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ORIGINAL PAPER

Analytical study of solitons in non-Kerr nonlinear negative-index materials

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Abstract We study the dynamics of optical solitons in negative-index materials with non-Kerr nonlinearity and third-order dispersion. Three types of non-Kerr law nonlinearities are considered. They are power law, parabolic law and dual-power law. With the help of the extended trial equation method, various families of solitons including bright, dark and singular solitons are derived. The presented results could provide a method and technique in ultra-short optical soliton control in various kinds of non-Kerr law nonlinear negative-index materials.

Keywords Solitons · Negative-index materials · Extended trial equation method

1 Introduction

Through independently engineering the shape, structure and size of metamaterial (an artificial electromagnetic material), the negative-index material can be achieved. The controllable magnetic permeability and dielectric permittivity make negative-index material an ideal medium for the propagation of new kinds of soliton [1–5]. While ultra-short optical pulse propagates in nonlinear negative-index material, it will be affected with additional non-Kerr law nonlinearity as well as higher-order dispersion [1]. At present, the theoretical study on optical soliton in various kinds of non-Kerr law nonlinear negative-index materials is at its infancy, and this paper will conduct a study on transmission properties of ultra-short optical solitons in three typical non-Kerr law nonlinear negative-index materials. They are power law, parabolic law and dual-power law.

The dynamics of solitons in negative-index material with non-Kerr nonlinearity and third-order dispersion is governed by the nonlinear Schrödinger's equation (NLSE) which in the dimensionless form is given by [1]

$$\begin{aligned} i q_t + a q_{xx} + b q_{xt} + c F(|q|^2) q = \\ -i\lambda q_x - is\left(|q|^2 q\right)_x - i\mu\left(|q|^2\right)_x q \\ -i\theta |q|^2 q_x - i\gamma q_{xxx} - \theta_1\left(|q|^2 q\right)_{xx} - \theta_2 |q|^2 q_{xx} \\ -\theta_3 q^2 q^*_{xx}, \end{aligned} \tag{1}$$

where $q(x,t)$ is the complex field amplitude. a, b and c are the coefficients of group velocity dispersion, spatiotemporal dispersion and non-Kerr nonlinearity, and λ, s, μ, θ and γ account for the inter-modal dispersion (IMD), self-steepening (SS), Raman effect, nonlinear

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dispersion (ND) and third-order dispersion (TOD). The last three terms appear in the context of negative-index material [1–5].

Very recently, some explicit soliton solutions are constructed based on the Riccati equation expansion method and the ansatz scheme [1]. In this work, we will report some new exact soliton solutions along with the corresponding existence conditions by extended trial equation method [6–10]. Therefore, this work is an extension of our previous results.

2 Exact solitons

In order to solve Eq. (1), we use the following wave transformation [11–16]

$$q(x,t)=U(\xi)\mathrm{e}^{i\Phi(x,t)} \tag{2}$$

where $U(\xi)$ represents the shape of the pulse and

$$\xi=x-vt, \tag{3}$$

$$\Phi(x,t)=-\kappa x+\omega t+\theta. \tag{4}$$

In Eq. (2), the function $\Phi(x,t)$ is the phase component of the soliton. Then, in Eq. (4), κ is the soliton frequency, while ω is the wave number of the soliton and θ is the phase constant. Finally in Eq. (3), v is the velocity of the soliton. After substituting (2) in (1) and decomposing into real and imaginary parts lead to

$$\begin{aligned}&(a-bv+3\kappa\gamma)U''\\&-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa+\gamma\kappa^3\right)U+cF(U^2)U\\&+\left(s\kappa+\theta\kappa-\theta_1\kappa^2-\theta_2\kappa^2-\theta_3\kappa^2\right)U^3\\&+6\theta_1U(U')^2+(3\theta_1+\theta_2+\theta_3)U^2U''=0,\end{aligned} \tag{5}$$

and

$$\begin{aligned}&(-v-2a\kappa+b\omega+b\kappa v+\lambda-3\gamma\kappa^2)U'\\&+(3s+2\mu+\theta-2\kappa(3\theta_1+\theta_2-\theta_3))U^2U'\\&+\gamma U'''=0.\end{aligned} \tag{6}$$

The imaginary part Eq. (6) implies the relations

$$v=-\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}, \tag{7}$$

$$\gamma=0, \tag{8}$$

$$3s+2\mu+\theta-2\kappa(3\theta_1+\theta_2-\theta_3)=0, \tag{9}$$

and

$$\begin{aligned}&(a-bv)U''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)U\\&+cF(U^2)U+\left(s\kappa+\theta\kappa-\kappa^2\theta_1-\kappa^2\theta_2-\kappa^2\theta_3\right)U^3\\&+(3\theta_1+\theta_2+\theta_3)U^2U''+6\theta_1U\left(U'\right)^2=0.\end{aligned} \tag{10}$$

To obtain the analytic solution, the transformations $\theta_1=0,\theta_2=-\theta_3$ and $s=-\theta$ are applied in Eq. (10) and give

$$\begin{aligned}&(a-bv)U''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)U\\&+cF(U^2)U=0,\end{aligned} \tag{11}$$

where

$$\kappa=\frac{\theta-\mu}{2\theta_3}. \tag{12}$$

2.1 Kerr law

For Kerr law nonlinearity

$$F(u)=u, \tag{13}$$

so that (1) reduces to

$$\begin{aligned}&iq_t+aq_{xx}+bq_{xt}+c|q|^2q=\\&-i\lambda q_x-is\left(|q|^2q\right)_x-i\mu\left(|q|^2\right)_xq-i\theta|q|^2q_x\\&-i\gamma q_{xxx}-\theta_1\left(|q|^2q\right)_{xx}-\theta_2|q|^2q_{xx}\\&-\theta_3q^2q^*_{xx},\end{aligned} \tag{14}$$

and Eq. (11) simplifies to

$$\begin{aligned}&(a-bv)U''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)\\&U+cU^3=0.\end{aligned} \tag{15}$$

In this subsection, we would like to extend the extended trial equation method [6–10] to solve the NLSE with Kerr law nonlinearity. Suppose that the solution of Eq. (15) can be given by

$$U=\sum_{i=0}^{\varsigma}\tau_i\Psi^i, \tag{16}$$

where

$$\begin{aligned}(\Psi')^2=\Lambda(\Psi)&=\frac{\Phi(\Psi)}{\Upsilon(\Psi)}\\&=\frac{\mu_\sigma\Psi^\sigma+\cdots+\mu_1\Psi+\mu_0}{\chi_\rho\Psi^\rho+\cdots+\chi_1\Psi+\chi_0}.\end{aligned} \tag{17}$$

Using the relations (16) and (17), we can derive the terms $(U')^2$ and U'' as

$$(U')^2 = \frac{\Phi(\Psi)}{\Upsilon(\Psi)}\left(\sum_{i=0}^{\varsigma} i\tau_i\Psi^{i-1}\right)^2, \tag{18}$$

and

$$U'' = \frac{\Phi'(\Psi)\Upsilon(\Psi)-\Phi(\Psi)\Upsilon'(\Psi)}{2\Upsilon^2(\Psi)}\left(\sum_{i=0}^{\varsigma} i\tau_i\Psi^{i-1}\right) + \frac{\Phi(\Psi)}{\Upsilon(\Psi)}\left(\sum_{i=0}^{\varsigma} i(i-1)\tau_i\Psi^{i-2}\right), \tag{19}$$

where $\Phi(\Psi)$ and $\Upsilon(\Psi)$ are polynomials of Ψ. We can reduce Eq. (17) to the elementary integral form as follows:

$$\pm(\xi-\xi_0) = \int\frac{\mathrm{d}\Psi}{\sqrt{\Lambda(\Psi)}} = \int\sqrt{\frac{\Upsilon(\Psi)}{\Phi(\Psi)}}\mathrm{d}\Psi. \tag{20}$$

Substituting Eqs. (16) and (19) in Eq. (15), and using the balance principle, we determine a relation of σ, ρ and ς as

$$\sigma = \rho + 2\varsigma + 2. \tag{21}$$

If we take $\sigma = 4$, $\rho = 0$ and $\varsigma = 1$ in Eq. (21), then

$$U = \tau_0 + \tau_1\Psi, \tag{22}$$

$$U'' = \frac{\tau_1(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0}, \tag{23}$$

where $\mu_4 \neq 0$, $\chi_0 \neq 0$. Substituting Eqs. (22) and (23) in Eq. (15), collecting the coefficients of Ψ and solving the resulting system, we have

$$\mu_0 = \mu_0, \quad \mu_1 = \mu_1, \quad \mu_2 = \frac{\mu_1\tau_1}{2\tau_0} - \frac{2c\tau_0^2\chi_0}{a-bv},$$
$$\mu_3 = -\frac{2c\tau_0\tau_1\chi_0}{a-bv}, \quad \mu_4 = -\frac{c\tau_1^2\chi_0}{2(a-bv)},$$
$$\chi_0 = \chi_0, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1,$$
$$\omega = \frac{\mu_1\tau_1(bv-a) - 2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}. \tag{24}$$

Substituting the solution set (24) in Eqs. (17) and (20), we find that

$$\pm(\xi-\xi_0) = W\int\frac{\mathrm{d}\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{25}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4},$$
$$W = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{26}$$

Integrating Eq. (25), and inserting the result in Eq. (22), we obtain the exact solutions to Eq. (15). Consequently, we have the traveling wave solutions to the NLSE with Kerr law nonlinearity (14) as the following:

When $\Lambda(\Psi) = (\Psi-\lambda_1)^4$, we obtain

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W}{x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0}\right\} \times \mathrm{e}^{i\left\{-\kappa x + \left(\frac{\mu_1\tau_1(bv-a) - 2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t + \theta\right\}}. \tag{27}$$

When $\Lambda(\Psi) = (\Psi-\lambda_1)^3(\Psi-\lambda_2)$ and $\lambda_2 > \lambda_1$, we get

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2 - \left[(\lambda_1-\lambda_2)\left(x + \left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t - \xi_0\right)\right]^2}\right\} \times \mathrm{e}^{i\left\{-\kappa x + \left(\frac{\mu_1\tau_1(bv-a) - 2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t + \theta\right\}}. \tag{28}$$

When $\Lambda(\Psi) = (\Psi-\lambda_1)^2(\Psi-\lambda_2)^2$, we have

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{(\lambda_2-\lambda_1)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t-\xi_0\right)\right]-1}\right\}$$
$$\times e^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \tag{29}$$

and

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{(\lambda_1-\lambda_2)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W}\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t-\xi_0\right)\right]-1}\right\}$$
$$\times e^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}. \tag{30}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$, we attain

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left(\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W}\left[x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right]\right)}\right\}$$
$$\times e^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}. \tag{31}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$, we find

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\,\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W}\left[x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t-\xi_0\right],l\right]}\right\}$$
$$\times e^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \tag{32}$$

where

$$l^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \tag{33}$$

Also, $\lambda_i(i=1,\ldots,4)$ are the roots of the polynomial equation

$$\Lambda(\Psi)=0. \tag{34}$$

When $\tau_0=-\tau_1\lambda_1$ and $\xi_0=0$, we can reduce the solutions (27)–(31) to plane wave solutions

$$q(x,t)=\left\{\pm\frac{\tau_1 W}{x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t}\right\}\times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \quad (35)$$

$$q(x,t)=\left\{\frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2-\left[(\lambda_1-\lambda_2)\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]^2}\right\}\times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \quad (36)$$

singular soliton solutions

$$q(x,t)=\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W}\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]\right\}\times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \quad (37)$$

and bright soliton solution

$$q(x,t)=\left\{\frac{A}{C+\cosh\left[B\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]}\right\}\times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \quad (38)$$

where

$$A=\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{\lambda_3-\lambda_2},\quad B=\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W},\quad C=\frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2}. \quad (39)$$

Here, A is the amplitude of the soliton, while B is the inverse width of the soliton. These solitons exist for $\tau_1<0$. Furthermore, when $\tau_0=-\tau_1\lambda_2$ and $\xi_0=0$, we can write the Jacobi elliptic function solution (32) as

$$q(x,t)=\left\{\frac{A_1}{C_1+\mathrm{sn}^2\left[B_j\left[x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right],\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]}\right\}\times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \quad (40)$$

where

$$A_1=\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4},\quad C_1=\frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4},\quad B_j=\frac{(-1)^j\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W},\quad (j=1,2). \quad (41)$$

Remark 1 When the modulus $l\to 1$, a second form of singular optical soliton solutions fall out:

$$q(x,t)=\left\{\frac{A_1}{C_1+\tanh^2\left[B_j\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]}\right\}\times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \quad (42)$$

where $\lambda_3=\lambda_4$.

Remark 2 However, if $l\to 0$, periodic singular solutions are listed as follows:

$$q(x,t)=\left\{\frac{A_1}{C_1+\sin^2\left[B_j\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]}\right\}\times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_1\tau_1(bv-a)-2\tau_0\chi_0\left[\kappa(\lambda-a\kappa)+c\tau_0^2\right]}{2\tau_0\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \quad (43)$$

where $\lambda_2=\lambda_3$.

2.2 Power law

In this case,

$$F(u)=u^n, \quad (44)$$

for power law nonlinear medium. Therefore, (1) takes the form

$$iq_t+aq_{xx}+bq_{xt}+c|q|^{2n}q=-i\lambda q_x-is\left(|q|^2q\right)_x-i\mu\left(|q|^2\right)_x q-i\theta|q|^2q_x-i\gamma q_{xxx}-\theta_1\left(|q|^2q\right)_{xx}-\theta_2|q|^2q_{xx}-\theta_3q^2q^*_{xx}. \quad (45)$$

In this case, Eq. (11) simplifies to

$$(a-bv)U''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)U+cU^{2n+1}=0. \quad (46)$$

Balancing U'' with U^{2n+1} in Eq. (46) gives $N = \frac{1}{n}$. In order to obtain closed-form solutions, we use the transformation

$$U = V^{\frac{1}{2n}}, \tag{47}$$

that will reduce Eq. (46) to the ODE

$$(a - bv)\left((1 - 2n)(V')^2 + 2nVV''\right) - 4n^2\left((1 - b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)V^2 + 4cn^2V^3 = 0. \tag{48}$$

In this subsection, we will utilize the extended trial equation method to construct the exact solutions to the NLSE with power law nonlinearity. Substituting Eqs. (16), (18) and (19) in Eq. (48), and using the balance principle, we determine a relation of σ, ρ and ς as

$$\sigma = \rho + \varsigma + 2. \tag{49}$$

Case 1: If we take $\sigma = 3$, $\rho = 0$ and $\varsigma = 1$ in Eq. (49), then

$$V = \tau_0 + \tau_1\Psi, \tag{50}$$

$$(V')^2 = \frac{\tau_1^2(\mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\chi_0}, \tag{51}$$

$$V'' = \frac{\tau_1(3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0}, \tag{52}$$

where $\mu_3 \neq 0$, $\chi_0 \neq 0$. Substituting Eqs. (50)–(52) in Eq. (48), and solving the resulting system of algebraic equations we have

$$\mu_1 = \frac{2\mu_0\tau_1}{\tau_0} - \frac{4cn^2\tau_0^2\chi_0}{\tau_1(1+n)(a-bv)},$$
$$\mu_2 = \frac{\mu_0\tau_1^2}{\tau_0^2} - \frac{8cn^2\tau_0\chi_0}{(1+n)(a-bv)},$$
$$\mu_3 = -\frac{4cn^2\tau_1\chi_0}{(1+n)(a-bv)},$$
$$\omega = \frac{-\mu_0\tau_1^2(1+n)(a-bv) + 4n^2\tau_0^2\chi_0\left[\kappa(1+n)(a\kappa-\lambda) - c\tau_0\right]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)},$$
$$\mu_0 = \mu_0, \quad \chi_0 = \chi_0, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1. \tag{53}$$

Substituting the solution set (53) in Eqs. (17) and (20), we find that

$$\pm(\xi - \xi_0) = \sqrt{W_1}\int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{54}$$

where

$$\Lambda(\Psi) = \Psi^3 + \frac{\mu_2}{\mu_3}\Psi^2 + \frac{\mu_1}{\mu_3}\Psi + \frac{\mu_0}{\mu_3}, \quad W_1 = \frac{\chi_0}{\mu_3}. \tag{55}$$

Consequently, we have the traveling wave solutions to the NLSE with power law nonlinearity (45) as the following:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3$, we find rational function solution as

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{4\tau_1W_1}{\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1 - b\kappa}\right)t - \xi_0\right)^2}\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(a-bv) + 4n^2\tau_0^2\chi_0[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{56}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, we obtain solitary wave solution as

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \tau_1(\lambda_1 - \lambda_2) \times \tanh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1 - b\kappa}\right)t - \xi_0\right)\right)\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(a-bv) + 4n^2\tau_0^2\chi_0[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{57}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)^2$ and $\lambda_1 > \lambda_2$, we have hyperbolic function solution as

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \tau_1(\lambda_1 - \lambda_2) \times \operatorname{cosech}^2\left[\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1 - b\kappa}\right)t\right)\right]\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(a-bv) + 4n^2\tau_0^2\chi_0[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{58}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, we get Jacobi elliptic function solutions as

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_3 + \tau_1(\lambda_2 - \lambda_3) \times \operatorname{sn}^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{W_1}}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1 - b\kappa}\right)t - \xi_0\right), l\right)\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(a-bv) + 4n^2\tau_0^2\chi_0[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)}\right)t + \theta\right\}}, \tag{59}$$

where

$$l^2 = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}. \tag{60}$$

Also, λ_i $(i = 1, 2, 3)$ are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \tag{61}$$

When $\tau_0 = -\tau_1\lambda_1$ and $\xi_0 = 0$, we can reduce the solutions (56)–(58) to plane wave solution

$$q(x,t) = \left\{\frac{\widetilde{A}}{x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t}\right\}^{\frac{1}{n}} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{-\mu_0\tau_1^2(1+n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(a\kappa-\lambda)-c\tau_0]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)}\right)t+\theta\right\}}, \tag{62}$$

1-soliton solution

$$q(x,t) = \left\{\frac{A_2}{\cosh^{\frac{1}{n}}\left[B_3\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]}\right\} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{-\mu_0\tau_1^2(1+n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(a\kappa-\lambda)-c\tau_0]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)}\right)t+\theta\right\}}, \tag{63}$$

and singular soliton solution

$$q(x,t) = \left\{\frac{A_3}{\sinh^{\frac{1}{n}}\left[B_3\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]}\right\} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{-\mu_0\tau_1^2(1+n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(a\kappa-\lambda)-c\tau_0]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)}\right)t+\theta\right\}}, \tag{64}$$

where

$$\widetilde{A} = 2\sqrt{\tau_1 W_1}, \quad A_2 = [\tau_1(\lambda_2-\lambda_1)]^{\frac{1}{2n}},$$
$$A_3 = [\tau_1(\lambda_1-\lambda_2)]^{\frac{1}{2n}}, \quad B_3 = \frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{W_1}}. \tag{65}$$

Here, A_2 and A_3 are respectively the amplitudes of 1-soliton and singular soliton, while B_3 is the inverse width of the solitons. These solitons exist for $\tau_1 > 0$. Furthermore, when $\tau_0 = -\tau_1\lambda_3$ and $\xi_0 = 0$, we can simplify the solution (59) as follows:

$$q(x,t) = A_4\,\mathrm{sn}^{\frac{1}{n}}\left[B_j\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right), \frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right] \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{-\mu_0\tau_1^2(1+n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(a\kappa-\lambda)-c\tau_0]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)}\right)t+\theta\right\}}, \tag{66}$$

where

$$A_4 = [\tau_1(\lambda_2-\lambda_3)]^{\frac{1}{2n}},$$
$$B_j = \frac{(-1)^j}{2}\sqrt{\frac{\lambda_1-\lambda_3}{W_1}}, \quad (j=4,5). \tag{67}$$

Remark 3 When the modulus $l \to 1$, dark soliton solutions fall out:

$$q(x,t) = A_4\tanh^{\frac{1}{n}}\left[B_j\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right] \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{-\mu_0\tau_1^2(1+n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(a\kappa-\lambda)-c\tau_0]}{4n^2\tau_0^2\chi_0(1+n)(b\kappa-1)}\right)t+\theta\right\}}, \tag{68}$$

where $\lambda_1 = \lambda_2$.

Case 2: If we take $\sigma = 4$, $\rho = 0$ and $\varsigma = 2$ in Eq. (49), then

$$V = \tau_0 + \tau_1\Psi + \tau_2\Psi^2, \tag{69}$$

$$(V')^2 = \frac{(\tau_1+2\tau_2\Psi)^2(\mu_4\Psi^4+\mu_3\Psi^3+\mu_2\Psi^2+\mu_1\Psi+\mu_0)}{\chi_0}, \tag{70}$$

$$V'' = \frac{(\tau_1+2\tau_2\Psi)(4\mu_4\Psi^3+3\mu_3\Psi^2+2\mu_2\Psi+\mu_1)}{2\chi_0} + \frac{2\tau_2(\mu_4\Psi^4+\mu_3\Psi^3+\mu_2\Psi^2+\mu_1\Psi+\mu_0)}{\chi_0}, \tag{71}$$

where $\mu_4 \neq 0$, $\chi_0 \neq 0$. Substituting Eqs. (69)–(71) in Eq. (48), and solving the resulting system of algebraic equations, we have

$$\mu_0 = -\frac{cn^2\tau_0^2\chi_0}{\tau_2(1+n)(a-bv)},$$
$$\mu_1 = -\frac{2cn^2\tau_0\tau_1\chi_0}{\tau_2(1+n)(a-bv)},$$
$$\mu_2 = -\frac{cn^2\chi_0\left(\tau_1^2+2\tau_0\tau_2\right)}{\tau_2(1+n)(a-bv)},$$
$$\mu_3 = -\frac{2cn^2\tau_1\chi_0}{(1+n)(a-bv)}, \quad \mu_4 = -\frac{cn^2\tau_2\chi_0}{(1+n)(a-bv)},$$
$$\omega = \frac{c\tau_1^2+4\tau_2[\kappa(1+n)(a\kappa-\lambda)-c\tau_0]}{4\tau_2(1+n)(b\kappa-1)},$$
$$\chi_0=\chi_0, \quad \tau_0=\tau_0, \quad \tau_1=\tau_1, \quad \tau_2=\tau_2. \tag{72}$$

Substituting the solution set (72) in Eqs. (17) and (20), we find that

$$\pm(\xi-\xi_0) = W_2\int\frac{\mathrm{d}\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{73}$$

where

$$\Lambda(\Psi) = \Psi^4+\frac{\mu_3}{\mu_4}\Psi^3+\frac{\mu_2}{\mu_4}\Psi^2+\frac{\mu_1}{\mu_4}\Psi+\frac{\mu_0}{\mu_4},$$
$$W_2 = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{74}$$

Consequently, taking $\xi_0 = 0$, we have the traveling wave solutions to the NLSE with power law nonlinearity (45) as the following:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, we obtain

$$q(x,t) = \left[\sum_{i=0}^{2} \tau_i \left(\lambda_1 \pm \frac{W_2}{x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t}\right)^i\right]^{\frac{1}{2n}} \times \mathrm{e}^{i\left\{-\kappa x + \left(\frac{c\tau_1^2 + 4\tau_2[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4\tau_2(1+n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{75}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, we get

$$q(x,t) = \left[\sum_{i=0}^{2} \tau_i \left(\lambda_1 + \frac{4W_2^2(\lambda_2 - \lambda_1)}{4W_2^2 - \left[(\lambda_1 - \lambda_2)\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t\right)\right]^2}\right)^i\right]^{\frac{1}{2n}} \times \mathrm{e}^{i\left\{-\kappa x + \left(\frac{c\tau_1^2 + 4\tau_2[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4\tau_2(1+n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{76}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, we have

$$q(x,t) = \left[\sum_{i=0}^{2} \tau_i \left(\lambda_2 + \frac{\lambda_2 - \lambda_1}{\exp\left[\frac{\lambda_1 - \lambda_2}{W_2}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t\right)\right] - 1}\right)^i\right]^{\frac{1}{2n}} \times \mathrm{e}^{i\left\{-\kappa x + \left(\frac{c\tau_1^2 + 4\tau_2[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4\tau_2(1+n)(b\kappa-1)}\right)t + \theta\right\}}, \tag{77}$$

and

$$q(x,t) = \left[\sum_{i=0}^{2} \tau_i \left(\lambda_1 + \frac{\lambda_1 - \lambda_2}{\exp\left[\frac{\lambda_1 - \lambda_2}{W_2}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t\right)\right] - 1}\right)^i\right]^{\frac{1}{2n}} \times \mathrm{e}^{i\left\{-\kappa x + \left(\frac{c\tau_1^2 + 4\tau_2[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4\tau_2(1+n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{78}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, we attain

$$q(x,t) = \left[\sum_{i=0}^{2} \tau_i \left(\lambda_1 - \frac{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2)\cosh\left(\frac{\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W_2}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t\right)\right)}\right)^i\right]^{\frac{1}{2n}} \times \mathrm{e}^{i\left\{-\kappa x + \left(\frac{c\tau_1^2 + 4\tau_2[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4\tau_2(1+n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{79}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, we achieve

$$q(x,t) = \left[\sum_{i=0}^{2} \tau_i \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4)\,\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t\right), l\right]}\right)^i\right]^{\frac{1}{2n}} \times \mathrm{e}^{i\left\{-\kappa x + \left(\frac{c\tau_1^2 + 4\tau_2[\kappa(1+n)(a\kappa-\lambda) - c\tau_0]}{4\tau_2(1+n)(b\kappa-1)}\right)t + \theta\right\}}, \tag{80}$$

where

$$l^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}. \tag{81}$$

Also, λ_i $(i = 1, \ldots, 4)$ are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \tag{82}$$

Remark 4 When the modulus $l \to 1$, we write the Jacobi elliptic function solutions (80) as

$$q(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\lambda_2+\frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\tanh^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_2}\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]}\right)^i\right]^{\frac{1}{2n}} \times e^{i\left\{-\kappa x+\left(\frac{c\tau_1^2+4\tau_2[\kappa(1+n)(a\kappa-\lambda)-c\tau_0]}{4\tau_2(1+n)(b\kappa-1)}\right)t+\theta\right\}}, \quad (83)$$

where $\lambda_3=\lambda_4$.

Remark 5 When the modulus $l\to 0$, we write the Jacobi elliptic function solutions (80) as

$$q(x,t) = \left[\sum_{i=0}^{2}\tau_i\left(\lambda_2+\frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\sin^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_2}\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]}\right)^i\right]^{\frac{1}{2n}} \times e^{i\left\{-\kappa x+\left(\frac{c\tau_1^2+4\tau_2[\kappa(1+n)(a\kappa-\lambda)-c\tau_0]}{4\tau_2(1+n)(b\kappa-1)}\right)t+\theta\right\}}, \quad (84)$$

where $\lambda_2=\lambda_3$.

2.3 Parabolic law

In this case,

$$F(u)=u+\eta u^2, \quad (85)$$

where η is a real-valued constant. Therefore, Eq. (1) takes the form

$$iq_t+aq_{xx}+bq_{xt}+c(|q|^2+\eta|q|^4)q = -i\lambda q_x - is\left(|q|^2q\right)_x - i\mu\left(|q|^2\right)_x q - i\theta|q|^2q_x - i\gamma q_{xxx} - \theta_1\left(|q|^2q\right)_{xx} - \theta_2|q|^2q_{xx} - \theta_3 q^2 q^*_{xx}. \quad (86)$$

In this case, Eq. (11) simplifies to

$$(a-bv)U'' - \left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)U + cU^3 + c\eta U^5 = 0. \quad (87)$$

Balancing U'' with U^5 gives $N=\frac{1}{2}$. In order to obtain closed-form solution, we use the transformation

$$U=V^{\frac{1}{2}}, \quad (88)$$

that will reduce Eq. (87) to the ODE

$$(a-bv)\left(2VV''-(V')^2\right) - 4\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)V^2 + 4cV^3 + 4c\eta V^4 = 0. \quad (89)$$

In this subsection, we will implement the extended trial equation method to obtain the exact solutions of the NLSE with parabolic law nonlinearity. Substituting Eqs. (16), (18) and (19) in Eq. (89), and using the balance principle, we determine a relation of σ, ρ and ς as

$$\sigma=\rho+2\varsigma+2. \quad (90)$$

If we take $\sigma=4$, $\rho=0$ and $\varsigma=1$ in Eq. (90), then

$$V=\tau_0+\tau_1\Psi, \quad (91)$$

$$(V')^2=\frac{\tau_1^2(\mu_4\Psi^4+\mu_3\Psi^3+\mu_2\Psi^2+\mu_1\Psi+\mu_0)}{\chi_0}, \quad (92)$$

$$V''=\frac{\tau_1(4\mu_4\Psi^3+3\mu_3\Psi^2+2\mu_2\Psi+\mu_1)}{2\chi_0}, \quad (93)$$

where $\mu_4 \neq 0$, $\chi_0 \neq 0$. Substituting Eqs. (91)–(93) in Eq. (89), collecting the coefficients of Ψ, and solving the resulting system, we have

$$\mu_0 = \frac{\tau_0}{\tau_1^2}\left(-\mu_2\tau_0 + \mu_1\tau_1 - \frac{2c\tau_0^2\chi_0(1+2\eta\tau_0)}{a-bv}\right),$$
$$\mu_3 = -\frac{2c\tau_1\chi_0(3+8\eta\tau_0)}{3(a-bv)},$$
$$\mu_4 = -\frac{4c\eta\tau_1^2\chi_0}{3(a-bv)},$$
$$\omega = \frac{\mu_2(bv-a) + 2\chi_0[2\kappa(a\kappa-\lambda) - c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)},$$
$$\mu_1 = \mu_1, \quad \mu_2 = \mu_2, \quad \chi_0 = \chi_0, \quad \tau_0 = \tau_0,$$
$$\tau_1 = \tau_1. \tag{94}$$

Substituting the solution set (94) in Eqs. (17) and (20), we find that

$$\pm(\xi - \xi_0) = W_3 \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{95}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4},$$
$$W_3 = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{96}$$

Consequently, we obtain the traveling wave solutions to the NLSE with parabolic law nonlinearity (86) as the following:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, we have

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W_3}{x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0}\right\}^{\frac{1}{2}} \times e^{i\left\{-\kappa x + \left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t + \theta\right\}}. \tag{97}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, we find

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{4W_3^2(\lambda_2-\lambda_1)\tau_1}{4W_3^2 - \left[(\lambda_1-\lambda_2)\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0\right)\right]^2}\right\}^{\frac{1}{2}} \times e^{i\left\{-\kappa x + \left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t + \theta\right\}}. \tag{98}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, we get

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \frac{(\lambda_2-\lambda_1)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W_3}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0\right)\right] - 1}\right\}^{\frac{1}{2}} \times e^{i\left\{-\kappa x + \left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t + \theta\right\}}, \tag{99}$$

and

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{(\lambda_1-\lambda_2)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W_3}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0\right)\right] - 1}\right\}^{\frac{1}{2}} \times e^{i\left\{-\kappa x + \left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t + \theta\right\}}. \tag{100}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, we attain

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 - \frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3-\lambda_2)\cosh\left(\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_3}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t\right)\right)}\right\}^{\frac{1}{2}} \times e^{i\left\{-\kappa x + \left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t + \theta\right\}}. \tag{101}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, we obtain

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1-\lambda_4)\,\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_3}\left[x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0\right], l\right]}\right\}^{\frac{1}{2}} \times e^{i\left\{-\kappa x + \left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t + \theta\right\}}, \tag{102}$$

where

$$l^2 = \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \tag{103}$$

Also, λ_i ($i = 1, \ldots, 4$) are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \tag{104}$$

When $\tau_0 = -\tau_1\lambda_1$ and $\xi_0 = 0$, we can reduce the solutions (97)–(101) to rational function solutions

$$q(x,t) = \left\{\pm\frac{\tau_1 W_3}{x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t}\right\}^{\frac{1}{2}} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \tag{105}$$

$$q(x,t) = \left\{\frac{4W_3^2(\lambda_2-\lambda_1)\tau_1}{4W_3^2-\left[(\lambda_1-\lambda_2)\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]^2}\right\}^{\frac{1}{2}} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \tag{106}$$

traveling wave solutions

$$q(x,t) = \left\{\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left(1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W_3}\times\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]\right)\right\}^{\frac{1}{2}} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \tag{107}$$

and soliton solution

$$q(x,t) = \left\{\frac{A_5}{\left(C_2+\cosh\left[B_6\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]\right)^{\frac{1}{2}}}\right\} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \tag{108}$$

where

$$A_5 = \left(\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{\lambda_3-\lambda_2}\right)^{\frac{1}{2}}, \quad B_6 = \frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_3}, \quad C_2 = \frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2}. \tag{109}$$

Here, A_5 is the amplitude of the soliton, while B_6 is the inverse width of the soliton. These solitons exist for $\tau_1 < 0$. On the other hand, when $\tau_0 = -\tau_1\lambda_2$ and $\xi_0 = 0$, we can write the Jacobi elliptic function solution (102) as

$$q(x,t) = \left\{\frac{A_6}{\left(C_3+\mathrm{sn}^2\left[B_j\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]\right)^{\frac{1}{2}}}\right\} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \tag{110}$$

where

$$A_6 = \left(\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}\right)^{\frac{1}{2}}, \quad C_3 = \frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4}, \quad B_j = \frac{(-1)^j\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_3}, \quad (j=7,8). \tag{111}$$

Remark 6 When the modulus $l \to 1$, hyperbolic function solutions fall out:

$$q(x,t) = \left\{\frac{A_6}{\left(C_3+\tanh^2\left[B_j\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]\right)^{\frac{1}{2}}}\right\} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \tag{112}$$

where $\lambda_3 = \lambda_4$.

Remark 7 However, if $l \to 0$, periodic wave solutions are as listed below:

$$q(x,t) = \left\{\frac{A_6}{\left(C_3+\sin^2\left[B_j\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]\right)^{\frac{1}{2}}}\right\} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{\mu_2(bv-a)+2\chi_0[2\kappa(a\kappa-\lambda)-c\tau_0(3+4\eta\tau_0)]}{4\chi_0(b\kappa-1)}\right)t+\theta\right\}}, \tag{113}$$

where $\lambda_2 = \lambda_3$.

2.4 Dual-power law

In this case,

$$F(u) = u^n + \eta u^{2n}, \tag{114}$$

where η is a real-valued constant. Therefore, Eq. (1) takes the form

$$iq_t + aq_{xx} + bq_{xt} + c(|q|^{2n} + \eta|q|^{4n})q = -i\lambda q_x - is\left(|q|^2 q\right)_x - i\mu\left(|q|^2\right)_x q - i\theta|q|^2 q_x - i\gamma q_{xxx} - \theta_1\left(|q|^2 q\right)_{xx} - \theta_2|q|^2 q_{xx} - \theta_3 q^2 q^*_{xx}. \quad (115)$$

In this case, Eq. (11) simplifies to

$$(a-bv)U'' - \left((1-b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)U + cU^{2n+1} + c\eta U^{4n+1} = 0. \quad (116)$$

Balancing U'' with U^{4n+1} gives $N = \frac{1}{2n}$. In order to obtain closed-form solutions, we use the transformation

$$U = V^{\frac{1}{2n}}, \quad (117)$$

that will reduce Eq. (116) to the ODE

$$(a-bv)\left((1-2n)(V')^2 + 2nVV''\right) - 4n^2\left((1-b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)V^2 + 4cn^2V^3 + 4cn^2\eta V^4 = 0. \quad (118)$$

In this subsection, we will apply the extended trial equation method to solve the NLSE with dual-power law nonlinearity. Substituting Eqs. (16), (18) and (19) in Eq. (118), and using the balance principle, we determine a relation of σ, ρ and ς as

$$\sigma = \rho + 2\varsigma + 2. \quad (119)$$

If we take $\sigma = 4$, $\rho = 0$ and $\varsigma = 1$ in Eq. (119), then

$$V = \tau_0 + \tau_1\Psi, \quad (120)$$

$$(V')^2 = \frac{\tau_1^2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\chi_0}, \quad (121)$$

$$V'' = \frac{\tau_1(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0}, \quad (122)$$

where $\mu_4 \neq 0$, $\chi_0 \neq 0$. Substituting Eqs. (120)–(122) in Eq. (118), collecting the coefficients of Ψ, and solving the resulting algebraic equations system, we have

$$\mu_1 = \frac{2\mu_0\tau_1}{\tau_0} - \frac{4cn^2\tau_0^2\chi_0\left[1 + 2n + 2\eta\tau_0(1+n)\right]}{\tau_1(1+n)(1+2n)(a-bv)},$$

$$\mu_2 = \frac{\mu_0\tau_1^2}{\tau_0^2} - \frac{4cn^2\tau_0\chi_0\left[2 + 4n + 5\eta\tau_0(1+n)\right]}{(1+n)(1+2n)(a-bv)},$$

$$\mu_3 = -\frac{4cn^2\tau_1\chi_0\left[1 + 2n + 4\eta\tau_0(1+n)\right]}{(1+n)(1+2n)(a-bv)},$$

$$\mu_4 = -\frac{4cn^2\eta\tau_1^2\chi_0}{(1+2n)(a-bv)},$$

$$\omega = \frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv) + 4n^2\tau_0^2\chi_0\left[\kappa(1+n)(1+2n)(a\kappa-\lambda) - c\tau_0\left(1+2n+\eta\tau_0(1+n)\right)\right]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)},$$

$$\mu_0 = \mu_0, \quad \chi_0 = \chi_0, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1. \quad (123)$$

Substituting the solution set (123) in Eqs. (17) and (20), we find that

$$\pm(\xi - \xi_0) = W_4\int\frac{\mathrm{d}\Psi}{\sqrt{\Lambda(\Psi)}}, \quad (124)$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4},$$

$$W_4 = \sqrt{\frac{\chi_0}{\mu_4}}. \quad (125)$$

Consequently, we have the traveling wave solutions to the NLSE with dual-power law nonlinearity (115) as the following:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, we get

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W_4}{x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0}\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv) + 4n^2\tau_0^2\chi_0\left[\kappa(1+n)(1+2n)(a\kappa-\lambda) - c\tau_0\left(1+2n+\eta\tau_0(1+n)\right)\right]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t + \theta\right\}}. \quad (126)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, we obtain

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{4W_4^2(\lambda_2-\lambda_1)\tau_1}{4W_4^2 - \left[(\lambda_1-\lambda_2)\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0\right)\right]^2}\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{127}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, we find

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \frac{(\lambda_2-\lambda_1)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W_4}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0\right)\right] - 1}\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t + \theta\right\}}, \tag{128}$$

and

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{(\lambda_1-\lambda_2)\tau_1}{\exp\left[\frac{\lambda_1-\lambda_2}{W_4}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0\right)\right] - 1}\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{129}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, we attain

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 - \frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3-\lambda_2)\cosh\left(\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_4}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t\right)\right)}\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t + \theta\right\}}. \tag{130}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, we have

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1-\lambda_4)\,\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_4}\left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1-b\kappa}\right)t - \xi_0\right), l\right]}\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t + \theta\right\}}, \tag{131}$$

where

$$l^2 = \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \quad (132)$$

Also, λ_i ($i = 1, \ldots, 4$) are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \quad (133)$$

When $\tau_0 = -\tau_1\lambda_1$ and $\xi_0 = 0$, we can reduce the solutions (126)–(130) to rational function solutions

$$q(x,t) = \left\{\pm\frac{\tau_1 W_4}{x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t}\right\}^{\frac{1}{2n}} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t+\theta\right\}}, \quad (134)$$

$$q(x,t) = \left\{\frac{4W_4^2(\lambda_2-\lambda_1)\tau_1}{4W_4^2-\left[(\lambda_1-\lambda_2)\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]^2}\right\}^{\frac{1}{2n}} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t+\theta\right\}}, \quad (135)$$

traveling wave solutions

$$q(x,t) = \left\{\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left(1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W_4}\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]\right)\right\}^{\frac{1}{2n}} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t+\theta\right\}}, \quad (136)$$

and soliton solution

$$q(x,t) = \left\{\frac{A_7}{\left(C_4+\cosh\left[B_9\left(x+\left(\frac{2a\kappa-b\omega-\lambda}{1-b\kappa}\right)t\right)\right]\right)^{\frac{1}{2n}}}\right\} \times \mathrm{e}^{i\left\{-\kappa x+\left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-bv)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)}\right)t+\theta\right\}}, \quad (137)$$

where

$$A_7 = \left(\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\tau_1}{\lambda_3-\lambda_2}\right)^{\frac{1}{2n}},$$

$$B_9 = \frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_4},$$

$$C_4 = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{\lambda_3 - \lambda_2}. \tag{138}$$

Here, A_7 is the amplitude of the soliton, while B_9 is the inverse width of the soliton. These solitons exist for $\tau_1 < 0$. Also, when $\tau_0 = -\tau_1\lambda_2$ and $\xi_0 = 0$, we can write the Jacobi elliptic function solution (131) as

$$q(x,t) = \left\{ \frac{A_8}{\left(C_5 + \mathrm{sn}^2\left[B_j \left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1 - b\kappa} \right) t \right), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)} \right] \right)^{\frac{1}{2n}}} \right\} \times e^{i\left\{ -\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-b\upsilon)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)} \right) t + \theta \right\}}, \tag{139}$$

where

$$A_8 = \left(\frac{\tau_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4} \right)^{\frac{1}{2n}},$$

$$C_5 = \frac{\lambda_4 - \lambda_2}{\lambda_1 - \lambda_4},$$

$$B_j = \frac{(-1)^j \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_4}, \quad (j = 10, 11). \tag{140}$$

Remark 8 When the modulus $l \to 1$, hyperbolic function solutions fall out:

$$q(x,t) = \left\{ \frac{A_8}{\left(C_5 + \tanh^2\left[B_j \left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1 - b\kappa} \right) t \right) \right] \right)^{\frac{1}{2n}}} \right\} \times e^{i\left\{ -\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-b\upsilon)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)} \right) t + \theta \right\}}, \tag{141}$$

where $\lambda_3 = \lambda_4$.

Remark 9 However, if $l \to 0$, periodic wave solutions are as listed below:

$$q(x,t) = \left\{ \frac{A_8}{\left(C_5 + \sin^2\left[B_j \left(x + \left(\frac{2a\kappa - b\omega - \lambda}{1 - b\kappa} \right) t \right) \right] \right)^{\frac{1}{2n}}} \right\} \times e^{i\left\{ -\kappa x + \left(\frac{-\mu_0\tau_1^2(1+n)(1+2n)(a-b\upsilon)+4n^2\tau_0^2\chi_0[\kappa(1+n)(1+2n)(a\kappa-\lambda)-c\tau_0(1+2n+\eta\tau_0(1+n))]}{4n^2\tau_0^2\chi_0(1+n)(1+2n)(b\kappa-1)} \right) t + \theta \right\}}, \tag{142}$$

where $\lambda_2 = \lambda_3$.

3 Conclusions

The nonlinear mathematical physical Eq. (1), which describes the propagation of ultra-short optical pulse in nonlinear negative-index material, is studied analytically by the extended trial equation method. Four types of nonlinearities including Kerr law, power law, parabolic law and dual-power law are taken into account. Also, the IMD, ND, SS, TOD and Raman effect are considered. Finally, some new soliton solutions are reported.

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