kinetic and the potential energy terms. In other words, there is no obvious identification of the kinetic and potential energy terms in NSH. The idea of nonstandard dynamics can be traced back to the work of Arnold [10]. The nonstandard Hamiltonian dynamics has its roots in the work of Feynman reported by Dyson [11] and its extension by Hojman and Shepley [9,12].

Recently, the researches about nonstandard Lagrangian(NSL) have obtained a series of important results, Musielak [13,14] studied the method of getting the NSL for dissipative system and its existence conditions, El-Nabulsi [15] introduced two nonstandard actions with NSL and obtained the dynamical equations. Zhang [16], studied the Routh method of reduction for dynamic systems with NSL. But there are few

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results about the nonstandard Hamiltonian (NSH) and its dynamical equations. NSH may take exponential forms, and so on, and it may or may not depend on time. In this work, we will introduce exponential forms of NSH and apply it in the nonlinear dynamical systems.

The outline of this paper is as follows: we obtain the equations of motion for the nonstandard Hamiltonian by adopting the contemporaneous variation for the action with an exponential Hamiltonian in Sect. 2. We show the application of nonstandard Hamiltonian equations for the nonlinear dynamical systems in Sect. 3, and conclusions are given in Sect. 4.

2 Action and equations of motion with exponential Hamiltonian

In this section, we will introduce the action with exponential Hamiltonian and give out the equations named nonstandard Hamiltonian equations for exponential Hamiltonian.

Definition 2.1 Defined the action with an exponential Hamiltonian as

$$S = \int_a^b e^{[p\dot{q} - H(p,q,t)]} dt \tag{1}$$

where $(p, q, t) \rightarrow H(p, q, t)$ is assumed to be a C^2 functions:

$$q, p \in C^1\left([a, b]; R^n\right)$$
$$H(p, q, t) \in C^2([a, b] \times R^n \times R^n; R)$$

Suppose that the admissible function $q \in C^1[a, b]$ for which the action functional (1) subject to the given boundary conditions $q(a) = q_a, q(b) = q_b$ has an extremum. So through the Hamiltonian principle

$$\delta S = \delta \int_a^b e^{[p\dot{q} - H(p,q,t)]} dt = 0 \tag{2}$$

we can obtain the nonstandard Hamiltonian equations (NSHE).

Theorem 2.1 *If $q(t)$ are the solutions of the action with an exponential Hamiltonian, then $q(t)$ satisfies the following NSHE:*

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p}\left(1 + p\frac{\partial H}{\partial p}\right) = -\frac{\partial H}{\partial q} - p^2 \frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + p\frac{dH}{dt} \end{cases} \tag{3}$$

Making use of the total differential of $H(p, q, t)$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} \tag{4}$$

Equations (3) can be expressed as

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} - p^2 \frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + p\frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + p\frac{\partial H}{\partial t} \end{cases} \tag{5}$$

Equations (5) are named as the NSHE for the exponential Hamiltonian.

Proof Adopting the contemporaneous variation to the action with an exponential Hamiltonian, and considering the boundary conditions $q(a) = q_a, q(b) = q_b$ and taking the conditions of extremum, the NSHE (5) can be obtained. □

If the Hamiltonian H has no explicit time dependence, that is $\partial H/\partial t = 0$, then the Eq. (5) can be reduced to

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} - p^2 \frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + p\frac{\partial H}{\partial q}\frac{\partial H}{\partial p} \end{cases} \tag{6}$$

Moreover, if we define

$$p\frac{d}{dt}\left(\frac{\partial H}{\partial P}\right) \equiv \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} \tag{7}$$

Then by computation, we can obtain that

$$\frac{d}{dt}\left(p\frac{\partial H}{\partial p}\right) = \left(\dot{p} + \frac{\partial H}{\partial q}\right)\frac{\partial H}{\partial p}$$

If taking $p\frac{\partial H}{\partial p} = K$, where K is a constant, the Eq. (6) are reduced to the standard Hamiltonian equations:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases} \tag{8}$$

Corollary 2.1 *When $H = H(p, q)$, the Hamiltonian H is not a constant of motion for the Eq. (6).*

Proof If we differentiate the Hamiltonian $H(p, q)$ with respect to time, we obtain

$$\begin{aligned} \frac{dH(p,q)}{dt} &= \frac{\partial H(p,q)}{\partial p}\dot{p} + \frac{\partial H(p,q)}{\partial q}\dot{q} \\ &= \frac{\partial H}{\partial p}\left[-\frac{\partial H}{\partial q} - p^2\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + p\frac{\partial H}{\partial q}\frac{\partial H}{\partial p}\right] \\ &\quad + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} \\ &= p\left(\frac{\partial H}{\partial p}\right)^2\frac{\partial H}{\partial q} - p^2\frac{\partial H}{\partial p}\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) \\ &= p\frac{\partial H}{\partial p}\left[\frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - p\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right)\right] \end{aligned}$$

So only when the expression (7) holds, that is, $p\frac{\partial H}{\partial p} = K$ holds, $dH(p,q)/dt = 0$, the Hamiltonian $H = H(p,q)$ is a conserved quantity of motion. □

Corollary 2.2 *When $H = H(p,t)$, the time-dependent canonical momentum p is not a constant of motion.*

Proof Because the NSH is independent of q, then Eq. (5) are reduced to

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -p^2 \frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + p\frac{\partial H}{\partial t} \end{cases} \tag{9}$$

Obviously, the expression $-p^2\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + p\frac{\partial H}{\partial t}$ is not always zero, so the momentum p is not a constant of motion. If we change the second formula of (9) into

$$\dot{p}\left(1 - p\frac{\partial H}{\partial p}\right) = -p\frac{d}{dt}\left(p\frac{\partial H}{\partial p}\right) + p\frac{\partial H}{\partial t} \tag{10}$$

only when the constraint $p\frac{\partial H}{\partial p} = K \neq 1$ and the Hamiltonian is independent of time t, then the Eq. (9) become the following equations

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = 0 \end{cases} \tag{11}$$

Here the momentum p is a conservative quantity. □

To show the interesting properties for NSH, we shall illustrate both time-dependent and time-independent Hamiltonians, and in all examples, the C_i, $i = 1, 2, \cdots$ are the constants of integration.

3 The application of NSHE in the nonlinear dynamical systems

3.1 Time-dependent Hamiltonians

Example 1 We consider the NSH as follows

$$H = pQ(t)q \tag{12}$$

Substituting the Hamiltonian (12) into equations (5), we can obtain the following NSHE

$$\begin{cases} \dot{q} = Q(t)q \\ \dot{p} = -pQ(t) \end{cases} \tag{13}$$

Taking $H(p,q,t) = pqt$ for Example, the equations of motion can be obtained easily by using equations(13).

$$\begin{cases} \dot{q} = qt \\ \dot{p} = -pt \end{cases} \tag{14}$$

where the analytic solutions are given by

$$\begin{cases} q = C_1 exp(t^2/2) \\ p = C_2 \exp(-t^2/2) \end{cases}, \tag{15}$$

Taking the initial conditions $q(0) = 1$, $p(0) = 1$, we show the variations of the solutions of equations (14) with time in Fig. 1a. Figure 1b illustrates the behavior of equations (15) on the plane of q–p.

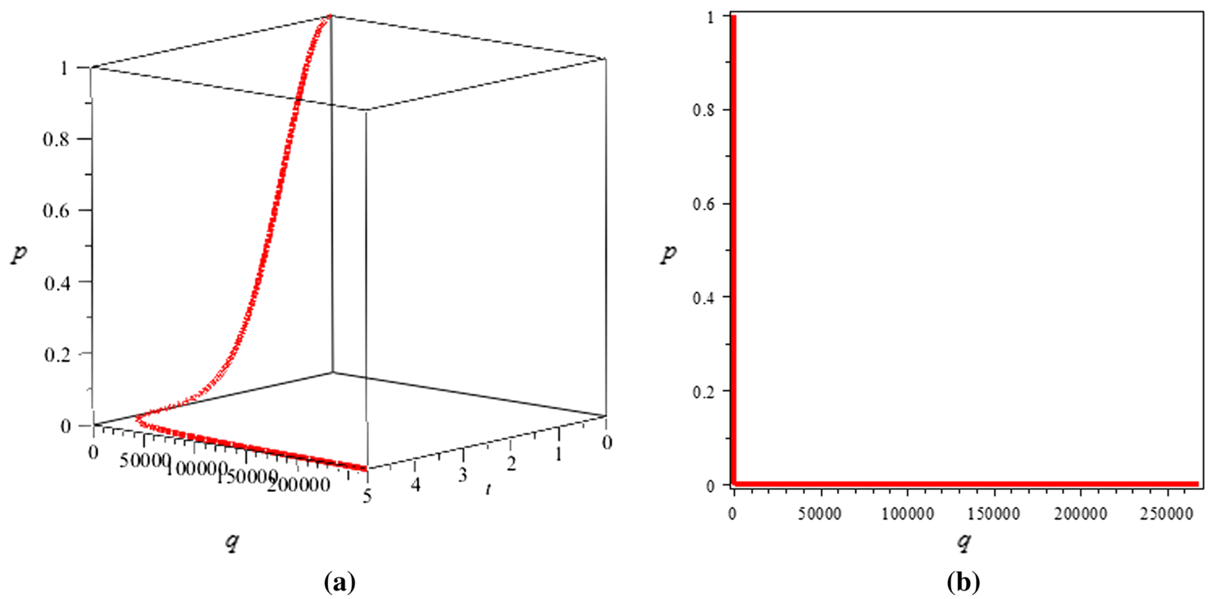


Fig. 1 **a** Variations of the solutions of equations (14) with time, **b** behavior of the motion equations (15) on the plane q–p

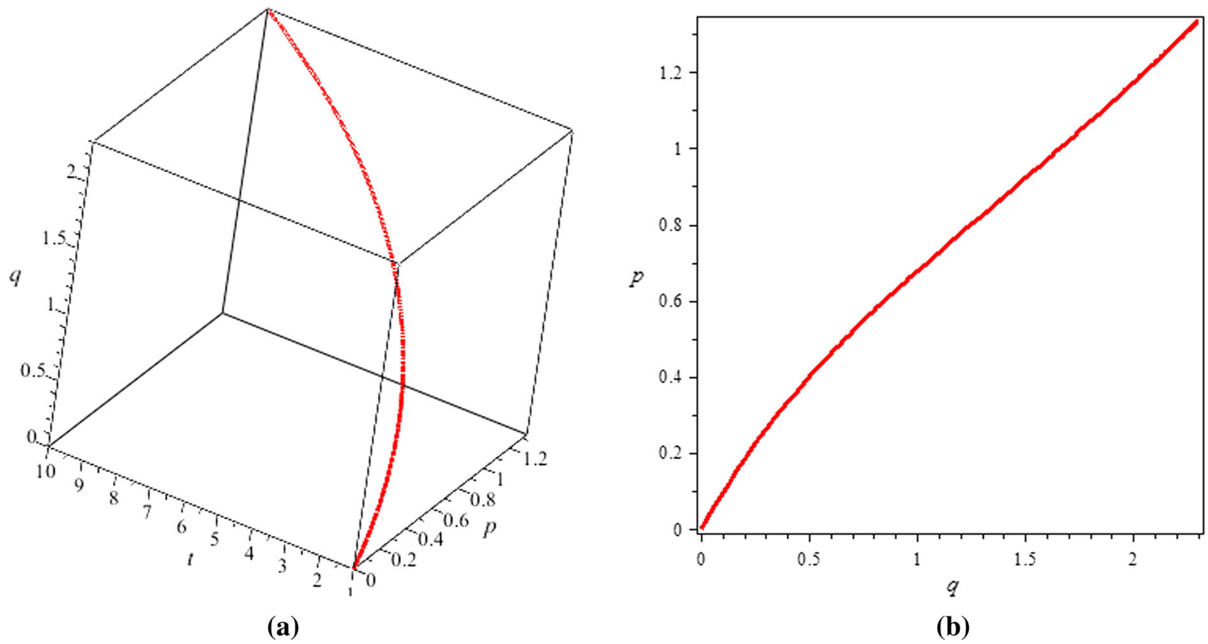


Fig. 2 **a** Variations of the solutions of equations (16) with time, **b** trajectory of the motion equations (17) on the plane q–p

Example 2 If we take the NSH to be $H = t^{-1}(p - q)$ with $t \neq 0$, the NSHE can be obtained by using the equations (5)

$$\begin{cases} \dot{q} = 1/t \\ \dot{p} = 1/t - p/t^2 - pq/t^2 \end{cases} \tag{16}$$

They have the analytic solutions

$$\begin{cases} q = \ln(t) + C_3 \\ p = \left(\int \frac{exp(\int (1+q)/t^2 dt)}{t} dt + C_4\right) exp\left(\int -(1+q)/t^2 dt\right) \end{cases} \tag{17}$$

Taking $q(1) = 0$, $p(1) = 1$ as the initial value, we can obtain the variations of the solutions of equations (16) with time in Fig. 2a and the trajectory of motion for the Eq. (17) in Fig. 2b.

Example 3 Taking the NSH $H(p, q, t) = p \sin t + q$, then from equations (5), we can obtain the equations of motion

$$\begin{cases} \dot{q} = \sin t \\ \dot{p} = -1 + p \sin t \end{cases} \tag{18}$$

Its analytic solutions are

$$\begin{cases} q(t) = -\cos(t) + C_5 \\ p(t) = -e^{-\cos(t)} \left(\int e^{\cos(t)} dt + C_6\right) \end{cases} \tag{19}$$

If taking $q(0) = -1$, $p(0) = 0$ as the initial value, the variations of the solutions with time and the trajectory on the plane q–p for the equations (19) are illustrated in Fig. 3a, b, respectively.

3.2 Time-independent Hamiltonians

Example 4 Taking the NSH $H(p, q) = pq$, by using the equations (6), we can obtain the NSHE as

$$\begin{cases} \dot{q} = q \\ \dot{p} = -p \end{cases} \tag{20}$$

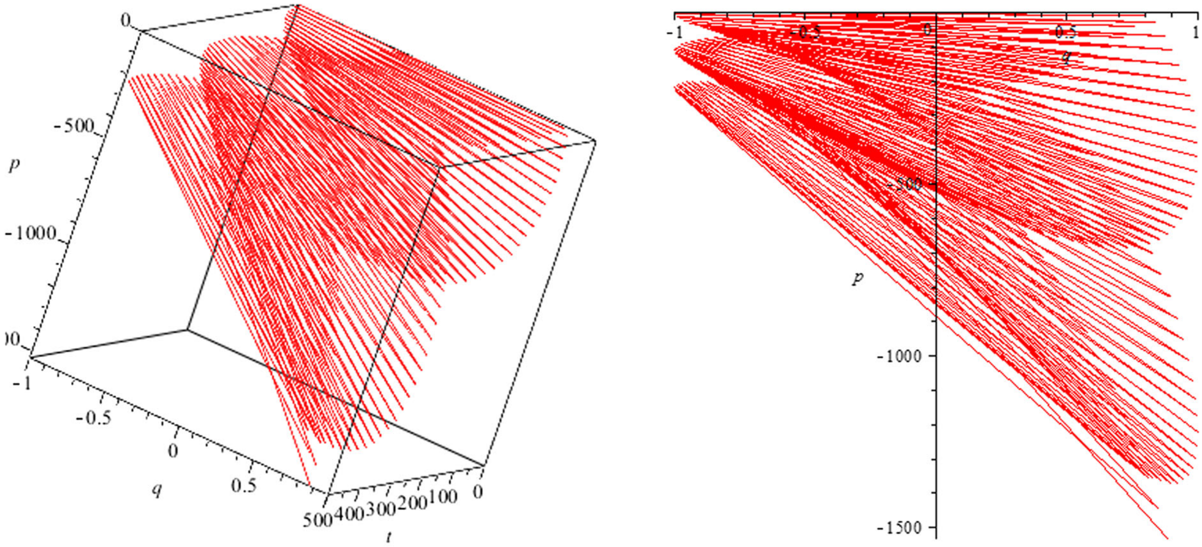
Its analytic solutions are

$$\begin{cases} q = C_7 e^t \\ p = C_8 e^{-t} \end{cases} \tag{21}$$

If taking $q\,(0) = 1$, $p\,(0) = 1$ as the initial-value for the equations (20), then the variations of the solutions with time and the trajectory on the plane q–p for the equations (20) can be shown in Fig. 4a, b, respectively.

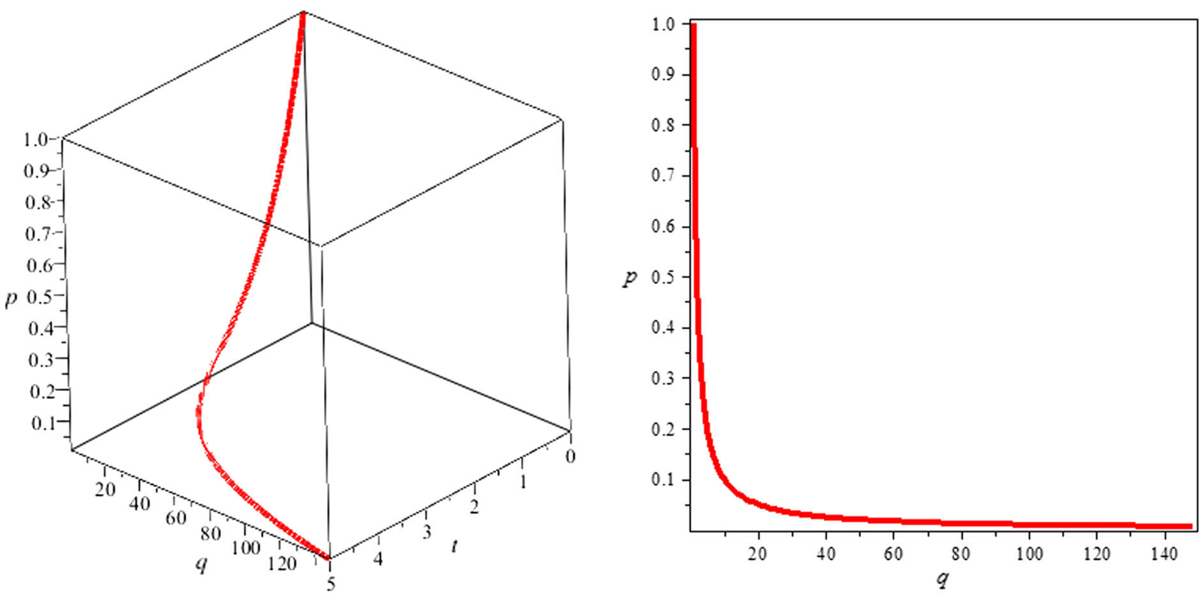
Example 5 We choose a special Hamiltonian which satisfies the condition $p\,(\partial H/\partial p) = K$ and its time independence is assumed to be verified. Let $H(p, q) = K \ln p + \sqrt{q}$ and subsequently from equation (8), we can find that

$$\begin{cases} \dot{q} = K/p \\ \dot{p} = \frac{-1}{2\sqrt{q}} \end{cases} \tag{22}$$



(a) **(b)**

Fig. 3 **a** Variations of the solutions of equations (18) with time, **b** trajectory of the motion equations (19) on the plane q–p



(a) **(b)**

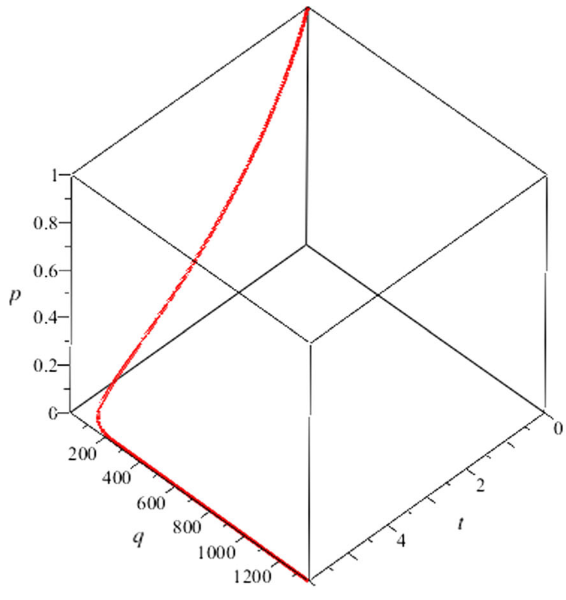
Fig. 4 **a** Variations of the solutions of equations (20) with time, **b** trajectory of the motion equations (21) on the plane q–p

there are analytic solutions for equations (22)

$$\begin{cases} q(t) = 4K^2\left(LambertW\left(\frac{C_9 t + C_{10}}{8K^2 e}\right) + 1\right)^2 \\ p = 2K/\dot{q} \end{cases} \quad (23)$$

where the *LambertW* represents the Lambert W function [17–19]. When taking the initial values $q(0) = 1$, $p(0) = 1$, and $K = 2$, we can obtain the variations of the solutions with time and the trajectory on the plane q–p for the equations (22) which are shown in Fig. 5a, b, respectively.

Example 6 We also choose the time-independent $H(p, q) = K \ln qp$ that satisfies the condition



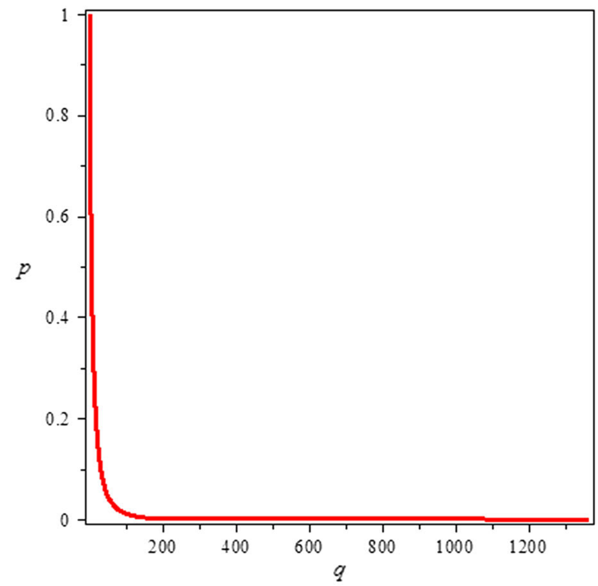


Fig. 5 **a** Variations of the solutions of equations (22) with time, **b** trajectory of the motion equations (23) on the plane q–p

$p\,(\partial H/\partial p) = K$, so equations (8) gives

$$\begin{cases} \dot{q} = K/p \\ \dot{p} = -K/q \end{cases} \tag{24}$$

There are analytic solutions for the equations (23)

$$\begin{cases} q = -2K/\dot{p} \\ p(t) = C_{11}\exp(C_{12}t) \end{cases} \tag{25}$$

When $K = 2$, and $q\,(0) = 1$, $p\,(0) = 1$ as the initial value, we obtain the variations of solutions in the three-dimensional time-space and the trajectory on the two-dimensional plane q–p for the system (24) in Fig. 6a, b, respectively.

3.3 q-Independent Hamiltonians

Example 7 Taking the NSH $H(p,t) = p\sin t + \sin t$, we have the NSHE by using the equations (9)

$$\begin{cases} \dot{q} = \sin t \\ \dot{p} = p\cos t \end{cases} \tag{26}$$

Equations (26) have the analytic solutions as follow

$$\begin{cases} q = -\cos t + C_{13} \\ p = C_{14}\exp(\sin t) \end{cases} \tag{27}$$

When taking $q(0) = 0$, $p(0) = 1$ as the conditions of initial value, we can obtain variations of solutions in the three-dimensional time-space and the trajectory on the two-dimensional plane q–p for system (26) which are shown in Fig. 7a, b, respectively.

Example 8 Taking the NSH $H(p,q,t) = p\sin t$, with the equations (9), we can obtain the equations

$$\begin{cases} \dot{q} = \sin t \\ \dot{p} = 0 \end{cases} \tag{28}$$

Equations (28) have the analytic solutions as follows

$$\begin{cases} q = -\cos t + C_{15} \\ p = C_{16} \end{cases} \tag{29}$$

Obviously, the solution of q varies as cosines, and p is a conservative quantity.

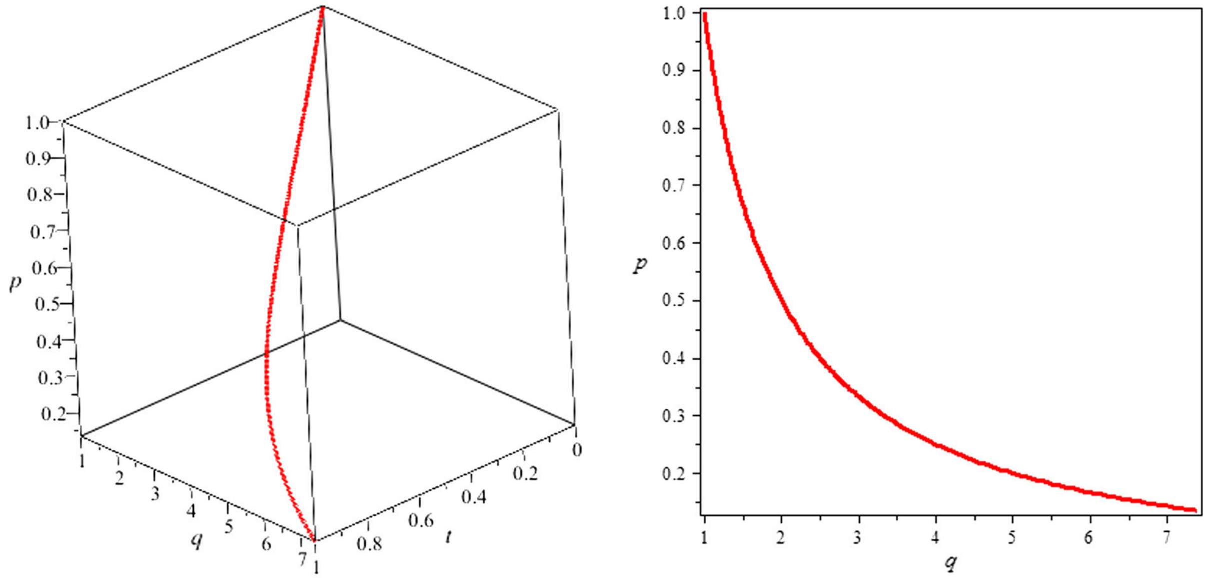
Example 9 Taking the NSH $H(p) = K\ln p$, through the equations (11), we can obtain the NSHE

$$\begin{cases} \dot{q} = K/p \\ \dot{p} = 0 \end{cases} \tag{30}$$

Equations (30) have the analytic solutions as follows

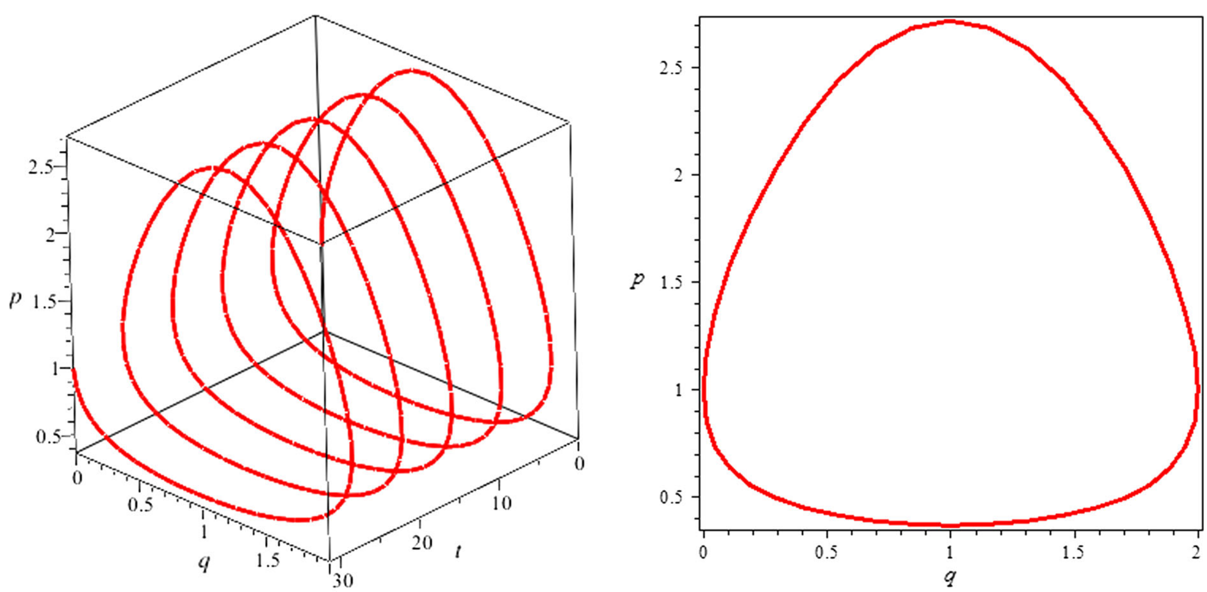
$$\begin{cases} q = \frac{Kt}{C_{18}} + C_{17} \\ p = C_{18} \end{cases} \tag{31}$$

So the solution of q varies linearly with time t, and p is a conservative quantity.



(a) **(b)**

Fig. 6 **a** Variations of the solutions of equations (24) with time, **b** trajectory of the motion equations (25) on the plane



(a) **(b)**

Fig. 7 **a** Variations of the solutions of equations (26) with time, **b** trajectory of the motion equations (27) on the plane

4 Conclusions

We successfully obtained the equations of motion for a kind of particular dynamical systems characterized by NSH. The NSHE differ completely from the standard Hamiltonian equations in essence, but under some special conditions, they can be simplified into the standard Hamiltonian equations. Particularly for the dissipative dynamical systems, the nonlinear evolution equations, etc., the NSHE can obviously give out an uncomplicated, simple way to solve the complex dynamical systems. It is noteworthy that when applying the standard Hamiltonian equations to the systems which hold a NSH, the equations of motion will be reasonable in some time as Example 1 and 4, but for the most of conditions, the equations of motion will be absolutely different from what we obtained and even sometimes nonreasonable, such as Example 3; it can be seen easily that the standard Hamiltonian equations is $p =$ constant, and hence, it is not realistic. We argue that the dynamical systems with NSH are important and they exist in some complex dynamical systems, such as the physics, mechanics and engineering applications. The main advantage of the NSHE is that they offer a new kind of dynamical systems to study and can be used to the nonlinear dynamical systems, dissipative dynamical systems, problem of quantization of a classical theory and cosmology as well.

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