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A new robust adaptive control method for modified function projective synchronization with unknown bounded parametric uncertainties and external disturbances

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Abstract This paper investigates the problem of modified function projective synchronization of two different chaotic systems in the presence of parametric uncertainties and external disturbances. A new robust adaptive control is proposed, which is able to attenuate all random uncertainties of the drive and response systems. Moreover, there is no need to know the norm-bounds of all random uncertainties, and the compensator gains can be automatically adapted to suitable constants. Numerical examples are provided to show the effectiveness of the proposed method.

Keywords Modified function projective synchronization · Parametric uncertainty · External disturbance · Robust control · Adaptive control

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1 Introduction

Chaos synchronization is a hot subject in the field of nonlinear science [1–4], which refers to a process wherein two (or many) chaotic systems adjust a given property of their motion to a common behavior due to a coupling or to a forcing. Since Pecora and Carroll presented a successful method to synchronize two identical chaotic systems in 1990 [5], chaos synchronization has been investigated greatly due to its potential applications in physical systems, biological networks, secure communications, etc. Up to now, various types of synchronization phenomena have been revealed to investigate chaos synchronization, including complete synchronization [5], phase synchronization [6], generalized synchronization [7], lag synchronization [8], anti-synchronization [9], projective synchronization [10], function projective synchronization [11], etc.

Recently, a new type of synchronization, termed as modified function projective synchronization (MFPS), has been extensively investigated in [12–21], in which the drive and response systems could be synchronized to a desired scaling function matrix. The novelty feature of this synchronization phenomenon is that the scaling functions can be arbitrarily designed to different state variables by means of control, while the unpredictability of the scaling functions can additionally enhance the security of communications. It is easy to see that MFPS is the more general definition of synchronization, which encompasses complete synchronization, anti-synchronization, projective synchronization, and function projective synchronization. In [12], the authors gave a MFPS scheme of two coupled Lorenz systems and a secure communication scheme by using MFPS. On the basis of the active control scheme, a general method of MFPS with time delay has been investigated in [13]. Ref. [14] investigated adaptive MFPS of hyper-chaotic systems with unknown parameters. MFPS in drive–response dynamical networks has been investigated by adaptive open-plus-closed-loop control method in [15]. Ref. [16] investigated two different hyper-chaotic secure communication schemes by using generalized function projective synchronization. More general forms of MFPS have been extensively investigated in [17–21].

Most of the researches mentioned above have concentrated on the chaotic systems without considering disturbances. However, disturbances are regrettably unavoidable in the practical situations. Ref. [22] dealed with the MFPS problem of different hyper-chaotic systems subject to external disturbances. Ref. [23] investigated MFPS in network with unknown parameters and mismatch parameters. In [22,23], the results presented are only applicable to the case when the disturbance bound is known. However, in general, it is difficult to have the full information about the disturbance in practice. Thus, designing control system to attenuate disturbance with unknown bound is important and useful in practical applications. The work in [24–27] studied the problems of robust and adaptive control for both linear and nonlinear systems with unknown bounded disturbances, while a novel control design method is proposed which can make the underlying system more adaptive and robust to parameter changes and system uncertainties. In [24–27], the system model is divided into two parts, i.e., the linear part and the uncertain term, and assumes that the uncertain term can be linearly parameterized. Obviously, most of the chaotic systems do not belong to this class of dynamical systems. Therefore, synchronization of two chaotic systems with unknown bounded disturbances is becoming an important issue. In [28,29], authors proposed a robust adaptive sliding mode controller to realize complete synchronization between two different chaotic systems with unknown bounded uncertainties and external disturbances. But proposed method in [28,29] is only applicable to complete synchronization. Thus, how to achieve MFPS of chaotic systems with unknown bounded disturbances is still an open and challenging problem. In this paper, a new robust adaptive control law of MFPS is proposed for two different chaotic systems with parametric uncertainties and external disturbances, in which the norm-bounds of all random uncertainties are not necessarily to be known, and the compensator gains can be automatically adjusted to suitable constants. To the best of authors' knowledge, the above issue has not been fully investigated yet.

The organization of this paper is as follows: In Sect. 2, some preliminaries are given. In Sect. 3, the MFPS scheme of a class of chaotic systems with parametric uncertainties and external disturbances is given. In Sect. 4, we will choose two groups of examples to show the effectiveness of the proposed method. The conclusion is finally drawn in Sect. 5.

2 Model description and preliminaries

Consider a class of chaotic systems with parametric uncertainties and external disturbances, which is described by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{x})(\boldsymbol{l}_1 + \boldsymbol{d}_1) + \boldsymbol{d}_2, \tag{1}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$ is the state vector, $\boldsymbol{f} : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a continuous vector function, $\boldsymbol{F} : \boldsymbol{R}^n \to \boldsymbol{R}^{n\times m}$ is a function matrix, $\boldsymbol{l}_1 \in \boldsymbol{R}^m$ is the parameter vector, $\boldsymbol{d}_1 \in \boldsymbol{R}^m$ and $\boldsymbol{d}_2 \in \boldsymbol{R}^n$ are the vectors of parametric uncertainty and external disturbance, respectively. Eq. (1) is considered as the drive system, and the controlled response system is given by

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{G}(\boldsymbol{y})(\boldsymbol{l}_2 + \boldsymbol{d}_3) + \boldsymbol{d}_4 + \boldsymbol{u}, \tag{2}$$

where $\boldsymbol{y} \in \boldsymbol{R}^n$ is the state vector of the response system, $\boldsymbol{g} : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a continuous vector function, $\boldsymbol{G} : \boldsymbol{R}^n \to \boldsymbol{R}^{n\times m}$ is a function matrix, $\boldsymbol{l}_2 \in \boldsymbol{R}^m$ is the parameter vector, $\boldsymbol{d}_3 \in \boldsymbol{R}^m$ and $\boldsymbol{d}_4 \in \boldsymbol{R}^n$ are the vectors of parametric uncertainty and external disturbance, respectively. $\boldsymbol{u} \in \boldsymbol{R}^n$ is the vector controller.

From a practical point of view, all of parametric uncertainties and external disturbances $\boldsymbol{d}_i$ $(i = 1, 2, 3, 4)$ are assumed to be bounded.

Assumption 1 *All of parametric uncertainties and external disturbances* $\boldsymbol{d}_i$ $(i = 1, 2, 3, 4)$ *are norm bounded, that is*

$$\|\boldsymbol{d}_i\| \le d_i^*, \qquad i = 1, 2, 3, 4, \tag{3}$$

where $d_i^* \in R^+$ $(i = 1, 2, 3, 4)$ *are the upper bound of the norm of* $\boldsymbol{d}_i$ $(i = 1, 2, 3, 4)$.

We define the error vector

$$\boldsymbol{e} = \boldsymbol{\Lambda}(t)\boldsymbol{x} - \boldsymbol{y}, \tag{4}$$

where $\boldsymbol{\Lambda}(t)$ is an n-order diagonal matrix, i.e., $\boldsymbol{\Lambda}(t) = diag(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$. The scaling functions $\alpha_i(t)$ $(i = 1, 2, \ldots, n)$ are continuously differentiable, bounded, and $\alpha_i(t) \neq 0$ for all t.

Definition 1 (MFPS) For the drive system (1) and the response system (2), it is said that the system (1) and the system (2) are modified function projective synchronization, if there exists a scaling function matrix $\boldsymbol{\Lambda}(t)$ such that $\lim_{t\to\infty} \|\boldsymbol{e}(t)\| = 0$.

Our objective in this paper is to design a controller $\boldsymbol{u}$ for system (2) under Assumption 1, such that the controlled response system (2) could be MFPS to the drive system (1), i.e., $\lim_{t\to\infty}\|\boldsymbol{e}\| = 0$.

3 Controller design

In this section, we investigate the problem of MFPS with unknown bounded parametric uncertainties and external disturbances.

Theorem 1 *For given synchronization scaling function matrix $\boldsymbol{\Lambda}(t)$ and any initial conditions $\boldsymbol{x}(0)$, $\boldsymbol{y}(0)$, if the Assumption 1 is held, the MFPS between the drive system (1) and the response system (2) will occur via the control law as below*

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2, \tag{5}$$

$$\begin{aligned}\boldsymbol{u}_1 &= \boldsymbol{\Lambda}(t)\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\boldsymbol{l}_1 \\ &\quad - \boldsymbol{G}(\boldsymbol{y})\boldsymbol{l}_2 + \dot{\boldsymbol{\Lambda}}(t)\boldsymbol{x},\end{aligned} \tag{6}$$

$$\begin{aligned}\boldsymbol{u}_2 &= \hat{d}_1\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\ &\quad + \hat{d}_2\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) + \hat{d}_3\boldsymbol{G}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{G}^T(\boldsymbol{y})\boldsymbol{e}) \\ &\quad + \hat{d}_4\mathrm{sgn}(\boldsymbol{e}) + k_1\boldsymbol{e},\end{aligned} \tag{7}$$

$$\dot{\hat{d}}_1 = k_2\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}), \tag{8}$$

$$\dot{\hat{d}}_2 = k_3\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}), \tag{9}$$

$$\dot{\hat{d}}_3 = k_4\boldsymbol{e}^T\boldsymbol{G}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{G}^T(\boldsymbol{y})\boldsymbol{e}), \tag{10}$$

$$\dot{\hat{d}}_4 = k_5\boldsymbol{e}^T\mathrm{sgn}(\boldsymbol{e}), \tag{11}$$

where $\boldsymbol{e} = \boldsymbol{\Lambda}(t)\boldsymbol{x} - \boldsymbol{y}$, k_1, k_2, k_3, k_4, k_5 are arbitrary positive constants and $\mathrm{sgn}(\cdot)$ denotes the sign function. In Eq. (5), $\boldsymbol{u}_1$ is the nonlinear controller, $\boldsymbol{u}_2$ is the compensator to be designed to compensate the uncertainties, where the gains of compensator can be automatically adapted to suitable constants.

Proof The time derivative of Eq. (4) is

$$\dot{\boldsymbol{e}} = \boldsymbol{\Lambda}(t)\dot{\boldsymbol{x}} - \dot{\boldsymbol{y}} + \dot{\boldsymbol{\Lambda}}(t)\boldsymbol{x}. \tag{12}$$

Substituting (1), (2) into (12), we have

$$\begin{aligned}\dot{\boldsymbol{e}} &= \boldsymbol{\Lambda}(t)(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{x})(\boldsymbol{l}_1 + \boldsymbol{d}_1) + \boldsymbol{d}_2) \\ &\quad - \boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{G}(\boldsymbol{y})(\boldsymbol{l}_2 + \boldsymbol{d}_3) - \boldsymbol{d}_4 - \boldsymbol{u} + \dot{\boldsymbol{\Lambda}}(t)\boldsymbol{x}.\end{aligned} \tag{13}$$

Substituting (5), (6) and (7) into (13), we have

$$\begin{aligned}\dot{\boldsymbol{e}} &= \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\boldsymbol{d}_1 + \boldsymbol{\Lambda}(t)\boldsymbol{d}_2 - \boldsymbol{G}(\boldsymbol{y})\boldsymbol{d}_3(\boldsymbol{t}) - \boldsymbol{d}_4 \\ &\quad - \hat{d}_1\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\ &\quad - \hat{d}_2\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) - \hat{d}_3\boldsymbol{G}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{G}^T(\boldsymbol{y})\boldsymbol{e}) \\ &\quad - \hat{d}_4\mathrm{sgn}(\boldsymbol{e}) - k_1\boldsymbol{e}.\end{aligned} \tag{14}$$

Construct the Lyapunov function

$$\begin{aligned}V &= \frac{1}{2}\boldsymbol{e}^T\boldsymbol{e} + \frac{1}{2k_2}(\hat{d}_1 - d_1^*)^2 + \frac{1}{2k_3}(\hat{d}_2 - d_2^*)^2 \\ &\quad + \frac{1}{2k_4}(\hat{d}_3 - d_3^*)^2 + \frac{1}{2k_5}(\hat{d}_4 - d_4^*)^2.\end{aligned} \tag{15}$$

With the choice of Eq. (8), (9), (10), and (11), the time derivative of $\boldsymbol{V}$ along the trajectories of Eq. (14) is

$$\begin{aligned}\dot{V} &= \boldsymbol{e}^T\dot{\boldsymbol{e}} + \frac{1}{k_2}(\hat{d}_1 - d_1^*)\dot{\hat{d}}_1 + \frac{1}{k_3}(\hat{d}_2 - d_2^*)\dot{\hat{d}}_2 \\ &\quad + \frac{1}{k_4}(\hat{d}_3 - d_3^*)\dot{\hat{d}}_3 + \frac{1}{k_5}(\hat{d}_4 - d_4^*)\dot{\hat{d}}_4 \\ &= \boldsymbol{e}^T[\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\boldsymbol{d}_1 + \boldsymbol{\Lambda}(t)\boldsymbol{d}_2 - \boldsymbol{G}(\boldsymbol{y})\boldsymbol{d}_3 - \boldsymbol{d}_4 \\ &\quad - \hat{d}_1\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\ &\quad - \hat{d}_2\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) - \hat{d}_3\boldsymbol{G}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{G}^T(\boldsymbol{y})\boldsymbol{e}) \\ &\quad - \hat{d}_4\mathrm{sgn}(\boldsymbol{e}) - k_1\boldsymbol{e}] \\ &\quad + (\hat{d}_1 - d_1^*)\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\ &\quad + (\hat{d}_2 - d_2^*)\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\ &\quad + (\hat{d}_3 - d_3^*)\boldsymbol{e}^T\boldsymbol{G}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{G}^T(\boldsymbol{y})\boldsymbol{e}) \\ &\quad + (\hat{d}_4 - d_4^*)\boldsymbol{e}^T\mathrm{sgn}(\boldsymbol{e}) \\ &= \boldsymbol{e}^T\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\boldsymbol{d}_1 + \boldsymbol{e}^T\boldsymbol{\Lambda}(t)\boldsymbol{d}_2 \\ &\quad - \boldsymbol{e}^T\boldsymbol{G}(\boldsymbol{y})\boldsymbol{d}_3 - \boldsymbol{e}^T\boldsymbol{d}_4 - k_1\boldsymbol{e}^T\boldsymbol{e} \\ &\quad - d_1^*\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\ &\quad - d_2^*\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\ &\quad - d_3^*\boldsymbol{e}^T\boldsymbol{G}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{G}^T(\boldsymbol{y})\boldsymbol{e}) \\ &\quad - d_4^*\boldsymbol{e}^T\mathrm{sgn}(\boldsymbol{e}) \\ &\le \|\boldsymbol{d}_1\|\|\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\| + \|\boldsymbol{d}_2\|\|\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\| \\ &\quad + \|\boldsymbol{d}_3\|\|\boldsymbol{e}^T\boldsymbol{G}(\boldsymbol{y})\| + \|\boldsymbol{d}_4\|\|\boldsymbol{e}^T\| \\ &\quad - d_1^*\|\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\| - d_2^*\|\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\| \\ &\quad - d_3^*\|\boldsymbol{e}^T\boldsymbol{G}(\boldsymbol{y})\| - d_4^*\|\boldsymbol{e}^T\| - k_1\boldsymbol{e}^T\boldsymbol{e}\end{aligned}$$

$$= (\|\boldsymbol{d}_1\| - d_1^*)\|\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\| + (\|\boldsymbol{d}_2\| - d_2^*)\|\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\|$$
$$(\|\boldsymbol{d}_3\| - d_3^*)\|\boldsymbol{e}^T\boldsymbol{G}(\boldsymbol{y})\| + (\|\boldsymbol{d}_4\| - d_4^*)\|\boldsymbol{e}^T\| - k_1\boldsymbol{e}^T\boldsymbol{e}$$
$$\leq -k_1\boldsymbol{e}^T\boldsymbol{e}. \tag{16}$$

According to the Lyapunov stability theorem, $\boldsymbol{e}, \hat{d}_1 - d_1^*$, $\hat{d}_2 - d_2^*$, $\hat{d}_3 - d_3^*$, and $\hat{d}_4 - d_4^*$ are bounded. Because chaos systems are bounded, Assumption 1 is held, $\dot{\boldsymbol{e}}$ is bounded, i.e., $\dot{\boldsymbol{e}} \in L_\infty$. According to Eq. (16), we have

$$\int_0^\infty \boldsymbol{e}^T\boldsymbol{e} \leq -\frac{1}{k_1}\int_0^\infty \dot{V} = \frac{1}{k_1}(V(0) - V(\infty)) < \infty. \tag{17}$$

So, $\boldsymbol{e} \in L_2$. According to Barbalat's Lemma, $\boldsymbol{e} \to \boldsymbol{0}$ with $t \to \infty$. The MFPS is achieved under the certain chosen controller $\boldsymbol{u}$ in Eq. (5). This completes the proof. □

Remark 1 It is easy to see that norm-boundeds of all parametric uncertainties and external disturbances have no effect on $\dot{V}$. Thus, proposed scheme can achieve synchronization when the bounds of parametric uncertainties and external disturbances are unknown.

Remark 2 In [28,29], a robust adaptive sliding mode controller was proposed to realize complete synchronization between two different chaotic systems with unknown bounded uncertainties and external disturbances. But the approach in [28,29] cannot be directly applied to MFPS. Thus, how to achieve MFPS between two different chaotic systems with unknown bounds of parametric uncertainties and external disturbances is essential and significant in both theory and applications.

If $\boldsymbol{g} = \boldsymbol{f}$, $\boldsymbol{G} = \boldsymbol{F}$, and $\boldsymbol{l}_1 = \boldsymbol{l}_2 = \boldsymbol{l}$, then the drive system (1) and the response system (2) are two identical chaotic systems with different parametric uncertainties and external disturbances, which are described as follows

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{x})(\boldsymbol{l} + \boldsymbol{d}_1) + \boldsymbol{d}_2, \tag{18}$$
$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{F}(\boldsymbol{y})(\boldsymbol{l} + \boldsymbol{d}_3) + \boldsymbol{d}_4 + \boldsymbol{u}, \tag{19}$$

where $\boldsymbol{d}_1$, $\boldsymbol{d}_3$ are the parametric uncertainties of the drive system and the response system, respectively. $\boldsymbol{d}_2$, $\boldsymbol{d}_4$ are the external disturbances of the drive system and the response system, respectively.

Corollary 1 *For given synchronization scaling function matrix $\boldsymbol{\Lambda}(t)$ and any initial conditions $\boldsymbol{x}(0)$, $\boldsymbol{y}(0)$, if the Assumption 1 is held, the MFPS between the drive system (18) and the response system (19) will occur via the control law as below*

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2, \tag{20}$$
$$\boldsymbol{u}_1 = \tilde{\boldsymbol{f}} + \tilde{\boldsymbol{F}}\boldsymbol{l} + \dot{\boldsymbol{\Lambda}}(t)\boldsymbol{x}, \tag{21}$$
$$\boldsymbol{u}_2 = \hat{d}_1\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) + \hat{d}_2\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) + \hat{d}_3\boldsymbol{F}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{y})\boldsymbol{e}) + \hat{d}_4\mathrm{sgn}(\boldsymbol{e}) + k_1\boldsymbol{e}, \tag{22}$$
$$\dot{\hat{d}}_1 = k_2\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}), \tag{23}$$
$$\dot{\hat{d}}_2 = k_3\boldsymbol{e}^T\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}), \tag{24}$$
$$\dot{\hat{d}}_3 = k_4\boldsymbol{e}^T\boldsymbol{F}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{y})\boldsymbol{e}), \tag{25}$$
$$\dot{\hat{d}}_4 = k_5\boldsymbol{e}^T\mathrm{sgn}(\boldsymbol{e}), \tag{26}$$

where $\tilde{\boldsymbol{f}} = \boldsymbol{\Lambda}(t)\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})$, $\tilde{\boldsymbol{F}} = \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{y})$, $\boldsymbol{e} = \boldsymbol{\Lambda}(t)\boldsymbol{x} - \boldsymbol{y}$, k_1, k_2, k_3, k_4, k_5 are arbitrary positive constants and $\mathrm{sgn}(\cdot)$ denotes the sign function.

Proof The time derivative of Eq. (4) is

$$\dot{\boldsymbol{e}} = \boldsymbol{\Lambda}(t)\dot{\boldsymbol{x}} - \dot{\boldsymbol{y}} + \dot{\boldsymbol{\Lambda}}(t)\boldsymbol{x}. \tag{27}$$

Substituting (18), (19) into (27), we have

$$\dot{\boldsymbol{e}} = \boldsymbol{\Lambda}(t)(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{x})(\boldsymbol{l} + \boldsymbol{d}_1) + \boldsymbol{d}_2) - \boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{F}(\boldsymbol{y})(\boldsymbol{l} + \boldsymbol{d}_3) - \boldsymbol{d}_4 - \boldsymbol{u} + \dot{\boldsymbol{\Lambda}}(t)\boldsymbol{x}. \tag{28}$$

Substituting (20), (21), and (22) into (28), we have

$$\dot{\boldsymbol{e}} = \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\boldsymbol{d}_1 + \boldsymbol{\Lambda}(t)\boldsymbol{d}_2 - \boldsymbol{F}(\boldsymbol{y})\boldsymbol{d}_3 - \boldsymbol{d}_4 - \hat{d}_1\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) - \hat{d}_2\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) - \hat{d}_3\boldsymbol{F}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{y})\boldsymbol{e}) - \hat{d}_4\mathrm{sgn}(\boldsymbol{e}) - k_1\boldsymbol{e}. \tag{29}$$

Construct the Lyapunov function

$$\boldsymbol{V} = \frac{1}{2}\boldsymbol{e}^T\boldsymbol{e} + \frac{1}{2k_2}(\hat{d}_1 - d_1^*)^2 + \frac{1}{2k_3}(\hat{d}_2 - d_2^*)^2 + \frac{1}{2k_4}(\hat{d}_3 - d_3^*)^2 + \frac{1}{2k_5}(\hat{d}_4 - d_4^*)^2. \tag{30}$$

With the choice of Eq. (23), (24), (25), and (26), the time derivative of $\boldsymbol{V}$ along the trajectories of Eq. (29) is

$$\dot{V} = \boldsymbol{e}^T\dot{\boldsymbol{e}} + \frac{1}{k_2}(\hat{d}_1 - d_1^*)\dot{\hat{d}}_1 + \frac{1}{k_3}(\hat{d}_2 - d_2^*)\dot{\hat{d}}_2 + \frac{1}{k_4}(\hat{d}_3 - d_3^*)\dot{\hat{d}}_3 + \frac{1}{k_5}(\hat{d}_3 - d_3^*)\dot{\hat{d}}_4$$
$$= \boldsymbol{e}^T[\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\boldsymbol{d}_1 + \boldsymbol{\Lambda}(t)\boldsymbol{d}_2 - \boldsymbol{F}(\boldsymbol{y})\boldsymbol{d}_3 - \boldsymbol{d}_4 - \hat{d}_1\boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) - \hat{d}_2\boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) - \hat{d}_3\boldsymbol{F}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{y})\boldsymbol{e})$$

$$
\begin{aligned}
&\quad - \hat{d}_4 \mathrm{sgn}(\boldsymbol{e}) - k_1 \boldsymbol{e}] \\
&\quad + (\hat{d}_1 - d_1^*)\boldsymbol{e}^T \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\
&\quad + (\hat{d}_2 - d_2^*)\boldsymbol{e}^T \boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\
&\quad + (\hat{d}_3 - d_3^*)\boldsymbol{e}^T \boldsymbol{F}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{y})\boldsymbol{e}) \\
&\quad + (\hat{d}_4 - d_4^*)\boldsymbol{e}^T \mathrm{sgn}(\boldsymbol{e}) \\
&= \boldsymbol{e}^T \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\boldsymbol{d}_1 + \boldsymbol{e}^T \boldsymbol{\Lambda}(t)\boldsymbol{d}_2 \\
&\quad - \boldsymbol{e}^T \boldsymbol{F}(\boldsymbol{y})\boldsymbol{d}_3 - \boldsymbol{e}^T \boldsymbol{d}_4 - k_1 \boldsymbol{e}^T \boldsymbol{e} \\
&\quad - d_1^* \boldsymbol{e}^T \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{x})\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\
&\quad - d_2^* \boldsymbol{e}^T \boldsymbol{\Lambda}(t)\mathrm{sgn}(\boldsymbol{\Lambda}(t)\boldsymbol{e}) \\
&\quad - d_3^* \boldsymbol{e}^T \boldsymbol{F}(\boldsymbol{y})\mathrm{sgn}(\boldsymbol{F}^T(\boldsymbol{y})\boldsymbol{e}) \\
&\quad - d_4^* \boldsymbol{e}^T \mathrm{sgn}(\boldsymbol{e}) \\
&\le \|\boldsymbol{d}_1\|\|\boldsymbol{e}^T \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\| + \|\boldsymbol{d}_2\|\|\boldsymbol{e}^T \boldsymbol{\Lambda}(t)\| \\
&\quad + \|\boldsymbol{d}_3\|\|\boldsymbol{e}^T \boldsymbol{F}(\boldsymbol{y})\| + \|\boldsymbol{d}_4\|\|\boldsymbol{e}^T\| \\
&\quad - d_1^* \|\boldsymbol{e}^T \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\| - d_2^* \|\boldsymbol{e}^T \boldsymbol{\Lambda}(t)\| \\
&\quad - d_3^* \|\boldsymbol{e}^T \boldsymbol{F}(\boldsymbol{y})\| - d_4^* \|\boldsymbol{e}^T\| - k_1 \boldsymbol{e}^T \boldsymbol{e} \\
&= (\|\boldsymbol{d}_1\| - d_1^*)\|\boldsymbol{e}^T \boldsymbol{\Lambda}(t)\boldsymbol{F}(\boldsymbol{x})\| + (\|\boldsymbol{d}_2\| - d_2^*)\|\boldsymbol{e}^T \boldsymbol{\Lambda}(t)\| \\
&\quad (\|\boldsymbol{d}_3\| - d_3^*)\|\boldsymbol{e}^T \boldsymbol{F}(\boldsymbol{y})\| + (\|\boldsymbol{d}_4\| - d_4^*)\|\boldsymbol{e}^T\| - k_1 \boldsymbol{e}^T \boldsymbol{e} \\
&\le -k_1 \boldsymbol{e}^T \boldsymbol{e}.
\end{aligned} \tag{31}
$$

According to the Lyapunov stability theorem, $\boldsymbol{e}, \hat{d}_1 - d_1^*$, $\hat{d}_2 - d_2^*$, $\hat{d}_3 - d_3^*$, and $\hat{d}_4 - d_4^*$ are bounded. Because chaos systems are bounded, Assumption 1 is held, $\dot{\boldsymbol{e}}$ is bounded, i.e., $\dot{\boldsymbol{e}} \in L_\infty$. According to Eq. (31), we have

$$
\begin{aligned}
\int_0^\infty \boldsymbol{e}^T \boldsymbol{e} &\le -\frac{1}{k_1} \int_0^\infty \dot{V} \\
&= \frac{1}{k_1}(V(0) - V(\infty)) < \infty.
\end{aligned} \tag{32}
$$

So, $\boldsymbol{e} \in L_2$. According to Barbalat theorem, $\boldsymbol{e} \to \boldsymbol{0}$ with $t \to \infty$. The MFPS is achieved under the certain chosen controller $\boldsymbol{u}$ in Eq. (20). This completes the proof. □

4 Illustrative examples

In this section, two groups of examples are provided to verify the effectiveness of the proposed schemes, which are chaotic Lorenz and Chen systems and two identical hyper-chaotic Lorenz systems.

4.1 MFPS of Lorenz and Chen systems

In this subsection, we will take chaotic Lorenz and Chen systems as example to verify the effectiveness of the proposed scheme in Theorem 1.

The Lorenz system is described as follows

$$
\begin{cases}
\dot{x}_1 = a(x_2 - x_1), \\
\dot{x}_2 = (b - x_3)x_1 - x_2, \\
\dot{x}_3 = x_1 x_2 - c x_3,
\end{cases} \tag{33}
$$

where x_1, x_2, and x_3 are state variables, a, b, and c are system parameters. When three real parameters $a = 10, b = 28, c = 8/3$, the system (33) shows chaotic behavior. Considering parametric uncertainties and external disturbances, rewriting Eq. (33) in the forms of Eq. (1), we can obtain the drive system as follows

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ -x_1 x_3 - x_2 \\ x_1 x_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} x_2 - x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&\quad + \begin{bmatrix} x_2 - x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{bmatrix} \begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \end{bmatrix} \\
&\quad + \begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \end{bmatrix}.
\end{aligned} \tag{34}
$$

The Chen system is described as follows

$$
\begin{cases}
\dot{y}_1 = d(y_2 - y_1), \\
\dot{y}_2 = -y_1 y_3 + (f - d)y_1 + f y_2, \\
\dot{y}_3 = y_1 y_2 - e y_3,
\end{cases} \tag{35}
$$

where y_1, y_2, and y_3 are state variables, d, e, and f are system parameters. When three real parameters $d = 35, e = 3, f = 28$, the system (35) shows chaotic behavior. The controlled Chen system with parametric uncertainties and external disturbances as the response system is described as

$$
\begin{aligned}
\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} y_2 - y_1 & 0 & 0 \\ -y_1 & 0 & y_1 + y_2 \\ 0 & -y_3 & 0 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\
&\quad + \begin{bmatrix} y_2 - y_1 & 0 & 0 \\ -y_1 & 0 & y_1 + y_2 \\ 0 & -y_3 & 0 \end{bmatrix} \begin{bmatrix} d_{31} \\ d_{32} \\ d_{33} \end{bmatrix} \\
&\quad + \begin{bmatrix} d_{41} \\ d_{42} \\ d_{43} \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.
\end{aligned} \tag{36}
$$

In the simulation, the parametric uncertainties and external disturbances of the drive system and the response system are, respectively, set as follows:

$$\begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \end{bmatrix} = \begin{bmatrix} \sin(\pi t/6) \\ \cos(\pi t/6) \\ 0.1\cos(\pi t/6) \end{bmatrix},$$

$$\begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \frac{\exp(t)}{1+\exp(t)} \\ -\sin(t) \end{bmatrix},$$

$$\begin{bmatrix} d_{31} \\ d_{32} \\ d_{33} \end{bmatrix} = \begin{bmatrix} \cos(\pi t/6) \\ 0.1\sin(\pi t/6) \\ \sin(\pi t/6) \end{bmatrix},$$

$$\begin{bmatrix} d_{41} \\ d_{42} \\ d_{43} \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\frac{\exp(t)}{1+\exp(t)} \\ -\cos(t) \end{bmatrix}.$$

It can be seen that the maximum norms of all parametric uncertainties and external disturbances are bounded.

We take the initial states as $\boldsymbol{x}(0) = [9\ 6\ 18]^T$, $\boldsymbol{y}(0) = [-6\ \ -8\ \ 9]^T$. We take $\hat{d}_1(0) = 1$, $\hat{d}_2(0) = 2$, $\hat{d}_3(0) = 3$, and $\hat{d}_4(0) = 4$. The scaling functions take $\alpha_1(t) = 1 + 0.5\sin(\pi t/6)$, $\alpha_2(t) = \exp(t)/(1 + \exp(t))$, and $\alpha_3(t) = -1 - 0.5\sin(\pi t/6)$. The controller $\boldsymbol{u}$ can be designed by Theorem 1, where $k_1 = 1$, $k_2 = 1/50$, $k_3 = 1/50$, $k_4 = 1/50$, and $k_5 = 1/50$. The simulation results are presented in Fig. 1, 2, and 3. Figure 1 displays the synchronized attractors in R^3. The time evolution of the MFPS errors is depicted in Fig. 2, which displays $\boldsymbol{e} \to \boldsymbol{0}$ with $t \to \infty$. Thus, the required synchronization has been achieved with our designed control law (5). Figure 3 displays the time evolution of the compensator gains $\hat{d}_1$, $\hat{d}_2$, $\hat{d}_3$, and $\hat{d}_4$.

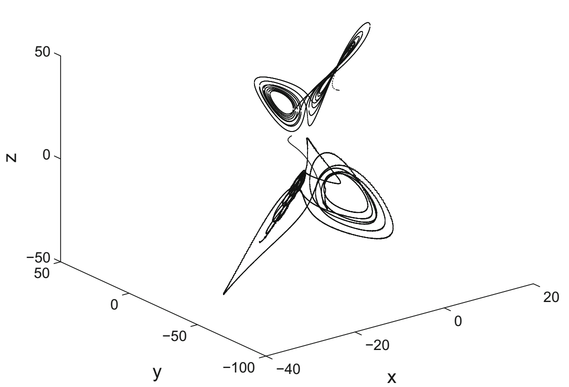

Fig. 1 Synchronized attractors in R^3

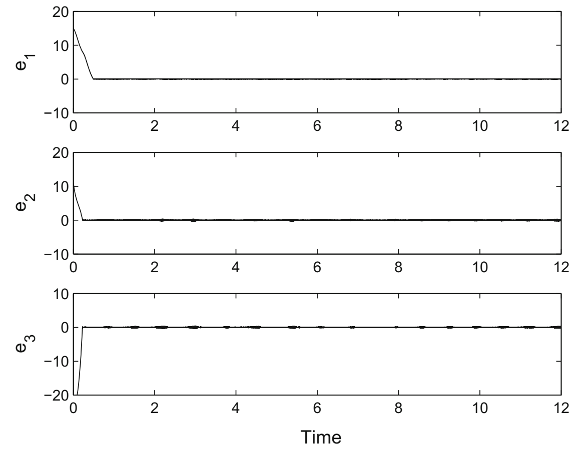

Fig. 2 Synchronization errors of chaotic Lorenz and Chen systems

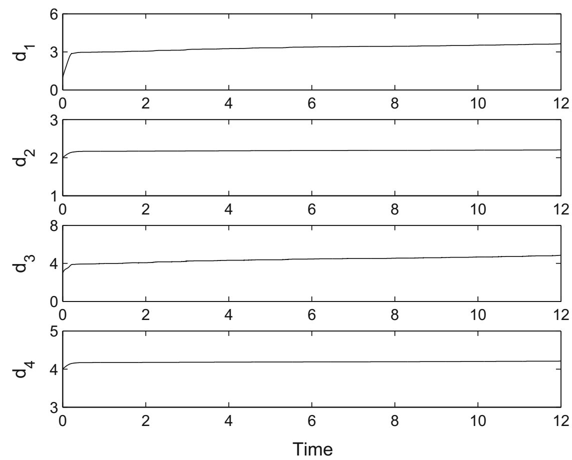

Fig. 3 Time evolution of the compensator gains

4.2 MFPS of two hyper-chaotic Lorenz systems

Because hyper-chaotic systems have more complex behavior than chaotic systems, which are more suitable for secure communication, we consider the example of two identical hyper-chaotic Lorenz systems with parametric uncertainties and external disturbances.

Hyper-chaotic Lorenz system is defined below

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + cx_4, \\ \dot{x}_2 = x_1(d - x_3) - x_2, \\ \dot{x}_3 = x_1x_2 - bx_3, \\ \dot{x}_4 = -x_1 - ax_4, \end{cases} \tag{37}$$

where x_1, x_2, x_3, and x_4 are state variables. When the four real parameters $a = 1$, $b = 0.7$, $c = 1.5$, and $d = 26$, the system (37) shows chaotic behavior.

Considering parametric uncertainties and external disturbances, rewriting Eq. (37) in the forms of Eq. (18) and Eq. (19), we can obtain the drive system and the response system as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1x_3 - x_2 \\ x_1x_2 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 & 0 & x_4 & 0 \\ 0 & 0 & 0 & x_1 \\ 0 & -x_3 & 0 & 0 \\ -x_4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 & 0 & x_4 & 0 \\ 0 & 0 & 0 & x_1 \\ 0 & -x_3 & 0 & 0 \\ -x_4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \\ d_{14} \end{bmatrix} + \begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \\ d_{24} \end{bmatrix}, \tag{38}$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -y_1y_3 - y_2 \\ y_1y_2 \\ -y_1 \end{bmatrix} + \begin{bmatrix} y_2 - y_1 & 0 & y_4 & 0 \\ 0 & 0 & 0 & y_1 \\ 0 & -y_3 & 0 & 0 \\ -y_4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} + \begin{bmatrix} y_2 - y_1 & 0 & y_4 & 0 \\ 0 & 0 & 0 & y_1 \\ 0 & -y_3 & 0 & 0 \\ -y_4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_{31} \\ d_{32} \\ d_{33} \\ d_{34} \end{bmatrix} + \begin{bmatrix} d_{41} \\ d_{42} \\ d_{43} \\ d_{44} \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}. \tag{39}$$

In the simulation, the parametric uncertainties and external disturbances of the drive system and the response system are, respectively, set as follows:

$$\begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \\ d_{14} \end{bmatrix} = \begin{bmatrix} 0.1\sin(t) \\ 0.1\cos(t) \\ \cos(t) \\ \sin(t) \end{bmatrix}, \quad \begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \\ d_{24} \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \cos(t) \\ 0.1\cos(t) \\ 0.1\sin(t) \end{bmatrix},$$

$$\begin{bmatrix} d_{31} \\ d_{32} \\ d_{33} \\ d_{34} \end{bmatrix} = \begin{bmatrix} 0.1\cos(t) \\ 0.1\sin(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}, \quad \begin{bmatrix} d_{41} \\ d_{42} \\ d_{43} \\ d_{44} \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0.1\sin(t) \\ 0.1\cos(t) \end{bmatrix}.$$

It is easy to see the maximum norms of parametric uncertainties and external disturbances are bounded, which do not need to know in numerical simulation.

We take the initial states as $\boldsymbol{x}(0) = [-15\ 6\ -10\ 2]^T$, $\boldsymbol{y}(0) = [8\ -3\ 15\ -6]^T$. We take $\hat{d}_1(0) = 1$, $\hat{d}_2(0) = 2$, $\hat{d}_3(0) = 3$, and $\hat{d}_4(0) = 4$. The scaling functions take $\alpha_1(t) = 1 - 0.5\sin(t)$, $\alpha_2(t) =$

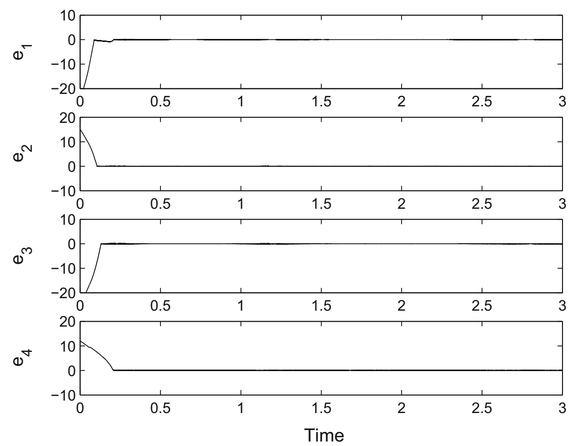

Fig. 4 Synchronization errors of two hyper-chaotic Lorenz systems

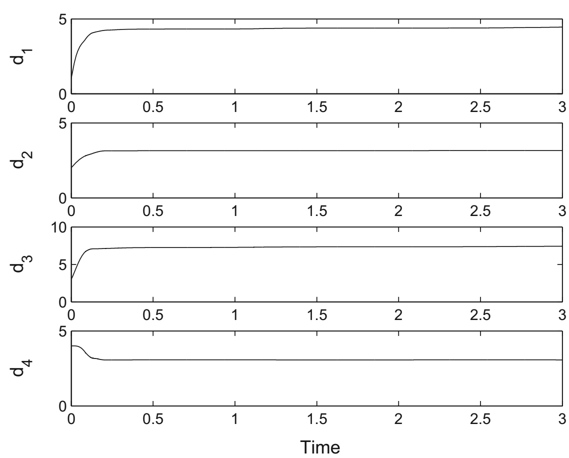

Fig. 5 Time evolution of the compensator gains

$\sin(t) + 2$, $\alpha_3(t) = 1$, and $\alpha_4(t) = \sin(t) + 3$. The controller $\boldsymbol{u}$ can be designed by Corrolary 1, where $k_1 = 1$, $k_2 = 1/20$, $k_3 = 1/5$, $k_4 = 1/20$, and $k_5 = 1$. Figure 4 displays the MFPS error $\boldsymbol{e} \to \mathbf{0}$ with $t \to \infty$, which implies that the required synchronization has been achieved with our designed control law (20). Figure 5 displays the time evolution of the compensator gains $\hat{d}_1$, $\hat{d}_2$, $\hat{d}_3$, and $\hat{d}_4$.

5 Conclusion

In this paper, a new MFPS scheme is proposed for a class of chaotic systems with parametric uncertainties and external disturbances. There is no need to know exactly bounds of parametric uncertainties and external disturbances, and the gains of the compensators can be automatically adapted to suitable constants. The proposed scheme can also be used in various types of synchronization. Numerical simulations are provided to show the effectiveness of the theoretical results obtained.

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