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On analysis of nonlinear dynamical systems via methods connected with λ-symmetry

Gülden Gün Polat · Teoman Özer

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Abstract This study focuses on the analysis of the first integrals, the integrating factors and the solutions for some classes of nonlinear dynamical systems represented by a mass–spring–damper model. The study consists of the applications of local–nonlocal transformations and the extended Prelle–Singer approach by considering the relation with Lie symmetry groups and λ-symmetry. In addition, the mathematical relations between these methods are presented, and time-dependent and time-independent first integrals and the corresponding exact solutions of nonlinear dynamical systems such as general Morse oscillator equation, the path equation, the harmonic oscillator equation and the displaced harmonic oscillator equation as special cases of Liénard-type equations are investigated. Furthermore, the general forms of Liénard-type equations including general nonlinear damping and nonlinear spring functions are studied.

G. G. Polat
Faculty of Science and Letters, Department of Mathematics, İstanbul Technical University, 34469 Maslak, Istanbul, Turkey

T. Özer (✉)
Division of Mechanics, Faculty of Civil Engineering, İstanbul Technical University, 34469 Maslak, Istanbul, Turkey
e-mail: tozer@itu.edu.tr

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1 Introduction

One can say that the concept of dynamical systems is very general and it includes many applications from engineering field to economy field. The dynamical system is about the interactions with the environment by means of its inputs and outputs. The modeling of a dynamical system, such as the general Morse oscillator equation, the path equation, the harmonic oscillator equation and the displaced harmonic oscillator equation, is given by differential equations including inputs, states and outputs. This means the behavior of dynamical systems is given by evolution its states; that is, if one has some information about the input of the systems and all its properties, one can predict the future behavior of the system. In addition, the behavior of the system is also characterized by its evolution over time.

The general procedure to analyze the behavior of dynamical systems is to construct a mathematical model described by a differential equation. In the literature, one of the representations of a dynamical system can be given by a mass–spring–damper model (Fig. 1). The Newton's second law leads to the following differential equation for this model

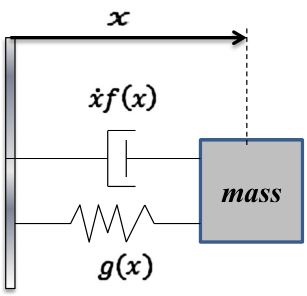

Fig. 1 A mass–spring–damper model

$$\ddot{x} + f(\dot{x}) + kx = 0, \tag{1.1}$$

where $x = x(t)$ denotes the position of the mass and varies with time t, $f(\dot{x})$ is the nonlinear damping force as a function of the velocity of the mass, k is the spring stiffness coefficient, and kx is the spring restoring force. For the case $f(\dot{x}) = c\dot{x}$, where c is the viscosity coefficient of the damper, then Eq. (1.1) becomes a differential equation representing a linear mass–spring–damper model. In addition, the case $f(\dot{x}) = c\dot{x}^2$ corresponds to a drag force on a mass moving in a fluid. In addition, one can consider the following second-order governing differential equation as a more general case of (1.1)

$$\ddot{x} + f(x, \dot{x}) = 0. \tag{1.2}$$

Thus, it is possible to write the following differential equations describing nonlinear mass–spring–damper model as specific cases of Eq. (1.2)

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.3}$$

and

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{1.4}$$

where $f(x)$ represents nonlinear damping and $g(x)$ represents nonlinear spring in the model. From the mathematical point of view, the differential equations given by (1.3) and (1.4) correspond to Liénard I-type and Liénard II-type equation, respectively, which have important properties in terms of mathematics, physics and mechanics [1–9]. In this study, we deal with the applications of mathematical methods such as Lie group theory related to the λ-symmetry approach, local and nonlocal transformations to analyze first integrals, Lagrangian–Hamiltonian forms and exact solutions of the nonlinear dynamical systems represented by the differential equations (1.3) and (1.4).

In the case of analysis of differential equations, Lie symmetry groups are one of the most effective methods in the literature [10–16]. Application of Lie groups has significant effects on all area of science including pure and applied mathematics, geometry as well as engineering, physics and social sciences. In 1883, S. Lie discovered that the general transformation formulas can be obtained by using a change of variables for certain classes of second-order ordinary differential equations to convert them to linear form, which is an application of Lie symmetry groups to nonlinear ordinary differential equations. He also indicated that the linearizable equations must be mostly cubic in the first-order derivative and its coefficients can be written as over-determined partial differential equations that should satisfy Lie linearization test [17,18]. The first requirement of Lie linearization test is to solve systems of partial differential equations. The second one is based on the fact that only second-order ordinary differential equations which have eight-dimensional Lie symmetry algebras can pass the Lie linearization test, which can be considered as a restriction property. In order to evaluate with wider perspective of linearization problems, new studies on nonlocal transformations has been developed in studies [18,19]. Sundman [18] proposes the general transformation formulas, which

$$X = F(x, t) \quad \mathrm{d}T = G(x, t)\mathrm{d}t, \tag{1.5}$$

are defined as a nonlocal transformation. If any nonlinear equation can be obtained in linear form via Sundman transformations, then it is called S-linearizable [3]. Duarte et al. [18] derive the second-order equations, which can be obtained as S-linearizable equations and demonstrate that they belong to the family of equations of the form

$$\ddot{x} + a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x) = 0. \tag{1.6}$$

Further studies on S-linearizable equations that involve an algorithm which enables finding the S-transformation are developed in [2,3]. They revaluate Duarte's results [18] and introduce some theorems for S-linearizable second-order ordinary differential equations as the form (1.6).

For Eq. (1.6), there are a list of three properties so that writing $a_2(t, x) = 0$ and $a_1(t, x) = a_0(t, x)$, and the function only depends on x, then (1.6) corresponds to Liénard I-type equation [6,7]. On the other hand, for a list of three properties so that writing $a_1(t, x) = 0$ and $a_2(t, x) = a_0(t, x)$, and the function only depends on x, then Eq. (1.6) becomes Liénard II-type equation [20]. In the literature, Lie point symmetry properties for Liénard I-type and II-type equations, which have eight-parameter Lie symmetry groups are investigated in detail in [6,7,20]. Some invariant solutions of the Liénard II-type equations representing some nonlinear dynamical systems are investigated by using the Lie point symmetry concept in [20]. Some solutions are presented and plotted. In the study [21], λ-symmetries are introduced as a new concept in the study of Lie groups. It is also shown that if any Lie point symmetry of second-order ordinary differential equation is known then the λ-symmetry of the same equation can be found by using a feasible algorithm. In fact, from the mathematical motivation point of view, in this study our aim is to consider more general mathematician concept in analyzing not only for Liénard II-type equation but for Liénard I-type equation and not only using Lie point symmetry but also using transformations and λ-symmetries approaches connected with Lie point symmetry and to represent some mathematical relations between these methods.

In addition to this, in the literature, the extended Prelle–Singer method is also used for investigating the time-dependent and time-independent first integrals of the ordinary differential equations. In fact, it is possible to show that there is a strong mathematical connection between λ-symmetry approach and the extended Prelle–Singer method based on the solution of null function and the integrating factors. We present that the use of Lie point symmetries and λ-symmetries in the concept of Prelle–Singer method procedure may give some important advantages to investigate first integrals of differential equations. Based on these facts, we analyze local and nonlocal transformations and λ-symmetries of some specific forms of Liénard I-type and Liénard II-type equations. In addition, we consider the relation between λ-symmetries and Prelle–Singer method for analyzing first integrals and integrating factors for general and spacial cases of Liénard-type equations.

This paper is organized as follows. In Sect. 2, we introduce some necessary preliminaries about Sundman transformation, theorems based on the conditions of being an S-linearizable equation. In Sect. 3 the linearizable cases of Liénard II-type and S-transformations, local transformations are presented by including some examples. Section 4 deals with λ-symmetries of given equation. Section 5 consists of analysis of Liénard I-type equation via S-linearizable conditions. Section 6 focuses on the application of extended Prelle–Singer method and the presentation of relations with the λ-symmetries. Section 7 includes conclusions and discussions.

2 Preliminaries

2.1 Nonlocal (S) transformations

Let us consider a second-order ordinary differential equation

$$\ddot{x} = M(t, x, \dot{x}). \tag{2.1}$$

Duarte et al. [18] illustrate that the equations of the form (2.1) can be written in the following linear equation form

$$X_{TT} = 0, \tag{2.2}$$

via generalized Sundman transformations

$$X = F(x, t) \quad \mathrm{d}T = G(x, t)\mathrm{d}t, \tag{2.3}$$

which are called *S-transformations* [3]. Besides, they state that if any equation can be linearized by means of S-transformations, then this equation can be called as *S-linearizable*. In fact, in order to express Eq. (2.1) as S-linearizable, it has to be in the form

$$\ddot{x} + a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x) = 0, \tag{2.4}$$

where

$$\begin{aligned}
a_2 &= \frac{GF_{xx} - F_x G_x}{GF_x} = \left(\frac{F_x}{G}\right)_x \cdot \left(\frac{F_x}{G}\right)^{-1},\\
a_1 &= \frac{2GF_{xx} - F_t G_x - F_x G_t}{GF_x}\\
&= \left[\left(\frac{F_t}{G}\right)_x + \left(\frac{F_x}{G}\right)_t\right]\left(\frac{F_x}{G}\right)^{-1},\\
a_0 &= \frac{GF_{tt} - F_t G_t}{GF_x} = \left(\frac{F_t}{G}\right)_t \cdot \left(\frac{F_x}{G}\right)^{-1}.
\end{aligned} \tag{2.5}$$

Theorem 2.1 [3] *The* Eq. (2.4) *is S-linearizable by means of S-transformation* (2.3) *if and only if its first integral can be defined as*

$$\omega\,(t,x,\dot{x}) = A(t,x)\dot{x} + B(t,x), \tag{2.6}$$

in which A and B are defined by

$$A(t,x) = \frac{F_x}{G} \quad B(t,x) = \frac{F_t}{G}. \tag{2.7}$$

It is clear that one can write

$$A(\ddot{x} + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x)) = D_t\,(\omega(t,x,\dot{x}))\,, \tag{2.8}$$

where D_{t} is the total derivative operator. Consequently, first integral and integrating factor corresponding to Eq. (2.4) *can be written as ω and $A = \omega_{\dot{x}} = F_x/G$, respectively. Conversely, if the first integral $\omega = A(t,x)\dot{x} + B(t,x)$ of* (2.4) *is known then a linearizing S-transformation can be determined by*

$$F(t,x) = \varphi(I(t,x)), \tag{2.9}$$

then G is uniquely given by

$$G(t,x) = \frac{F_x}{A} \quad \text{or} \quad G(t,x) = \frac{F_t}{B} \quad \text{if} \quad B{\neq}0, \tag{2.10}$$

in which $I(x,y)$ is the first integral of

$$\dot{x} = -\frac{B}{A}. \tag{2.11}$$

In the study [3], a new characterization method in terms of coefficient a_0, a_1, a_2 is developed to obtain S-linearizable form of Eq. (2.4). The following theorem summarizes the construction of the method.

Theorem 2.2 [3] *Let us consider an equation of the form* (2.4) *and let S_1 and S_2 be the functions defined by*

$$S_1(t,x) = a_{1x} - 2a_{2t}, \tag{2.12}$$

$$S_2(t,x) = (a_0a_2 + a_{0x})_x + (a_{2t} + a_{1x})_t + (a_{2t} - a_{1x})\,a_1. \tag{2.13}$$

Hence, the following alternatives hold:

- *If $S_1 = 0$ then* Eq. (2.4) *is S-linearizable if and only if $S_2 = 0$.*
- *If $S_1 \neq 0$, let S_3 and S_4 be the functions defined by*

$$S_3(t,x) = \left(\frac{S_2}{S_1}\right)_x - (a_{2t} + a_{1x})\,, \tag{2.14}$$

$$S_4(t,x) = \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_0a_2 + a_{0x}, \tag{2.15}$$

then the Eq. (2.4) *is said to be S-linearizable if and only if $S_3{=}0$ and $S_4 = 0$.*

Theorems 2.1 and 2.2 present the conditions for any equation to be S-linearizable by using S-transformation. Now we consider the following theorem consisting of the methods to obtain F and G functions for the Sundman transformation.

Theorem 2.3 [3] *Suppose that* Eq. (2.4) *is* S-*linearizable. If $S_1 = S_2 = 0$, let $\tau(t)$ be the function defined by*

$$\tau(t) = a_{0x} - \frac{1}{2}a_{1t} + a_0a_2 - \frac{1}{4}a_1^2. \tag{2.16}$$

Let $\omega(t)$ be a solution of the system of equations

$$\omega_t + \omega^2 + \tau = 0, \qquad \omega_x = 0, \tag{2.17}$$

and $C(t,x)$ is a solution of the system

$$C_t = a_0 - C\left(\frac{a_1}{2} + \omega\right), \tag{2.18}$$

$$C_x = \left(\frac{a_1}{2} + \omega\right) - Ca_2, \tag{2.19}$$

and $P(t,x)$ is a solution of

$$P_x = a_2, \quad P_t = \frac{a_1}{2}, \tag{2.20}$$

then if $F(t,x)$ is a nonconstant solution of equation

$$F_t = CF_x, \tag{2.21}$$

and G is given by

$$G = F_x \cdot \exp\left(-P - \int \omega(t)\mathrm{d}t\right), \tag{2.22}$$

thus the pair F and G define, via Eq. (2.3)*, a linearizing* S*-transformation. If $S_1 \neq 0$ then the equation* S*-linearizable if and only if $S_3 = S_4 = 0$ then $C(t,x)$ is a solution of the system*

$$C_t = a_0 - C(a_1 + u), \tag{2.23}$$

$$C_x = -u - Ca_2, \tag{2.24}$$

where $u = S_2/S_1$. And let $P(t,x)$ be a solution of

$$P_x = a_2, \quad P_t = \frac{S_1}{S_2} + a_1. \tag{2.25}$$

If F is a nonconstant solution of Eq. (2.21) *and G is defined by*

$$G = F_x \cdot \exp(-P), \tag{2.26}$$

then the pair F and G define via Eq. (2.3) *a linearizing S-transformation.*

2.2 Local (L) transformations

Let us consider Eq. (2.4), which is linearizable to the Eq. (2.2) via local transformations

$$X = R(t,x), \quad T = S(t,x), \tag{2.27}$$

then Eq. (2.1) has the form

$$\ddot{x} + a_3(t,x)\dot{x}^3 + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0, \tag{2.28}$$

where the coefficients $a_i(t,x)$, $0 \le i \le 3$ can be expressed in terms of R and S and their derivatives [8]

$$\begin{aligned} a_3(t,x) &= \frac{S_x R_{xx} - S_{xx} R_x}{S_t R_x - S_x R_t}, \\ a_2(t,x) &= \frac{S_t R_{xx} - R_t S_{xx} + 2\left(S_x R_{tx} - R_x S_{tx}\right)}{S_t R_x - S_x R_t}, \\ a_1(t,x) &= \frac{S_x R_{tt} - R_x S_{tt} + 2\left(S_t R_{tx} - R_t S_{tx}\right)}{S_t R_x - S_x R_t}, \\ a_0(t,x) &= \frac{S_t R_{tt} - R_t S_{tt}}{S_t R_x - S_x R_t}. \end{aligned} \tag{2.29}$$

Thus, if Eq. (2.28) can be linearized by means of local transformations (2.27) then it is called *L-linearizable*.

Theorem 2.4 [8] *L-linearizable equation must be at most cubic in the first-order derivative and its coefficients satisfy the following equations*

$$3(a_3)_{tt} - 2(a_2)_{tt} + (a_1)_{xx} = \left(3a_1a_3 - a_2^2\right)_t - 3(a_0a_3)_x - 3a_3a_{0x} + a_2a_{1x}$$

$$3a_{0xx} - 2(a_1)_{tx} + (a_2)_{tt} = 3(a_0a_3)_t + \left(a_1^2 - a_0a_2\right)_x + 3a_0(a_3)_t - a_1(a_2)_t. \tag{2.30}$$

It can be said that the conditions of L-linearizable equations are also valid for S-linearizable equations. The following procedure can be given to obtaining local transformation for the condition $S_1 = S_2 = 0$. Let's consider a nonzero solution $\sigma(t)$ of the equation

$$\sigma'(t) = \omega(t)\sigma(t), \tag{2.31}$$

where ω is a solution of (2.17). If P satisfies (2.20) then the function H can be defined such as $H = \sigma e^P$. Since $S_2 = 0$, there exists a function N such that

$$N_t = a_0 H, \quad N_x = \left(\frac{1}{2}a_1 - \frac{\sigma'}{\sigma}\right) H. \tag{2.32}$$

With the solution of (2.16), the function $F(t,x)$ can be found from the integration of the differential equations

$$F_x = \frac{H}{\sigma^2}, \quad F_t = \frac{N}{\sigma^2}. \tag{2.33}$$

By using Eq. (2.31) ω is written in terms of σ and by substituting in (2.22) then the expression

$$G = F_x e^{-P}\sigma^{-1}, \tag{2.34}$$

is obtained. And G can be defined in terms of σ as below

$$G = F_x e^{-P}\sigma^{-1} = \frac{H}{\sigma^2} e^{-P} \frac{1}{\sigma} = \frac{\sigma e^P e^{-P}}{\sigma^3} = \frac{1}{\sigma^2}. \tag{2.35}$$

Let $\bar{S} = \bar{S}(t)$ be such that $\bar{S}'(t) = 1/\sigma^2$. The local transformations (2.27) that linearize (2.4) can be defined as

$$R(t,x) = F(t,x), \quad S(t,x) = \bar{S}(t). \tag{2.36}$$

Theorem 2.5 [3] *An equation of the form* (2.4) *is L-linearizable and S-linearizable if and only if $S_1 = S_2 = 0$.*

3 S and L-transformations of Liénard II-type equation

We deal with the second-order nonlinear differential equation, which is called Liénard II-type equation [22, 23]

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{3.1}$$

where $f(x)$ and $g(x)$ are arbitrary functions of x. In this section, we investigate S-linearizable conditions of equation of type (3.1). From Eqs. (2.4) and (3.1) one can write

$$a_2(t,x) = f(x), \quad a_1(t,x) = 0, \quad a_0(t,x) = g(x). \tag{3.2}$$

3.1 S-transformations

Proposition 3.1 *Liénard II-type equation is called L-linearizable and S-linearizable if and only if with the condition $g'(x) + f(x)g(x) = \gamma$, $\gamma > 0$ is satisfied.*

Proof Substituting the coefficients a_2, a_1, a_0 to the Eq. (2.12), $S_1(t,x) = 0$ is obtained. Based on Theorem (2.2) if $S_2(t,x) = 0$ satisfies, then Eq. (3.1) is called S-linearizable. Let us assume that $S_2(t,x) = 0$, then the differential equation in terms of $f(x)$ and $g(x)$ is found

$$g''(x) + (f(x)g(x))' = 0. \tag{3.3}$$

If we integrate Eq. (3.3), then the relationship between $f(x)$ and $g(x)$ is given by

$$g'(x) + f(x)g(x) = \gamma, \quad \gamma > 0, \tag{3.4}$$

where γ is a constant. □

Equation (3.4) describes *isochronous condition*, which is an important result from physical and mathematical point of view [20]. In addition, a (classical) dynamical system is called *isochronous* if it features an open (hence fully dimensional) region in its phase space in which all its solutions are completely periodic (i.e., periodic in all degrees of freedom) with the same fixed period (independent of the initial data, provided they are inside the isochrony region) [24]. The integration of Eq. (3.4) enables us to define $g(x)$ function in terms of given $f(x)$. Hence, the special form of (3.1) can be written as

$$\ddot{x} + f(x)\dot{x}^2 + \gamma \mathrm{e}^{-\int f(x)\mathrm{d}x} \int \mathrm{e}^{\int f(x)\mathrm{d}x}\mathrm{d}x + \gamma_1 \mathrm{e}^{-\int f(x)\mathrm{d}x} = 0. \tag{3.5}$$

Here we deal with obtaining F and G functions to define S-transformations. For this purpose, in terms of the coefficients a_2, a_1, a_0 Eq. (2.16) is written to find $\tau(t)$ as below

$$\tau(t) = g'(x) + f(x)g(x). \tag{3.6}$$

Since $\tau(t)$ only should depend on t, one can see that the Eq. (3.6) must be equal to a constant such that

$$\tau(t) = \gamma. \tag{3.7}$$

Based on Eq. (3.4) it is possible to see that two different cases of γ value should be considered since if γ is equal to zero then Eq. (3.4) becomes homogeneous but it is different from zero then Eq. (3.4) becomes nonhomogeneous.

Case 1. $\gamma = 0$ From Eq. (2.17) we find

$$\omega(t) = 0. \tag{3.8}$$

A particular solution of the system (2.18) is

$$C(t,x) = tg(x). \tag{3.9}$$

Then, $P(t,x)$ via Eq. (2.20) is obtained

$$P(t,x) = -\ln(g(x)). \tag{3.10}$$

By solving Eqs. (2.21) and (2.22), the pair F and G defines, via Eq. (2.3), S-transformations

$$F(t,x) = \varphi\left(\frac{1}{2}\left(t^2 + 2\int \frac{\mathrm{d}x}{g(x)}\right)\right),$$
$$G(t,x) = \varphi'\left(\frac{1}{2}\left(t^2 + 2\int \frac{\mathrm{d}x}{g(x)}\right)\right), \tag{3.11}$$

that linearize Eq. (3.1). From the (2.6) and (2.7) the first integral of (3.1) can be written as

$$I = \frac{1}{g(x)}\dot{x} + t. \tag{3.12}$$

Case 2. $\gamma \neq 0$. The solution of (2.17) gives

$$\omega(t) = -\sqrt{\gamma}\tan\left(\sqrt{\gamma}t\right). \tag{3.13}$$

If we solve the system (2.18), then

$$C(t,x) = \frac{g(x)}{\sqrt{\gamma}}\tan\left(\sqrt{\gamma}t\right), \tag{3.14}$$

is obtained. Thus, $P(t,x)$ is found in the following form

$$P(t,x) = \int f(x)\mathrm{d}x. \tag{3.15}$$

From the (2.21) and (2.22) the functions F and G, which define S-transformations, yield

$$F(t,x) = \varphi\left(\frac{\mathrm{d}x}{\int g(x)} - \frac{1}{\gamma}\ln\left(\cos(\sqrt{\gamma}t)\right)\right), \tag{3.16}$$

$$G(t,x) = \frac{1}{g(x)}\exp\left(-\int f(x)\mathrm{d}x\right)\sec\left(\sqrt{\gamma}t\right)\varphi' \times\left(\frac{\mathrm{d}x}{\int g(x)} - \frac{1}{\gamma}\ln\left(\cos\left(\sqrt{\gamma}t\right)\right)\right), \tag{3.17}$$

which linearize Eq. (3.1). From the (2.6) and (2.7) the first integral for (3.1) can be written as

$$I = \exp\left(\int f(x)\mathrm{d}x\right) \times\left(\cos\left(\sqrt{\gamma}t\right)\dot{x} + \frac{1}{\sqrt{\gamma}}g(x)\sin\left(\sqrt{\gamma}t\right)\right). \tag{3.18}$$

3.2 L-transformations

Based on the Theorem 2.5, Eq. (3.1) is linearizable by a transformation of the form (2.36). It can easily be checked that Liénard II-type equation (quadratic Liénard type) satisfies Lie's linearization test given by (2.30). If we substitute the coefficients of (3.5) in Eq. (2.30), then one can see that these two equations are satisfied. Now let us consider the case of $\omega(t) = -\gamma\tan(\gamma t)$, then Eq. (2.31) yields

$$\sigma = \cos\left(\sqrt{\gamma}t\right). \tag{3.19}$$

From the equation $H = \sigma e^{P}$, we find

$$H(t,x) = \exp\left(\int f(x)\mathrm{d}x\right)\cos\left(\sqrt{\gamma}t\right). \tag{3.20}$$

Solution of Eq. (2.32) yields

$$N(t,x) = \frac{1}{\gamma}\exp\left(\int f(x)\mathrm{d}x\right)\sin\left(\sqrt{\gamma}t\right)g(x). \tag{3.21}$$

Consequently from Eqs. (2.33) and (2.35), we can write the pair

$$R(t,x) = \frac{g(x)}{\gamma}\sec\left(\sqrt{\gamma}t\right)\exp\left(\int f(x)\mathrm{d}x\right),$$
$$S(t,x) = \frac{1}{\sqrt{\gamma}}\tan\left(\sqrt{\gamma}t\right), \tag{3.22}$$

which define a local L-transformation that linearizes Eq. (3.1). The examples of the application of S and L-transformations to Liénard II-type equations having important mechanical and physical meaning are studied below.

Example 1 The differential equation describing the path of minimum drag work has been defined such that

$$\ddot{x} - \frac{f'(x)}{f(x)}\dot{x}^2 - \frac{f'(x)}{f(x)} = 0, \tag{3.23}$$

where $x = x(t)$ is the altitude function [25]. In a fluid medium, drag forces are the major sources of energy loss for moving objects. Fuel consumption may have to be reduced to minimize the drag work. This can be achieved by the selection of the optimum path. It is a fact that the path equation (3.23) is a type of quadratic Liénard equation. It can be seen that it satisfies the isochronous condition (3.4) and S-linearizable conditions $S_1 = S_2 = 0$ are satisfied if and only if $f(x)$ has only the form $f(x) = \beta\mathrm{sech}(\sqrt{\gamma}(x - c_1))$, where β and c_1 are constants. If Eq. (3.23) is rewritten by means of $f(x)$, then one can find

$$\ddot{x} + \sqrt{\gamma}\tanh\left(\sqrt{\gamma}(x - c_1)\right)\left(1 + \dot{x}^2\right) = 0. \tag{3.24}$$

By using Eqs. (3.16)–(3.18), for the Eq. (3.24) pair of S-transformations are

$$F(t,x) = \varphi\left(\frac{1}{\gamma}\left(\ln\left(\frac{\cos(\sqrt{\gamma}t)}{\sin\left(\sqrt{\gamma}(x - c_1)\right)}\right)\right)\right), \tag{3.25}$$

$$G(t,x) = -\frac{\operatorname{csch}\left(\sqrt{\gamma}(x-c_1)\right)\sec\left(\sqrt{\gamma}t\right)}{\sqrt{\gamma}}\varphi' \times\left(\frac{1}{\gamma}\left(\ln\left(\frac{\cos\left(\sqrt{\gamma}t\right)}{\sin\left(\sqrt{\gamma}(x-c_1)\right)}\right)\right)\right), \tag{3.26}$$

and the first integral is

$$I = \cos\left(\sqrt{\gamma}t\right)\cosh\left(\sqrt{\gamma}(x-c_1)\right)\dot{x} + \sin\left(\sqrt{\gamma}t\right)\sinh\left(\sqrt{\gamma}(x-c_1)\right), \tag{3.27}$$

where gives the invariant solution

$$x(t) = c_1 \pm \frac{1}{\sqrt{\gamma}}\operatorname{arcsinh}\left(\frac{1}{2}\left(-\sqrt{\gamma}c_2\cos\left(\sqrt{\gamma}t\right) \pm 2c_3\sin\left(\sqrt{\gamma}t\right)\right)\right), \tag{3.28}$$

where c_2 and c_3 are arbitrary constants. In addition, the L-transformations for Eq. (3.23) are

$$R(t,x) = \frac{-1}{\sqrt{\gamma}}\sinh\left(\sqrt{\gamma}(x-c_1)\right)\sec\left(t\sqrt{\gamma}\right),$$
$$S(t,x) = \frac{1}{\sqrt{\gamma}}\tan\left(\sqrt{\gamma}t\right), \tag{3.29}$$

which transform Eq. (3.24) to the linear equation $R'' = 0$. The general solution of this equation can be written as

$$R + aS + b = 0, \tag{3.30}$$

where a and b are constants. By applying Eq. (3.29) to the (3.30), we find the general solution of (3.24)

$$x(t) = c_1 - \frac{1}{\sqrt{\gamma}}\operatorname{arcsinh}\left(\sqrt{\gamma}a\cos\left(t\sqrt{\gamma}\right) + b\sin\left(t\sqrt{\gamma}\right)\right), \tag{3.31}$$

which is the same as the solution given in (3.28) for $c_2 = a$, $c_3 = b$. Furthermore, if the first integral of an equation is known, then the relation

$$I(x,\dot{x}) = H(x,p) = p\dot{x} - L(x,\dot{x}), \tag{3.32}$$

can be given, in which H is the Hamiltonian, L is the Lagrangian, and p is the conjugate momentum. Thus, the relation

$$\frac{\partial I}{\partial \dot{x}} = \frac{\partial H}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x} + p - \frac{\partial L}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x}, \tag{3.33}$$

can be written. In addition, the p conjugate momentum is expressed of the form

$$p = \int \frac{I_{\dot{x}}}{\dot{x}} + k(x), \tag{3.34}$$

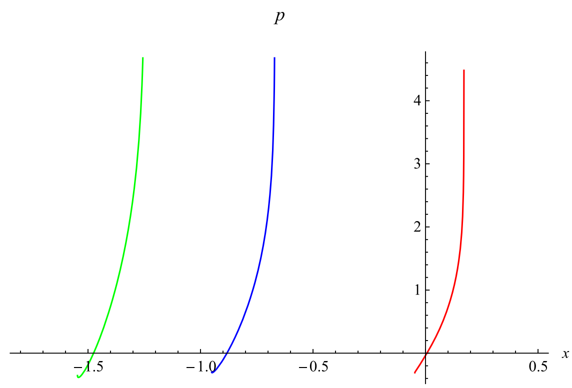

Fig. 2 Phase portrait of (3.28) for three different integration constants

where $k(x)$ is an arbitrary function of x. For the convenience, $k(x)$ can be assumed to be zero. Hence, it is easily checked that Lagrangian formulation can be given

$$L = p\dot{x} - I(x,\dot{x}). \tag{3.35}$$

For Eq. (3.24), the Lagrangian is obtained from (3.27) and (3.35)

$$L = \dot{x}\cos\left(\sqrt{\gamma}t\right)\cosh\left(\sqrt{\gamma}(x-c_1)\right)(\ln(\dot{x}) - 1) - \sin\left(\sqrt{\gamma}t\right)\sinh\left(\sqrt{\gamma}(x-c_1)\right). \tag{3.36}$$

From (3.32) the Hamiltonian

$$H = \cos\left(\sqrt{\gamma}t\right)\cosh\left(\sqrt{\gamma}(x-c_1)\right) \times\left(\mathrm{e}^{p\sec(\sqrt{\gamma}t)\operatorname{sech}(\sqrt{\gamma}(x-c_1))} + \tan\left(\sqrt{\gamma}t\right)\tanh\left(\sqrt{\gamma}(x-c_1)\right)\right), \tag{3.37}$$

and the conjugate momentum p.

$$p = \cos\left(\sqrt{\gamma}t\right)\cosh\left(\sqrt{\gamma}(x-c_1)\right)\ln(\dot{x}). \tag{3.38}$$

are found. Figure 2 presents the variation of the conjugate momentum p (3.34) versus position for three different values of integration constants for the parameter $\gamma = 0.4$.

Additionally, Fig. 3 shows the states of simulations for the position, velocity and acceleration.

Example 2 If one substitutes $f(x) = \lambda_1$ to Eq. (3.5), then generalized Morse oscillator

$$\ddot{x} + \lambda_1\dot{x}^2 + \frac{\gamma}{\lambda_1}\left(1 - \mathrm{e}^{-\lambda_1 x}\right) = 0, \tag{3.39}$$

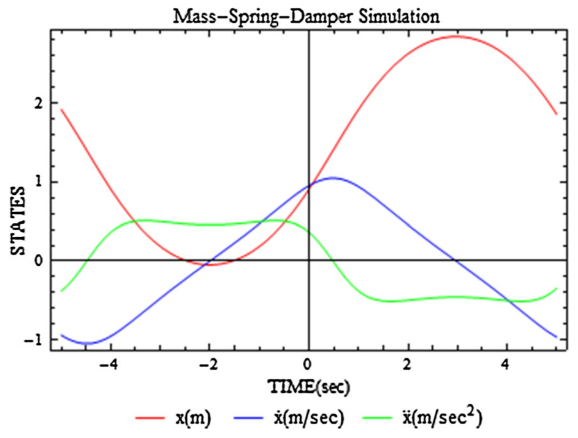

Fig. 3 State simulation representing position $x(t)$, velocity $\dot{x}$ and acceleration $\ddot{x}$ of (3.28)

is obtained, where λ_1 and γ are constants. For the limit $\lambda_1 \to 0$, the above equation corresponds to the linear harmonic oscillator. Then, S-transformations are

$$F(t,x) = \varphi\left(\frac{1}{\gamma}\ln\left(\sec\left(t\sqrt{\gamma}\right)\left(1-e^{\lambda_1 x}\right)\right)\right), \quad (3.40)$$

$$G(t,x) = \frac{\lambda_1}{\gamma\left(e^{\lambda_1 x}-1\right)}\sec\left(t\sqrt{\gamma}\right)\varphi' \times\left(\frac{1}{\gamma}\ln\left(\sec\left(t\sqrt{\gamma}\right)\left(1-e^{\lambda_1 x}\right)\right)\right), \quad (3.41)$$

and the first integral of (3.39) from (3.16) to (3.18)

$$I = e^{\lambda_1 x}\cos\left(t\sqrt{\gamma}\right)\dot{x} + \frac{\sqrt{\gamma}}{\lambda_1}\left(e^{\lambda_1 x}-1\right)\sin\left(t\sqrt{\gamma}\right). \quad (3.42)$$

is obtained. Furthermore, L-transformations can be expressed by using (3.22) as below

$$R(t,x) = \frac{\sec\left(t\sqrt{\gamma}\right)}{\lambda_1}\left(e^{\lambda_1 x}-1\right)$$

$$S(t,x) = \frac{1}{\sqrt{\gamma}}\tan\left(\sqrt{\gamma}t\right). \quad (3.43)$$

With the similar process, the general solution can be obtained by utilizing L-transformations (3.43) in the following form

$$x(t) = \frac{1}{\lambda_1}\ln\left(1 - b\lambda_1\cos\left(t\sqrt{\gamma}\right) - \frac{a\lambda_1}{\sqrt{\gamma}}\sin\left(t\sqrt{\gamma}\right)\right). \quad (3.44)$$

In addition, the corresponding p, H and L functions

$$p = e^{\lambda_1 x}\cos\left(t\sqrt{\gamma}\right)\ln\left(\dot{x}\right), \quad (3.45)$$

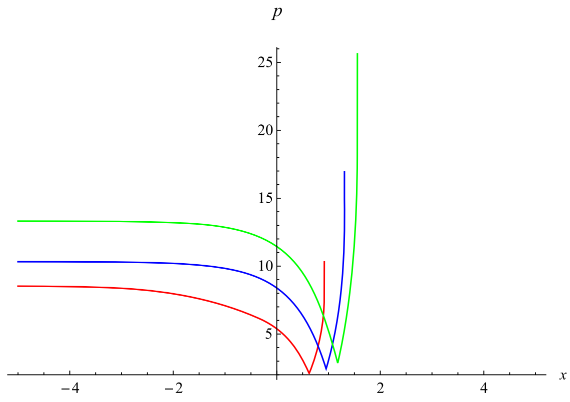

Fig. 4 Phase portrait for (3.44) with three different integration constants

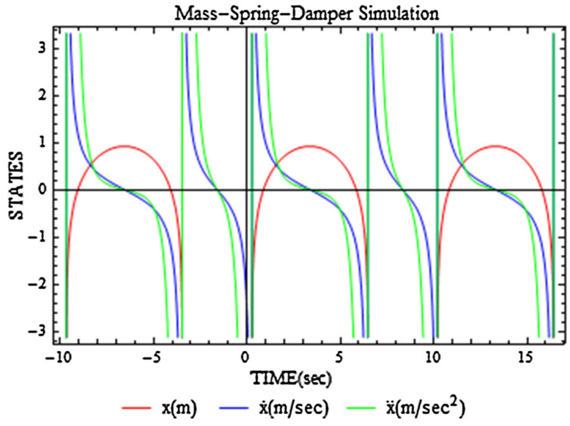

Fig. 5 State simulation representing position $x(t)$, velocity $\dot{x}$ and acceleration $\ddot{x}$ of (3.44)

$$L = e^{\lambda_1 x}\cos\left(t\sqrt{\gamma}\right)\dot{x}\left(\ln\left(\dot{x}\right)-1\right) - \frac{\sqrt{\gamma}}{\lambda_1}\left(e^{\lambda_1 x}-1\right)\sin\left(t\sqrt{\gamma}\right), \quad (3.46)$$

$$H = e^{x\lambda_1 + e^{-x\lambda_1} p \sec\left(t\sqrt{\gamma}\right)}\cos\left(t\sqrt{\gamma}\right) + \frac{\sqrt{\gamma}}{\lambda_1}\left(e^{\lambda_1 x}-1\right)\sin\left(t\sqrt{\gamma}\right). \quad (3.47)$$

can be obtained. Similarly, Fig. 4 presents the variation of the conjugate momentum p (3.34) versus position for three different values of integration constants and Fig. 5 shows the states of simulations for position, velocity and acceleration.

Example 3 For the case of $f(x) = -\frac{2\lambda_1}{1+x\lambda_1}$, Eq. (3.5) becomes

$$\ddot{x} - \frac{2\lambda_1}{1+x\lambda_1}\dot{x}^2 + \gamma\left(x+\lambda_1 x^2\right) = 0. \quad (3.48)$$

For the limit $\lambda_1 \to 0$, Eq. (3.48) reduces to the linear harmonic oscillator. Pair of S-transformations become

$$F(t,x) = \varphi\left(-\frac{1}{\gamma}\ln\left(\cos\left(\sqrt{\gamma}t\right)\left(\lambda_1 + \frac{1}{x}\right)\right)\right), \tag{3.49}$$

$$G(t,x) = \frac{(\lambda_1 x + 1)}{x\gamma}\sec\left(\sqrt{\gamma}t\right)\varphi' \times\left(-\frac{1}{\gamma}\ln\left(\cos\left(\sqrt{\gamma}t\right)\left(\lambda_1 + \frac{1}{x}\right)\right)\right). \tag{3.50}$$

The corresponding first integral is

$$I = \frac{1}{\lambda_1 x + 1}\left(\frac{\cos\left(\sqrt{\gamma}t\right)}{\lambda_1 x + 1}\dot{x} + x\sin\left(\sqrt{\gamma}t\right)\sqrt{\gamma}\right), \tag{3.51}$$

which yields the invariant solution of (3.48)

$$x(t) = \frac{1}{\lambda_1}\left(\frac{(1+b)\cos\left(t\sqrt{\gamma}\right) + a\sin(t\sqrt{\gamma})}{1-(b+1)\cos(t\sqrt{\gamma}) + a\sin\left(t\sqrt{\gamma}\right)}\right). \tag{3.52}$$

And L-transformations are

$$R(t,x) = -\frac{\sec\left(t\sqrt{\gamma}\right)}{\lambda_1}\left(1 + \frac{1}{\lambda_1 x + 1}\right),$$
$$S(t,x) = \frac{1}{\sqrt{\gamma}}\tan\left(\sqrt{\gamma}t\right), \tag{3.53}$$

which linearize Eq. (3.48). Conjugate momentum p, Lagrangian L and Hamiltonian for Eq. (3.48) are represented by

$$p = \frac{\cos\left(\sqrt{\gamma}t\right)}{(\lambda_1 x + 1)^2}\ln(\dot{x}), \tag{3.54}$$

$$L = \frac{\dot{x}\cos\left(t\sqrt{\gamma}\right)(\ln(\dot{x}) - 1) - x\sqrt{\gamma}(\lambda_1 x + 1)\sin\left(t\sqrt{\gamma}\right)}{(\lambda_1 x + 1)^2}, \tag{3.55}$$

$$H = \frac{1}{(\lambda_1 x + 1)^2}e^{p(\lambda_1 x + 1)^2\sec(t\sqrt{\gamma})}\cos\left(t\sqrt{\gamma}\right) + x\sqrt{\gamma}(\lambda_1 x + 1)\sin\left(t\sqrt{\gamma}\right). \tag{3.56}$$

Figure 6 presents the variation of the conjugate momentum p (3.34) versus time for three different values of integration constants, Fig. 7 shows the states of simulations for position, velocity and acceleration, and Fig. 8 is about the comparison of the graphs between the solution (3.52) and the linear case $f(x) = g(x) =$ constant.

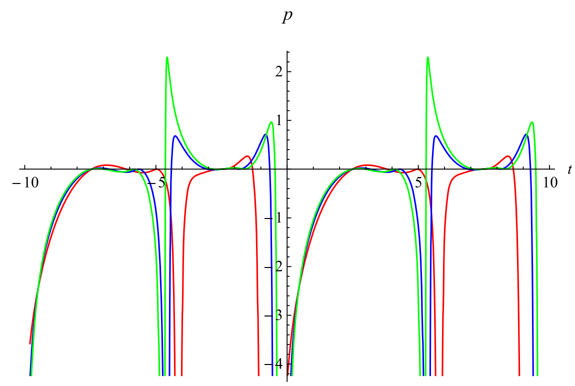

Fig. 6 Phase portrait of(3.52) with three different integration constants

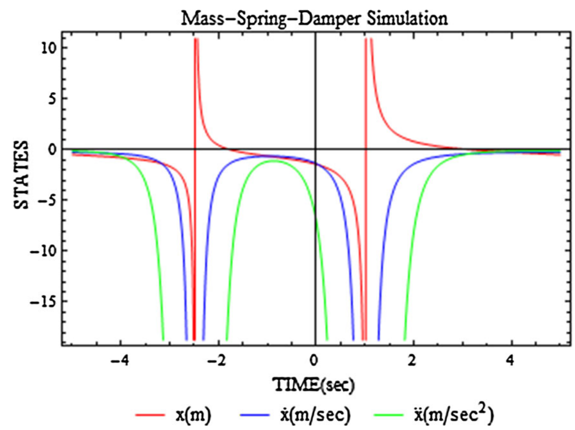

Fig. 7 Graph of state simulation representing position $x(t)$, velocity $\dot{x}$ and acceleration $\ddot{x}$ for (3.52)

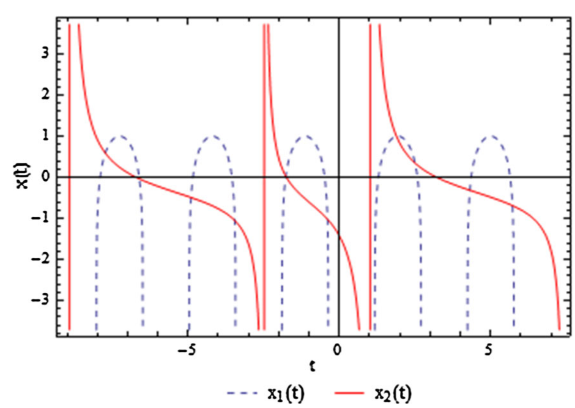

Fig. 8 Plot of two solutions corresponding to (3.52) as $x_2(t)$ and corresponding for $f(x) = g(x) =$ constant in (3.1) as $x_1(t)$

4 Relationships between S, L transformations and λ-symmetry

We have so far investigated the first integrals and invariant solutions of Liénard II-type equation by using S and L-transformations. In this section, we consider λ-symmetry approach and its relation with these transformations. Now let us consider the vector field $\upsilon = \partial_x$ is a λ-symmetry of (2.4) if and only if λ is a solution of the equation

$$M_x + \lambda M_{\dot{x}} = \lambda_t + \dot{x}\lambda_x + M\lambda_{\dot{x}} + \lambda^2, \tag{4.1}$$

and assume that the coefficients a_0, a_1, a_2 satisfy the conditions $S_1 = S_2 = 0$. If Eq. (4.1) has the solutions of the linear form of $\lambda = \alpha(t,x)\dot{x} + \beta(t,x)$, then the following system has particular solutions for α and β

$$\alpha_x + \alpha^2 + a_2\alpha + a_{2x} = 0, \tag{4.2}$$
$$\beta_x + 2(a_2 + \alpha)\beta + a_{1x} + \alpha_t = 0, \tag{4.3}$$
$$\beta_t + \beta^2 + a_1\beta + a_{0x} - a_0\alpha = 0. \tag{4.4}$$

It is easy to see that the solution of (4.2) is $\alpha(t,x) = -a_2(t,x)$. If we substitute α in Eqs. (4.3) and (4.4), then these equations can be rewritten such that

$$\beta_x + a_{1x} - a_{2t} = 0, \tag{4.5}$$
$$\beta_t + \beta^2 + a_1\beta + a_{0x} + a_0a_2 = 0. \tag{4.6}$$

We consider here two different cases:
Case 1. If $S_1 = S_2 = 0$, the function τ which is defined by (2.16) only depends on independent variable t. Suppose that $h(t)$ is any solution of the Riccati equation given below

$$h'(t) + h^2(t) + \tau(t) = 0. \tag{4.7}$$

Therefore, the solution of (4.7) which satisfies the Eqs. (4.5) and (4.6) is defined by $\beta(t,x) = h(t) - \frac{1}{2}a_1(t,x)$. As a result, for the λ function

$$\lambda = -a_2\dot{x} + \beta, \tag{4.8}$$

∂_x is a λ-symmetry of (2.4).
Case 2. If $S_1 \neq 0$ and $S_3 = S_4 = 0$, then solution of Eqs. (4.5) and (4.6) gives $\beta(t,x) = S_2/S_1$. Hence, ∂_x is a λ-symmetry of (2.4) for

$$\lambda = -a_2\dot{x} + S_2/S_1. \tag{4.9}$$

Proposition 4.1 [9] *The Liénard equation of the second kind $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$ admits a λ-symmetry of the form $\lambda = a(t,x)\dot{x} + b(t,x)$, with $a(t,x) = -f(x)$ and $b(t,x) = \sqrt{\gamma}\cot(\sqrt{\gamma}t)$ when the functions f and g satisfy the Sabatini (isochroniticity) condition $g'(x) + f(x)g(x) = \gamma$, $\gamma > 0$.*

In addition to the method for obtaining λ-symmetry, there is an alternative approach based on the relation between Lie point symmetry and λ-symmetry [1,21,26]. Here, a way to derive λ-symmetries associated with Lie symmetries directly is considered. If Lie symmetries of any equation are known, then λ symmetries can be found by a simple algorithm as explained below.

Let us consider the second-order differential equation of the form

$$\ddot{x} = \phi(t, x, \dot{x}), \tag{4.10}$$

and total derivative operator for above equation can be written in the form

$$A = \partial_t + \dot{x}\partial_x + \phi(t,x,\dot{x})\,\partial_{\dot{x}}. \tag{4.11}$$

In addition, let

$$\upsilon = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x}, \tag{4.12}$$

be a Lie point symmetry of (4.10) and then the characteristic of υ is

$$Q = \eta - \xi y', \tag{4.13}$$

and the λ-symmetry for the given equation can be determined as below

$$\lambda = \frac{A(Q)}{Q}. \tag{4.14}$$

The method for obtaining the first integrals and integrating factors from λ-symmetries are given below step by step [21]:

1. Substitute λ-symmetry (4.14) to the differential equation
$$\omega_x + \lambda\omega_{\dot{x}} = 0. \tag{4.15}$$
2. Evaluate $A(\omega)$ and derive $A(\omega)$ in terms of (t,ω) as $A(\omega) = F(t,\omega)$.
3. Find a first integral G of $\partial_t + F(t,\omega)\partial_\omega$.
4. Evaluate $I(t,x,\dot{x}) = G(t,\omega(t,x,\dot{x}))$ and find the integrating factor, which is given in the form

$$\mu = G_\omega \cdot \omega_{\dot{x}}. \tag{4.16}$$

Remark 1 In proposition 4.1 by Guha [9] for S and L transformations it is claimed that λ function must be in the form $\lambda = -f(x)\dot{x} + \sqrt{\gamma}\cot(\sqrt{\gamma}t)$. However, it is possible to show that λ symmetries with respect to S and L transformations can be different forms from the λ function mentioned in Proposition 4.1. The proof is given in the following subsection.

4.1 First integral and invariant solutions of Liénard II-type equations via λ-symmetries

Now, we deal with the λ-symmetry analysis of Liénard II-type equations with respect to Lie point symmetries for the cases considered in Examples 2 and 3. For Example 1, the detailed information about λ-symmetry analysis to the path equation can be found in the reference [27]. We here consider some of Lie point symmetries to present λ-symmetry procedure of the generalized Morse oscillator equation (3.39) for the form that is given in Eq. (3.5) of Example 2, which has an eight-parameter symmetry groups (please see Sect. 6 for the detail).

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\sin\left(\sqrt{\gamma}t\right)\left(e^{\lambda_1 x}-1\right)}{\lambda_1}\frac{\partial}{\partial t}\\
&\quad+\frac{\sqrt{\gamma}\cos\left(\sqrt{\gamma}t\right)\left(e^{\lambda_1 x}-2\right)}{\lambda_1^2}\frac{\partial}{\partial x},\\
X_3 &= \frac{\cos\left(\sqrt{\gamma}t\right)\left(e^{\lambda_1 x}-1\right)}{\lambda_1}\frac{\partial}{\partial t}\\
&\quad-\frac{\sqrt{\gamma}\sin\left(\sqrt{\gamma}t\right)\left(e^{\lambda_1 x}-2\right)}{\lambda_1^2}\frac{\partial}{\partial x},\\
X_4 &= \frac{1-e^{-\lambda_1 x}}{\lambda_1}\frac{\partial}{\partial x},\\
X_5 &= -\frac{\sin\left(2\sqrt{\gamma}t\right)}{2\lambda_1}\frac{\partial}{\partial t}-\frac{\cos\left(2\sqrt{\gamma}t\right)\left(1-e^{-\lambda_1 x}\right)}{2\lambda_1\sqrt{\gamma}}\frac{\partial}{\partial x},\\
X_6 &= -\frac{\cos\left(2\sqrt{\gamma}t\right)}{2\lambda_1}\frac{\partial}{\partial t}+\frac{\sin\left(2\sqrt{\gamma}t\right)\left(1-e^{-\lambda_1 x}\right)}{2\lambda_1\sqrt{\gamma}}\frac{\partial}{\partial x},\\
X_7 &= \sin\left(\sqrt{\gamma}t\right)e^{-\lambda_1 x}\frac{\partial}{\partial x},\\
X_8 &= \cos\left(\sqrt{\gamma}t\right)e^{-\lambda_1 x}\frac{\partial}{\partial x}.
\end{aligned} \tag{4.17}$$

Here, we only consider Lie point symmetry generators X_1, X_4 and X_7 in (4.17).

Case 1. Let us consider X_1 vector field. Via this vector field, the infinitesimals are

$$\xi = 1, \quad \eta = 0. \tag{4.18}$$

By using Eq. (4.5), we obtain the λ-symmetry

$$\lambda = -\lambda_1\dot{x} - \frac{\gamma\left(1-e^{-\lambda_1 x}\right)}{\lambda_1\dot{x}}, \tag{4.19}$$

and then the integration factor becomes

$$\mu = 2e^{2x\lambda_1}\dot{x}. \tag{4.20}$$

We can write the first integral of the form

$$I = \frac{e^{x\lambda_1}}{\lambda^2}\left(e^{x\lambda_1}\lambda_1^2\dot{x}^2 + e^{x\lambda_1}\gamma - 2\gamma\right). \tag{4.21}$$

Case 2. With the similar procedure, we utilize X_4 vector field and then corresponding infinitesimals are

$$\xi = 0, \quad \eta = \frac{1-e^{-\lambda_1 x}}{\lambda_1}. \tag{4.22}$$

The λ-symmetry is

$$\lambda = \frac{\lambda_1\dot{x}}{e^{\lambda_1 x}-1}, \tag{4.23}$$

and then the integration factor is written as

$$\mu = \frac{e^{x\lambda_1}\left(1-e^{x\lambda_1}\right)}{\left(e^{x\lambda_1}-1\right)^2\gamma + e^{2x\lambda_1}{\lambda_1}^2\dot{x}^2}. \tag{4.24}$$

Thus, the first integral is found as below

$$D_t\left(-\frac{1}{\sqrt{\gamma}\lambda_1}\left(t\sqrt{\gamma}+\arctan\left(\frac{e^{x\lambda_1}\dot{x}\lambda_1}{\left(e^{x\lambda_1}-1\right)\sqrt{\gamma}}\right)\right)\right)=0, \tag{4.25}$$

and the invariant solution is given by

$$x(t) = \frac{1}{\lambda_1}\ln\left(1-e^{c_1}\cos\left(t\sqrt{\gamma}+c_2\right)\right), \tag{4.26}$$

where c_1 and c_2 are arbitrary constants. The Lagrangian L is written in the following form from (3.35)

$$\begin{aligned}
L &= \frac{1}{\sqrt{\gamma}\lambda_1}\left(t\sqrt{\gamma}+\arctan\left(\frac{e^{x\lambda_1}\dot{x}\lambda_1}{\left(e^{x\lambda_1}-1\right)\sqrt{\gamma}}\right)\right)\\
&\quad+\frac{e^{x\lambda_1}\dot{x}}{2\gamma\left(e^{x\lambda_1}-1\right)}\ln\left(\frac{\left(1-e^{x\lambda_1}\right)^2\gamma+e^{2x\lambda_1}\dot{x}^2\lambda_1^2}{\dot{x}^2}\right),
\end{aligned} \tag{4.27}$$

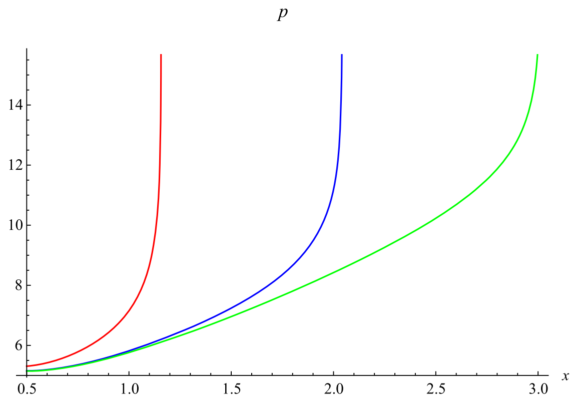

Fig. 9 Phase portrait for Eq. (4.26) for the three different values of integration constants

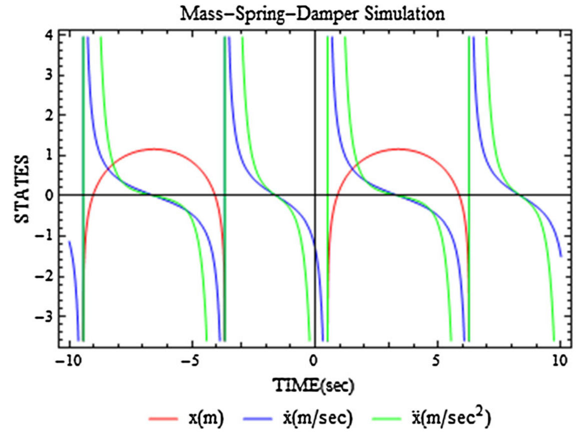

Fig. 10 Graph represents position $x(t)$, velocity $\dot{x}$ and acceleration $\ddot{x}$ of (4.26)

Hamiltonian is via (3.32)

$$
\begin{aligned}
H = &-\frac{1}{\sqrt{\gamma}\lambda_1} \\
&\times \left(t\sqrt{\gamma} + \arctan\left(\frac{e^{x\lambda_1}\lambda_1}{\sqrt{e^{2(1-e^{-x\lambda_1})p\gamma} - e^{2x\lambda_1}\lambda_1^2}}\right)\right),
\end{aligned}
\tag{4.28}
$$

and then conjugate momentum is written by (3.34)

$$
p = \frac{e^{x\lambda_1}}{2\gamma\left(e^{x\lambda_1}-1\right)} \ln\left(\frac{\left(1-e^{x\lambda_1}\right)^2\gamma + e^{2x\lambda_1}\dot{x}^2\lambda_1^2}{\dot{x}^2}\right). \tag{4.29}
$$

Figure 9 shows the plot of p versus x for Eq. (3.39) are given for the parametric choices of $\gamma = 0.4$ and $\lambda_1 = 1.4$ for the three different values of integration constants c_1 and c_2. Figure 10 presents the simulation for the position, velocity and deceleration.

Case 3. Another vector field X_7 gives the following infinitesimals

$$
\xi = 0, \quad \eta = \sin\left(\sqrt{\gamma}t\right)e^{-\lambda_1 x}. \tag{4.30}
$$

λ-symmetry is equal to

$$
\lambda = \sqrt{\gamma}\cot\left(t\sqrt{\gamma}\right) - \lambda_1\dot{x}, \tag{4.31}
$$

and the integration factor is written as

$$
\mu = e^{x\lambda_1}\sin\left(t\sqrt{\gamma}\right). \tag{4.32}
$$

The first integral of Eq. (3.39) is

$$
\begin{aligned}
I = &\frac{1}{\lambda_1}\sin\left(t\sqrt{\gamma}\right)\left(\sqrt{\gamma}\cot\left(t\sqrt{\gamma}\right)\right. \\
&\left.+ e^{\lambda_1 x}\left(-\sqrt{\gamma}\cot\left(t\sqrt{\gamma}\right) + \lambda_1\dot{x}\right)\right),
\end{aligned}
\tag{4.33}
$$

which gives the invariant solution of (3.39)

$$
\begin{aligned}
x(t) = &\frac{1}{\lambda_1}\ln\left(\sin(t\sqrt{\gamma})\right. \\
&\left.\times\left(c_1 - \frac{c_2\cot\left(t\sqrt{\gamma}\right)}{\sqrt{\gamma}} + \csc\left(t\sqrt{\gamma}\right)\right)\right),
\end{aligned}
\tag{4.34}
$$

where c_1 and c_2 are arbitrary constants. Similarly, with respect to the previous case of the Lagrangian, the Hamiltonian and the conjugate momentum can be derived as below, respectively

$$
\begin{aligned}
L = &\frac{\left(e^{x\lambda_1}-1\right)\sqrt{\gamma}\cos\left(t\sqrt{\gamma}\right)}{\lambda_1} \\
&+ e^{x\lambda_1}\dot{x}\left(\ln(\dot{x}) - 1\right)\sin\left(t\sqrt{\gamma}\right), \\
H = &\frac{\left(e^{x\lambda_1}-1\right)\sqrt{\gamma}\cos\left(t\sqrt{\gamma}\right)}{\lambda_1} \\
&+ e^{x\lambda_1 + e^{-x\lambda_1}p\csc(t\sqrt{\gamma})}\sin\left(t\sqrt{\gamma}\right), \\
p = &\, e^{x\lambda_1}\ln(\dot{x})\sin\left(t\sqrt{\gamma}\right).
\end{aligned}
\tag{4.35}
$$

Using the same parametric values in the previous case, the plots p versus x and numerical simulation for the position, velocity and acceleration are presented in Figs. 11 and 12, respectively. Similarly for the harmonic oscillator equation (3.48) of Example 3, we follow the similar approach by considering its Lie point symmetries. It can be shown that Lie point symmetries of Eq. (3.48) (please see Sect. 6 for the detail) are

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{x\sin\left(\sqrt{\gamma}t\right)}{1+\lambda_1 x}\frac{\partial}{\partial t} + \frac{\sqrt{\gamma}\cos\left(\sqrt{\gamma}t\right)\left(1+2\lambda_1 x\right)}{\lambda_1^2}\frac{\partial}{\partial x},
\end{aligned}
$$

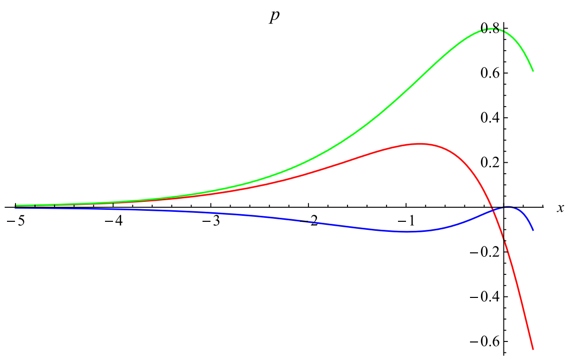

Fig. 11 Phase portrait for the conjugate momentum for Eq. (4.34) for the three different values of integration constants

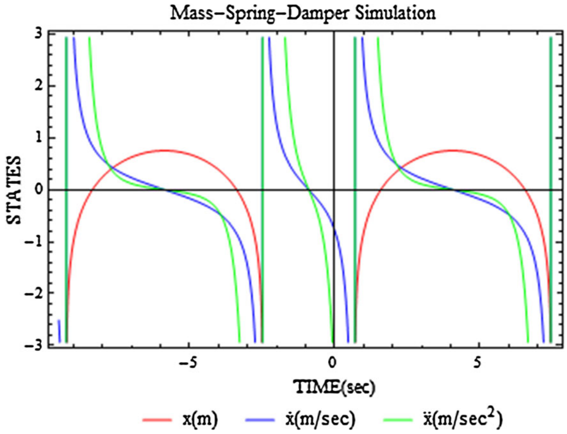

Fig. 12 State simulation representing position $x(t)$, velocity $\dot{x}$ and acceleration $\ddot{x}$ of (4.34)

$$X_3 = \frac{x\cos\left(\sqrt{\gamma}t\right)}{1+\lambda_1 x}\frac{\partial}{\partial t} - \frac{\sqrt{\gamma}\sin\left(\sqrt{\gamma}t\right)(1+2\lambda_1 x)}{\lambda_1^2}\frac{\partial}{\partial x},$$
$$X_4 = \left(x+\lambda_1 x^2\right)\frac{\partial}{\partial x},$$
$$X_5 = -\frac{\sin\left(2\sqrt{\gamma}t\right)}{2\gamma}\frac{\partial}{\partial t} - \frac{\left(x+\lambda_1 x^2\right)\cos\left(2\sqrt{\gamma}t\right)}{2\sqrt{\gamma}}\frac{\partial}{\partial x},$$
$$X_6 = -\frac{\cos\left(2\sqrt{\gamma}t\right)}{2\gamma}\frac{\partial}{\partial t} + \frac{\left(x+\lambda_1 x^2\right)\sin\left(2\sqrt{\gamma}t\right)}{2\sqrt{\gamma}}\frac{\partial}{\partial x},$$
$$X_7 = \sin\left(\sqrt{\gamma}t\right)(1+\lambda_1 x)^2\frac{\partial}{\partial x},$$
$$X_8 = \cos\left(\sqrt{\gamma}t\right)(1+\lambda_1 x)^2\frac{\partial}{\partial x}. \tag{4.36}$$

For this example, we only deal with the application of the symmetries X_1, X_4 and X_8.

Case 1. By utilizing the similar process, the infinitesimal functions corresponding to X_1 vector field are

$$\xi = 1, \quad \eta = 0. \tag{4.37}$$

From (4.14) λ-symmetry is found

$$\lambda = \frac{2\dot{x}\lambda_1}{1+\lambda_1 x} - \frac{x\sqrt{\gamma}(1+\lambda_1 x)}{\dot{x}}, \tag{4.38}$$

and the integration factor becomes

$$\mu = \frac{2\dot{x}}{(1+\lambda_1 x)^4}. \tag{4.39}$$

Thus, one can find the first integral

$$I = \frac{1}{\lambda_1^2(1+\lambda_1 x)^4} \times \left(-\gamma - 4\gamma x\lambda_1 - 5\gamma x^2\lambda_1^2 + \dot{x}^2\lambda_1^2 - 2\gamma x^3\lambda_1^3\right). \tag{4.40}$$

Case 2. Now based on X_4 vector field, the infinitesimals are obtained as the following form

$$\xi = 0, \quad \eta = x + \lambda_1 x^2. \tag{4.41}$$

The λ-symmetry is

$$\lambda = \frac{\dot{x}(2x\lambda_1+1)}{x(1+x\lambda_1)}, \tag{4.42}$$

which yields the integration factor

$$\mu = \frac{x(1+x\lambda_1)}{\dot{x}^2+\gamma x^2(1+x\lambda_1)^2}. \tag{4.43}$$

First integral of (3.48) can be written

$$I = t + \frac{1}{\sqrt{\gamma}}\arctan\left(\frac{\dot{x}}{\sqrt{\gamma}x(1+x\lambda_1)}\right), \tag{4.44}$$

which gives the invariant solution of (3.48)

$$x(t) = \frac{e^{c_1}\cos\left(\sqrt{\gamma}(c_2-t)\right)}{e^{c_1}\lambda_1\cos\left(\sqrt{\gamma}(c_2-t)\right)-1}, \tag{4.45}$$

where c_1 and c_2 are arbitrary constants. Via Eq. (3.35), Lagrangian formulation is obtained as

$$L = -t - \frac{1}{\sqrt{\gamma}}\arctan\left(\frac{\dot{x}}{\sqrt{\gamma}x(1+x\lambda_1)}\right) - \frac{\dot{x}}{2x\gamma(1+x\lambda_1)}\ln\left(\frac{\dot{x}^2+x^2\gamma(1+x\lambda_1)^2}{\dot{x}^2}\right), \tag{4.46}$$

and by using Eq. (3.34), the conjugate momentum is found in the time-independent form

$$p = \frac{1}{2x\gamma(1+x\lambda_1)}\ln\left(\frac{\dot{x}^2}{\dot{x}^2+x^2\gamma(1+x\lambda_1)^2}\right). \tag{4.47}$$

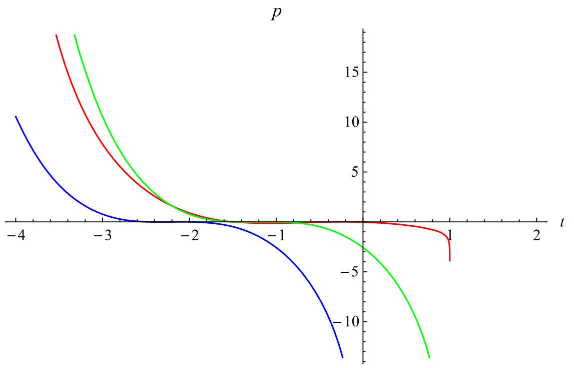

Fig. 13 Phase portrait for the conjugate momentum for Eq. (4.45) with respect to t

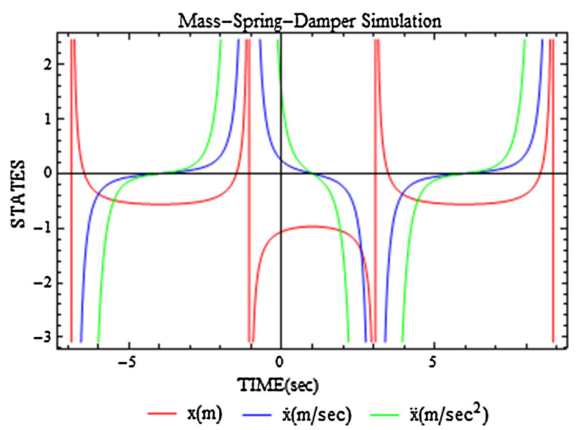

Fig. 14 State simulation representing position $x(t)$, velocity $\dot{x}$ and acceleration $\ddot{x}$ of (4.45)

Figures 13 and 14 present the plots of p versus t for the parameters $\gamma = 0.4$ and $\lambda_1 = 1.4$ for the three different values of integration constants c_1 and c_2 and the numerical simulation for position, velocity and acceleration, respectively. Figure 15 shows the comparison of the graphs between the solution (4.45) and the linear case $f(x) = g(x) =$ constant.

Case 3. Finally, by means of X_8 vector field, the infinitesimals of (3.48) are written

$$\xi = 0, \quad \eta = \cos\left(t\sqrt{\gamma}\right)(x + \lambda_1 x)^2. \tag{4.48}$$

The corresponding λ-symmetry is

$$\lambda = \frac{\dot{x}}{(x\lambda_1 + 1)} - \sqrt{\gamma}\tan\left(t\sqrt{\gamma}\right), \tag{4.49}$$

and integration factor from Eq. (4.16) is obtained

$$\mu = \frac{\cos\left(t\sqrt{\gamma}\right)}{(x + \lambda_1 x)^2}. \tag{4.50}$$

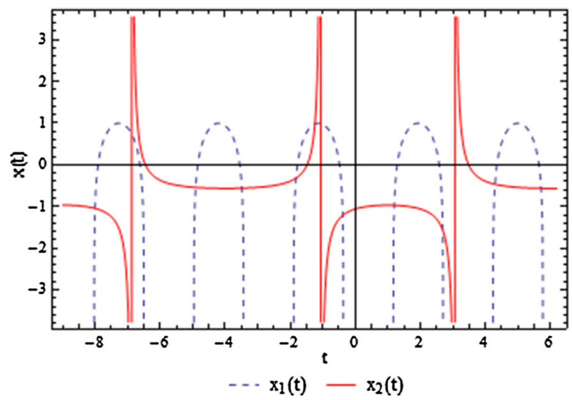

Fig. 15 Plot of two solutions corresponding to (4.45) as $x_2(t)$ and corresponding for $f(x) = g(x) = constant$ in (3.1) as $x_1(t)$

Additionally, one can find first integral of equation

$$I = \frac{\dot{x}\cos\left(t\sqrt{\gamma}\right) + \sqrt{\gamma}x(x\lambda_1 + 1)\sin\left(t\sqrt{\gamma}\right)}{(1 + \lambda_1 x)^2}, \tag{4.51}$$

which gives the invariant solution

$$x(t) = \frac{-\sqrt{\gamma} - c_2\gamma\lambda_1 c_1 - c_2\lambda_1\tan\left(t\sqrt{\gamma}\right)}{c_1 c_2\gamma^{3/2}{\lambda_1}^2 - 2\sqrt{\gamma}\lambda_1\sec\left(t\sqrt{\gamma}\right)\sin\left(\frac{t\sqrt{\gamma}}{2}\right)^2 + c_2{\lambda_1}^2\tan\left(t\sqrt{\gamma}\right)}, \tag{4.52}$$

where c_1 and c_2 are arbitrary constants. Lagrangian and time-dependent conjugate momentum are expressed by

$$L = \frac{\dot{x}\cos\left(t\sqrt{\gamma}\right)(\ln(\dot{x}) - 1) - \sqrt{\gamma}x\,(x\lambda_1 + 1)\sin\left(t\sqrt{\gamma}\right)}{(1 + \lambda_1 x)^2}, \tag{4.53}$$

$$p = \frac{\cos\left(t\sqrt{\gamma}\right)\ln\left(\dot{x}\right)}{(1 + \lambda_1 x)^2}. \tag{4.54}$$

Similarly, Fig. 16 presents the plot p versus t with the same values for parameters γ and λ_1. Three different colors correspond to the three different values of integration constants. Figure 17 presents the state simulation for the position $x(t)$, velocity $\dot{x}$ and acceleration $\ddot{x}$ of (4.52).

5 Liénard I-type equation, the corresponding S and L transformations and λ-symmetry

In this part, we study Liénard I-type differential equation based on S and L transformations and the relation

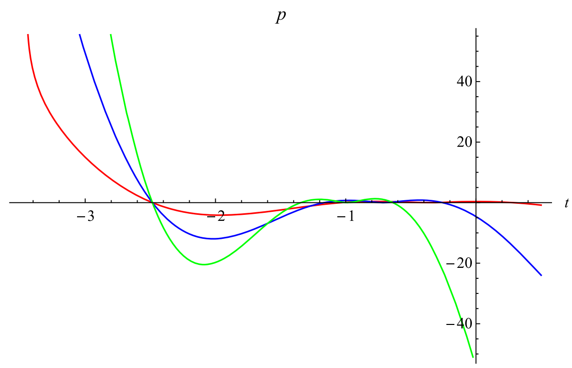

Fig. 16 Phase portrait of conjugate momentum for the Eq. (4.52) with respect to t

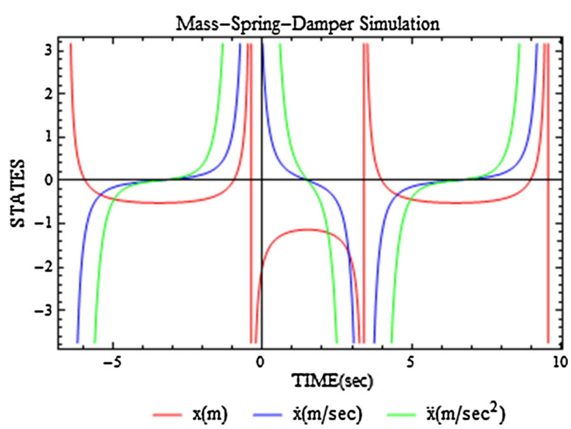

Fig. 17 State simulation for the position $x(t)$, velocity $\dot{x}$ and acceleration $\ddot{x}$ of (4.52)

with λ-symmetry and some related properties. In addition to Liénard II-type equations, a great number of mathematical models of physical systems give rise to differential equation of the Liénard I-type equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{5.1}$$

where $f(x)$ and $g(x)$ are arbitrary real functions of x. Linearizable conditions for Liénard equation consist of two cases, which are $S_1 = S_2 = 0$ and $S_3 = S_4 = 0$.

Proposition 5.1 *Liénard I-type equation is S-linearizable and L-linearizable with the necessary condition $S_1 = S_2 = 0$ satisfies if and only if $f(x) = k$ and $g(x) = \lambda_1 x + \lambda_2$ where λ_1 and λ_2 are constants.*

In this case, the Liénard I-type equation gets the form

$$\ddot{x} + k\dot{x} + \lambda_1 x + \lambda_2 = 0, \tag{5.2}$$

which represents the displaced damped harmonic oscillator. Equation (2.16) gives

$$\tau(t) = -\frac{k^2}{4} + \lambda_1. \tag{5.3}$$

Then, a particular solution of the system (2.18)

$$C(t,x) = \frac{\lambda_1 x + \lambda_2}{2\lambda_1}\left(2\alpha + 2\beta\tan(\beta t)\right), \tag{5.4}$$

can be found, where $\alpha = \frac{k}{2}$, $\beta = \frac{1}{2}(4\lambda_1 - k^2)^{1/2}$. Hence, $P(t,x)$ via Eq. (2.20) can be written as

$$P(t,x) = \alpha t. \tag{5.5}$$

Solving Eqs. (2.21) and (2.22), the pair F and G define a transformation (2.3)

$$\begin{aligned} F(t,x) &= \varphi\left(\frac{e^{kt}}{\lambda_1}(\lambda_1 x + \lambda_2)\sec(\beta t)\right), \\ G(t,x) &= \sec(\beta t)^2\varphi'\left(\frac{e^{kt}}{\lambda_1}(\lambda_1 x + \lambda_2)\sec(\beta t)\right), \end{aligned} \tag{5.6}$$

that linearize Eq. (5.2). From (2.6), first integral of (5.2) can be written as

$$I = \frac{e^{kt}}{2}\cos(\beta t)\dot{x} + \frac{e^{kt/2}}{2\lambda_1}(\lambda_1 x + \lambda_2) \cos(\beta t)(k + 2\beta\tan(\beta t)). \tag{5.7}$$

The local transformations are

$$\begin{aligned} R(t,x) &= \frac{e^{kt/2}}{\lambda_1}(\lambda_1 x + \lambda_2)\sec(\beta t), \\ S(t,x) &= \frac{\tan(\beta t)}{\beta}. \end{aligned} \tag{5.8}$$

Thus, the λ-symmetry for Eq. (5.2) can be found by the formula (4.8) in the following form

$$\lambda = -\alpha - \beta\tan(\beta t). \tag{5.9}$$

Proposition 5.2 *Another case of* Eq. (5.1) *$S_1 \neq 0$ and $S_3 = S_4 = 0$ if and only if the relation between $f(x)$ and $g(x)$ is as $f(x) = \frac{1}{\lambda_1} + g'(x)$, then* (5.1) *is called S-linearizable.*

Equation (5.1) can be rewritten

$$\ddot{x} + \left(\frac{1}{\lambda_1} + g'(x)\right)\dot{x} + g(x) = 0. \tag{5.10}$$

We recall the formula $u = \frac{S_1}{S_2}$, then in this case $u = -\lambda_1 g'(x)$ is obtained. Equations (2.23) and (2.25) yield

$$C(t,x) = \lambda_1 g(x), \quad P(t,x) = \frac{t}{\lambda_1}. \tag{5.11}$$

Consequently, the pair of S-transformations are evaluated as below

$$F = \varphi\left(\lambda_1 t + \int \frac{1}{g(x)}\mathrm{d}x\right),$$
$$G = \frac{\mathrm{e}^{-t/\lambda_1}}{g(x)}\varphi'\left(\lambda_1 t + \int \frac{1}{g(x)}\mathrm{d}x\right), \tag{5.12}$$

and the corresponding first integral is

$$I = \mathrm{e}^{-t/\lambda_1}\left(g(x) + \dot{x}\right). \tag{5.13}$$

The λ-symmetry can be found by Eq. (4.9) of the form

$$\lambda = -\lambda_1 g(x). \tag{5.14}$$

Similar to the Liénard II-type equations, one can deal with λ-symmetry by considering the connection of these equations with the Lie point symmetry generators. For example, for the Lie point symmetry $X_6 = (x + \frac{\lambda_2}{\lambda_1})\frac{\partial}{\partial x}$ of Eq. (5.10) the infinitesimals yield

$$\xi = 0, \quad \eta = x + \frac{\lambda_2}{\lambda_1}. \tag{5.15}$$

From Eq. (4.14) the λ-symmetry is found

$$\lambda = \frac{\dot{x}\lambda_1}{\lambda_1 x + \lambda_2}, \tag{5.16}$$

and the integration factor from (4.16) is equal to

$$\mu = \frac{\lambda_1 x + \lambda_2}{\dot{x}^2\lambda_1 + k\dot{x}(\lambda_1 x + \lambda_2) + (\lambda_1 x + \lambda_2)^2}, \tag{5.17}$$

thus the time-dependent first integral of Eq. (5.10) becomes

$$I = t + \frac{1}{\beta}\arctan\left(\frac{2\dot{x}\lambda_1 + k(\lambda_1 x + \lambda_2)}{2\beta(\lambda_1 x + \lambda_2)}\right). \tag{5.18}$$

6 Extended Prelle–Singer method and λ-symmetry connection

In fact, the extended Prelle–Singer method can also be considered to study Liénard I-type and Liénard II-type equations with the relation between λ-symmetry approach and the extended Prelle–Singer method [5] for the investigation of new first integrals, integrating factors and invariant solutions. Let us consider the second-order ODEs of the form

$$\ddot{x} = \frac{P}{Q} = \phi, \quad P, Q \in \mathbb{C}[t, x, \dot{x}], \tag{6.1}$$

where over dot denotes differentiation with respect to time and P and Q are analytic functions of the variables t, x and $\dot{x}$. Let us assume that (6.1) admits a first integral $I(t, x, \dot{x}) = C$, where C is a constant, then the total differentials are

$$\mathrm{d}I = I_t\mathrm{d}t + I_x\mathrm{d}x + I_{\dot{x}}\mathrm{d}\dot{x} = 0, \tag{6.2}$$

where the subscript denotes partial differentiation with respect to that variable. Reconsidering Eq. (6.1) in the form $(P/Q)\mathrm{d}t - \mathrm{d}\dot{x} = 0$ and adding a null term $\Omega(t, x, \dot{x})\dot{x}\mathrm{d}t - \Omega(t, x, \dot{x})\mathrm{d}x$ to the latter, we obtain that on the solutions the one-form

$$\left(\frac{P}{Q} + \Omega\dot{x}\right)\mathrm{d}t - \Omega\mathrm{d}x - \mathrm{d}\dot{x} = 0. \tag{6.3}$$

Hence, on the solutions, the one-forms (6.2) and (6.3) must be proportional. Multiplying (6.3) by the factor $\Psi(t, x, \dot{x})$, which acts as the integrating factor for (6.3), we have

$$\mathrm{d}I = \Psi(\phi + \Omega\dot{x})\mathrm{d}t - \Psi\Omega\mathrm{d}x - \Psi\mathrm{d}\dot{x} = 0, \tag{6.4}$$

where $\phi = P/Q$. Comparing Eqs. (6.2) and (6.4), we obtain three relations relating the integral, integrating factor and the null term,

$$I_t = \Psi(\phi + \Omega\dot{x}),$$
$$I_x = -\Psi\Omega,$$
$$I_{\dot{x}} = -\Psi. \tag{6.5}$$

Then, the differential equations in terms of Ψ and Ω are given as below

$$\Omega_t + \dot{x}\Omega_x + \phi\Omega_{\dot{x}} = -\phi_x + \Omega\phi_{\dot{x}} + \Omega^2, \tag{6.6}$$
$$\Psi_t + \dot{x}\Psi_x + \phi\Psi_{\dot{x}} = \Psi(\phi + \Omega\dot{x}), \tag{6.7}$$
$$\Psi_x = \Psi_{\dot{x}}\Omega + \Psi\Omega_{\dot{x}}. \tag{6.8}$$

The last equation can be considered as a compatibility equation. In fact, in addition to the classical investigation procedure of Lie point symmetries of differential equations [10–16], one can present a connection between Lie point symmetries and the extended Prelle–Singer method discussed above. For this purpose, it can be proved that the first equation (6.6) is related to the Lie point symmetries of a given differential equation. Then, we can define the following statements.

Remark 2 For the generalized Morse oscillator (3.39), the solution of Eq. (6.6) by considering $\phi = -\lambda_1\dot{x}^2 - \frac{\gamma}{\lambda_1}(1-\mathrm{e}^{-\lambda_1 x})$ and $\Omega = -\frac{A(Q)}{Q}$ for $Q = \eta - \xi y'$ and $A = \partial_t + \dot{x}\partial_x + \phi(t,x,\dot{x})\partial_{\dot{x}}$ gives the infinitesimal functions ξ and η, which construct an eight-parameter symmetry groups given in (4.17).

Remark 3 Similarly, for $\phi = \frac{2\lambda_1}{1+\lambda_1}\dot{x}^2 - \gamma(x+\lambda_1 x^2)$ the solution of Eq. (6.6) gives symmetry groups given in (4.36) corresponding to the harmonic oscillator (3.48).

If the solutions for functions Ψ and Ω, which satisfy Eqs. (6.7) and (6.8), are found, then the first integral $I(t,x,\dot{x})$ can be found by the relation

$$I(t,x,\dot{x}) = r_1 - r_2 - \int\left[\Psi + \frac{\mathrm{d}}{\mathrm{d}\dot{x}}(r_1 - r_2)\right]\mathrm{d}\dot{x}, \qquad (6.9)$$

where

$$r_1 = \int \Psi(\phi + \Omega\dot{x})\,\mathrm{d}t \quad \text{and}$$
$$r_2 = \int\left(\Psi\Omega + \frac{\mathrm{d}}{\mathrm{d}x}r_1\right)\mathrm{d}x. \qquad (6.10)$$

In the following subcases, we analyze the integrating factors and first integrals by considering these mathematical relations between λ-symmetry and null Ω function and integrating factor μ and Ψ function by comparing the results found in the previous sections.

Proposition 6.1 *The function Ψ is equal to negative of integrating factor μ, and null function Ω is equal to negative of λ.*

6.1 Application to the generalized morse oscillator

We deal with the generalized morse oscillator equation as given in (3.39) as an example of Liénard II-type equation

$$\ddot{x} + \lambda_1\dot{x}^2 + \frac{\gamma}{\lambda_1}\left(1 - \mathrm{e}^{-\lambda_1 x}\right) = 0. \qquad (6.11)$$

From (4.10), we have

$$\phi = -\frac{\left(1-\mathrm{e}^{-\lambda_1 x}\right)\gamma}{\lambda_1} - \lambda_1\dot{x}^2, \qquad (6.12)$$

and the null form in the general case is

$$\Omega = -\frac{\phi}{\dot{x}} = \frac{\dot{x}^2\lambda_1^2 - \mathrm{e}^{-\lambda_1 x}\gamma + \gamma}{\dot{x}\lambda_1}. \qquad (6.13)$$

To obtain the integrating factor Ψ, one can consider the ansatz in the following form [5],

$$\Psi = \frac{\dot{x}}{\left(A(x) + B(x)\dot{x} + C(x)\dot{x}^2\right)^r}, \qquad (6.14)$$

where A, B and C are functions of only variable x and r is constant. If we substitute (6.14) in Eqs. (6.7) and (6.8), we obtain a set of equations in terms of $\dot{x}$ and its powers. From these equations the explicit solution forms of A, B and C functions can be found. Following this procedure, the integrating factor Ψ can be found in the form

$$\Psi = \dot{x}\left(b\mathrm{e}^{-2x\lambda_1/r} + a\mathrm{e}^{2x(-\lambda_1+r\lambda_1)/r}\dot{x}^2 + \frac{a\left(\mathrm{e}^{x\lambda_1} - 2\right)\gamma\mathrm{e}^{x\lambda_1 - 2x\lambda_1/r}}{\lambda_1^2}\right)^{-r}, \qquad (6.15)$$

where a and b are arbitrary constants. In addition, by using the relation (6.9) the time-independent first integral

$$I = \frac{\mathrm{e}^{(2(1-r)x\lambda_1)/r}}{2a(r-1)} \times\left(\frac{\mathrm{e}^{-2x\lambda_1/r}}{\lambda_1^2}\left(b\lambda_1^2 + a\mathrm{e}^{x\lambda_1}\left(-2\gamma + \mathrm{e}^{x\lambda_1}\left(\gamma + \dot{x}^2\lambda_1^2\right)\right)\right)\right), \quad r \neq 1, \qquad (6.16)$$

is derived. In the following cases, we present new integrating factors and first integrals of generalized morse oscillator by using the relation λ-symmetry with the extended Prelle–Singer approach for each corresponding Lie point symmetry.

Case 1. By using X_1 from Eq. (4.17), the infinitesimals ξ and η correspond to

$$\xi = 1, \quad \eta = 0. \qquad (6.17)$$

This case coincides with the general case obtained by using the extended Prelle–Singer procedure discussed above. It is easy to see that for the case $a = 0$, $b = 1/2$ and $r = 1$ the integration factor Ψ in (4.20) can be easily obtained from (6.16).

Case 2. If we consider X_2 from (4.17) the infinitesimals ξ and η are written as

$$\xi = \frac{\sin\left(\sqrt{\gamma}t\right)\left(e^{\lambda_1 x}-1\right)}{\lambda_1},$$
$$\eta = \frac{\sqrt{\gamma}\cos\left(\sqrt{\gamma}t\right)\left(e^{\lambda_1 x}-2\right)}{\lambda_1^2}. \tag{6.18}$$

By using the relation $\lambda = -\Omega$, we have

$$\Omega = -\lambda = \frac{\lambda_1^2\left(\frac{\dot{x}\cos(\sqrt{\gamma}t)}{\lambda_1} + \sin\left(\sqrt{\gamma}t\right)\left(\frac{e^{-\lambda_1 x}\gamma}{\lambda_1^2} - \dot{x}^2\right)\right)}{\left(e^{\lambda_1 x}-1\right)\dot{x}\sin(\sqrt{\gamma}t) - \left(e^{\lambda_1 x}-2\right)\sqrt{\gamma}\cos\left(\sqrt{\gamma}t\right)}. \tag{6.19}$$

If Ψ may be defined as the following form

$$\Psi = \frac{Sd}{(A(t,x)+B(t,x)\dot{x})^r}, \tag{6.20}$$

where $A(t,x)$ and $B(t,x)$ are functions of x and t and Sd is the denominator of Ω and we substitute (6.20) in (6.7) by considering Sd, then the functions $A(t,x)$ and $B(t,x)$ are determined and Ψ is written in the form

$$\Psi = -\mu = \frac{e^{2x\lambda_1}\lambda_1^3\left(\left(e^{\lambda_1 x}-2\right)\sqrt{\gamma}\cos\left(\sqrt{\gamma}t\right) - \left(e^{\lambda_1 x}-1\right)\dot{x}\lambda_1\sin\left(\sqrt{\gamma}t\right)\right)}{\left(e^{\lambda_1}\dot{x}\lambda_1\cos\left(\sqrt{\gamma}t\right) + \left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\sin\left(\sqrt{\gamma}t\right)\right)^3}. \tag{6.21}$$

It is easy to see that Ω and Ψ also satisfy the compatibility condition (6.8). From Eq. (6.9) we find time-dependent first integral by considering (6.10)

$$I = \frac{e^{2x\lambda_1}\lambda_1^2\left(-2\gamma + e^{x\lambda_1}\left(\gamma + \dot{x}^2\lambda_1^2\right)\right)}{2\sqrt{\gamma}\left(e^{\lambda_1}\dot{x}\lambda_1\cos\left(\sqrt{\gamma}t\right) + \left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\sin\left(\sqrt{\gamma}t\right)\right)^2}. \tag{6.22}$$

Case 3. By using X_3 from Eq. (4.17) the vector field for infinitesimals ξ and η corresponds to

$$\xi = \frac{\cos\left(\sqrt{\gamma}t\right)\left(e^{\lambda_1 x}-1\right)}{\lambda_1},$$
$$\eta = -\frac{\sqrt{\gamma}\sin\left(\sqrt{\gamma}t\right)\left(e^{\lambda_1 x}-2\right)}{\lambda_1^2}. \tag{6.23}$$

For this case

$$\Omega = -\lambda = \frac{e^{-x\lambda_1}\left(-\gamma\cos\left(\sqrt{\gamma}t\right) + e^{x\lambda_1}\dot{x}\lambda_1\left(\dot{x}\lambda_1\cos\left(\sqrt{\gamma}t\right) + \sqrt{\gamma}\sin\left(\sqrt{\gamma}t\right)\right)\right)}{\left(e^{\lambda_1 x}-1\right)\dot{x}\cos\left(\sqrt{\gamma}t\right) - \left(e^{\lambda_1 x}-2\right)\sqrt{\gamma}\sin\left(\sqrt{\gamma}t\right)}, \tag{6.24}$$

is obtained. Similar to the first case, from solutions of $A(t,x)$ and $B(t,x)$, the solution of Ψ can be written as

$$\Psi = -\mu = \frac{e^{2x\lambda_1}\lambda_1^3\left(\left(e^{\lambda_1 x}-2\right)\sqrt{\gamma}\sin\left(\sqrt{\gamma}t\right) + \left(e^{\lambda_1 x}-1\right)\dot{x}\lambda_1\cos\left(\sqrt{\gamma}t\right)\right)}{\left(e^{\lambda_1}\dot{x}\lambda_1\sin\left(\sqrt{\gamma}t\right) - \left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\cos\left(\sqrt{\gamma}t\right)\right)^3}, \tag{6.25}$$

and the time-dependent first integral is obtained from (6.9) and (6.10)

$$I = \frac{e^{2x\lambda_1}\lambda_1^2\left(-2\gamma + e^{x\lambda_1}\left(\gamma + \dot{x}^2\lambda_1^2\right)\right)}{2\sqrt{\gamma}\left(-e^{\lambda_1}\dot{x}\lambda_1\sin\left(\sqrt{\gamma}t\right) + \left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\cos\left(\sqrt{\gamma}t\right)\right)^2}. \tag{6.26}$$

Case 4. For symmetry X_4 from Eq. (4.17), the vector field is

$$\xi = 0, \quad \eta = \frac{1-e^{\lambda_1 x}}{\lambda_1}. \tag{6.27}$$

For this case, the null function

$$\Omega = -\lambda = \frac{\dot{x}\lambda_1}{1-e^{\lambda_1 x}}, \tag{6.28}$$

is obtained. Because ξ, η and Ω are time-independent we can write Ψ as below if it is assumed to be in the form (6.14)

$$\Psi = \dot{x}\left(\frac{ce^{-2\lambda_1 x}\dot{x}^2}{\left(e^{\lambda_1 x}-1\right)^2}\right)^{-1/2}, \tag{6.29}$$

where c is an arbitrary constant. Thus, it is easy to see that Ω and Ψ satisfy (6.6) and (6.7). However, it can be checked that the last equation as compatibility condition (6.8) is not satisfied. To overcome this difficulty, we consider the modified form of Ψ function in the following form

$$\hat{\Psi} = F(I) \cdot \Psi, \tag{6.30}$$

where $F(I)$ is an arbitrary function of the first integral I. Here, the first integral I can obtained from Eq. (4.21) and one can write

$$\hat{\Psi} = F\left(\frac{e^{x\lambda_1}}{\lambda^2}\left(e^{x\lambda_1}\lambda_1^2\dot{x}^2 + e^{x\lambda_1}\gamma - 2\gamma\right)\right) \times \dot{x}\left(\frac{ce^{-2\lambda_1 x}\dot{x}^2}{\left(e^{\lambda_1 x}-1\right)^2}\right)^{-1/2}. \tag{6.31}$$

Then, substituting (6.31) in (6.30) into the compatible condition (6.8) a first-order differential equation in terms of I

$$F(I) + \left(-\frac{\gamma}{\lambda_1^2} + I\right)F'(I) = 0, \tag{6.32}$$

is obtained by yielding the solution

$$\Omega = -\lambda = \frac{\left(2e^{x\lambda_1}-1\right)\dot{x}\sqrt{\gamma}\lambda_1\cos\left(2t\sqrt{\gamma}\right) - \left(\gamma + e^{x\lambda_1}\left(-\gamma + \dot{x}^2\lambda_1^2\right)\right)\sin\left(2t\sqrt{\gamma}\right)}{\left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\cos\left(2t\sqrt{\gamma}\right) - e^{\lambda_1 x}\dot{x}\lambda_1\sin\left(2t\sqrt{\gamma}\right)}, \tag{6.39}$$

$$F(I) = \frac{1}{\gamma - I\lambda_1^2}. \tag{6.33}$$

Thus, the modifying form (6.30) has the form

$$\hat{\Psi} = -\mu = \left(\left(e^{\lambda_1 x}-1\right)^2\gamma + e^{2\lambda_1 x}\dot{x}^2\lambda_1^2\right)\dot{x} \times \left(\frac{ce^{-2\lambda_1 x}\dot{x}^2}{\left(e^{\lambda_1 x}-1\right)^2}\right)^{-1/2}, \tag{6.34}$$

which now satisfies the compatible condition (6.8). Finally, by considering (6.10), one can write

$$r_1 = -\frac{e^{-\lambda_1 x}t\dot{x}}{(e^{\lambda_1 x}-1)\lambda_1}\left(\frac{ce^{-2\lambda_1 x}\dot{x}^2}{(e^{\lambda_1 x}-1)^2}\right)^{-1/2}, \tag{6.35}$$

$$r_2 = -\frac{e^{-\lambda_1 x}\dot{x}\arctan\left(\frac{e^{x\lambda_1}\lambda_1^2\dot{x}^2 + e^{x\lambda_1}\gamma - \gamma}{\dot{x}\lambda_1\sqrt{\gamma}}\right)}{\sqrt{\gamma}(e^{\lambda_1 x}-1)\lambda_1} \times \left(\frac{ce^{-2\lambda_1 x}\dot{x}^2}{(e^{\lambda_1 x}-1)^2}\right)^{-1/2}. \tag{6.36}$$

Hence, the following time-dependent first integral in the form

$$I = -\frac{e^{-\lambda_1 x}\dot{x}\left(t\sqrt{\gamma} + \arctan\left(\frac{\dot{x}\lambda_1}{\sqrt{\gamma}} - \arctan\left(\frac{e^{x\lambda_1}\lambda_1^2\dot{x}^2 + e^{x\lambda_1}\gamma - \gamma}{\dot{x}\lambda_1\sqrt{\gamma}}\right)\right)\right)}{\sqrt{\gamma}\left(e^{\lambda_1 x}-1\right)\lambda_1} \times \left(\frac{ce^{-2\lambda_1 x}\dot{x}^2}{\left(e^{\lambda_1 x}-1\right)^2}\right)^{-1/2}, \tag{6.37}$$

is found.

Case 5. For X_5 from Eq. (4.17), the infinitesimals ξ and η are

$$\xi = \frac{-\sin\left(2\sqrt{\gamma}t\right)}{2\gamma}, \quad \eta = -\frac{\cos\left(\sqrt{\gamma}t\right)\left(1-e^{-\lambda_1 x}\right)}{2\sqrt{\gamma}\lambda_1}. \tag{6.38}$$

For this case the null function

is obtained. If we solve the determining equations depending on the functions $A(t, x)$ and $B(t, x)$, then we have

$$\Psi = -\mu = e^{\lambda_1 x}\left(\left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\cos\left(2t\sqrt{\gamma}\right) - e^{\lambda_1 x}\dot{x}\lambda_1\sin\left(2t\sqrt{\gamma}\right)\right). \tag{6.40}$$

Thus, time-dependent first integral is found to be

$$I = -\frac{1}{2\lambda_1}\left(2e^{\lambda_1 x}\left(e^{\lambda_1 x}-1\right)\dot{x}\sqrt{\gamma}\lambda_1\cos\left(2t\sqrt{\gamma}\right) + \left(\left(e^{\lambda_1 x}-1\right)^2\gamma - e^{2x\lambda_1}\dot{x}^2\lambda_1^2\right)\sin\left(2t\sqrt{\gamma}\right)\right). \tag{6.41}$$

Case 6. For X_6 from Eq. (4.17) we have

$$\xi = \frac{-\cos\left(2\sqrt{\gamma}t\right)}{2\gamma}, \quad \eta = -\frac{\sin\left(\sqrt{\gamma}t\right)\left(1-e^{-\lambda_1 x}\right)}{2\sqrt{\gamma}\lambda_1}. \tag{6.42}$$

For this case

$$\Omega = -\lambda = \frac{\left(2e^{x\lambda_1}-1\right)\dot{x}\sqrt{\gamma}\lambda_1\sin\left(2t\sqrt{\gamma}\right) - \left(-\gamma + e^{x\lambda_1}\left(\gamma + \dot{x}^2\lambda_1^2\right)\right)\cos\left(2t\sqrt{\gamma}\right)}{\left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\cos\left(2t\sqrt{\gamma}\right) - e^{\lambda_1 x}\dot{x}\lambda_1\sin\left(2t\sqrt{\gamma}\right)}, \tag{6.43}$$

is obtained. Based on Eq. (6.20), the integrating factor is

$$\begin{aligned}\Psi &= -\mu \\ &= e^{\lambda_1 x}\left(\left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\sin\left(2t\sqrt{\gamma}\right)+e^{\lambda_1 x}\dot{x}\lambda_1\cos\left(2t\sqrt{\gamma}\right)\right).\end{aligned} \tag{6.44}$$

The corresponding time-dependent first integral is

$$\begin{aligned}I = \frac{1}{2\lambda_1}\Big(&-2e^{\lambda_1 x}\left(e^{\lambda_1 x}-1\right)\dot{x}\sqrt{\gamma}\lambda_1\sin\left(2t\sqrt{\gamma}\right) \\ &+\left(\left(e^{\lambda_1 x}-1\right)^2\gamma-e^{2x\lambda_1}\dot{x}^2\lambda_1^2\right)\cos\left(2t\sqrt{\gamma}\right)\Big).\end{aligned} \tag{6.45}$$

Case 7. By using X_7 from Eq. (4.17) vector field infinitesimals for ξ and η correspond to

$$\xi = 0, \quad \eta = \sin\left(\sqrt{\gamma}t\right)e^{-\lambda_1 x}. \tag{6.46}$$

For this case

$$\Omega = -\lambda = \dot{x}\lambda_1 - \sqrt{\gamma}\cos\left(\sqrt{\gamma}t\right), \tag{6.47}$$

is obtained. From Eq. (6.20)

$$\Psi = -\mu = e^{\lambda_1 x}\sin\left(t\sqrt{\gamma}\right), \tag{6.48}$$

is found. The first integral can be written as

$$I = \frac{\left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\cos\left(t\sqrt{\gamma}\right)}{\lambda_1} - e^{\lambda_1 x}\dot{x}\sin\left(t\sqrt{\gamma}\right), \tag{6.49}$$

and the invariant solution for this case overlaps with Eq. (4.34).

Case 8. As the cast case we consider X_8 from Eq. (4.17), which corresponds to

$$\xi = 0, \quad \eta = \cos\left(\sqrt{\gamma}t\right)e^{-\lambda_1 x}. \tag{6.50}$$

For this case, the null function

$$\Omega = -\lambda = \dot{x}\lambda_1 + \sqrt{\gamma}\tan\left(\sqrt{\gamma}t\right), \tag{6.51}$$

is obtained. By using similar calculations, the integration factor

$$\Psi = -\mu = e^{\lambda_1 x}\cos\left(t\sqrt{\gamma}\right), \tag{6.52}$$

and the time-dependent first integral

$$I = -\frac{\left(e^{\lambda_1 x}-1\right)\sqrt{\gamma}\sin\left(t\sqrt{\gamma}\right)}{\lambda_1} - e^{\lambda_1 x}\dot{x}\cos\left(t\sqrt{\gamma}\right), \tag{6.53}$$

are found.

6.2 Application to the displaced harmonic oscillator

In this section, we consider the displaced harmonic oscillator as given in (5.2), which is an example of Liénard I-type equation

$$\ddot{x} + k\dot{x} + \lambda_1 x + \lambda_2 = 0, \tag{6.54}$$

First of all, we consider a similar procedure to calculate Lie point symmetries of Eq. (6.54) discussed in the previous section by considering the following remark.

Remark 4 For $\phi = -k\dot{x} - \lambda_1 x - \lambda_2$ and $\Omega = -\dfrac{A(Q)}{Q}$ the solution of Eq. (6.6) in terms of ξ and η by considering $Q = \eta - \xi y'$ and $A = \partial_t + \dot{x}\partial_x + \phi(t, x, \dot{x})\partial_{\dot{x}}$ gives the following symmetry groups corresponding to the displaced harmonic oscillator (6.54).

$$\begin{aligned}
X_1 &= \frac{e^{t(\alpha-\beta)}(x\lambda_1+\lambda_2)}{\lambda_1}\frac{\partial}{\partial t} \\
&\quad+\frac{e^{t(\alpha-\beta)}(x\lambda_1+\lambda_2)^2\left(\alpha(\alpha+\beta)-\lambda_1\right)}{\beta\lambda_1^2}\frac{\partial}{\partial x}, \\
X_2 &= \frac{e^{t(\alpha+\beta)}(x\lambda_1+\lambda_2)}{\lambda_1}\frac{\partial}{\partial t} \\
&\quad+\frac{e^{t(\alpha+\beta)}(x\lambda_1+\lambda_2)^2\left(\alpha(-\alpha+\beta)+\lambda_1\right)}{\beta\lambda_1^2}\frac{\partial}{\partial x}, \\
X_3 &= \left(x+\frac{\lambda_2}{\lambda_1}\right)\frac{\partial}{\partial x}, \\
X_4 &= \frac{e^{2t\beta}}{2\beta}\frac{\partial}{\partial t}+\frac{e^{2t\beta}(x\lambda_1+\lambda_2)(\beta-\alpha)}{2\beta\lambda_1^2}\frac{\partial}{\partial x}, \\
X_5 &= -\frac{e^{-2t\beta}}{2\beta}\frac{\partial}{\partial t}+\frac{e^{-2t\beta}(x\lambda_1+\lambda_2)\left(\alpha(-\alpha+\beta)+\lambda_1\right)}{\beta\lambda_1^2}\frac{\partial}{\partial x}, \\
X_6 &= \frac{\partial}{\partial t}-\frac{\alpha(x\lambda_1+\lambda_2)}{\lambda_1}\frac{\partial}{\partial x}, \\
X_7 &= e^{-t(\alpha+\beta)}\frac{\partial}{\partial x}, \\
X_8 &= e^{t(-\alpha+\beta)}\frac{\partial}{\partial x},
\end{aligned} \tag{6.55}$$

where $\frac{k}{2} = \alpha$, $\beta = \frac{1}{2}(k^2 - 4\lambda_1)^{1/2}$. Finally for each symmetry, from the solution of Eq. (6.54) the null function and the corresponding λ function are found by the formula (4.14) based on the relation $\Omega = -\lambda$. It can be seen easily that symmetries X_1, X_2, X_3, symmetries X_4 and X_8 and symmetries X_5 and X_7 have same forms of Ω. Therefore, we only consider the symmetry generators X_3, X_4, X_5 and X_6.

Case 1. To obtain Ω by using λ function, we use the X_3 symmetry generator

$$\xi = 0, \quad \eta = x + \frac{\lambda_2}{\lambda_1}. \tag{6.56}$$

For this case, the null function is

$$\Omega = -\frac{\dot{x}\lambda_1}{x\lambda_1 + \lambda_2}. \tag{6.57}$$

Using by the same ansatz (6.20), then $A(x) = 0$, $B(x) = 0$ and $C(x) = 0$ are found, then the integration factor related function

$$\Psi = \lambda_1^2 (x\lambda_1 + \lambda_2) \left((2\dot{x}\lambda_1 + 2\alpha (x\lambda_1 + \lambda_2)) \cosh\beta - 2\beta (x\lambda_1 + \lambda_2) \sinh\beta\right)^{-2}, \tag{6.58}$$

is derived and the first integral via Eqs. (6.9) and (6.10) can be defined as

$$\begin{aligned} I = {} & -(\lambda_1 \Big(-2\lambda_1 \Big(\dot{x}^2\lambda_1 + 2\alpha\dot{x}(x\lambda_1 + \lambda_2) + (x\lambda_1 + \lambda_2)^2 \\ & -e^{2t\beta} \Big(4\alpha^2 (x\lambda_1 + \lambda_2)^2 + 2\alpha(x\lambda_1 + \lambda_2) \\ & \times (2\dot{x}\lambda_1 + 2\beta(x\lambda_1 + \lambda_2)) + 2\lambda_1 \Big(\dot{x}^2\lambda_1 + \dot{x}2\beta \\ & (x\lambda_1 + \lambda_2) - (x\lambda_1 + \lambda_2)^2\Big)\Big)\Big)\Big) \\ & / \Big(4\beta\ (\lambda_1 \Big(\dot{x}^2\lambda_1 + 2\alpha\dot{x}(x\lambda_1 + \lambda_2) + (x\lambda_1 + \lambda_2)^2\Big) \\ & +e^{4t\beta}\lambda_1 \Big(\dot{x}^2\lambda_1 + 2\alpha\dot{x}(x\lambda_1 + \lambda_2) + (x\lambda_1 + \lambda_2)^2 \\ & +e^{2t\beta} \Big(4\alpha\dot{x}\lambda_1(x\lambda_1 + \lambda_2) + 4\alpha^2(x\lambda_1 + \lambda_2)^2 \\ & -2\lambda_1 \Big(-\dot{x}^2\lambda_1 + (x\lambda_1 + \lambda_2)^2\Big)\Big)\Big)\Big)\Big). \end{aligned} \tag{6.59}$$

Case 2. Infinitesimals corresponding to X_4 vector field are

$$\xi = \frac{e^{2t\beta}}{2\beta}, \quad \eta = \frac{e^{2t\beta}(x\lambda_1 + \lambda_2)(\beta - \alpha)}{2\beta\lambda_1^2}. \tag{6.60}$$

For this case, one can get the null function Ω of the form

$$\Omega = \alpha - \beta. \tag{6.61}$$

Similarly, the function Ψ is given by

$$\Psi = \frac{1}{\lambda_1} 2e^{t(\alpha+\beta)} \Big(\lambda_1 + e^{t(\alpha+\beta)} (\dot{x}\lambda_1 + (\alpha - \beta)(x\lambda_1 + \lambda_2))\Big), \tag{6.62}$$

and the first integral

$$I = -\frac{2e^{t(\alpha+\beta)} \left(\dot{x}(\alpha + \beta) + x\lambda_1 + \lambda_2)(2\lambda_1 + e^{t(\alpha+\beta)}(\dot{x}\lambda_1 + (\alpha - \beta)(x\lambda_1 + \lambda_2))\right)}{(\alpha + \beta)\lambda_1}. \tag{6.63}$$

is obtained.

Case 3. Infinitesimals corresponding to X_5 vector field are

$$\xi = -\frac{e^{-2t\beta}}{2\beta}, \quad \eta = \frac{e^{-2t\beta}(x\lambda_1 + \lambda_2)\ (\alpha(-\alpha + \beta) + \lambda_1)}{\beta\lambda_1^2}. \tag{6.64}$$

For this case, the relation

$$\Omega = \alpha + \beta, \tag{6.65}$$

gives

$$\Psi = -\frac{e^{t(\alpha-2\beta)}}{\lambda_1} \left(e^{t\beta}\lambda_1 + e^{t\alpha}(\dot{x}\lambda_1 + (\alpha + \beta)(x\lambda_1 + \lambda_2))\right). \tag{6.66}$$

Then, it is easy to see that the first integral is in the following form

$$I = -\frac{e^{t(\alpha-2\beta)} \left(\dot{x}(\alpha - \beta) + x\lambda_1 + \lambda_2)(2\lambda_1 e^{t\beta} + e^{t\alpha}(\dot{x}\lambda_1 + (\alpha + \beta)(x\lambda_1 + \lambda_2))\right)}{2(\alpha - \beta)\lambda_1}. \tag{6.67}$$

Case 4. As the last case, the infinitesimals for X_6 vector field can be written of the form

$$\xi = 1, \quad \eta = -\frac{\alpha(x\lambda_1 + \lambda_2)}{\lambda_1}, \tag{6.68}$$

then the null function and

$$\Omega = \frac{\lambda_1\,(\alpha\dot{x} + (x\lambda_1 + \lambda_2))}{\dot{x}\lambda_1 + (x\lambda_1 + \lambda_2)}, \tag{6.69}$$

and integrating factor-related function

$$\Psi = \mathrm{e}^{2\alpha t} 2\,(\dot{x}\lambda_1 + \alpha(x\lambda_1 + \lambda_2))\,. \tag{6.70}$$

are found. As a result, one can write the time-dependent first integral in the following form

$$I = -\mathrm{e}^{2\alpha t}\left(\dot{x}^2\lambda_1 + 2\alpha\dot{x}(x\lambda_1 + \lambda_2) + (x\lambda_1 + \lambda_2)^2\right). \tag{6.71}$$

7 Concluding remarks and discussions

In this study, we analyze the nonlinear dynamical system including general nonlinear damping and nonlinear spring functions by considering relations between the local–nonlocal transformations, Lie point symmetries and λ-symmetries. From the mathematical point of view, we consider the nonlinear differential equations related to the nonlinear dynamical systems called Liénard I-type and II-type equations corresponding to many physical examples well known in the literature. Then, we consider linearizable conditions for each case of Liénard I-type and II-type equations. We prove that the nonlinear dynamical system with general forms in the form of Liénard II-type equation can be linearizable using Lie point transformations and the necessary corresponding conditions are introduced. For this kind of equations, we determine whether the condition of S_1 exists. Then, if the condition $S_2 = 0$ is satisfied, then we call them as S and L-linearizable. With the relation of S_2 condition, we derive the property defined as *isochronicity* condition for Liénard II-type equations. Based on the S and L-transformations, the first integrals and integrating factors for the general and special types of Liénard I-type and II-type equations are analyzed. Some problems such as the path equation, the generalized Morse equation and the linear harmonic oscillator equations are considered. For these equations, we prove that the *isochronous condition* is satisfied for some cases based on the local and nonlocal transformations. After that, the corresponding conserved forms and the exact solutions are obtained. Additionally, via the local and nonlocal transformations we obtain the necessary relation, which is based on the fact that the nonlinear damping term must be constant and the nonlinear spring term must be in a linear form in terms of the position of the mass. As a physical example for this case, we consider the displaced damped harmonic oscillator equation and obtain the transformation pair to linearize this equation. Also, we prove that there is an additional condition between nonlinear damping function and the nonlinear spring function in order to have S-linearizable property for the displaced damped harmonic oscillator problem. Similar to the first case, we obtain the corresponding λ-symmetries related to the corresponding Lie point symmetries, new conserved forms and the integrating factors of the equation. For each case, the exact solutions are obtained and some corresponding graphs for conjugate momentum versus time and conjugate momentum versus position are presented for different values of equation parameters and integration constants of the solutions. With respect to these solutions, the mass–spring–damper simulations including position, velocity and acceleration relations are presented and some linear and nonlinear cases are compared.

In addition to the local and nonlocal transformations, we present that it is possible to find new conserved forms and exact solutions of the nonlinear dynamical systems by using the relation between the Lie point symmetry and λ-symmetry. In this concept, it is shown that one can obtain a new exact solution for the problem by considering each Lie point symmetry. For example, in the case of the general Morse oscillator having an eight-parameter symmetry groups, for each symmetry group the corresponding λ-symmetry is obtained and then using the standard procedure the corresponding integrating factors and first integrals are identified. In addition for the linear harmonic oscillator having an eight-parameter symmetry groups, the same procedure is carried out and the new conserved forms are obtained. The results corresponding to each Lie point symmetry are given and discussed in the study since they are important not only from mathematical point of view but also from physical point of view.

Furthermore, in the literature, the extended Prelle–Singer method is used for the investigation of first integrals and integrating factors of differential equations. In this approach, in order to obtain the integrating factors and first integrals, one has to solve a system of partial differential equations given by (6.6)–(6.8). In general, it is not easy to get nontrivial solutions from this system of differential equations. However, in this study, we introduce a new approach related to λ-symmetry, and by using the relation between the λ-symmetry and the extended Prelle–Singer method, we prove that the time-dependent and the time-independent first integrals of Liénard-type equations can be obtained. It is also important to mention that there is an important relation between equations Ω, Ψ functions, which are solutions of three differential equations (6.6)–(6.8) and λ-function (4.14). In the literature, Ω and Ψ functions are determined by using some proper ansatz functions. In fact, the present study can be considered as an application of the Lie point symmetry—λ-symmetry approach together to find Ω and Ψ functions as a different approach. By using this idea, for the cases of Liénard I-type and II-type equations, new, nontrivial integrating factors, nontrivial first integrals and nontrivial λ-symmetries corresponding to each Lie point symmetry are determined. For the general Morse equation and linear harmonic oscillator, for each Lie point symmetry for the equations, new first integrals, and invariant solutions are obtained and presented. In fact, Sects. 4 and 6 concentrate on the derivation and the interrelation of the symmetries and the methods. These parts represent one of the main important results and contributions in the study.

We also deal with a different approach for investigating classical Lie point symmetries of differential equations, which is related to the application of the extended Prelle–Singer approach. It can be shown that the first Eq. (6.6) in terms of Ω corresponds to the *determining equations* in which the solutions of them give Lie point symmetries. It means that the solution of Eq. (6.6) by using the expression (4.14) gives all Lie point symmetries of the corresponding equation. Thus, instead of solving null function from Eq. (6.6) we consider the solutions of Ω-function considering the corresponding Lie point symmetries since it is difficult to get a solution from equations (6.6) to (6.8). In general, it is also possible to prove that every Lie point symmetry corresponds to a null Ω-function; thus, one can say that Ω function is related to λ-function. Then, the connection between λ-symmetry and null function can be given based on the fact that $\lambda = -\Omega$ and the relation between the integrating factor μ and Ψ is given as $\mu = -\Psi$. Therefore, as mentioned in Sect. 6.2, first, one can consider the characteristic Q function in terms of ξ and η and then substitute the Q function and the λ-function (4.14) in Eq. (6.6). Finally, the determining equations in terms of infinitesimal functions ξ and η are obtained. From the solution of infinitesimal functions ξ and η, the corresponding nontrivial λ-functions and nontrivial null functions are determined. In the study, this method is applied for investigation for Lie point symmetries of the generalized Morse oscillator equation, the harmonic oscillator equation and the displaced harmonic oscillator equation and we prove that their Lie symmetry groups found by this approach are equivalent to the symmetries obtained by the classical method known in the literature.

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