Nonlinear Dyn (2016) 85:1217–1222
DOI 10.1007/s11071-016-2755-8

ORIGINAL PAPER

Study of lump dynamics based on a dimensionally reduced Hirota bilinear equation

Xing Lü · Wen-Xiu Ma

Received: 21 December 2015 / Accepted: 20 March 2016 / Published online: 4 April 2016

Abstract With symbolic computation, two classes of lump solutions to the dimensionally reduced equations in (2+1)-dimensions are derived, respectively, by searching for positive quadratic function solutions to the associated bilinear equations. To guarantee analyticity and rational localization of the lumps, two sets of sufficient and necessary conditions are presented on the parameters involved in the solutions. Localized characteristics and energy distribution of the lump solutions are also analyzed and illustrated.

Keywords Lump solution · Dimensionally reduced Hirota bilinear equation · Lump dynamics · Symbolic computation

Mathematics Subject Classification 35Q51 · 35Q55 · 37K40

X. Lü (✉)
Department of Mathematics, Beijing Jiao Tong University, Beijing 100044, China
e-mail: xlv@bjtu.edu.cn; xinglv655@aliyun.com

W.-X. Ma
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

W.-X. Ma
Department of Mathematical Sciences, International Institute for Symmetry Analysis and Mathematical Modelling, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, South Africa

1 Introduction

In soliton theory [1–14], lump solutions have attracted more and more attention [15–20]. As a kind of rational function solutions, lump solutions localized in all directions in the space. Such integrable equations as the KPI equation [16,17], the BKP equation [18], the three-dimensional three wave resonant interaction equation [19], the Davey–Stewartson-II equation [17] and the Ishimori I equation [20] have been found to possess lump solutions.

Lump solutions can be studied based on the Hirota bilinear equations and their generalized counterparts. For example, a class of lump solutions to the KPI equation has been presented by making use of its Hirota bilinear form [16]. The resulting lump solutions contain six free parameters, two of which are due to the translation invariance of the KP equation and the other four of which satisfy a nonzero determinant condition guaranteeing analyticity and rational localization of the solutions. Further, based on generalized bilinear forms, lump solutions to dimensionally reduced p-gKP and p-gBKP equations in (2+1) dimensions have been computed [15]. The sufficient and necessary conditions to guarantee analyticity and rational localization of the solutions have been given.

In our previous work [21], a new Hirota bilinear equation has been proposed and studied, which reads

$$(D_t D_y - D_x^3 D_y - 3\,D_x^2 + 3\,D_z^2) f \cdot f = 0, \tag{1}$$

that is,

$$2\,[(f f_{ty} - f_t f_y + f_{xxx} f_y + 3\, f_{xxy} f_x - 3\, f_{xx} f_{xy} - f f_{xxxy} - 3(f f_{xx} - f_x^2) + 3(f f_{zz} - f_z^2)] = 0, \tag{2}$$

where $f = f(x, y, z, t)$, and the derivatives $D_t D_y$, $D_x^3 D_y$, D_x^2 and D_z^2 are the Hirota bilinear operators [22] defined by

$$D_x^\alpha D_t^\beta (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^\alpha \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^\beta \times\ f(x,t) g(x',t')\Big|_{x'=x, t'=t.}$$

Bell polynomial theories (see, e.g., Refs. [23–27]) motivate us to consider a dependent variable transformation

$$u = 2\Big[\ln f(x, y, z, t)\Big]_x = 2\frac{f_x(x, y, z, t)}{f(x, y, z, t)}, \tag{3}$$

and map Eq. (2) into

$$u_{yt} - u_{xxxy} - 3\,(u_x u_y)_x - 3\,u_{xx} + 3\,u_{zz} = 0. \tag{4}$$

Eq. (4) is a (3+1)-dimensional model, and it is clear that if f solves Eq. (2), then $u = u(x, y, z, t)$ is a solution to Eq. (4) through the transformation (3).

Via applying to Eq. (2) the linear superposition principle [28,29], two types of resonant N-wave solutions have been found and illustrated [21]. In this paper, we will search for positive quadratic function solutions to the dimensionally reduced Hirota bilinear Eq. (2) via taking $z = y$ or $z = t$ cases[1] , and begin with

$$f = g^2 + h^2 + a_9, \tag{5}$$

and

$$g = a_1 x + a_2 y + a_3 t + a_4,$$
$$h = a_5 x + a_6 y + a_7 t + a_8,$$

where a_i $(1 \le i \le 9)$ are all real parameters to be determined. To determine the lump solutions, we note that the conditions guaranteeing the well-definedness of f, positiveness of f and localization of u in all directions in the space need to be satisfied.

[1] We found no lump solutions in the from of (5) to Eq. (2) via taking $z = x$.

2 Lump solutions to reduction with $z = y$

With $z = y$, the dimensionally reduced form of the Hirota bilinear Eq. (2) turns out to be

$$2\,\Big[(f f_{ty} - f_t f_y + f_{xxx} f_y + 3\, f_{xxy} f_x - 3\, f_{xx} f_{xy} - f f_{xxxy} - 3(f f_{xx} - f_x^2) + 3(f f_{yy} - f_y^2)\Big] = 0, \tag{6}$$

which is transformed into

$$u_{yt} - u_{xxxy} - 3\,(u_x u_y)_x - 3\,u_{xx} + 3\,u_{yy} = 0. \tag{7}$$

through the link between f and u:

$$u = 2\Big[\ln f(x, y, t)\Big]_x = 2\frac{f_x(x, y, t)}{f(x, y, t)}. \tag{8}$$

Symbolic computation manipulation on a direct substitution of f in Eq. (5) into Eq. (6) leads to the following set of constraining equations for the parameters:

$$\Big\{ a_1 = a_1, a_2 = a_2,\; a_3 = \frac{3[a_2(a_1^2 - a_2^2 - a_5^2 - a_6^2) + 2a_1 a_5 a_6]}{a_2^2 + a_6^2},\; a_4 = a_4, a_5 = a_5, a_6 = a_6,\; a_7 = -\frac{3[a_6(a_1^2 + a_2^2 - a_5^2 + a_6^2) - 2a_1 a_2 a_5]}{a_2^2 + a_6^2}, a_8 = a_8,\; a_9 = -\frac{(a_2^2 + a_6^2)(a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2} \Big\}, \tag{9}$$

which needs to satisfy the conditions

$$a_1 a_6 - a_2 a_5 \neq 0, \tag{10}$$
$$a_1 a_2 + a_5 a_6 < 0, \tag{11}$$

to guarantee the well-definedness of f, the positiveness of f and the localization of u in all directions in the space. The parameters in the set (9) yield a class of positive quadratic function solution to Eq. (6) as

$$f = \left(a_1 x + a_2 y + \frac{3[a_2(a_1^2 - a_2^2 - a_5^2 - a_6^2) + 2a_1 a_5 a_6]}{a_2^2 + a_6^2} t + a_4\right)^2 + \left(a_5 x + a_6 y - \frac{3[a_6(a_1^2 + a_2^2 - a_5^2 + a_6^2) - 2a_1 a_2 a_5]}{a_2^2 + a_6^2} t + a_8\right)^2 - \frac{(a_2^2 + a_6^2)(a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2}, \tag{12}$$

which, in turn, generates a class of lump solutions to the dimensionally reduced Eq. (7) through transformation (8) as

$$u^{(\mathrm{I})} = \frac{4(a_1 g + a_5 h)}{f}, \tag{13}$$

where the function f is defined by Eq. (12), and the functions g and h are given as follows:

$$\begin{aligned} g &= a_1 x + a_2 y \\ &\quad + \frac{3[a_2(a_1^2 - a_2^2 - a_5^2 - a_6^2) + 2a_1 a_5 a_6]}{a_2^2 + a_6^2} t + a_4, \\ h &= a_5 x + a_6 y \\ &\quad - \frac{3[a_6(a_1^2 + a_2^2 - a_5^2 + a_6^2) - 2a_1 a_2 a_5]}{a_2^2 + a_6^2} t + a_8. \end{aligned}$$

Note here that *six* parameters a_1, a_2, a_4, a_5, a_6 and a_8 are involved in the solution u, among which, a_4 and a_8 are arbitrary, while the rests are demanded to satisfy conditions (10) and (11) to guarantee $u^{(\mathrm{I})}$ to be lump solutions.

3 Lump solutions to reduction with $z = t$

With $z = t$, the dimensionally reduced form of the Hirota bilinear Eq. (2) reads

$$\begin{aligned} &2\,[(f f_{ty} - f_t f_y + f_{xxx} f_y + 3\, f_{xxy} f_x - 3\, f_{xx} f_{xy} \\ &\quad - f f_{xxxy} - 3(f f_{xx} - f_x^2) + 3(f f_{yy} - f_y^2)] = 0, \end{aligned} \tag{14}$$

which is cast into

$$u_{yt} - u_{xxxy} - 3\,(u_x u_y)_x - 3\,u_{xx} + 3\,u_{tt} = 0. \tag{15}$$

through the link between f and u, that is transformation (8).

For Eq. (14), a direct substitution of f gives rise to the following set of constraining equations for the parameters:

$$\begin{aligned} &\Big\{ a_1 = a_1,\ a_2 = \frac{3[a_3(a_1^2 - a_3^2 - a_5^2 - a_7^2) + 2a_1 a_5 a_7]}{a_3^2 + a_7^2}, \\ &a_3 = a_3,\ a_4 = a_4,\ a_5 = a_5, \\ &a_6 = -\frac{3[a_7(a_1^2 + a_3^2 - a_5^2 + a_7^2) - 2a_1 a_3 a_5]}{a_3^2 + a_7^2}, \\ &a_7 = a_7,\ a_8 = a_8, \\ &a_9 = -\frac{3(a_1^2 + a_5^2)(a_1 a_3 + a_5 a_7)(a_1^2 - a_3^2 + a_5^2 - a_7^2)}{(a_1 a_7 - a_3 a_5)^2} \Big\}, \end{aligned} \tag{16}$$

which needs to satisfy the conditions

$$a_1 a_7 - a_3 a_5 \neq 0, \tag{17}$$

$$(a_1 a_3 + a_5 a_7)(a_1^2 - a_3^2 + a_5^2 - a_7^2) < 0, \tag{18}$$

to guarantee the well-definedness of f, the positiveness of f and the localization of u in all directions in the space. The parameters in the set (16) yield a class of positive quadratic function solution to Eq. (14) as

$$\begin{aligned} f &= \Big(a_1 x + \frac{3[a_3(a_1^2 - a_3^2 - a_5^2 - a_7^2) + 2a_1 a_5 a_7]}{a_3^2 + a_7^2} y \\ &\quad + a_3 t + a_4 \Big)^2 \\ &\quad + \Big(a_5 x - \frac{3[a_7(a_1^2 + a_3^2 - a_5^2 + a_7^2) - 2a_1 a_3 a_5]}{a_3^2 + a_7^2} y \\ &\quad + a_7 t + a_8 \Big)^2 \\ &\quad - \frac{3(a_1^2 + a_5^2)(a_1 a_3 + a_5 a_7)(a_1^2 - a_3^2 + a_5^2 - a_7^2)}{(a_1 a_7 - a_3 a_5)^2}, \end{aligned} \tag{19}$$

which, in turn, generates a class of lump solutions to the dimensionally reduced Eq. (15) through transformation (8) as

$$u^{(\mathrm{II})} = \frac{4(a_1 g + a_5 h)}{f}, \tag{20}$$

where the function f is defined by Eq. (19), and the functions g and h are given as follows:

$$\begin{aligned} g &= a_1 x + \frac{3[a_3(a_1^2 - a_3^2 - a_5^2 - a_7^2) + 2a_1 a_5 a_7]}{a_3^2 + a_7^2} y \\ &\quad + a_3 t + a_4, \\ h &= a_5 x - \frac{3[a_7(a_1^2 + a_3^2 - a_5^2 + a_7^2) - 2a_1 a_3 a_5]}{a_3^2 + a_7^2} y \\ &\quad + a_7 t + a_8. \end{aligned}$$

Note here that *six* parameters a_1, a_3, a_4, a_5, a_7 and a_8 are involved in the solution u, among which, a_4 and a_8 are arbitrary, while the rests are demanded to satisfy conditions (17) and (18) to guarantee $u^{(\mathrm{II})}$ to be lump solutions.

4 Lump dynamics and energy distribution

For the exact solution $u(x, y, t)$ to Eqs. (7) and (15) to be lump ones, it is required that

$$\lim_{x^2+y^2\to\infty} u(x, y, t) = 0, \ \forall\, t \in \mathbb{R}.$$

By virtue of transformation (8), a sufficient condition for $u(x, y, t)$ to be a lump solution constraining on $f(x, y, t)$ is

$$\lim_{x^2+y^2\to\infty} f(x, y, t) = \infty, \ \forall\, t \in \mathbb{R}.$$

All the solutions derived in this paper ($u^{(\mathrm{I})}$ and $u^{(\mathrm{II})}$) satisfy this criterion, and they are rationally localized in all directions in the space.

The amplitude of a lump solution u is defined as max $|u|$, and the location of a lump solution is then defined as the place where the max $|u|$ is attained. With these definitions, we know that the amplitude of $u^{(\mathrm{I})}$ is $\frac{2|a_1a_6-a_2a_5|}{\sqrt{-(a_1a_2+a_5a_6)(a_2^2+a_6^2)}}$ and initially located at

$$\left(\frac{\epsilon\sqrt{a_1^2+a_5^2}(a_2a_8-a_4a_6)\pm\sqrt{a_9}(a_1a_6-a_2a_5)}{\epsilon\sqrt{a_1^2+a_5^2}(a_1a_6-a_2a_5)}, \frac{a_4a_5-a_1a_8}{a_1a_6-a_2a_5}\right),$$

where a_i $(1 \le i \le 9)$ are given in (9) and the amplitude of $u^{(\mathrm{II})}$ is $\frac{2|a_1a_7-a_3a_5|}{\sqrt{-3(a_1a_3+a_5a_7)(a_1^2-a_3^2+a_5^2-a_7^2)}}$ and initially located at

$$\left(\frac{\epsilon\sqrt{a_1^2+a_5^2}(a_2a_8-a_4a_6)\pm\sqrt{a_9}(a_1a_6-a_2a_5)}{\epsilon\sqrt{a_1^2+a_5^2}(a_1a_6-a_2a_5)}, \frac{a_4a_5-a_1a_8}{a_1a_6-a_2a_5}\right),$$

where a_i $(1 \le i \le 9)$ are given in (16), and

$$\epsilon = \begin{cases} 1, & a_5 > 0, \\ -1, & a_5 < 0. \end{cases}$$

The localized characteristics and energy distribution of the lump solutions can be seen clearly in Figs. 1 and 2 including 3-dimensional plots, density plots and

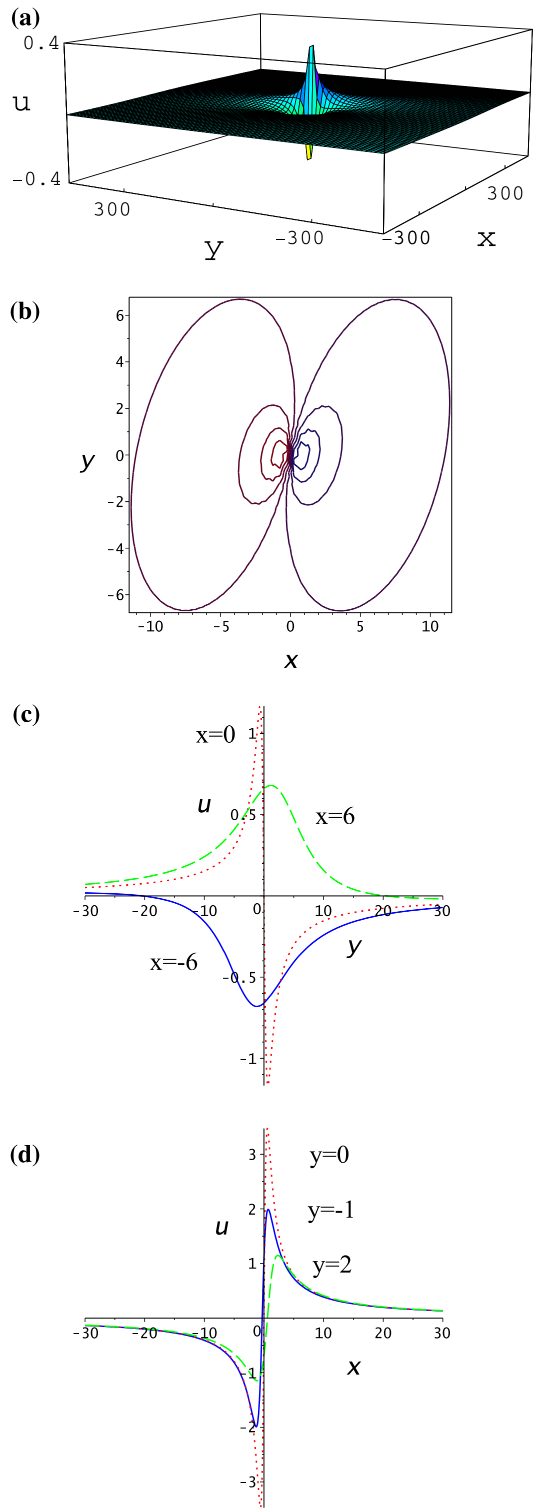

Fig. 1 Lump dynamic characteristics of $u^{(\mathrm{I})}$ via Eq. (13) with $a_1 = 1, a_2 = 3, a_4 = 0, a_5 = -4, a_6 = 2, a_8 = 0$ and $t = 0$: **a** 3-dimensional plot; **b** density plot; **c** x-curves and **d** y-curves

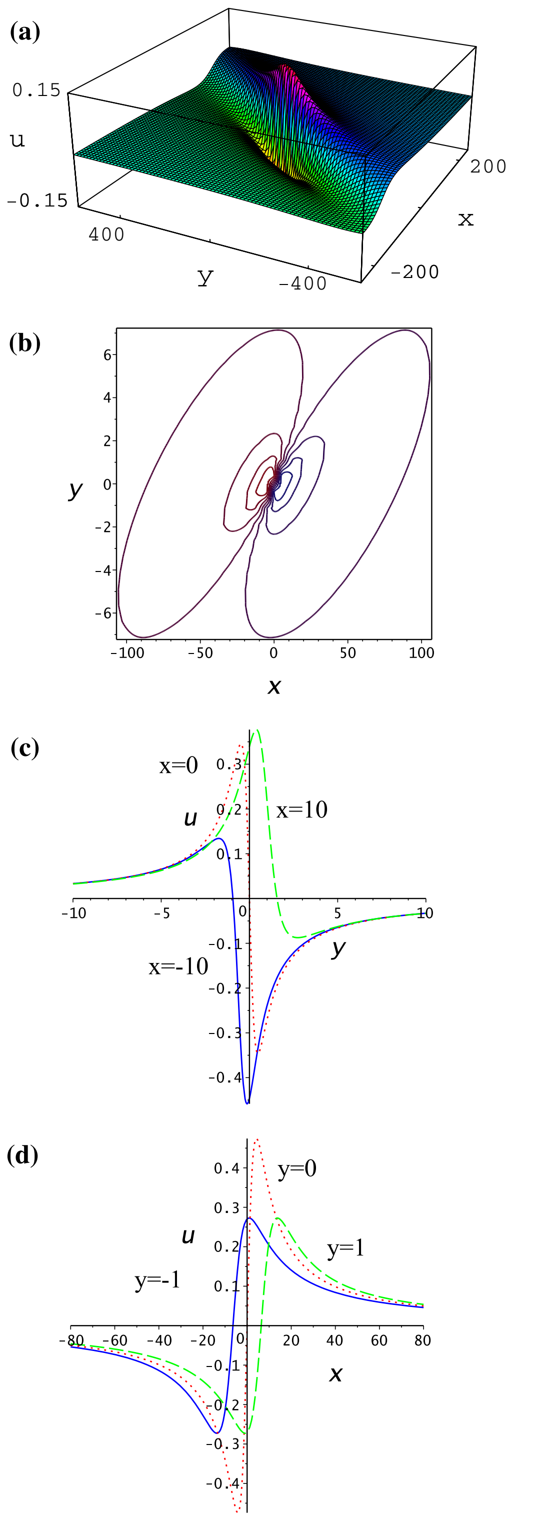

Fig. 2 Lump dynamic characteristics of $u^{(\mathrm{II})}$ via Eq. (20) with $a_1 = 2$, $a_3 = 3$, $a_4 = 0$, $a_5 = 1$, $a_7 = 6$, $a_8 = 0$ and $t = 0$: **a** 3-dimensional plot; **b** density plot; **c** x-curves and **d** y-curves

2-dimensional curves with particular choices of the involved parameters in the potential function u.

5 Concluding remarks

Lump solution is a type of rational solution, and another type of exact solution with rational function amplitudes is rogue wave solution, which attracts recent attention in describing nonlinear wave phenomena in oceanography and nonlinear optics [30,31]. In this paper, we have derived two classes of lump solutions (see Eqs. 13 and 20) to the dimensionally reduced Eqs. (7) and (15), respectively, by searching for positive quadratic function solutions to the associated bilinear equations, i.e., Eqs. (6) and (14). This method can be used to search for rogue wave solutions, that is to say, rogue wave solutions could be generated as well in terms of positive polynomial solutions to the associated bilinear equations. Work of this aspect will be proceeded in our future papers.

It should be noticed that we have studied lump solutions to two types of dimensional reductions with $z = y$ and $z = t$ for Eq. (2), or correspondingly for Eq. (4). For the reduction with $z = x$, Eq. (2) is reduced to

$$2\,[(ff_{ty} - f_t f_y + f_{xxx} f_y + 3\,f_{xxy} f_x - 3\,f_{xx} f_{xy} - ff_{xxxy}] = 0, \tag{21}$$

which is linked to

$$u_{yt} - u_{xxxy} - 3\,(u_x u_y)_x = 0, \tag{22}$$

through the transformation (8). We have found no positive quadratic function solutions in terms of Eqs. (5)–(21) such that no lump solutions to Eq. (22) either. How to derive lump solutions or how to prove its nonexistence to Eq. (22) is a further question.

Acknowledgments This work is supported by the 111 Project of China (B16002), the National Natural Science Foundation of China under Grant No. 61308018, by China Postdoctoral Science Foundation under Grant No. 2014T70031, by the Fundamental Research Funds for the Central Universities of China (2015JBM111). The second author is supported in part by the National Natural Science Foundation of China under Grant Nos. 11371326 and 11271008, Natural Science Foundation of Shanghai under Grant No. 11ZR1414100, Zhejiang Innovation Project of China under Grant No. T200905, the First-class Discipline of Universities in Shanghai and the Shanghai University Leading Academic Discipline Project (No. A13-0101-12-004) and the Distinguished Professorship at Shanghai University of Electric Power.

References

1. Rajan Mani, M.S., Mahalingam, A.: Nonautonomous solitons in modified inhomogeneous Hirota equation: soliton control and soliton interaction. Nonlinear Dyn. **79**, 2469 (2015)
2. Biswas, A.: Solitary waves for power-law regularized long-wave equation and R (m, n) equation. Nonlinear Dyn. **59**, 423 (2010)
3. Biswas, A.: Solitary wave solution for KdV equation with power-law nonlinearity and time-dependent coefficients. Nonlinear Dyn. **58**, 345 (2009)
4. Biswas, A., Khalique, C.M.: Stationary solutions for nonlinear dispersive Schrödinger equation. Nonlinear Dyn. **63**, 623 (2011)
5. Ablowitz, M.J., Clarkson, P.A.: Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1991)
6. Biswas, A.: Optical solitons in magneto-optic waveguides with spatio-temporal dispersion. Frequenz **68**, 445 (2014)
7. Savescu, M., Bhrawy, A.H., Hilal, E.M., Alshaery, A.A., Biswas, A.: Optical solitons in magneto-optic waveguides with spatio-temporal dispersion. Frequenz **68**, 445 (2014)
8. Michelle, S., Alshaery, A.A., Hilal, E.M., Bhrawy, A.H., Zhou, Q., Biswas, A.: Optical solitons in DWDM system with four-wave mixing. Optoelectron. Adv. Mater. **9**, 14 (2015)
9. Guzman, J.V., Hilal, E.M., Alshaery, A.A., Bhrawy, A.H., Mahmood, M.F., Moraru, L., Biswas, A.: Thirring optical solitons with spatio-temporal dispersion. Proc. Rom. Acad. Ser. A **16**, 41 (2015)
10. Wang, D.S., Yin, S., Tian, Y., Liu, Y.F.: Integrability and bright soliton solutions to the coupled nonlinear Schrödinger equation with higher-order effects. Appl. Math. Comput. **229**, 296 (2014)
11. Wang, D.S., Ma, Y.Q., Li, X.G.: Prolongation structures and matter-wave solitons in F1 spinor Bose-Einstein condensate. Commun. Nonlinear Sci. Numer. Simul. **19**, 3556 (2014)
12. Dai, C.Q., Wang, X.G., Zhou, G.Q.: Stable light-bullet solutions in the harmonic and parity-time-symmetric potentials. Phys. Rev. A **89**, 013834 (2014)
13. Dai, C.Q., Wang, Y.Y., Zhang, X.F.: Controllable Akhmediev breather and Kuznetsov-Ma soliton trains in PT-symmetric coupled waveguides. Opt. Express **22**, 29862 (2014)
14. Lü, X., Ma, W.X., Yu, J., Lin, F.H., Khalique, C.M.: Envelope bright- and dark-soliton solutions for the Gerdjikov–Ivanov model. Nonlinear Dyn. **82**, 1211 (2015)
15. Ma, W.X., Qin, Z.Y., Lü, X.: Lump solutions to dimensionally reduced p-gKP and p-gBKP equations. Nonlinear Dyn. **84**, 923 (2016)
16. Ma, W.X.: Lump solutions to the Kadomtsev–Petviashvili equation. Phys. Lett. A **379**, 1975 (2015)
17. Satsuma, J., Ablowitz, M.J.: Two-dimensional lumps in nonlinear dispersive systems. J. Math. Phys. **20**, 1496 (1979)
18. Gilson, C.R., Nimmo, J.J.C.: Lump solutions of the BKP equation. Phys. Lett. A **147**, 472 (1990)
19. Kaup, D.J.: The lump solutions and the Bäklund transformation for the three-dimensional three-wave resonant interaction. J. Math. Phys. **22**, 1176 (1981)
20. Imai, K.: Dromion and lump solutions of the Ishimori-I equation. Prog. Theor. Phys. **98**, 1013 (1997)
21. Yu, J. C., Lü, X.: Study on the resonant behavior of N-wave solutions to a Hirota bilinear equation. Appl. Math. Comp. submitted, (2015)
22. Hirota, R.: The Direct Method in Soliton Theory. Cambridge University Press, Cambridge (2004)
23. Ma, W.X.: Bilinear equations and resonant solutions characterized by Bell polynomials. Rep. Math. Phys. **72**, 41 (2013)
24. Ma, W.X.: Trilinear equations, bell polynomials, and resonant solutions. Front. Math. China **8**, 1139 (2013)
25. Bell, E.T.: Exponential polynomials. Ann. Math. **35**, 258 (1934)
26. Lü, X., Tian, B., Sun, K., Wang, P.: Bell-polynomial manipulations on the Bäcklund transformations and Lax pairs for some soliton equations with one Tau-function. J. Math. Phys. **51**, 113506 (2010)
27. Lü, X., Tian, B., Qi, F.H.: Bell-polynomial construction of Bäcklund transformations with auxiliary independent variable for some soliton equations with one Tau-function. Nonlinear Anal. Real World Appl. **13**, 1130 (2012)
28. Ma, W.X., Fan, E.G.: Linear superposition principle applying to Hirota bilinear equations. Comput. Math. Appl. **61**, 950 (2011)
29. Ma, W.X., Zhang, Y., Tang, Y.N., Tu, J.Y.: Hirota bilinear equations with linear subspaces of solutions. Appl. Math. Comput. **218**, 7174 (2012)
30. Müller, P., Garrett, C., Osborne, A.: Roguewaves. Oceanography **18**, 66 (2005)
31. Solli, D.R., Ropers, C., Koonath, P., Jalali, B.: Optical rogue waves. Nature **450**, 1054 (2007)