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Some new integrable systems of two-component fifth-order equations

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Abstract In this work, we develop some fifth-order integrable coupled systems of weight 0 and 1 which possess seventh-order symmetry. We establish four new systems, where in some cases, related recursion operator and bi-Hamiltonian formulations are given. We also investigate the integrability of the developed systems.

Keywords Fifth-order integrable coupled systems · Recursion operator · Bi-Hamiltonian formulations

1 Introduction

Integrable systems of equations that possess sufficiently large number of conservation laws and give rise to multiple soliton solutions play a major role in theoretical physics and in propagation of waves. The work on integrable systems of equations is flourishing because these systems have richer phenomena in scientific applications than the regular systems.

An evolution equation is defined to be integrable in symmetry sense if it admits infinitely many symmetries. Integrable systems are nonlinear differential equations which can be solved analytically. Exactly solvable models and integrable evolution equations in nonlinear science play an essential role in many branches of science and engineering. The useful findings in integrable systems of equations have stimulated much research activity.

The study of constructing integrable systems of equations by using methods, such as recursion operator, symmetries, and bi-Hamiltonian, is an interesting topic of growing interest and has gained large interest recently. Magri [5] studied the connection between conservation laws and symmetries from the geometric point of view, where he proved that some systems admitted two distinct but compatible Hamiltonian structures, now known as bi-Hamiltonian system [2,3,15].

In recent years, studies on fifth-order systems of two-component nonlinear evolution equations have received considerable attention [4,11,12]. Multi-component generalizations of fifth-order Kaup–Kupershmidt equation

$$u_t = u_{5x} + 10uu_{3x} + 25u_xu_{xx} + 20u^2u_x, \tag{1}$$

Sawada–Kotera equation

$$u_t = u_{5x} + 5uu_{3x} + 5u_xu_{xx} + 5u^2u_x, \tag{2}$$

and Kupershmidt equation

$$u_t = u_{5x} + 5u_xu_{3x} + 5u_{xx}^2 - 5u^2u_{3x} - 20uu_xu_{xx} - 5u_x^3 + 5u^4u_x. \tag{3}$$

have been the subject of systematic integrability study. Among these, only five homogeneous systems of two-component cases have been found [7,9,12] so far. Here

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we mention papers pertaining to multi-component generalizations of fifth-order systems only. For the other integrable systems and their properties, we refer the readers to the useful papers [1,4,6,8,13] and the some of the references therein.

So far the only known integrable systems of fifth-order two-component equations are as follows

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -\frac{5}{3}u_{5x} - 10vv_{3x} + 10uu_{3x} + 25u_xu_{xx} - 15v_xv_{xx} - 12u^2u_x \\ +6v^2u_x + 12uvv_x - 6v^2v_x \\ 15v_{5x} - 10vu_{3x} - 30uv_{3x} - 35v_xu_{xx} + 30v_xv_{xx} - 45u_xv_{xx} \\ +6v^2u_x - 6v^2v_x + 12uvu_x + 12u^2v_x \end{pmatrix}, \tag{4}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{5x} + 10uu_{3x} + 25u_xu_{xx} + 20u^2u_x + v^2v_x \\ u_{3x}v + u_{xx}v_x + 8uvu_x + 4u^2v_x \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{8}u_{5x} - 2uu_{3x} - 2u_xu_{xx} - \frac{32}{5}u^2u_x + v_x \\ \frac{9}{8}v_{5x} + 6uv_{3x} + 6u_xv_{xx} + 4u_{xx}v_x + \frac{32}{5}u^2v_x \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_5 + \frac{5}{2}v_5 + 6u_3u + 18u_3v + 12v_3u + 42v_3v + 12u_2u_1 + 24u_2v_1 + 21v_2u_1 \\ +42v_2v_1 + \frac{54}{5}u_1u^2 + \frac{108}{5}u_1uv - 18u_1v^2 + \frac{72}{5}v_1u^2 - \frac{72}{5}v_1uv - 144v_1v^2 \\ \frac{5}{4}u_5 + \frac{7}{2}v_5 + 3u_3u + 6v_3u - 6v_3v + \frac{3}{2}u_2u_1 - 6u_2v_1 - 3v_2u_1 - 33v_2v_1 \\ +\frac{36}{5}u_1v^2 - \frac{18}{5}v_1u^2 - \frac{36}{5}v_1uv + \frac{126}{5}v_1v^2 \end{pmatrix}, \tag{7}$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{5x} - 30vv_{4x} + 5u_xu_{3x} - 5u^2u_{3x} + 15v^2u_{3x} - 75v_xv_{3x} \\ +60uvv_{3x} + 90v^2v_{3x} + 5u_{xx}^2 - 20uu_xu_{xx} + 60vv_xu_{xx} - 45v_{xx}^2 \\ +90vu_xv_{xx} + 90uv_xv_{xx} + 540vv_xv_{xx} + 30u^2vv_{xx} - 180uv^2v_{xx} \\ -90v^3v_{xx} - 5u_x^3 + 45u_xv_x^2 + 60uvu_xv_x - 180v^2u_xv_x + 5u^4u_x \\ -90u^2v^2u_x + 45v^4u_x + 180v_x^3 + 30u^2v_x^2 - 360uvv_x^2 - 270v^2v_x^2 \\ -60u^3vv_x + 180uv^3v_x \\ -9v_{5x} + 10vu_{4x} + 25v_xu_{3x} + 20uvu_{3x} + 30v^2u_{3x} + 15u_xv_{3x} \\ +90v_xv_{3x} + 15u^2v_{3x} + 15v^2v_{3x} + 30u_{xx}v_{xx} + 50vu_xu_{xx} \\ -10u^2vu_{xx} + 50uv_xu_{xx} + 60vv_xu_{xx} + 60uv^2u_{xx} + 30v^3u_{xx} \\ +90v_{xx}^2 + 60uu_xv_{xx} + 60vv_xv_{xx} + 45u_x^2v_x - 20uvu_x^2 \\ +60v^2u_x^2 - 10u^2u_xv_x + 90v^2u_xv_x - 20u^3vu_x + 120uvu_xv_x \\ +60uv^3u_x + 15v_x^3 - 5u^4v_x + 90u^2v^2v_x - 45v^4v_x \end{pmatrix}. \tag{8}$$

Bi-Hamiltonian structures and recursion operators for the aforementioned systems are discussed in [5,7,9–14] and in some of the references therein. Systems (4) and (5) admit a reduction $v = 0$ to the Kaup–Kupershmidt eqnarray. By setting $v = 0$, system (7) reduces to the Sawada–Kotera equation. By setting $v = 0$ the well-known Kupershmidt equation is an obvious reduction of system (8).

2 New homogeneous fifth-order integrable systems

In the literature, all of classified integrable systems are second- and third-order generalization of the KdV and Burgers equations, or equations related to the KdV and Burgers equations. In the case of fifth-order systems, because of the very big number of arbitrary terms that must be considered, the act of classification of such systems is very complicated. Motivated by some existing examples of bi-Hamiltonian two-component generalization of fifth-order equations, we considered a narrow class of fifth-order two-component systems with specific Jordan matrix for integrability.

From a practical point of view, we observed that in second- and third- order integrable systems, when there is a 2-homogeneous integrable equation in a specific Jordan form, then there is certainly at least one $1, 0$-homogeneous system in that Jordan form. Then using the sense of 2-homogeneous fifth-order systems introduced in [5,7,9–14], we aim to develop new integrable $1, 0$-homogeneous systems in the same Jordan form. Our analysis found four new integrable systems, where some of these systems allow us to write Magri schemes which contain the new systems proving it complete integrability. In what follows, we introduce the new fifth- order two-component systems with the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 4u_5 + 5v_5 + 20u_4u_1 + 10u_4v_1 + 40u_1v_4 + 20v_4v_1 + 20u_3u_2 - 40u_3u_1^2 \\ +140u_3u_1v_1 + 70u_3v_2 + 80u_3v_1^2 + 40u_2^2u_1 + 80u_2^2v_1 - 80u_2u_1^3 \\ +360u_2u_1^2v_1 + 400u_2u_1v_2 + 600u_2u_1v_1^2 + 70u_2v_3 + 260u_2v_2v_1 + 200u_2v_1^3 \\ +24u_1^5 - 240u_1^4v_1 - 160u_1^3v_2 + 360u_1^3v_1^2 + 40u_1^2v_3 + 720u_1^2v_2v_1 \\ +1200u_1^2v_1^3 + 220u_1v_3v_1 + 370u_1v_2^2 + 1200u_1v_2v_1^2 + 600u_1v_1^4 + 110v_3v_2 \\ +100v_3v_1^2 + 200v_2^2v_1 + 400v_2v_1^3 \\ 10u_5 + 14v_5 - 40u_4u_1 - 20u_4v_1 - 20u_1v_4 + 20u_3u_2 - 40u_3u_1^2 \\ -40u_3u_1v_1 + 100u_3v_2 - 100u_3v_1^2 - 200u_2^2u_1 - 40u_2^2v_1 \\ +160u_2u_1^3 - 720u_2u_1^2v_1 - 560u_2u_1v_2 - 1200u_2u_1v_1^2 + 100u_2v_3 \\ -400u_2v_2v_1 - 400u_2v_1^3 + 600u_1^4v_1 + 80u_1^3v_2 + 1200u_1^3v_1^2 - 200u_1^2v_3 \\ -360u_1^2v_2v_1 + 360u_1^2v_1^3 - 380u_1v_3v_1 - 320u_1v_2^2 - 600u_1v_2v_1^2 \\ -240u_1v_1^4 - 10v_4v_1 + 140v_3v_2 - 320v_3v_1^2 - 370v_2^2v_1 - 200v_2v_1^3 + 24v_1^5 \end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 4u_{5x} + 4v_{4x}v + 20u_{3x}u_x - 20u_{3x}u^2 - 8u_{3x}v^2 + 16v_{3x}v_x - 8v_{3x}uv + 20u_{xx}^2 \\ -80u_{xx}u_xu - 8u_{xx}v_xv - 6u_{xx}uv^2 + 12v_{xx}^2 - 12v_{xx}u_xv - 24v_{xx}v_xu \\ -4v_{xx}u^2v - 8v_{xx}v^3 - 20u_x^3 - 12u_x^2v^2 - 12u_xv_x^2 - 8u_xv_xuv + 20u_xu^4 \\ +24u_xu^2v^2 + u_xv^4 - 4v_x^2u^2 - 12v_x^2v^2 + 8v_xu^3v + 10v_xuv^3 \\ -2u_{4x}v + 4u_{3x}v_x + 2u_{3x}uv - 2v_{3x}v^2 - 10u_{xx}u_xv - 4u_{xx}v_xu + 8u_{xx}u^2v \\ +4u_{xx}v^3 - 2v_{xx}v_xv + 6v_{xx}uv^2 + 12u_x^2v_x + 16u_x^2uv - 16u_xv_xu^2 \\ +6u_xv_xv^2 - 8u_xu^3v + 8u_xuv^3 + 4v_x^3 - 6v_x^2uv + 4v_xu^4 + 5v_xv^4 \end{pmatrix}, \tag{10}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{5x} + v_{5x} + 2u_x u_{4x} - 2v_x u_{4x} + 6u_x v_{4x} - 6v_x v_{4x} - 16u_{xx}u_{3x} \\ -4v_{xx}u_{3x} - 54u_x^2 u_{3x} - 20u_x v_x u_{3x} - 6v_x^2 u_{3x} - 4u_{xx}v_{3x} \\ -16v_{xx}v_{3x} - 22u_x^2 v_{3x} - 52u_x v_x v_{3x} - 6v_x^2 v_{3x} - 52u_x u_{xx}^2 - 4v_x u_{xx}^2 \\ -32u_x u_{xx} v_{xx} - 16v_x u_{xx} v_{xx} - 12u_x^3 u_{xx} + 4u_x^2 v_x u_{xx} - 4u_x v_x^2 u_{xx} \\ +12v_x^3 u_{xx} - 44u_x v_{xx}^2 - 12v_x v_{xx}^2 - 36u_x^3 v_{xx} + 12u_x^2 v_x v_{xx} \\ -12u_x v_x^2 v_{xx} + 36v_x^3 v_{xx} + 72u_x^5 + 96u_x^4 v_x + 176u_x^3 v_x^2 + 96u_x^2 v_x^3 \\ +72u_x v_x^4 \\ u_{5x} + v_{5x} - 6u_x u_{4x} + 6v_x u_{4x} - 2u_x v_{4x} + 2v_x v_{4x} - 16u_{xx}u_{3x} \\ -4v_{xx}u_{3x} - 6u_x^2 u_{3x} - 52u_x v_x u_{3x} - 22v_x^2 u_{3x} - 4u_{xx}v_{3x} \\ -16v_{xx}v_{3x} - 6u_x^2 v_{3x} - 20u_x v_x v_{3x} - 54v_x^2 v_{3x} - 12u_x u_{xx}^2 \\ -44v_x u_{xx}^2 - 16u_x u_{xx} v_{xx} - 32v_x u_{xx} v_{xx} + 36u_x^3 u_{xx} \\ -12u_x^2 v_x u_{xx} + 12u_x v_x^2 u_{xx} - 36v_x^3 u_{xx} - 4u_x v_{xx}^2 - 52v_x v_{xx}^2 \\ +12u_x^3 v_{xx} - 4u_x^2 v_x v_{xx} + 4u_x v_x^2 v_{xx} - 12v_x^3 v_{xx} + 72u_x^4 v_x \\ +96u_x^3 v_x^2 + 176u_x^2 v_x^3 + 96u_x v_x^4 + 72v_x^5 \end{pmatrix}, \tag{11}$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{5x} + v_{5x} + u_x u_{4x} - v_x u_{4x} + 3u_x v_{4x} - 3v_x v_{4x} + 7u_{xx}u_{3x} \\ +13v_{xx}u_{3x} - 36u_x^2 u_{3x} - 20u_x v_x u_{3x} - 24v_x^2 u_{3x} + 13u_{xx}v_{3x} \\ +7v_{xx}v_{3x} - 28u_x^2 v_{3x} - 28u_x v_x v_{3x} - 24v_x^2 v_{3x} - 28u_x u_{xx}^2 \\ -16v_x u_{xx}^2 - 8u_x u_{xx} v_{xx} - 64v_x u_{xx} v_{xx} - 24u_x^3 u_{xx} + 8u_x^2 v_x u_{xx} \\ -8u_x v_x^2 u_{xx} + 24v_x^3 u_{xx} + 4u_x v_{xx}^2 - 48v_x v_{xx}^2 - 72u_x^3 v_{xx} \\ +24u_x^2 v_x v_{xx} - 24u_x v_x^2 v_{xx} + 72v_x^3 v_{xx} + 72u_x^5 + 96u_x^4 v_x \\ +176u_x^3 v_x^2 + 96u_x^2 v_x^3 + 72u_x v_x^4 \\ u_{5x} + v_{5x} - 3u_x u_{4x} + 3v_x u_{4x} - u_x v_{4x} + v_x v_{4x} + 7u_{xx}u_{3x} \\ +13v_{xx}u_{3x} - 24u_x^2 u_{3x} - 28u_x v_x u_{3x} - 28v_x^2 u_{3x} + 13u_{xx}v_{3x} \\ +7v_{xx}v_{3x} - 24u_x^2 v_{3x} - 20u_x v_x v_{3x} - 36v_x^2 v_{3x} - 48u_x u_{xx}^2 \\ +4v_x u_{xx}^2 - 64u_x u_{xx} v_{xx} - 8v_x u_{xx} v_{xx} + 72u_x^3 u_{xx} - 24u_x^2 v_x u_{xx} \\ +24u_x v_x^2 u_{xx} - 72v_x^3 u_{xx} - 16u_x v_{xx}^2 - 28v_x v_{xx}^2 + 24u_x^3 v_{xx} \\ -8u_x^2 v_x v_{xx} + 8u_x v_x^2 v_{xx} - 24v_x^3 v_{xx} + 72u_x^4 v_x + 96u_x^3 v_x^2 \\ +176u_x^2 v_x^3 + 96u_x v_x^4 + 72v_x^5 \end{pmatrix}. \tag{12}$$

To find the second set of systems, we use a classification that we restricted to the case $\lambda = 0$ homogeneous symmetrically coupled systems. We determine all equations of the form as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} A[u_x, v_x] \\ A[v_x, u_x] \end{pmatrix}. \tag{13}$$

with the class of two-component 0-homogeneous symmetrically coupled systems with undetermined constant coefficients γ_i have the form

$$\begin{aligned} A = &\gamma_1 u_{5x} + \gamma_2 v_{5x} + \alpha_1 u_x u_{4x} + \alpha_2 v_x u_{4x} \\ &+ \alpha_3 u_x v_{4x} + \alpha_4 v_x v_{4x} + \alpha_5 u_{xx} u_{3x} \\ &+ \alpha_6 v_{xx} u_{3x} + \alpha_7 u_x^2 u_{3x} \\ &+ \alpha_8 u_x v_x u_{3x} + \alpha_9 v_x^2 u_{3x} \\ &+ \alpha_{10} u_{xx} v_{3x} + \alpha_{11} v_{xx} v_{3x} \\ &+ \alpha_{12} u_x^2 v_{3x} + \alpha_{13} u_x v_x v_{3x} + \alpha_{14} v_x^2 v_{3x} \\ &+ \alpha_{15} u_x u_{xx}^2 + \alpha_{16} v_x u_{xx}^2 \\ &+ \alpha_{17} u_x v_{xx} u_{xx} + \alpha_{18} v_x v_{xx} u_{xx} \\ &+ \alpha_{19} u_x^3 u_{xx} + \alpha_{20} u_x^2 v_x u_{xx} \\ &+ \alpha_{21} u_x v_x^2 u_{xx} \\ &+ \alpha_{22} v_x^3 u_{xx} + \alpha_{23} u_x v_{xx}^2 \\ &+ \alpha_{24} v_x v_{xx}^2 + \alpha_{25} u_x^3 v_{xx} \\ &+ \alpha_{26} u_x^2 v_x v_{xx} \\ &+ \alpha_{27} u_x v_x^2 v_{xx} \\ &+ \alpha_{28} v_x^3 v_{xx} \\ &+ \alpha_{29} u_x^5 + \alpha_{30} u_x^4 v_x \\ &+ \alpha_{31} u_x^3 v_x^2 + \alpha_{32} u_x^2 v_x^3 \\ &+ \alpha_{33} u_x v_x^4 + \alpha_{34} v_x^5 \end{aligned} \tag{14}$$

possessing an admissible generator of form (13) with the main matrix of these systems is $\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix}$. By a linear change of variables, the matrix (13) can be reduced to following canonical Jordan form $\begin{pmatrix} \gamma_1 + \gamma_2 & 0 \\ 0 & \gamma_1 - \gamma_2 \end{pmatrix}$. Because of properties of symmetric systems, we will restrict our attention to $\gamma_2 \leq \gamma_1$, $\gamma_{1,2} = 0, 1$. Similarly, we will deal with two canonical Jordan form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Imposing compatibility condition among the classes of systems and an arbitrary seventh-order 0-homogeneous, we obtain a system of equations among the undetermined constants. If we separate out the coefficients of powers of u and v in this equation, then in some condition the coefficients of $u_{nx}^m * v_{n'x}^{m'}$ all vanish identically.

Solutions of the compatibility condition are given in the following theorems.

Theorem 2.1 *A coupled fifth-order system of two-component evolution equations of the forms (13) and (14) that possesses a seventh-order generalized symmetry of form (13) with $\gamma_1 = \gamma_2 = 1$ has a lower order symmetry or can be transformed by a linear change of variables to one of the following two systems (11) and (12).*

Theorem 2.2 *Every coupled fifth-order system of two-component evolution equations of form (13) and (14) that possesses a seventh-order generalized symmetry of form (13) with $\gamma_1 = 1, \gamma_2 = 0$ has a lower order symmetry.*

2.1 Integrability of the system (9)

System (9) possesses a symplectic operator as

$$S = \begin{pmatrix} 2D_x & D_x \\ D_x & 2D_x \end{pmatrix}. \tag{15}$$

Second Hamiltonian or symplectic operator for this system is an open question for us.

2.2 Integrability of the system (10)

Our main concern now is to show the integrability of the system (2.2). To achieve this goal, we set

$$R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix} \tag{16}$$

where

$$\begin{aligned} R_1 &= \alpha_1 D_x^6 + \alpha_2 D_x^4 + \alpha_3 D_x^3 + \alpha_4 D_x^2 \\ &\quad + \alpha_5 D_x + \alpha_6 + \alpha_{01} D_x^{-1} \alpha_{07} + \alpha_{02} D_x^{-1} \alpha_{05} \\ R_2 &= \alpha_7 D_x^5 + \alpha_8 D_x^4 + \alpha_9 D_x^3 + \alpha_{10} D_x^2 \\ &\quad + \alpha_{11} D_x + \alpha_{12} + \alpha_{01} D_x^{-1} \alpha_{08} + \alpha_{02} D_x^{-1} \alpha_{06} \\ R_3 &= \alpha_{13} D_x^5 + \alpha_{14} D_x^4 + \alpha_{15} D_x^3 + \alpha_{16} D_x^2 \\ &\quad + \alpha_{17} D_x + \alpha_{18} + \alpha_{03} D_x^{-1} \alpha_{07} + \alpha_{04} D_x^{-1} \alpha_{05} \\ R_4 &= \alpha_{19} D_x^4 + \alpha_{20} D_x^3 + \alpha_{21} D_x^2 + \alpha_{22} D_x + \alpha_{23} \\ &\quad + \alpha_{03} D_x^{-1} \alpha_{08} + \alpha_{04} D_x^{-1} \alpha_{06} \\ \alpha_1 &= 12 \\ \alpha_2 &= 72u_1 - 72u^2 - 30v^2 \\ \alpha_3 &= 180u_2 - 360u_1 u - 18uv^2 - 60v_1 v \end{aligned}$$

$$\alpha_4 = 168u_3 - 480u_2u - 372u_1^2 - 72u_1u^2 - 126u_1v^2 + 108u^4 + 114u^2v^2 - 48uv_1v - 72v_2v - 72v_1^2 + 12v^4$$

$$\alpha_5 = 72u_4 - 360u_3u - 756u_2u_1 - 108u_2u^2 - 126u_2v^2 - 216u_1^2u + 648u_1u^3 + 342u_1uv^2 - 180u_1v_1v + 18u^3v^2 + 180u^2v_1v - 72uv_2v - 72uv_1^2 + 9uv^4 - 48v_3v - 144v_2v_1 + 54v_1v^3$$

$$\alpha_6 = 12u_5 - 144u_4u - 276u_3u_1 - 36u_3u^2 - 36u_3v^2 - 180u_2^2 - 456u_2u_1u + 456u_2u^3 + 210u_2uv^2 - 96u_2v_1v - 72u_1^3 + 888u_1^2u^2 + 132u_1^2v^2 + 108u_1u^2v^2 + 288u_1uv_1v - 84u_1v_2v - 84u_1v_1^2 + 18u_1v^4 - 48u^6 - 84u^4v^2 + 48u^3v_1v + 156u^2v_2v + 156u^2v_1^2 - 30u^2v^4 - 72uv_3v - 216uv_2v_1 + 78uv_1v^3 - 12v_4v - 48v_3v_1 - 36v_2^2 + 24v_2v^3 + 36v_1^2v^2$$

$$\alpha_7 = 12v$$

$$\alpha_8 = -24uv + 60v_1$$

$$\alpha_9 = -48u_1v - 24u^2v - 96uv_1 + 120v_2 - 30v^3$$

$$\alpha_{10} = -60u_2v - 72u_1uv - 144u_1v_1 + 48u^3v - 72u^2v_1 - 144uv_2 + 42uv^3 + 120v_3 - 150v_1v^2$$

$$\alpha_{11} = -72u_3v - 72u_2uv - 120u_2v_1 - 84u_1^2v + 216u_1u^2v - 144u_1uv_1 - 144u_1v_2 + 54u_1v^3 + 12u^4v + 96u^3v_1 - 72u^2v_2 + 30u^2v^3 - 96uv_3 + 156uv_1v^2 + 60v_4 - 162v_2v^2 - 192v_1^2v + 12v^5$$

$$\alpha_{12} = -48u_4v - 24u_3uv - 72u_3v_1 - 228u_2u_1v + 180u_2u^2v - 72u_2uv_1 - 60u_2v_2 + 54u_2v^3 + 240u_1^2uv - 84u_1^2v_1 + 24u_1u^3v + 216u_1u^2v_1 - 72u_1uv_2 + 66u_1uv^3 - 48u_1v_3 + 114u_1v_1v^2 - 24u^5v + 12u^4v_1 + 48u^3v_2 - 42u^3v^3 - 24u^2v_3 + 66u^2v_1v^2 - 24uv_4 + 114uv_2v^2 + 144uv_1^2v - 15uv^5 + 12v_5 - 78v_3v^2 - 276v_2v_1v - 72v_1^3 + 66v_1v^4$$

$$\alpha_{13} = -6v$$

$$\alpha_{14} = 6uv + 12v_1$$

$$\alpha_{15} = -36u_1v + 30u^2v - 12uv_1 + 15v^3$$

$$\alpha_{16} = -54u_2v + 150u_1uv + 84u_1v_1 - 30u^3v - 60u^2v_1 + 21uv^3 + 24v_1v^2$$

$$\alpha_{17} = -30u_3v + 144u_2uv + 24u_2v_1 + 78u_1^2v - 54u_1u^2v - 180u_1uv_1 + 45u_1v^3 - 24u^4v + 60u^3v_1 - 24u^2v^3 + 30uv_1v^2 + 18v_2v^2 - 30v_1^2v - 6v^5$$

$$\alpha_{18} = -6u_4v + 66u_3uv + 36u_3v_1 + 90u_2u_1v - 36u_2u^2v - 108u_2uv_1 + 18u_2v^3 + 48u_1^2uv + 24u_1^2v_1 - 144u_1u^3v - 72u_1u^2v_1 - 72u_1uv^3 + 18u_1v_1v^2 + 24u^5v + 48u^4v_1 - 12u^3v^3 - 48u^2v_1v^2 + 18uv_2v^2 - 42uv_1^2v - 12uv^5 + 6v_3v^2 + 30v_2v_1v + 12v_1^3 - 18v_1v^4$$

$$\alpha_{19} = -6v^2$$

$$\alpha_{20} = 18uv^2 - 12v_1v$$

$$\alpha_{21} = 12u_1v^2 - 6u^2v^2 + 18uv_1v - 36v_2v + 36v_1^2 + 15v^4$$

$$\alpha_{22} = 18u_2v^2 + 18u_1uv^2 + 36u_1v_1v - 18u^3v^2 + 54uv_2v - 72uv_1^2 - 9uv^4 - 24v_3v + 36v_2v_1 + 54v_1v^3$$

$$\alpha_{23} = 18u_3v^2 - 30u_2v_1v + 30u_1^2v^2 - 66u_1u^2v^2 - 42u_1uv_1v + 12u_1v_2v + 12u_1v_1^2 - 15u_1v^4 + 12u^4v^2 + 18u^3v_1v - 6u^2v_2v + 12u^2v_1^2 - 6u^2v^4 + 18uv_3v - 36uv_2v_1 - 27uv_1v^3 - 6v_4v + 12v_3v_1 + 33v_2v^3 - 6v_1^2v^2 - 6v^6$$

$$\alpha_{01} = -24u_5 - 120u_3u_1 + 120u_3u^2 + 48u_3v^2 - 120u_2^2 + 480u_2u_1u + 36u_2uv^2 + 48u_2v_1v + 120u_1^3 + 72u_1^2v^2 - 120u_1u^4 - 144u_1u^2v^2 + 48u_1uv_1v + 72u_1v_2v + 72u_1v_1^2 - 6u_1v^4 - 48u^3v_1v + 24u^2v_2v + 24u^2v_1^2 + 48uv_3v + 144uv_2v_1 - 60uv_1v^3 - 24v_4v - 96v_3v_1 - 72v_2^2 + 48v_2v^3 + 72v_1^2v^2$$

$$\alpha_{02} = -3u_1$$

$$\alpha_{03} = 12u_4v - 12u_3uv - 24u_3v_1 + 60u_2u_1v - 48u_2u^2v + 24u_2uv_1 - 24u_2v^3 - 96u_1^2uv - 72u_1^2v_1 + 48u_1u^3v + 96u_1u^2v_1 - 48u_1uv^3 - 36u_1v_1v^2 - 24u^4v_1 - 36uv_2v^2 + 36uv_1^2v + 12v_3v^2 + 12v_2v_1v - 24v_1^3 - 30v_1v^4$$

$$\alpha_{04} = -3v_1$$

$$\alpha_{05} = 8u_4 + 8v_3v + 40u_2u_1 - 40u_2u^2 - 4u_2v^2 + 24v_2v_1 - 16v_2uv - 40u_1^2u - 8u_1v_1v - 16v_1^2u - 8v_1u^2v - 4v_1v^3 + 8u^5 + 8u^3v^2 + 2uv^4$$

$$\alpha_{06} = -8u_3v - 16u_2uv - 8v_2v^2 - 12u_1^2v + 8u_1u^2v + 4u_1v^3 - 8v_1^2v + 4u^4v + 4u^2v^3 + v^5$$

$$\alpha_{07} = u$$

$$\alpha_{08} = \frac{v}{2}$$

This shows that the system (10) passes the integrability test.

2.3 Integrability of system (11)

We proceed as before to show the integrability of the system (11). By change of dependent variables

$$u \to \frac{1}{2}\int (w-z)\mathrm{d}x \tag{17}$$

$$v \to \frac{1}{2}\int (w+z)\mathrm{d}x \tag{18}$$

system (11) can be written in its canonical form as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} w_{5x} - 2zz_{4x} - 10w_x w_{3x} - 20w^2 w_{3x} - 2z^2 w_{3x} - 8z_x z_{3x} \\ -8wzz_{3x} - 10w_{xx}^2 - 80ww_x w_{xx} - 8zz_x w_{xx} - 6z_{xx}^2 \\ -12zw_x z_{xx} - 24wz_x z_{xx} + 8w^2 zz_{xx} + 4z^3 z_{xx} - 20w_x^3 - 12w_x z_x^2 \\ +16wzw_x z_x + 80w^4 w_x + 48w^2 z^2 w_x + 4z^4 w_x + 8w^2 z_x^2 \\ +12z^2 z_x^2 + 32w^3 zz_x + 16wz^3 z_x \\ 4zw_{4x} + 4z_x w_{3x} - 16wzw_{3x} - 8z^2 z_{3x} - 40zw_x w_{xx} \\ -16wz_x w_{xx} - 16w^2 zw_{xx} - 8z^3 w_{xx} - 32zz_x z_{xx} - 12w_x^2 z_x \\ -32wzw_x^2 - 16w^2 w_x z_x - 24z^2 w_x z_x + 64w^3 zw_x + 32wz^3 w_x \\ -8z_x^3 + 16w^4 z_x + 48w^2 z^2 z_x + 20z^4 z_x \end{pmatrix} \tag{19}$$

Proposition *The infinite hierarchy of the system (19) can be written in two different ways*

$$\begin{pmatrix} w_t \\ z_t \end{pmatrix} = J \begin{pmatrix} \delta_w \\ \delta_z \end{pmatrix} \int \rho_1 \,\mathrm{d}x = K \begin{pmatrix} \delta_w \\ \delta_z \end{pmatrix} \int \rho_0 \,\mathrm{d}x \tag{20}$$

with the compatible pair of Hamiltonian operators

$$J = \begin{pmatrix} D_x & 0 \\ 0 & 2D_x \end{pmatrix}, \quad K = \begin{pmatrix} K_1^1 & K_2^1 \\ K_3^1 & K_4^1 \end{pmatrix} \tag{21}$$

where

$$\begin{aligned}
K_1^1 &= D_x^7 + \omega_1 D_x^5 + D_x^5 \omega_1 + \omega_2 D_x^3 + D_x^3 \omega_2 \\
&\quad + \omega_3 D_x + D_x \omega_3 + 8w_x D_x^{-1} w_t \\
&\quad + 8w_t D_x^{-1} w_x \\
K_2^1 &= D_x^6 \omega_4 + D_x^5 \omega_5 + D_x^4 \omega_6 + D_x^3 \omega_7 \\
&\quad + D_x^2 \omega_8 + D_x \omega_9 + \omega_{10} \\
&\quad + 8w_x D_x^{-1} z_t \\
&\quad + 8w_t D_x^{-1} z_x \\
K_3^1 &= -\omega_4 D_x^6 + \omega_5 D_x^5 - \omega_6 D_x^4 \\
&\quad + \omega_7 D_x^3 - \omega_8 D_x^2 + \omega_9 D_x - \omega_{10} + 8z_x D_x^{-1} w_t \\
&\quad + 8z_t D_x^{-1} w_x \\
k_4^1 &= \omega_{11} D_x^5 + D_x^5 \omega_{11} + \omega_{12} D_x^3 \\
&\quad + D_x^3 \omega_{12} + \omega_{13} D_x + D_x \omega_{13} + 8z_x D_x^{-1} z_t \\
&\quad + 8z_t D_x^{-1} z_x
\end{aligned} \tag{22}$$

and the coefficients satisfy

$$\begin{aligned}
\omega_1 &= -6w_x - 12w^2 - 2z^2 \\
\omega_2 &= 16w_{3x} + 40ww_{xx} + 8zz_{xx} + 58w_x^2 \\
&\quad + 24w^2 w_x + 12z^2 w_x + 8z_x^2 + 16wzz_x \\
&\quad + 72w^4 + 40w^2 z^2 + 18z^4 \\
\omega_3 &= -10w_{5x} - 24ww_{4x} - 4zz_{4x} - 100w_x w_{3x} \\
&\quad - 24w^2 w_{3x} - 12z^2 w_{3x} \\
&\quad - 16z_x z_{3x} \\
&\quad - 84w_{xx}^2 - 64ww_x w_{xx} - 64zz_x w_{xx} \\
&\quad - 128w^3 w_{xx} - 64wz^2 w_{xx} - 12z_{xx}^2 \\
&\quad - 56zw_x z_{xx} - 16w^2 zz_{xx} \\
&\quad - 152z^3 z_{xx} - 48w_x^3 - 704w^2 w_x^2 - 80z^2 w_x^2 \\
&\quad - 56w_x z_x^2 - 288wzw_x z_x - 96z^4 w_x \\
&\quad - 16w^2 z_x^2 - 216z^2 z_x^2 \\
&\quad - 64w^3 zz_x + 96wz^3 z_x \\
&\quad - 128w^6 - 128w^4 z^2 - 96w^2 z^4 - 16z^6
\end{aligned} \tag{23}$$

$$\begin{aligned}
\omega_4 &= -4z \\
\omega_5 &= 4z_x - 16wz \\
\omega_6 &= +48zw_x + 16wz_x + 32w^2 z \\
\omega_7 &= -72zw_{xx} - 32w_x z_x - 160wzw_x \\
&\quad - 32w^2 z_x + 96z^2 z_x + 128w^3 z \\
\omega_8 &= +40zw_{3x} + 40z_x w_{xx} + 192wzw_{xx} \\
&\quad +208zw_x^2 + 96ww_x z_x - 576w^2 zw_x \\
&\quad - 320zz_x^2 - 128w^3 z_x - 64w^4 z
\end{aligned}$$

$$
\begin{aligned}
\omega_9 = &-8zw_{4x} - 128wzw_{3x} - 104z^2z_{3x} \\
&- 240zw_xw_{xx} - 96wz_xw_{xx} + 480w^2zw_{xx} \\
&- 96z^3w_{xx} - 152zz_xz_{xx} + 576wz^2z_{xx} - 112w_x^2z_x \\
&+ 640wzw_x^2 + 192w^2w_xz_x \\
&- 192z^2w_xz_x + 384w^3zw_x + 384wz^3w_x \\
&+ 160z_x^3 + 448wzz_x^2 + 64w^4z_x \\
&- 768w^2z^2z_x - 96z^4z_x - 256w^5z \\
\omega_{10} = &+8z_xw_{4x} + 32wzw_{4x} + 40z^2z_{4x} + 32wz_xw_{3x} \\
&- 128w^2zw_{3x} + 48z^3w_{3x} \\
&+ 224zz_xz_{3x} - 288wz^2z_{3x} - 80w_xz_xw_{xx} \\
&- 320wzw_xw_{xx} - 288w^2z_xw_{xx} \\
&- 128w^3zw_{xx} - 192wz^3w_{xx} + 304z^2z_xw_{xx} \\
&+ 120zz_{xx}^2 + 64z^2w_xz_{xx} \\
&+ 192z_x^2z_{xx} - 1376wzz_xz_{xx} + 384w^2z^2z_{xx} \\
&+ 48z^4z_{xx} - 256ww_x^2z_x \\
&- 256w^2zw_x^2 + 64z^3w_x^2 + 160zw_xz_x^2 \\
&- 128w^3w_xz_x - 192wz^2w_xz_x \\
&+ 512w^4zw_x - 512wz_x^3 \\
&+ 1216w^2zz_x^2 + 272z^3z_x^2 + 256w^5z_x \\
\omega_{11} = &-12z^2 \\
\omega_{12} = &+100zz_{xx} - 72z^2w_x \\
&+ 80z_x^2 - 96wzz_x + 144w^2z^2 + 36z^4 \\
\omega_{13} = &-68zz_{4x} + 8z^2w_{3x} - 292z_xz_{3x} \\
&+ 176wzz_{3x} + 232zz_xw_{xx} + 64wz^2w_{xx} \\
&- 264z_{xx}^2 + 512zw_xz_{xx} + 816wz_xz_{xx} \\
&- 896w^2zz_{xx} - 88z^3z_{xx} - 64z^2w_x^2 \\
&+ 280w_xz_x^2 - 1056wzw_xz_x + 96z^4w_x \\
&- 752w^2z_x^2 - 380z^2z_x^2 + 768w^3zz_x + 96wz^3z_x \\
&- 384w^4z^2 - 192w^2z^4 - 32z^6
\end{aligned}
$$

The first few conserved densities of the hierarchy are listed as follows

$$
\begin{aligned}
\rho_0 &= \alpha \\
\rho_1 &= 2w^2 + z^2 \\
\rho_2 &= +3ww_{4x} + 10w^2w_{3x} \\
&\quad + 6z^2w_{3x} - 20w^3w_{xx} - 6wz^2w_{xx} \\
&\quad - 18w^2zz_{xx} - 4z^3z_{xx} \\
&\quad - 18w^2z_x^2 + 16w^3zz_x + 24wz^3z_x \\
&\quad + 16w^6 + 24w^4z^2 + 12w^2z^4 + 2z^6 \\
&\quad \vdots
\end{aligned} \tag{24}
$$

These densities are sufficient to write two Magri schemes with the same Hamiltonian operators such that one of them contains the new system, and this confirms the integrability of the system (27).

2.4 Integrability of system (12)

In a manner parallel to the analysis presented earlier, and to prove the integrability of the system (12), we use the change of dependent variables

$$u \to \frac{1}{2}\int (w - z)\mathrm{d}x \tag{25}$$

$$v \to \frac{1}{2}\int (w + z)\mathrm{d}x \tag{26}$$

which carries the system (12) to its canonical form as

$$
\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix}
w_{5x} - zz_{4x} + 10w_xw_{3x} - 20w^2w_{3x} - 8z^2w_{3x} - 4z_xz_{3x} - 2wzz_{3x} \\
+10w_{xx}^2 - 80ww_xw_{xx} - 32zz_xw_{xx} - 3z_{xx}^2 - 18zw_xz_{xx} - 6wz_xz_{xx} \\
+16w^2zz_{xx} + 8z^3z_{xx} - 20w_x^3 - 18w_xz_x^2 + 32wzw_xz_x + 80w^4w_x \\
+48w^2z^2w_x + 4z^4w_x + 16w^2z_x^2 + 24z^2z_x^2 + 32w^3zz_x + 16wz^3z_x \\
2zw_{4x} + 2z_xw_{3x} - 4wzw_{3x} - 2z^2z_{3x} + 20zw_xw_{xx} - 4wz_xw_{xx} \\
-32w^2zw_{xx} - 16z^3w_{xx} - 8zz_xz_{xx} + 12w_x^2z_x - 64wzw_x^2 \\
-48z^2w_xz_x + 64w^3zw_x - 32w^2w_xz_x + 32wz^3w_x - 2z_x^3 \\
+16w^4z_x + 48w^2z^2z_x + 20z^4z_x
\end{pmatrix} \tag{27}
$$

Proposition *The infinite hierarchy of system (27) can be written in not just one but two different ways*

$$\begin{pmatrix} w_t \\ z_t \end{pmatrix} = J \begin{pmatrix} \delta_w \\ \delta_z \end{pmatrix} \int \rho_1 \, \mathrm{d}x = K \begin{pmatrix} \delta_w \\ \delta_z \end{pmatrix} \int \rho_1 \, \mathrm{d}x \tag{28}$$

with the compatible pair of Hamiltonian operators

$$J = \begin{pmatrix} D_x & 0 \\ 0 & 2D_x \end{pmatrix}, \quad K^2 = \begin{pmatrix} K_1^2 & K_2^2 \\ K_3^2 & K_4^2 \end{pmatrix} \tag{29}$$

where

$$\begin{aligned}
K_1^2 &= D_x^7 + \psi_1 D_x^5 + D_x^5 \psi_1 \\
&\quad + \psi_2 D_x^3 + D_x^3 \psi_2 + \psi_3 D_x \\
&\quad + D_x \psi_3 + 8w_x D_x^{-1} w_t + 8w_t D_x^{-1} w_x \\
K_2^2 &= D_x^6 \psi_4 + D_x^5 \psi_5 + D_x^4 \psi_6 + D_x^3 \psi_7 \\
&\quad + D_x^2 \psi_8 + D_x \psi_9 + \psi_{10} + 8w_x D_x^{-1} z_t \\
&\quad + 8w_t D_x^{-1} z_x \\
K_3^2 &= -\psi_4 D_x^6 + \psi_5 D_x^5 - \psi_6 D_x^4 \\
&\quad + \psi_7 D_x^3 - \psi_8 D_x^2 + \psi_9 D_x - \psi_{10} \\
&\quad + 8z_x D_x^{-1} w_t + 8z_t D_x^{-1} w_x \\
K_4^2 &= \psi_{11} D_x^5 + D_x^5 \psi_5 + \psi_{12} D_x^3 + D_x^3 \psi_{12} \\
&\quad + \psi_{13} D_x + D_x \psi_{13} + 8z_x D_x^{-1} z_t \\
&\quad + 8z_t D_x^{-1} z_x
\end{aligned} \tag{30}$$

where the coefficients satisfy

$$\begin{aligned}
\psi_1 &= 6w_x - 12w^2 - 5z^2 \\
\psi_2 &= -16w_{3x} + 40ww_{xx} + 26zz_{xx} + 58w_x^2 \\
&\quad - 24w^2 w_x - 12z^2 w_x + 26z_x^2 \\
&\quad + 20wzz_x + 72w^4 + 52w^2 z^2 + 18z^4
\end{aligned} \tag{31}$$

$$\begin{aligned}
\psi_3 &= 10w_{5x} - 24ww_{4x} - 16zz_{4x} - 100w_x w_{3x} \\
&\quad + 24w^2 w_{3x} + 12z^2 w_{3x} - 64z_x z_{3x} \\
&\quad - 12wzz_{3x} - 84w_{xx}^2 + 64ww_x w_{xx} \\
&\quad + 28zz_x w_{xx} - 128w^3 w_{xx} - 40wz^2 w_{xx} \\
&\quad - 48z_{xx}^2 - 16zw_x z_{xx} - 36wz_x z_{xx} \\
&\quad - 88w^2 zz_{xx} - 134z^3 z_{xx} + 48w_x^3 \\
&\quad - 704w^2 w_x^2 - 104z^2 w_x^2 - 16w_x z_x^2 \\
&\quad - 288wzw_x z_x + 24z^4 w_x - 88w^2 z_x^2 \\
&\quad - 216z^2 z_x^2 - 128w^3 zz_x - 24wz^3 z_x \\
&\quad - 128w^6 - 128w^4 z^2 - 96w^2 z^4 - 16z^6 \\
\psi_4 &= -2z \\
\psi_5 &= 2z_x - 4wz \\
\psi_6 &= -24zw_x + 4wz_x + 40w^2 z \\
\psi_7 &= +36zw_{xx} + 28w_x z_x - 200wzw_x \\
&\quad - 40w^2 z_x + 120z^2 z_x + 80w^3 z \\
\psi_8 &= -20zw_{3x} - 8z_x w_{xx} + 192wzw_{xx} \\
&\quad + 104zw_x^2 + 120ww w_x z_x - 144w^2 zw_x \\
&\quad - 80w^3 z_x - 424zz_x^2 - 128w^4 z \\
\psi_9 &= +4zw_{4x} + 12z_x w_{3x} - 88wzw_{3x} \\
&\quad - 154z^2 z_{3x} - 120zw_x w_{xx} - 72wz_x w_{xx} \\
&\quad + 96w^2 zw_{xx} + 48z^3 w_{xx} - 238zz_x z_{xx} \\
&\quad + 360wz^2 z_{xx} + 16w_x^2 z_x - 128wzw_x^2 \\
&\quad - 96w^2 w_x z_x + 312z^2 w_x z_x \\
&\quad + 768w^3 zw_x - 96wz^3 w_x + 224z_x^3 + 232wzz_x^2 \\
&\quad + 128w^4 z_x - 672w^2 z^2 z_x - 192z^4 z_x - 256w^5 z
\end{aligned} \tag{32}$$

$$\begin{aligned}
\psi_{10} &= +8z_x w_{4x} + 16wzw_{4x} + 62z^2 z_{4x} + 16wz_x w_{3x} \\
&\quad - 32w^2 zw_{3x} - 24z^3 w_{3x} \\
&\quad - 180wz^2 z_{3x} + 364zz_x z_{3x} \\
&\quad + 80w_x z_x w_{xx} + 160wzw_x w_{xx} - 296z^2 z_x w_{xx} \\
&\quad - 256w^3 zw_{xx} - 192w^2 z_x w_{xx} + 48wz^3 w_{xx} \\
&\quad + 186zz_{xx}^2 - 176z^2 w_x z_{xx} \\
&\quad + 348z_x^2 z_{xx} - 812wzz_x z_{xx} \\
&\quad + 336w^2 z^2 z_{xx} + 96z^4 z_{xx} - 64ww_x^2 z_x \\
&\quad - 512w^2 zw_x^2 + 128z^3 w_x^2 - 656zw_x z_x^2 \\
&\quad - 256w^3 w_x z_x + 512w^4 zw_x \\
&\quad + 912wz^2 w_x z_x - 272wz_x^3 \\
&\quad + 1136w^2 zz_x^2 + 544z^3 z_x^2 + 256w^5 z_x \\
\psi_{11} &= -12z^2 \\
\psi_{12} &= +190zz_{xx} - 144z^2 w_x \\
&\quad + 74z_x^2 - 120wzz_x + 144w^2 z^2 + 36z^4 \\
\psi_{13} &= -146zz_{4x} + 88z^2 w_{3x} - 514z_x z_{3x} \\
&\quad + 244wzz_{3x} + 560zz_x w_{xx} - 224wz^2 w_{xx} \\
&\quad - 480z_{xx}^2 + 904zw_x z_{xx} + 1092wz_x z_{xx} \\
&\quad - 824w^2 zz_{xx} - 160z^3 z_{xx} + 512w_x z_x^2 \\
&\quad - 520z^2 w_x^2 - 1632wzw_x z_x + 864w^2 z^2 w_x \\
&\quad + 192z^4 w_x - 632w^2 z_x^2 - 464z^2 z_x^2 \\
&\quad + 672w^3 zz_x + 192wz^3 z_x \\
&\quad - 384w^4 z^2 - 192w^2 z^4 - 32z^6
\end{aligned}$$

The first few conserved densities of the system (27) *are listed as follows*

$$\rho_0 = \alpha$$

$$\rho_1 = 2w^2 + z^2$$

$$\rho_2 = +3ww_{4x} - 10w^2w_{3x} + 3z^2w_{3x} - 20w^3w_{xx} - 24wz^2w_{xx} + 18w^2zz_{xx} - z^3z_{xx} + 18w^2z_x^2 + 32w^3zz_x + 48wz^3z_x + 16w^6 + 24w^4z^2 + 12w^2z^4 + 2z^6 \\ \vdots \tag{33}$$

These densities suffice to write two Magri schemes with same Hamiltonian operators that one of them contains the new system, and this in turn emphasizes the integrability of the system (27).

3 Concluding remarks

In this work, we established four fifth-order integrable coupled systems of weight 0 and 1. We examined the related recursion operator and bi-Hamiltonian formulations for the developed systems. We used the compatible pair of Hamiltonian operators to formally prove the integrability of the developed systems. The obtained results will add valuable findings to the existing integrable systems of fifth-order two-component equation. It is expected that other works will be conducted for recovering the scientific features of these systems of equations.

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