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Generalization of approximate partial Noether approach in phase space

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Abstract The approximate partial Noether theorem proposed earlier for the ordinary differential equations (ODEs) (Naeem and Mahomed in Nonlinear Dyn 57(1–2):303–311, 2009) is generalized in phase space for the perturbed Hamiltonian-type systems. The notion of approximate partial Hamiltonian is developed. An approximate partial Hamiltonian gives rise to an approximate Hamiltonian-type perturbed dynamical system of first-order ODEs. An approximate Legendre-type transformation connects the approximate partial Lagrangian and what we term as approximate partial Hamiltonian. The formulas for approximate partial Hamiltonian operators determining equations and first integrals are provided explicitly. We have explained our approach with the help of simple illustrative example. Then, it is applied to establish the approximate first integrals, reductions and exact solutions of two perturbed cubically coupled Duffing–Van der Pol oscillators. Both resonant and nonresonant cases are considered.

Keywords Approximate partial Lagrangian · Approximate partial Hamiltonian · Approximate first integrals · Approximate partial Noether theorem · Perturbed cubically coupled Duffing–Van der Pol oscillators

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1 Introduction

Many differential equations arising in mathematical physics, fluid mechanics, economic growth theory, epidemics, physics, engineering and many other fields of applied mathematics do not admit nontrivial exact Lie symmetries. The Lie's integration theorems (see, e.g., [1]) cannot be applied to establish the group-invariant solutions. Noether's theorem [2] is also not applicable for these models as no nontrivial exact symmetries exist. These types of differential equations can be analyzed by decomposing them into the unperturbed and perturbed parts provided the former admits exact symmetries.

Baikov et al [3,4] developed perturbation methods in group analysis for differential equations which do not admit nontrivial exact Lie symmetries. The approximate Lie's theorems were proposed by Baikov et al [3,4]. Baikov et al [5] investigated the approximate symmetries of closed orbit problems. Govinder et al [6] proposed the approximate version of Noether's theorem to derive the first integrals of ODEs. Later on, Feroze and Kara [7] provided group theoretic methods for approximate invariants and Lagrangian of a special class of second-order perturbed ODEs. The partial Lagrangian approach [8] was developed to construct first integrals for unperturbed ODEs which do not have standard Lagrangian. Naeem and Mahomed [9,10] provided notions of the approximate partial Lagrangians and approximate Euler–Lagrange equations for perturbed ODEs. Naz [11] utilized the approximate partial Lagrangian approach to study the approximate first integrals for cubically coupled nonlinear Duffing oscillators subject to a periodically driven force.

A separate strand of literature looked at the approximate symmetries and conservation laws of perturbed partial differential equations (PDEs) [12–15]. The variational formulation of approximate symmetries and conservation laws was provided by Johnpillai and Kara [13]. Johnpillai and Kara [15] also extended the notions of Noether-type symmetries and conservation laws via partial Lagrangian developed for the PDEs [16] for the perturbed case. All different approaches to construct first integrals/conservation laws for the unperturbed differential equations are presented in [17,18]. Wang [19] studied the perturbation to symmetry and adiabatic invariants of discrete nonholonomic nonconservative mechanical system.

A dynamic optimization problem involves the determination of the extremal of the functional involving time, dependent, independent variables and their derivatives up to a finite order. The calculus of variation provides a set of equations known as Euler–Lagrange equations [20,21] for a standard Lagrangian. The optimal control theory which is an extension of calculus of variation utilizes Hamiltonian systems and is developed by Lev Semyonovich Pontryagin [22]. The Legendre transformation [23,24] connects the Hamiltonian and Lagrangian. This connection between Hamiltonian and Lagrangian motivated Dorodnitsyn and Kozlov [25] to reformulate the celebrated Noether's theorem in terms of Hamiltonian. Naz et al [26] provided a partial Legendre transformation which connects the current value Hamiltonian and discount-free Lagrangian arising in economic growth theory. Naz et al [27,28] developed current value Hamiltonian approach for economic growth models, and this elegant approach provides a foundation of connection between Lie group theory and economic growth theory. The approximate version of Noether's theorem was formulated in terms of the approximate Hamiltonian function and approximate symmetry operators by Ünal [29]. It was successfully applied to construct the approximate first integrals of weakly nonlinear, damped-driven oscillators [30] and galaxy model [31]. A natural question arises can we generalize the approximate partial Noether approach [9] to the phase space in terms of an approximate Hamiltonian which we call as an approximate partial Hamiltonian. According to the best of our knowledge the approximate first integrals of a mechanical or dynamical approximate Hamiltonian-type systems via approximate partial Hamiltonian are not studied before. We will analyze these in this paper.

The generalization of approximate partial Noether theorem in phase space is presented here. The notion of the approximate partial Hamiltonian is developed. A relation which connects approximate partial Lagrangian to approximate partial Hamiltonian is proposed, and it is termed as 'approximate Legendre-type transformation.' The approximate partial Hamiltonian operators determining equation is deduced from the formulas of approximate partial Lagrangian operator determining equation. Also formulas are provided to construct approximate first integrals. We explain our approach by simple illustrative example. The utility of method is explained by applying it to establish the approximate first integrals, reductions and exact solutions of two perturbed cubically coupled Duffing–Van der Pol oscillators.

The layout of the paper is as follows. In Sect. 2, the overview of approximate partial Lagrangian approach is given. The generalization of approximate partial Noether theorem in phase space is presented in Sect. 3. An approximate Legendre-type transformation is introduced which connects approximate partial Lagrangian and approximate partial Hamiltonian. The formulas are provided to find the approximate partial Hamiltonian operators and approximate first integrals. In Sect. 4, the approximate first integral of perturbed orbit equation is derived to explain how our approach works. In Sect. 5, the approximate first integrals of two perturbed cubically coupled Duffing–Van der Pol oscillators are established to show effectiveness of approach proposed here. The reductions and exact solutions for the model understudy are provided in Sect. 6. Finally, conclusions are presented in Sect. 7.

2 Approximate partial Lagrangian approach

An overview of the approximate partial Lagrangian approach as proposed in [9] is provided here for the second-order ODEs. Also we have provided formulas for the modified approximate partial Lagrangian approach.

2.1 Overview of approximate partial Lagrangian approach

Consider a second-order perturbed system of ODEs involving a small parameter ϵ

$$E_\alpha(t, q, \dot{q}, \ddot{q}; \epsilon) = O(\epsilon^2), \ \alpha = 1, 2, \ldots n \tag{1}$$

where t is the independent variable and $q = (q^1, q^2, \ldots, q^n)$ is the vector of dependent variables. The total derivative operator D_t with respect to t is defined by

$$D_t = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} + \cdots . \tag{2}$$

The following definitions are adopted from [9].

Definition 1 The first-order approximate Lie-B$\ddot{a}$cklund symmetry operator is given by

$$X = X_0 + \epsilon X_1 \tag{3}$$

where

$$X_0 = \xi_0 \frac{\partial}{\partial t} + \eta_0^i \frac{\partial}{\partial q^i} + [D_t(\eta_0^i) - \dot{q}^i D_t(\xi_0)] \frac{\partial}{\partial \dot{q}^i}, \tag{4}$$

$$X_1 = \xi_1 \frac{\partial}{\partial t} + \eta_i^1 \frac{\partial}{\partial q^i} + [D_t(\eta_1^i) - \dot{q}^i D_t(\xi_1)] \frac{\partial}{\partial \dot{q}^i}. \tag{5}$$

Definition 2 A differential function I is an approximate first integral of system (1) if

$$D_t I = O(\epsilon^2), \tag{6}$$

holds for every solution of system (1) where

$$I = I_0 + \epsilon I_1. \tag{7}$$

Suppose that Eq. (1) can be expressed as

$$E_\alpha = E_\alpha^0 + \epsilon E_\alpha^1, \ \alpha = 1, 2 \ldots n. \tag{8}$$

Definition 3 Suppose that $L = L(t, q^i, \dot{q}^i)$ is a differential function and there exists nonzero functions f_α^β such that (8) can be expressed as

$$\delta L / \delta q^\alpha = \epsilon^k f_{1\alpha}^\beta E_\beta^k, \ \alpha = \beta = 1, 2, \tag{9}$$

or more generally $\delta L / \delta q^\alpha = -\Gamma^\alpha$. Then L is said to be an approximate partial Lagrangian of system (8). Equation (9) is termed as an approximate partial Euler–Lagrange equation. In (9) $\delta / \delta q^\alpha$ is defined by

$$\frac{\delta}{\delta q^\alpha} = \frac{\partial}{\partial q} - D_t \frac{\partial}{\partial \dot{q}} + D_t^2 \frac{\partial}{\partial \ddot{q}} + \cdots \tag{10}$$

and is known as the Euler–Lagrange operators.

Definition 4 The approximate Lie-B$\ddot{a}$cklund operator X defined in (3) is said to be an approximate partial Noether operator corresponding to an approximate partial Lagrangian L if it satisfies

$$X(L) + L D_t(\xi) = (\eta^i - \dot{q}^i \xi) \frac{\delta L}{\delta q^i} + D_t(B), \tag{11}$$

with respect to a suitable function B. Notice that in (11)

$$B = B_0 + \epsilon B_1, \ \xi = \xi_0 + \epsilon \xi_1, \ \eta^1 = \eta_0^i + \epsilon \eta_1^i. \tag{12}$$

Theorem 1 *If X is an approximate partial Noether operator associated with an approximate partial Lagrangian $L = L(t, q, \dot{q})$, then the formula for the approximate first integrals is*

$$I = B - \xi L - (\eta^i - \dot{q}^i \xi) \frac{\delta L}{\delta \dot{q}^i} + O(\epsilon^2). \tag{13}$$

2.2 Modified approximate partial Lagrangian approach

One can modify the approximate partial Lagrangian as $L(t, q^i, \dot{q}^i; \epsilon)$. We have provided the modified definitions of approximate partial Lagrangian, approximate partial Euler–Lagrange equations, approximate partial Noether operators determining equations and approximate first integral.

Definition 5 Suppose that $L(t, q^i, \dot{q}^i; \epsilon) = L_0(t, q^i, \dot{q}^i) + \epsilon L_1(t, q^i, \dot{q}^i)$ is a differential function and there exists nonzero functions Γ^i such that (8) can be expressed as

$$\delta L / \delta q^\alpha = -\Gamma^i, \tag{14}$$

where $\Gamma^i = \Gamma_0^i + \epsilon \Gamma_1^i$. Then L is said to be an approximate partial Lagrangian of system (8). Equation (14) is termed as an approximate partial Euler–Lagrange equation. The approximate Euler–Lagrange equation (14) for ϵ^0 and ϵ^1, results in

$$\frac{\delta L_0}{\delta q^i} = -\Gamma_0^i, \tag{15}$$

$$\frac{\delta L_1}{\delta q^i} = -\Gamma_1^i. \tag{16}$$

Definition 6 The approximate Lie-B$\ddot{a}$cklund operator X defined in (3) is said to be an approximate partial Noether operator corresponding to an approximate partial Lagrangian $L(t, q^i, \dot{q}^i; \epsilon) = L_0(t, q^i, \dot{q}^i) + \epsilon L_1(t, q^i, \dot{q}^i)$ if it satisfies

$$X(L) + L D_t(\xi) = (\eta^i - \dot{q}^i \xi) \frac{\delta L}{\delta q^i} + D_t(B), \tag{17}$$

with respect to a suitable function B. Then the formula for the approximate first integrals is

$$I = B - \xi L - (\eta^i - \dot{q}^i \xi)\frac{\delta L}{\delta \dot{q}^i} + O(\epsilon^2). \quad (18)$$

The approximate partial Euler–Lagrange Eq. (17) for ϵ^0 and ϵ^1, yields

$$X_0(L_0)+L_0 D_t(\xi_0)=(\eta_0^i-\dot{q}^i\xi_0)\frac{\delta L_0}{\delta q^i}+D_t(B_0), \quad (19)$$

$$X_1(L_0) + X_0 L_1 + L_0 D_t(\xi_1) + L_1 D_t(\xi_0)$$
$$= (\eta_0^i - \dot{q}^i \xi_0)\frac{\delta L_1}{\delta q^i} + (\eta_1^i - \dot{q}^i \xi_1)\frac{\delta L_0}{\delta q^i} + D_t(B_1). \quad (20)$$

The approximate first integrals formula (18) for ϵ^0 and ϵ^1 results in the following form:

$$I_0 = B_0 - \xi_0 L_0 - (\eta_0^i - \dot{q}^i \xi_0)\frac{\delta L_0}{\delta \dot{q}^i}, \quad (21)$$

$$I_1 = B_1 - \xi_0 L_1 - \xi_1 L_0 - (\eta_0^i - \dot{q}^i \xi_0)\frac{\delta L_1}{\delta \dot{q}^i}$$
$$-(\eta_1^i - \dot{q}^i \xi_1)\frac{\delta L_0}{\delta \dot{q}^i}. \quad (22)$$

Remark 1 It is worthy to mention here that by setting $L_1 = 0$, the approximate partial Noether operators determining equations and approximate first integrals formulas (19)–(22) reduce to the ones given by Naeem and Mahomed [9].

3 Generalization of approximate partial Noether theorem in phase space

In this section first, we present the notions of approximate partial Hamiltonian function, approximate Legendre-type transformation and approximate Hamiltonian-type system. Then, we generalize the approximate partial Noether theorem proposed by Naeem and Mahomed [9] and its modified form presented in previous section in phase space for an approximate partial Hamiltonian.

3.1 Notion of an approximate partial Hamiltonian

Let t be the independent variable which is usually time and $(q, p) = (q^1, \dots, q^n, p^1, \dots, p^n)$ the phase space coordinates. The Euler operator $\delta/\delta q^i$ and the variational operator $\delta/\delta p^i$ are defined as (see, e.g., [25,27])

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D\frac{\partial}{\partial \dot{q}^i}, \ i = 1, 2, \dots, n, \quad (23)$$

and

$$\frac{\delta}{\delta p^i} = \frac{\partial}{\partial p^i} - D\frac{\partial}{\partial \dot{p}^i}, i = 1, 2, \dots, n, \quad (24)$$

where

$$D = \frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + \dot{p}^i\frac{\partial}{\partial p^i} + \cdots \quad (25)$$

is the total derivative operator with respect to the time t. The summation convention applies for repeated indices here and in the sequel.

The variables q^i, p^i satisfy the differential relations

$$\dot{p}^i = D(p^i), \ \dot{q}^i = D(q^i), \ i = 1, 2, \dots, n. \quad (26)$$

Definition 7 (*Approximate Legendre-type transformation)* Consider a perturbed system of second-order ODEs

$$\ddot{q}^i = f_i(t, q^i, \dot{q}^i; \epsilon). \quad (27)$$

If $L(t, q^i, \dot{q}^i; \epsilon) = L_0(t, q^i, \dot{q}^i)+\epsilon L_1(t, q^i, \dot{q}^i)$ be the approximate partial Lagrangian for system (27) satisfying $\frac{\delta L}{\delta q_i} = -\Gamma^i$, then there exists an approximate partial Hamiltonian $H(t, q^i, p^i; \epsilon) = H_0(t, q^i, p^i) + \epsilon H_1(t, q^i, p^i)$ satisfying

$$H(t, q^i, p^i; \epsilon) = p^i \dot{q}^i - L(t, q^i, \dot{q}^i; \epsilon), \quad (28)$$

where $p^i = \frac{\partial L}{\partial \dot{q}^i}$ and $\Gamma^i = \Gamma_0^i + \epsilon \Gamma_1^i$. The relation (28) is termed as an approximate Legendre-type transformation.

Remark 2 It is important to mention here that the variables p and q are related as $p^i = \frac{\partial L}{\partial \dot{q}^i}$. So perturbations in p will arise as a consequence. Also in solution we will in general have perturbation in p.

Proposition 1 *Consider a perturbed system of second-order ODEs* (27) *and* $L(t, q_i, \dot{q}_i; \epsilon)$ *be the approximate partial Lagrangian satisfying* $\frac{\delta L}{\delta q_i} = -\Gamma^i$. *If there exists a function* $H(t, p_i, q^i; \epsilon)$ *satisfying Legendre-type transformation* (28)*, then the second-order perturbed system* (27) *can be re-cast as a first-order perturbed dynamical system*

$$\dot{q}^i = \frac{\partial H}{\partial p_i},$$
$$\dot{p}^i = -\frac{\partial H}{\partial q^i} + \Gamma^i, \quad i = 1, \dots, n, \quad (29)$$

where $\Gamma^i = \Gamma_0^i + \epsilon\Gamma_1^i$ *is a nonzero function of* t, p^i, q^i. *The function* H *is termed as an approximate par-*

tial Hamiltonian for the perturbed system of second-order ODEs (27). *The system* (29) *is the perturbed Hamiltonian-type system.*

Proof The action of variational operator $\delta/\delta p_i$ on approximate Legendre-type transformation (28) yields

$$\frac{\delta H}{\delta p_i} = \dot{q}_i - \frac{\delta L}{\delta p_i}, \tag{30}$$

as $\frac{\delta L}{\delta p_i} = 0$, and this provides the first equation of system (29). Applying the variational operator $\delta/\delta q_i$ to (28) results in

$$\frac{\delta H}{\delta q^i} = -\dot{p}_i - \frac{\delta L}{\delta q_i}, \tag{31}$$

and this yields second equation of system (29) as $\frac{\delta L}{\delta q_i} = -\Gamma^i$. This completes proof. □

Remark 3 It is important to mention here whether a model has an approximate standard Hamiltonian formulations. The approximate partial Hamiltonian too exists that results in the approximate first integrals. Also the partial Hamiltonian is not unique.

Now some examples are provided to explain how to write an approximate partial Hamiltonian.

1. Consider following perturbed system of first-order ODEs.

$$\begin{aligned}
\dot{q}_1 &= p_1,\\
\dot{q}_2 &= p_2,\\
\dot{p}_1 &= -\omega_1^2 q_1 + \epsilon\left(d_1\dot{q}_1(1-q_1^2) - \alpha_1 q_1^3 - \delta q_1 q_2^2\right),\\
\dot{p}_2 &= -\omega_2^2 q_2 + \epsilon\left(d_2\dot{q}_2(1-q_2^2) - \alpha_2 q_2^3 - \delta q_1^2 q_2\right).
\end{aligned} \tag{32}$$

An approximate partial Hamiltonian for system (32) is

$$\begin{aligned}
H &= \frac{1}{2}\left(p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2\right)\\
&\quad + \frac{1}{4}\epsilon(\alpha_1 q_1^4 + \alpha_2 q_2^4) + \epsilon\delta\frac{1}{2}q_1^2 q_2^2,
\end{aligned} \tag{33}$$

and the functions Γ^i are

$$\begin{aligned}
\Gamma^1 &= \epsilon d_1 p_1(1-q_1^2),\\
\Gamma^2 &= \epsilon d_2 p_2(1-q_2^2).
\end{aligned} \tag{34}$$

Another approximate partial Hamiltonian for system (32) is

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2\right), \tag{35}$$

and now the functions Γ^i are

$$\begin{aligned}
\Gamma^1 &= \epsilon\left(d_1 p_1(1-q_1^2) - \alpha_1 q_1^3 - \delta q_1 q_2^2\right),\\
\Gamma^2 &= \epsilon\left(d_2 p_2(1-q_2^2) - \alpha_2 q_2^3 - \delta q_1^2 q_2\right).
\end{aligned} \tag{36}$$

Note that the approximate partial Hamiltonian (33) is of the form $H = H_0 + \epsilon H_1$, whereas the approximate partial Hamiltonian (35) is of simple form $H = H_0$. Also, it is worthy to mention here that function Γ_i are defined according to approximate partial Hamiltonian.

2. The equations of motion in a mechanical system are

$$\begin{aligned}
\dot{q}_1 &= p_1,\\
\dot{q}_2 &= p_2,\\
\dot{p}_1 &= -p_1 - \epsilon t,\\
\dot{p}_2 &= -\epsilon p_1(2-t).
\end{aligned} \tag{37}$$

A partial Hamiltonian for system (37) is

$$H = \frac{1}{2}(p_1^2 + p_2^2), \tag{38}$$

and the functions $\Gamma_1 = -p_1 - \epsilon t,\ \Gamma_2 = -\epsilon p_1(2-t)$. Note that here $\Gamma_1,\ \Gamma_2$ are nonpotential generalized forces or generalized constrained forces of mechanical system.

3. The Hamiltonian dynamical system for the galaxy model reads as (see e.g., [29])

$$\begin{aligned}
\dot{q}_1 &= p_1,\\
\dot{q}_2 &= p_2,\\
\dot{p}_1 &= -\omega_1^2 q_1 + \epsilon q_2^2,\\
\dot{p}_2 &= -\omega_2^2 q_2 + 2\epsilon q_1 q_2.
\end{aligned} \tag{39}$$

The approximate standard Hamiltonian for system (39) is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2) - \epsilon q_1 q_2^2. \tag{40}$$

A partial Hamiltonian for system (39) exists too and is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2), \tag{41}$$

with $\Gamma_1 = \epsilon q_2^2$ and $\Gamma_2 = 2\epsilon q_1 q_2$.

Next, we deduce the approximate partial Hamiltonian operators determining equation from the formulas of approximate partial Lagrangian operator determining equation. Also formulas are provided to construct approximate first integrals upto $o(\epsilon)$.

3.2 Generalization of approximate Noether-like theorem

The generators of point symmetries (see, e.g., [25,27,28]) in the space (t, q, p) are operators of the form $X = X_0 + \epsilon X_1$ where

$$X_0 = \xi_0(t, q, p)\frac{\partial}{\partial t} + \eta_0^i(t, q, p)\frac{\partial}{\partial q^i} + \zeta_0^i(t, q, p)\frac{\partial}{\partial p_i}, \tag{42}$$

$$X_1 = \xi_1(t, q, p)\frac{\partial}{\partial t} + \eta_1^i(t, q, p)\frac{\partial}{\partial q^i} + \zeta_1^i(t, q, p)\frac{\partial}{\partial p_i}. \tag{43}$$

Proposition 2 *An operator $X = X_0 + \epsilon X_1$ is said to be an approximate partial Hamiltonian operator corresponding to an approximate partial Hamiltonian $H(t, q^i, p^i; \epsilon) = H_0(t, q^i, p^i) + \epsilon H_1(t, q^i, p^i)$, if there exists a function $B(t, q^i, p^i; \epsilon) = B_0(t, q^i, p^i) + \epsilon B_1(t, q^i, p^i)$ such that*

$$\zeta_0^i\frac{\partial H_0}{\partial p^i} + p^i D(\eta_0^i) - X_0(H_0) - H_0 D(\xi_0) = D(B_0) + \left(\eta_0^i - \xi_0\frac{\partial H_0}{\partial p^i}\right)(-\Gamma_0^i), \tag{44}$$

$$\zeta_0^i\frac{\partial H_1}{\partial p^i} + \zeta_1^i\frac{\partial H_0}{\partial p^i} + p^i D(\eta_1^i) - X_0(H_1) - X_1(H_0) - H_0 D(\xi_1) - H_1 D(\xi_0) = D(B_0) + \left(\eta_0^i - \xi_0\frac{\partial H_1}{\partial p^i}\right)(-\Gamma_1^i) + \left(\eta_1^i - \xi_1\frac{\partial H_0}{\partial p^i}\right)(-\Gamma_0^i), \tag{45}$$

hold.

Proof The partial Lagrangian operator determining Eq. (17) with the aid of approximate Legendre-type transformation (28) yields

$$X[p^i\dot{q}^i - H(t, p^i, q^i)] + [p^i\dot{q}^i - H(t, p^i, q^i)]D_t(\xi) = D_t(B) + (\eta^i - \xi\dot{q}^i)(-\Gamma_i). \tag{46}$$

On expanding Eq. (46) and utilizing the first equation of system (29), we have

$$p^i[D_t(\eta^i) - \dot{q}^i D_t(\xi)] - \xi\frac{\partial H}{\partial t} - \eta^i\frac{\partial H}{\partial q^i} + p^i\dot{q}^i D_t(\xi) - H D_t(\xi) = D_t(B) + (\eta^i - \xi\frac{\partial H}{\partial p^i})(-\Gamma_i), \tag{47}$$

and this on separation with respect to ϵ yields Eqs. (44) and (45). This completes the proof. □

The following theorem is essential for the construction of approximate first integrals for system (29) and is analogous to one presented in [25,27].

Proposition 3 *The approximate first integral $I = I_0 + \epsilon I_1$ corresponding to system (29) associated with an approximate partial Hamiltonian operator $X = X_0 + \epsilon X_1$ of the approximate partial Hamiltonian $H(t, q, p; \epsilon)$ is determined from*

$$I_0 = p^i\eta_0^i - \xi_0 H_0 - B_0, \tag{48}$$

$$I_1 = p^i\eta_1^i - \xi_0 H_1 - \xi_1 H_0 - B_1, \tag{49}$$

where $B(t, q^i, p^i; \epsilon) = B_0(t, q^i, p^i) + \epsilon B_1(t, q^i, p^i)$ is a gauge-like function.

Proof The formula for first integral (18) with the aid of approximate Legendre-type transformation (28) and $p^i = \frac{\partial L}{\partial \dot{q}^i}$ yields

$$I = \xi[p^i\dot{q}^i - H(t, p^i, q^i; \epsilon)] + (\eta^i - \xi\dot{q}^i)p^i - B, \tag{50}$$

and this after separation with respect to ϵ provides formulas (48) and (49). This completes the proof. □

4 Illustrative example: perturbed orbit equation

In this section, we explain our approach with the help of a simple example of perturbed orbit equation. An approximate standard Hamiltonian also exists for this model. Consider the following perturbed orbit equation [6]

$$\ddot{q} + q = \epsilon\frac{A}{q^3}. \tag{51}$$

In [6], the approximate Noether symmetries and first integrals were established for Eq. (51). A standard Hamiltonian also exists for this equation. We apply our newly developed approximate partial Hamiltonian approach to derive approximate first integrals of Eq. (51).

An approximate partial Lagrangian for Eq. (51) is

$$L(t,q,\dot{q})=\frac{\dot{q}^2}{2}. \tag{52}$$

An approximate partial Hamiltonian with the aid of an approximate Legendre-type transformation (28) is

$$H=\frac{p^2}{2}, \tag{53}$$

where $p=\frac{\partial L}{\partial \dot{q}}$. With the aid of approximate partial Hamiltonian (53), the second-order ODE (51) can be re-cast into following first-order system:

$$\begin{aligned}\dot{q}&=p,\\ \dot{p}&=\epsilon\frac{A}{q^3}-q,\end{aligned} \tag{54}$$

with $\Gamma=\epsilon\frac{A}{q^3}-q$. Now we apply our newly developed approximate partial Hamiltonian approach to construct first integrals of system (54). The approximate partial Hamiltonian operators determining Eqs. (44) and (45) yield the following determining equations corresponding to zeroth- and first-order approximations of ϵ, respectively:

$$\begin{aligned}\epsilon^0&: p(\eta_{0t}+p\eta_{0q})-\frac{p^2}{2}(\xi_{0t}+p\xi_{0q})\\ &=B_{0t}+pB_{0q}+q(\eta_0-\xi_0 p),\end{aligned} \tag{55}$$

$$\begin{aligned}\epsilon&: p(\eta_{1t}+p\eta_{1q})-\frac{p^2}{2}(\xi_{1t}+p\xi_{1q})\\ &=B_{1t}+pB_{1q}+q(\eta_1-\xi_1 p)-\frac{A}{q^3}(\eta_0-\xi_0 p),\end{aligned} \tag{56}$$

in which $\xi_0(t,q)$, $\xi_1(t,q)$, $\eta_0(t,q)$, $\eta_1(t,q)$, $B_0(t,q)$ and $B_1(t,q)$. Separating (55), (56), after expansion, with respect to the different combinations of derivatives of p, we obtain the following systems:

Zeroth-order approximation:

$$\begin{aligned}p^3&: \xi_{0q}=0,\\ p^2&: \eta_{0q}-\frac{1}{2}\xi_{0t}=0,\\ p&: B_{0q}-\eta_{0t}-\xi_{0q}=0,\\ p^0&: B_{0t}+q\eta_0=0.\end{aligned} \tag{57}$$

First-order approximation:

$$\begin{aligned}p^3&: \xi_{1q}=0,\\ p^2&: \eta_{1q}-\frac{1}{2}\xi_{1t}=0,\\ p&: B_{1q}-\eta_{1t}-\xi_{1q}+\frac{A}{q^3}\xi_0=0,\\ p^0&: B_{1t}+q\eta_1-\frac{A}{q^3}\eta_0=0.\end{aligned} \tag{58}$$

The solution of systems (57) and (58) yields following approximate partial Hamiltonian operators and gauge terms:

$$\begin{aligned}X^1&=\frac{\partial}{\partial t},\ B^1=\frac{q^2}{2}+\epsilon\frac{A}{2q^2},\\ X^2&=\sin(2t)\frac{\partial}{\partial t}+q\cos(2t)\frac{\partial}{\partial q},\\ B^2&=-\frac{q^2}{2}\sin(2t)+\epsilon\frac{A}{2q^2}\sin(2t),\\ X^3&=\cos(2t)\frac{\partial}{\partial t}-q\sin(2t)\frac{\partial}{\partial q},\\ B^3&=-\frac{q^2}{2}\cos(2t)+\epsilon\frac{A}{2q^2}\cos(2t),\\ X^4&=\epsilon\frac{\partial}{\partial t},\ \ B^4=\epsilon\frac{q^2}{2}\\ X^5&=\epsilon\left(\sin(2t)\frac{\partial}{\partial t}+q\cos(2t)\frac{\partial}{\partial q}\right),\\ B^5&=-\epsilon\frac{q^2}{2}\sin(2t),\\ X^6&=\epsilon\left(\cos(2t)\frac{\partial}{\partial t}-q\sin(2t)\frac{\partial}{\partial q}\right),\\ B^6&=-\epsilon\frac{q^2}{2}\cos(2t),\\ X^7&=\epsilon\cos t\frac{\partial}{\partial q},\ \ B^7=-\epsilon q\sin(t),\\ X^8&=\epsilon\sin t\frac{\partial}{\partial q},\ \ B^8=\epsilon q\cos t.\end{aligned} \tag{59}$$

In [6], the same Hamiltonian operators were derived for the standard Hamiltonian by the approximate Noether's theorem in phase space.

Next, we utilize formulas (48) and (49), to establish the approximate first integrals $I=I_0+\epsilon I_1$ associated with the approximate partial Hamiltonian operators (59). The approximate first integrals are

$$\begin{aligned}I^1&=\frac{p^2+q^2}{2}+\epsilon\frac{A}{2q^2},\\ I^2&=pq\cos(2t)-\frac{p^2}{2}\sin(2t)+\frac{q^2}{2}\sin(2t)\\ &\quad-\epsilon\frac{A}{2q^2}\sin(2t),\end{aligned}$$

$$I^3 = pq\sin(2t) + \frac{p^2}{2}\cos(2t) - \frac{q^2}{2}\cos(2t)$$
$$+\epsilon\frac{A}{2q^2}\cos(2t),$$
$$I^4 = \frac{1}{2}\epsilon(p^2+q^2),$$
$$I^5 = \epsilon\left(pq\cos(2t) - \frac{p^2}{2}\sin(2t) + \frac{q^2}{2}\sin(2t)\right),$$
$$I^6 = \epsilon\left(-pq\sin(2t) - \frac{p^2}{2}\cos(2t) + \frac{q^2}{2}\cos(2t)\right),$$
$$I^7 = \epsilon\left(p\cos t + q\sin t\right),$$
$$I^8 = \epsilon\left(p\sin t - q\cos t\right). \tag{60}$$

The approximate partial Hamiltonian approach provided three stable and five unstable approximate first integrals. It is important to mention here that the approximate first integrals I^1, I^2 and I^3 are stable, whereas $I^4 \cdots I^8$ are unstable.

4.1 Exact solutions of perturbed orbit equation

We compute the exact solutions of perturbed orbit equation by utilizing the stable approximate first integrals I_1, I_2 and I_3 given in (60). Setting $I_1 = c_1$, we have

$$\frac{p^2+q^2}{2} + \epsilon\frac{A}{2q^2} = c_1, \tag{61}$$

which results in

$$p = \pm\frac{\sqrt{2\,c_1q^2 - q^4 - \epsilon\, A}}{q}. \tag{62}$$

The substitution of (62) in Eq. (54) yields

$$\pm\frac{1}{2}\arctan\left(\frac{q^2-c_1}{\sqrt{2\,c_1q^2-q^4-\epsilon\, A}}\right) = t + c_2. \tag{63}$$

Equations (62) and (63) form solution of system (54).

Next we utilize I_1 and I_2 to construct the exact solution. Setting $I_1 = c_1$, we arrive at (62). Setting $I_2 = c_2$ gives

$$pq\cos(2t) - \frac{1}{2}p^2\sin(2t) + \frac{1}{2}q^2\sin(2\,t)$$
$$-\frac{1}{2}\frac{\epsilon\, A\sin(2\,t)}{q^2} = c_2. \tag{64}$$

Equations (61) and (64) finally give rise to

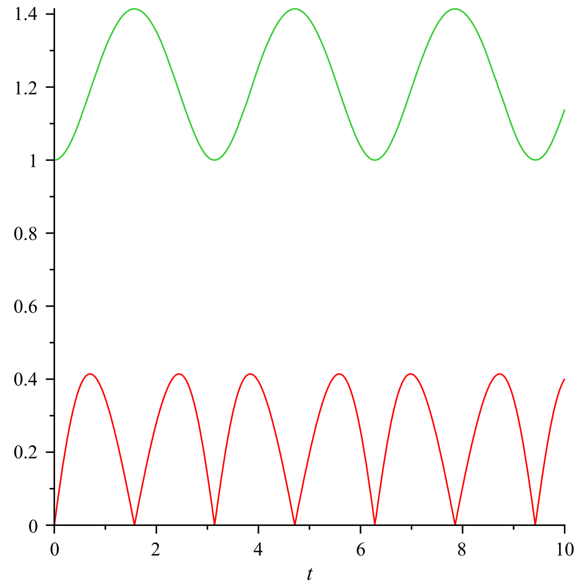

Fig. 1 Graphical behavior of perturbed system (54) for $\epsilon = 1, A = 2, a = 1$

$$p(t)^2 = -c_1 - 2c_2\sin t\cos t - (1-2\cos^2 t)\sqrt{{c_1}^2-\epsilon A - {c_2}^2}+2c_1 \tag{65}$$
$$-\frac{\epsilon A}{c_1+2c_2\sin t\cos t+(1-2\cos^2 t)\sqrt{{c_1}^2-\epsilon A-{c_2}^2}}, \tag{66}$$

$$q(t) = \sqrt{c_1+2c_2\sin t\cos t+(1-2\cos^2 t)\sqrt{{c_1}^2-\epsilon\, A-{c_2}^2}}. \tag{67}$$

Using the initial conditions $q(0) = a$ and $\dot{q}(0) = 0$ in system (54) provides $p(0) = 0$ and $\dot{p}(0) = \frac{\epsilon A}{a^3} - a$. Solutions (66) and (67) with the help of initial conditions result in $c_1 = 0$ and $c_2 = \frac{\epsilon\, A+a^4}{2a^2}$.
The solutions in (66) and (67) finally give rise to

$$p(t) = \frac{\sin t\cos t\,\left(\epsilon A - a^4\right)}{a\sqrt{\epsilon\, A + a^4\cos^2 t - \epsilon\, A\cos^2 t}}, \tag{68}$$

and

$$q(t) = \frac{\sqrt{\epsilon\, A + a^4\cos^2 t - \epsilon\, A\cos^2 t}}{a}. \tag{69}$$

The exact solutions (68) and (69) are graphically represented in Fig. 1.

Remark 4 Note that the combination of I_1 and I_3 provides the same solutions as in Eqs. (68) and (69).

5 Approximate first integrals of two perturbed cubically coupled Duffing–Van der Pol oscillators

In this section, approximate partial Hamiltonian approach is utilized for the construction of approximate first integrals of two perturbed cubically coupled Duffing–Van der Pol oscillators given in (32). Both resonant and nonresonant cases were investigated in detail. We obtain stable and unstable approximate first integrals for both resonant and nonresonant cases. System (32) can be alternatively expressed as a system of two second-order ODEs which represents two perturbed cubically coupled Duffing–Van der Pol oscillators [32]. The parameter ω_i denotes the natural frequency, d_i is damping coefficient, $\alpha_i > 0$ is stiffness term, and $\delta > 0$ represents the coupling strength between the oscillators. It is important to mention here that the damping coefficients $d_1 \geq 0$ and $d_2 \geq 0$.

We consider the approximate partial Hamiltonian given in (35), and functions Γ_i are given in (36). The approximate partial Hamiltonian operators determining Eqs. (44) and (45) for the partial Hamiltonian (35) yield the following determining equations corresponding to zeroth-order and first-order approximations of ϵ:

$$\begin{aligned} & p_1[\eta^1_{0t}+\eta^1_{0q_1}\dot{q}_1+\eta^1_{0q_2}\dot{q}_2]+p_2[\eta^2_{0t}+\eta^2_{0q_1}\dot{q}_1+\eta^2_{0q_2}\dot{q}_2], \\ & \quad -\eta^1_0\omega_1^2 q_1 - \eta^2_0\omega_2^2 q_2 - \frac{1}{2}[\xi_{0t} + \xi_{0q_1}\dot{q}_1 \\ & \quad +\xi_{0q_2}\dot{q}_2][p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2] \\ & = B_{0t} + B_{0q_1}\dot{q}_1 + B_{0q_2}\dot{q}_2 \end{aligned} \tag{70}$$

and

$$\begin{aligned} & p_1[\eta^1_{1t}+\eta^1_{1q_1}\dot{q}_1+\eta^1_{1q_2}\dot{q}_2]+p_2[\eta^2_{1t}+\eta^2_{1q_1}\dot{q}_1+\eta^2_{1q_2}\dot{q}_2], \\ & \quad -\eta^1_1\omega_1^2 q_1 - \eta^2_1\omega_2^2 q_2 - \frac{1}{2}[\xi_{1t} + \xi_{1q_1}\dot{q}_1 + \xi_{1q_2}\dot{q}_2] \\ & \quad \times[p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2] \\ & = B_{1t} + B_{1q_1}\dot{q}_1 + B_{1q_2}\dot{q}_2 + [\eta^1_0 - \xi_0 p_1] \\ & \quad \times[-d_1 p_1(1-q_1^2) + \alpha_1 q_1^3 + \delta q_1 q_2^2] \\ & \quad +[\eta^2_0 - \xi_0 p_2][-d_2 p_2(1-q_2^2) + \alpha_2 q_2^3 + \delta q_1^2 q_2] \end{aligned} \tag{71}$$

in which $\xi_0(t, q_1, q_2)$, $\xi_1(t, q_1, q_2)$, $\eta^1_0(t, q_1, q_2$, $\eta^1_1(t, q_1, q_2)$, $\eta^2_0(t, q_1, q_2)$, $\eta^2_1(t, q_1, q_2)$, $B_0(t, q_1, q_2)$ and $B_1(t, q_1, q_2)$. Separating Eqs. (70) and (71), after expansion, with respect to the different combinations of derivatives of p_1 and p_2, we obtain the following systems:

Zeroth-order approximation:

$$\begin{aligned} & p_1^3: \xi_{0q_1} = 0, \\ & p_2^3: \xi_{0q_2} = 0, \\ & p_1^2: \eta^1_{0q_1} - \frac{1}{2}\xi_{0t} = 0, \\ & p_2^2: \eta^2_{0q_2} - \frac{1}{2}\xi_{0t} = 0, \\ & p_1 p_2: \eta^1_{0q_2} + \eta^2_{0q_1} = 0, \\ & p_1: \eta^1_{0t} = B_{0q_1}, \\ & p_2: \eta^2_{0t} = B_{0q_2}, \\ & rest: -\eta^1_0\omega_1^2 q_1 - \eta^2_0\omega_2^2 q_2 \\ & \qquad -\frac{1}{2}\xi_{0t}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2) = B_{0t}. \end{aligned} \tag{72}$$

First-order approximation:

$$\begin{aligned} & p_1^3: \xi_{1q_1} = 0, \\ & p_2^3: \xi_{1q_2} = 0, \\ & p_1^2: \eta^1_{1q_1} - \frac{1}{2}\xi_{1t} = d_1\xi_0(1-q_1^2), \\ & p_2^2: \eta^2_{1q_2} - \frac{1}{2}\xi_{1t} = d_2\xi_0(1-q_2^2), \\ & p_1 p_2: \eta^1_{1q_2} + \eta^2_{1q_1} = 0, \\ & p_1: \eta^1_{1t} = B_{1q_1} - \xi_0(\alpha_1 q_1^3 + \delta q_1 q_2^2) - d_1(1-q_1^2)\eta^1_0, \\ & p_2: \eta^2_{1t} = B_{1q_2} - \xi_0(\alpha_2 q_2^3 + \delta q_1^2 q_2) - d_2(1-q_2^2)\eta^2_0, \\ & rest: -\eta^1_1\omega_1^2 q_1 - \eta^2_1\omega_2^2 q_2 - \frac{1}{2}\xi_{1t}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2) \\ & \qquad = B_{1t} + \eta^1_0(\alpha_1 q_1^3 + \delta q_1 q_2^2) + \eta^2_0(\alpha_2 q_2^3 + \delta q_1^2 q_2). \end{aligned} \tag{73}$$

The solution of systems (72) and (73) is established for two cases (i) $\omega_1 = \omega_2$ and (ii) $\omega_1 \neq \omega_2$. For the sake of brevity the details of long calculation will not be reproduced here, and we will only give approximate partial Hamiltonian operators, gauge terms and first integrals for both cases.

5.1 Resonant case: $\omega_1 = \omega_2$

For the resonant case $\omega_1 = \omega_2$, following four subcases arise.

Subcase 1: $d_1 \neq 0,\ d_2 \neq 0$.

Subcase 2: When $d_1 = 0,\ d_2 = 0,\ \delta = \alpha_1 = \alpha_2$.

Subcase 3: When $d_1 \neq 0,\ d_2 = 0$.

Subcase 4: When $d_1 = 0,\ d_2 \neq 0$.

Next, we provide the approximate partial Hamiltonian operators and gauge terms for all these subcases.

Subcase 1: $d_1 \neq 0,\ d_2 \neq 0.$
The partial Hamiltonian operators and gauge terms for this case are given as follows:

$$\begin{aligned}
X^1 &= \epsilon\frac{\partial}{\partial t},\ B^1 = 0,\\
X^2 &= \epsilon\Big(\sin 2\omega_1 t\frac{\partial}{\partial t} + q_1\omega_1\cos 2\omega_1 t\frac{\partial}{\partial q_1}\\
&\quad + q_2\omega_1\cos 2\omega_1 t\frac{\partial}{\partial q_2}\Big),\\
B^2 &= -\epsilon(q_1^2+q_2^2)\omega_1^2\sin 2\omega_1 t,\\
X^3 &= \epsilon\Big(\cos 2\omega_1 t\frac{\partial}{\partial t} - q_1\omega_1\sin 2\omega_1 t\frac{\partial}{\partial q_1}\\
&\quad - q_2\omega_1\sin 2\omega_1 t\frac{\partial}{\partial q_2}\Big),\\
B^3 &= -\epsilon(q_1^2+q_2^2)\omega_1^2\cos 2\omega_1 t,\\
X^4 &= \epsilon\Big(q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2}\Big),\ B^4 = 0,\\
X^5 &= \epsilon\sin\omega_1 t\frac{\partial}{\partial q_1},\ B^5 = \epsilon\omega_1 q_1\cos\omega_1 t,\\
X^6 &= \epsilon\cos\omega_1 t\frac{\partial}{\partial q_1},\ B^6 = -\epsilon\omega_1 q_1\sin\omega_1 t,\\
X^7 &= \epsilon\sin\omega_1 t\frac{\partial}{\partial q_2},\ B^7 = \epsilon\omega_1 q_2\cos\omega_1 t,\\
X^8 &= \epsilon\cos\omega_1 t\frac{\partial}{\partial q_2},\ B^8 = -\epsilon\omega_1 q_2\sin\omega_1 t.
\end{aligned}\tag{74}$$

Subcase 2: When $d_1 = 0,\ d_2 = 0, \delta = \alpha_1 = \alpha_2.$

The approximate partial Hamiltonian approach for this case yields the following two stable partial Hamiltonian operators and gauge terms in addition to the unstable ones given in (74):

$$\begin{aligned}
X^9 &= \frac{\partial}{\partial t},\ B^9 = \epsilon\left(\frac{\alpha_1}{4}q_1^4 + \frac{\delta}{2}q_1^2q_2^2 + \frac{\alpha_2}{4}q_2^4\right),\\
X^{10} &= q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2},\ B^{10} = 0.
\end{aligned}\tag{75}$$

Subcase 3: When $d_1 \neq 0,\ d_2 = 0.$
For this case, we obtain only unstable approximate partial Hamiltonian operators (74).
Subcase 4: When $d_1 = 0,\ d_2 \neq 0.$
For this case, in addition to the unstable partial Hamiltonian operators given in (74), we also obtain one stable approximate partial Hamiltonian operator

$$\begin{aligned}
X^{11} &= q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2} + \frac{\epsilon}{2}\Big(-q_2d_2t\frac{\partial}{\partial q_1}\\
&\quad + d_2q_1t\frac{\partial}{\partial q_2}\Big),\ B^{11} = -\frac{\epsilon}{2}d_2q_1q_2.
\end{aligned}\tag{76}$$

5.1.1 Approximate first integrals

Next, we utilize formulas (48) and (49), to establish the approximate first integrals $I = I_0 + \epsilon I_1$ associated with the approximate partial Hamiltonian operators (74)–(76) for different cases.

Subcase 1: $d_1 \neq 0,\ d_2 \neq 0.$

With the aid of formulas (48) and (49), we obtain the following eight approximate first integrals $I = I_0 + \epsilon I_1$ associated with the approximate partial Hamiltonian operators (74):

$$\begin{aligned}
I^1 &= -\frac{\epsilon}{2}\Big[p_1^2 + p_2^2 + \omega_1^2(q_1^2+q_2^2)\Big],\\
I^2 &= \epsilon\Big[\omega_1\cos 2\omega_1 t(p_1q_1 + p_2q_2)\\
&\quad - \frac{1}{2}\sin 2\omega_1 t(p_1^2 + p_2^2 - \omega_1^2(q_1^2+q_2^2))\Big],\\
I^3 &= -\epsilon\Big[\omega_1\sin 2\omega_1 t(p_1q_1 + p_2q_2)\\
&\quad + \frac{1}{2}\cos 2\omega_1 t(p_1^2 + p_2^2 - \omega_1^2(q_1^2+q_2^2))\Big],\\
I^4 &= \epsilon\Big[p_1q_2 - p_2q_1\Big],\\
I^5 &= \epsilon\Big[p_1\sin\omega_1 t - \omega_1q_1\cos\omega_1 t\Big],\\
I^6 &= \epsilon\Big[p_1\cos\omega_1 t + \omega_1q_1\sin\omega_1 t\Big],\\
I^7 &= \epsilon\Big[p_2\sin\omega_1 t - \omega_1q_2\cos\omega_1 t\Big],\\
I^8 &= \epsilon\Big[p_2\cos\omega_1 t + \omega_1q_2\sin\omega_1 t\Big].
\end{aligned}\tag{77}$$

Note that all of these approximate first integrals are multiplied by ϵ and thus all are unstable.

Subcase 2 When $d_1 = 0,\ d_2 = 0, \delta = \alpha_1 = \alpha_2.$

For this case in addition to unstable approximate first integrals $I^1, \ldots I^8$, following two stable approximate first integrals are derived:

$$\begin{aligned}
I^9 &= \frac{1}{2}\Big[p_1^2 + p_2^2 + \omega_1^2(q_1^2+q_2^2) + \epsilon\alpha_1\\
&\quad \left(\frac{q_1^4}{2} + q_1^2q_2^2 + \frac{q_2^4}{2}\right)\Big],\\
I^{10} &= p_1q_2 - p_2q_1.
\end{aligned}\tag{78}$$

Subcase 3 When $d_1 \neq 0,\ d_2 = 0.$
For this case, we obtain only unstable approximate first integrals $I^1, \ldots I^8$ given in Eq. (77).

Subcase 4 When $d_1 = 0,\ d_2 \neq 0$.
For this case, in addition to the unstable approximate first integrals $I^1, \ldots I^8$ given in (77), we obtain the following stable approximate first integral:

$$I^{11} = p_1q_2 - p_2q_1 + \frac{\epsilon d_2}{2}\Big[- p_1q_2t + p_2q_1t + q_1q_2\Big]. \tag{79}$$

For the resonant case, the approximate partial Hamiltonian approach yields eight unstable approximate first integrals for $d_1 \neq 0,\ d_2 \neq 0$ and for the case $d_1 \neq 0,\ d_2 = 0$. When $d_1 = 0,\ d_2 = 0,\ \delta = \alpha_1 = \alpha_2$ then in addition to eight unstable approximate first integrals, we have derived two stable approximate first integrals I^9 and I^{10}. For the case $d_1 = 0,\ d_2 \neq 0$, one stable approximate first integral I^{11} is attained in addition to eight unstable approximate first integrals.

5.2 Nonresonant case: $\omega_1 \neq \omega_2$

For the nonresonant case $\omega_1 \neq \omega_2$, following three subcases arise.

Subcase 1: $d_1 \neq 0,\ d_2 \neq 0$.

Subcase 2: $d_1 = 0,\ d_2 = 0$.

Subcase 3: $d_1 \neq 0,\ d_2 = 0$.

Now, we provide the approximate partial Hamiltonian operators and gauge terms for these three subcases.

Subcase 1: $d_1 \neq 0,\ d_2 \neq 0$.
The partial Hamiltonian operators and gauge terms for this case are given as follows:

$$\begin{aligned}
X^1 &= \epsilon\frac{\partial}{\partial t},\ B^1 = 0,\\
X^2 &= \epsilon \sin\omega_1 t\frac{\partial}{\partial q_1},\ B^2 = \epsilon\omega_1 q_1 \cos\omega_1 t,\\
X^3 &= \epsilon \cos\omega_1 t\frac{\partial}{\partial q_1},\ B^3 = -\epsilon\omega_1 q_1 \sin\omega_1 t,\\
X^4 &= \epsilon \sin\omega_2 t\frac{\partial}{\partial q_2},\ B^4 = \epsilon\omega_2 q_2 \cos\omega_2 t,\\
X^5 &= \epsilon \cos\omega_2 t\frac{\partial}{\partial q_2},\ B^5 = -\epsilon\omega_2 q_2 \sin\omega_2 t.
\end{aligned} \tag{80}$$

Subcase 2: $d_1 = 0,\ d_2 = 0$.
When $d_1 = 0,\ d_2 = 0$, the approximate partial Hamiltonian approach yields the following one stable approximate partial Hamiltonian operator and gauge term in addition to the unstable ones provided in (80) :

$$X^6 = \frac{\partial}{\partial t},\ B^6 = \epsilon(\frac{\alpha_1}{4}q_1^4 + \frac{\delta}{2}q_1^2q_2^2 + \frac{\alpha_2}{4}q_2^4). \tag{81}$$

Subcase 3: $d_1 \neq 0,\ d_2 = 0$.
For the case, when $d_1 \neq 0,\ d_2 = 0$ then we obtain only unstable approximate partial Hamiltonian operators (80).

It is worthy to mention here that the subcase $d_1 = 0,\ d_2 \neq 0$ does not arise for nonresonant case.

5.2.1 Approximate first integrals

With the aid of formulas (48) and (49), we establish approximate first integrals $I = I_0 + \epsilon I_1$ associated with the approximate partial Hamiltonian operators and gauge terms given in Eqs. (80) and (81).

Subcase 1: $d_1 \neq 0,\ d_2 \neq 0$.
The approximate first integrals $I = I_0 + \epsilon I_1$ associated with the approximate partial Hamiltonian operators and gauge terms given in Eq. (80) are given by

$$\begin{aligned}
I^1 &= -\frac{\epsilon}{2}\Big[p_1^2 + p_2^2 + \omega_1^2q_1^2 + \omega_2^2q_2^2\Big],\\
I^2 &= \epsilon\Big[p_1 \sin\omega_1 t - \omega_1 q_1 \cos\omega_1 t\Big],\\
I^3 &= \epsilon\Big[p_1 \cos\omega_1 t + \omega_1 q_1 \sin\omega_1 t\Big],\\
I^4 &= \epsilon\Big[p_2 \sin\omega_2 t - \omega_2 q_2 \cos\omega_2 t\Big],\\
I^5 &= \epsilon\Big[p_2 \cos\omega_2 t + \omega_2 q_2 \sin\omega_2 t\Big].
\end{aligned} \tag{82}$$

Note that all of these approximate first integrals are multiplied by ϵ and thus all are unstable.

Subcase 2: $d_1 = 0,\ d_2 = 0$.
The approximate first integral associated with the approximate partial Hamiltonian operator and gauge term (81) is given by

$$\begin{aligned}
I^6 = \frac{1}{2}\Big[&p_1^2 + p_2^2 + \omega_1^2q_1^2 + \omega_2^2q_2^2\\
&+\epsilon\left(\frac{\alpha_1 q_1^4}{2} + \delta q_1^2q_2^2 + \frac{\alpha_2 q_2^4}{2}\right)\Big].
\end{aligned} \tag{83}$$

Subcase 3: $d_1 \neq 0,\ d_2 = 0$.
The approximate first integrals are the same as for the subcase 1 and are given in Eq. (82).

The approximate partial Hamiltonian approach provided five unstable approximate first integrals for $d_1 \neq 0,\ d_2 \neq 0$ and for $d_1 \neq 0,\ d_2 = 0$. For the case

$d_1 = 0,\ d_2 = 0$, one stable first integral I^6 is attained in addition to five unstable first integrals.

The approximate partial Hamiltonian approach is utilized for the construction of approximate first integrals of two perturbed cubically coupled Duffing–Van der Pol oscillators. Both resonant and nonresonant cases are investigated in detail. For the resonant case approximate partial Hamiltonian approach provided three stable and eight unstable approximate partial Hamiltonian operators. We established two stable approximate first integrals for the resonant case when the damping coefficients $d_1 = 0, d_2 = 0$ and $\alpha_1 = \alpha_2 = \delta$. The condition $\alpha_1 = \alpha_2 = \delta$ means that the coefficient of Duffing term and coefficients of coupling terms of both the oscillators are the same. Another stable approximate first integral for the resonant case is derived for the case $d_1 = 0,\ d_2 \neq 0$. We obtained one stable and five unstable approximate first integrals for the nonresonant case. The stable first integral for the nonresonant case arises when the damping coefficients are zero, i.e., $d_1 = 0, d_2 = 0$. It is important to mention here that these parameter restrictions naturally arise in process of construction of approximate partial Hamiltonian operators.

6 Reductions and exact solutions of two perturbed cubically coupled Duffing–Van der Pol oscillators

In this section, we provide the reductions and exact solutions of two perturbed cubically coupled Duffing–Van der Pol oscillators governed by first-order system of ODEs given in (32) with the aid of the stable first integrals derived in the preceding section. We only consider those cases for which stable first integrals have been established.

6.1 Exact solutions for the resonant case when $d_1 = 0,\ d_2 = 0, \delta = \alpha_1 = \alpha_2$

The approximate partial Hamiltonian approach provided eight unstable approximate first integrals $I^1 \cdots I^8$ given in Eq. (77) and two stable approximate first integrals $I^9,\ I^{10}$ provided in Eq. (78). The stable approximate first integrals I^9 and I^{10} are utilized to derive the solution of first-order system of ODEs given in (32). Setting $I^{10} = c_1$, we have

$$p_1 q_2 - p_2 q_1 = c_1 \tag{84}$$

and this yields

$$q_1(t) = q_2 \int \frac{c_1}{q_2^2} \mathrm{d}t + c_2 q_2, \tag{85}$$

where c_1 and c_2 are arbitrary constants. Setting $I^9 = c_3$, we have

$$\frac{1}{2}\left[\dot{q}_1^2 + \dot{q}_2^2 + \omega_1^2 (q_1^2 + q_2^2) + \frac{\epsilon \alpha_1}{2} (q_1^2 + q_2^2)^2\right] = c_3. \tag{86}$$

Equations (85) and (86) give following exact solution provided $c_1 = 0$

$$q_2(t) = \frac{2 Jacobi SN\left(\frac{1}{\sqrt{2}}\sqrt{\omega_1^2 + \sqrt{\omega_1^4 + 4 c_3 \alpha_1 \epsilon t}} + c_4, \sqrt{\frac{2\epsilon c_3 \alpha_1}{\epsilon c_3 \alpha + \omega_1^4 + \omega^2 \sqrt{\omega_1^4 + 4\epsilon c_3 \alpha}}}\right)\sqrt{c_3}}{\sqrt{(c_2^2 + 1)\omega_1^2 + (c_2^2 + 1)\sqrt{\omega_1^4 + 4\epsilon c_3 \alpha_1}}}, \tag{87}$$

and

$$q_1(t) = c_2 q_2(t). \tag{88}$$

From the first two equations of (32), we have $\dot{q}_1 = p_1$ and $\dot{q}_2 = p_2$. Thus, we have provided exact solution for the first-order system of perturbed ODEs given in (32) for the resonant case under parameter restriction $d_1 = 0,\ d_2 = 0, \delta = \alpha_1 = \alpha_2$. This solution is new in the literature and not reported before.

6.2 Reduction for the resonant case when $d_1 = 0,\ d_2 \neq 0$

The approximate partial Hamiltonian approach provided eight unstable approximate first integrals listed in Eq. (77) and one stable given in Eq. (79) approximate first integrals for this case. The stable approximate first integral I^{11} is utilized to provide a reduction to the first-order system of ODEs given in (32). Setting $I^{11} = c_1$, we have

$$p_1 q_2 - p_2 q_1 + \frac{\epsilon d_2}{2}\left[- p_1 q_2 t + p_2 q_1 t + q_1 q_2\right] = c_1, \tag{89}$$

where c_1 is arbitrary constant. Equation (89) yields

$$q_1(t) = \left(\int \frac{-2c_1}{q_2^2(\epsilon d_2 t - 2)^2} \mathrm{d}t + c_2 \right)(\epsilon d_2 t - 2) q_2, \tag{90}$$

where c_2 is arbitrary constant.

6.3 Exact solutions for the nonresonant case when $d_1 = 0,\ d_2 = 0$

The approximate partial Hamiltonian approach provided five unstable approximate first integrals presented in Eq. (82) and one stable given in Eq. (83) approximate first integrals for this case. Setting $I^6 = c_1$, we have

$$\frac{1}{2}\Big[p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2 + \epsilon \left(\frac{\alpha_1 q_1^4}{2} + \delta q_1^2 q_2^2 + \frac{\alpha_2 q_2^4}{2} \right) \Big] = c_1, \tag{91}$$

and this provides reduction of first-order system of ODEs given in (32).

7 Conclusions

The approximate partial Noether theorem proposed earlier for ODEs is generalized in phase space, and notion of approximate partial Hamiltonian is developed. An approximate Legendre-type transformation connects the approximate partial Lagrangian and approximate partial Hamiltonian. The formulas for approximate partial Hamiltonian operators determining equations and first integrals are provided explicitly.

First, it is applied to derive approximate first integrals of perturbed orbit equation as an illustrative example. The approximate partial Hamiltonian approach provides three stable and five unstable approximate first integrals. It is worthy to mention here that for the perturbed orbit equation approximate Noether's theorem also provided three stable and five unstable approximate first integrals corresponding to an approximate standard Lagrangian.

In order to show effectiveness of approach developed here, it is applied to derive approximate first integrals, reductions and exact solutions of two perturbed cubically coupled Duffing–Van der Pol oscillators. Both resonant and nonresonant cases were investigated in detail. For the resonant case, four subcases arise. The approximate partial Hamiltonian approach provided eight unstable approximate first integrals for all cases. Two stable approximate first integrals are derived provided the coefficient of Duffing term and coupling terms of both the oscillators are the same. Moreover, the effects of damping term are neglected. The third stable approximate first integral exists for the case when the effect of damping term is neglected for the first oscillator, but it is taken into account for the second oscillator. For the nonresonant case three subcases arise. We obtained five unstable approximate first integrals for all cases. For the nonresonant case, the stable approximate first integral is obtained when the effects of damping term are neglected. The parameter restriction for both resonant and nonresonant cases naturally arises in process of construction of approximate partial Hamiltonian operators. Then reductions and exact solutions for two perturbed cubically coupled Duffing–Van der Pol oscillators are constructed with the help of these derived approximate first integrals. We have considered only those cases for which stable approximate first integrals are established.

References

1. Olver, P.J.: Applications of Lie Group to Differential Equations. Berlin (1986)
2. Noether, E.: Invariante Variationsprobleme. Nachr. König. Gesell. Wissen., Göttingen, Math.-Phys. Kl., Heft **2**, 235–257 (1918). (English translation in Transport Theory and Statistical Physics 1(3) 1971 186–207.)
3. Baikov, V.A., Gazizov, R.K., Ibragimov, N.H.: Approximate symmetries. Math. Sbornik, 136 (178), No. 3: 435450, 1988. English transl. Math. USSR Sb **64**, 427–441 (1989)
4. Baikov, V.A., Gazizov, R.K., Ibragimov, N.K.: Perturbation methods in group analysis. J. Soviet. Math. **55**(1), 1450–1490 (1991)
5. Baikov, V.A., Gazizov, R.K., Ibragimov, N.H., Mahomed, F.M.: Closed orbits and their stable symmetries. J. Math. Phys. **35**(12), 6525–6535 (1994)
6. Govinder, K.S., Heil, T.G., Uzer, T.: Approximate Noether symmetries. Phys. Lett. A. **240**(3), 127–131 (1998)
7. Feroze, T., Kara, A.H.: Group theoretic methods for approximate invariants and Lagrangians for some classes of $y'' + \varepsilon F(t) y' + y = f(y, y')$. Int. J. Nonlinear Mech. **37**(2), 275–280 (2002)
8. Kara, A.H., Mahomed, F.M., Naeem, I., Wafo Soh, C.: Partial Noether operators and first integrals via partial Lagrangians. Math. Method. Appl. Sci. **30**(16), 2079–2089 (2007)
9. Naeem, I., Mahomed, F.M.: Approximate partial Noether operators and first integrals for coupled nonlinear oscillators. Nonlinear Dyn. **57**(1–2), 303–311 (2009)

10. Naeem, I., Mahomed, F.M.: Approximate first integrals for a system of two coupled van der Pol oscillators with linear diffusive coupling. Math. Comput. Appl. **15**(4), 720 (2010)
11. Naz, R.: Approximate partial Noether operators and first integrals for cubically coupled nonlinear Duffing oscillators subject to a periodically driven force. J. Math. Anal. Appl. **380**(1), 289–298 (2011)
12. Kara, A.H., Mahomed, F.M., Ünal, G.: Approximate symmetries and conservation laws with applications. Int. J. Theor. Phys. **38**(9), 2389–2399 (1999)
13. Johnpillai, A.G., Kara, A.H.: Variational formulation of approximate symmetries and conservation laws. Int. J. Theor. Phys. **40**(8), 1501–1509 (2001)
14. Gan, Y., Qu, C.: Approximate conservation laws of perturbed partial differential equations. Nonlinear Dyn. **61**(1–2), 217–228 (2010)
15. Johnpillai, A.G., Kara, A.H., Mahomed, F.M.: Approximate Noether-type symmetries and conservation laws via partial Lagrangians for PDEs with a small parameter. J. Comput. Appl. Math. **223**(1), 508–518 (2009)
16. Kara, A.H., Mahomed, F.M.: Noether-type symmetries and conservation laws via partial Lagrangians. Nonlinear Dyn. **45**(3–4), 367–383 (2006)
17. Naz, R., Mahomed, F.M., Mason, D.P.: Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics. Appl. Math. Comput. **205**(1), 212–230 (2008)
18. Naz, R., Freire, I.L., Naeem, I.: Comparison of Different Approaches to Construct First Integrals for Ordinary Differential Equations. Abstr. Appl. Anal. (2014). doi:10.1155/2014/978636
19. Wang, P.: Perturbation to symmetry and adiabatic invariants of discrete nonholonomic nonconservative mechanical system. Nonlinear Dyn. **68**, 53–62 (2012)
20. de Lagrange, J.L.: Méchanique analitique. Paris: Desaint 512 p.; in 8.; DCC. 4.403, 1 (1788)
21. Euler, L.: Discovery of a new principle of mechanics. Mémoires de lAcademie Royale des science (1750)
22. Pontryagin, L.S.: Mathematical Theory of Optimal Processes. CRC Press, Boca Raton (1987)
23. Legendre, A.M.: Réflexions sur differéntes manières de démostrer la théorie des paralléles ou le théorème sur la somme des trois angles du triangle. Mém. del Acad. des Sci. de Paris **13**, 213–220 (1833)
24. Arnol'd, V.I.: Mathematical methods of classical mechanics. Springer, New York Mémoires de lAcademie des Sciences de Paris **13**, 213–220 (1989)
25. Dorodnitsyn, V., Kozlov, R.: Invariance and first integrals of continuous and discrete Hamiltonian equations. J. Eng. Math. **66**, 253–270 (2010)
26. Naz, R., Mahomed, F.M., Chaudhry, A.: A partial Lagrangian method for dynamical systems. Nonlinear Dyn. (2016). doi:10.1007/s11071-016-2605-8
27. Naz, R., Mahomed, F.M., Chaudhry, A.: A partial Hamiltonian approach for current value Hamiltonian systems. Commun. Nonlinear Sci. Numer. Simul. **19**(10), 3600–3610 (2014)
28. Naz, R., Chaudhry, A., Mahomed, F.M.: Closed-form solutions for the Lucas Uzawa model of economic growth via the partial Hamiltonian approach. Commun. Nonlinear Sci. Numer. Simul. **30**(1), 299–306 (2016)
29. Ünal, G.: Approximate generalized symmetries, normal forms and approximate first integrals. Phys. Lett. A. **269**(1), 13–30 (2000)
30. Ünal, G.: Approximate first integrals of weakly nonlinear, damped-driven oscillators with one degree of freedom. Nonlinear Dyn. **26**(4), 309–329 (2001)
31. Ünal, G., Gorali, G.: Approximate first integrals of a galaxy model. Nonlinear Dyn. **28**(2), 195–211 (2002)
32. Rajasekar, S., Murali, K.: Resonance behaviour and jump phenomenon in a two coupled Duffing–Van der Pol oscillators. Chaos Solitons. Frac. **19**(4), 925–934 (2004)