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Synchronization of complex dynamical networks with time-varying inner coupling

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Abstract The complex dynamical networks have attracted increasing attention in various fields. The previous models investigated are commonly time invariant; however, the complex networks with time-varying inner coupling are widespread in real world, and their synchronization problems have been rarely studied. We introduce a complex dynamical network model with time-varying inner coupling in this paper. We give the sufficient condition to achieve the exponential synchronization. The numerical simulation results are presented to illustrate the effectiveness of the proposed synchronization criteria.

Keywords Complex dynamical networks · Time-varying inner coupling · Exponential synchronization

1 Introduction

The last decade has witnessed the evolution of complex dynamical networks in various fields of humanities and science, such as the biological networks, smart power grids, information science, secure communization, etc [1,2]. In complex networks, the elements are modeled as nodes and the links represent the interaction among the elements. Complex networks often exhibit complex and interesting synchronization behavior [3]; synchronization is essentially a kind of collective behavior and plays more vital role in real-life systems. The synchronization problems of some fractional-order modified chaotic systems have been studied and applied in the field of secure communication, such as digital signature and message recovery [4], image encryption and decryption [5] and affine cipher with high-level security [6]. In order to exploit the relationship between the power network model and coupled oscillators, the synchronization problem for the network-reduced model of a power grid system with non-trivial transfer conductances was presented in [7].

The synchronization problems of complex dynamical networks include the local synchronization, global synchronization, lag synchronization, exponential synchronization, cluster synchronization, etc. Several criteria on local and global exponential synchronization were derived in [8] for the proposed delayed network model with output coupling. The problem of finite-time generalized function matrix projective lag synchronization between two different coupled dynamical networks with different dimensions was presented in [9]; the method was based on the double power function nonlinear feedback control. The exponential synchronization problems for complex dynamical networks with time-varying coupling delay and sampled data were considered in [10], and the sampling period was assumed to be time-varying but bounded. Some pinning synchronization criteria are established to ensure the global pinning synchronization of a class of complex

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dynamical networks in [11]. The adaptive projective synchronization of dynamical network with unknown topology and with both unknown topology and system parameters was considered in [12], respectively, and the sufficient conditions were obtained by using Lyapunov stability theory and LaSalle invariance principle.

To achieve synchronization in complex dynamical networks, the nodes should try to synchronize to its neighbors via a sufficient information exchange among the interconnections. Various control schemes have been proposed for the stabilization and synchronization of complex dynamical networks, such as adaptive control, impulsive control and pinning control, to name just a few. The generalized function matrix projective lag synchronization of uncertain complex dynamical networks via adaptive control was investigated in [13]; the adaptive controller was obtained based on Lyapunov stability theory. By constructing appropriate Lyapunov functions, several adaptive feedback synchronization criteria are derived in [14] for achieving globally exponentially asymptotic synchronization. The globally exponential synchronization of the delayed complex dynamical networks was studied in [15] by using concept of "average impulsive interval"; the networks were subject to impulsive topology and stochastic perturbations. The impulsive distributed control was designed to achieve the exponential synchronization of complex dynamical networks with system delay and coupling delays in [16], and the sufficient conditions for global exponential synchronization were derived. Pinning control can reduce the number of controllers and improve control efficiency. The adaptive pinning synchronization in complex networks with non-delay and variable couplings was investigated in [17].

However, many real-world networks, particularly biological networks, are not static but more likely to be time-varying evolving. A robust adaptive synchronization approach based on LaSalle–Yoshizawa theorem was proposed in [18], and the coupling matrix were unknown but only a time-varying coupling strength was used. The stability of synchronized state of dynamical complex networks with time-varying couplings, which were not restricted to the symmetric and irreducible connections or the non-negative off-diagonal links, was presented in [19]. The above works mainly focus on the network with time invariant inner coupling, the synchronization results of complex networks with time-varying inner coupling are seriously lacking; this is our main contribution in this paper. The [20] is about the time-varying inner coupling matrix case, however, the requirement on the coupling matrix is about every instant; and cannot be zero at any instant, meanwhile, the vector-valued function in the dynamical system model must be differentiable. The system models considered in our work contain ones that cannot be handled by the existing similar works.

The rest of this work is organized as follows. Some graph theory and mathematical preliminary knowledge are given in Sect. 2. In Sect. 3, we give the problem description and our objective. This is followed by our main results in Sect. 4 where we give the theoretical analysis of exponential synchronization for such a network with time-varying inner coupling. Numerical simulations are shown in Sect. 5. The conclusion is finally drawn in Sect. 6.

2 Graph theory and mathematical preliminary

In this section, we give some graph theory and mathematical preliminary knowledge which will be used in the later analysis, the interested readers can refer to [21] for more details.

Some basic concepts and results from algebraic graph theory are introduced here. For a system of n connected agents, its network topology can be modeled as a digraph defined as a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is a nonempty finite set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set, in which an edge is represented by an ordered pair of distinct nodes. Note that $\mathcal{G}$ is said to be undirected if $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$ for arbitrary $v_i, v_j \in \mathcal{V}$. The neighbors of agent i are denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$, and $v_j \in \mathcal{N}_i$ means that node i can access the state information of agent j. A directed path is a sequence of edges in a directed graph of the form $(v_1, v_2), (v_2, v_3), \ldots$, where $v_i \in \mathcal{V}$. An undirected path in an undirected graph is defined analogously. A directed path has a directed spanning tree if there exists at least one node having a directed path to all other nodes. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ associated with the graph $\mathcal{G}$ is defined such that a_{ij} is a positive weight if $(v_j, v_i) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. Suppose that each node has no self-edge, i.e., $a_{ii} = 0$.

The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$ is defined as follows

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{j=1, j\neq i}^{n} a_{ij}, & i = j \end{cases}$$

For the undirected graph, both $\mathcal{A}$ and $\mathcal{L}$ are symmetric.

Lemma 1 *Construct a new matrix*

$$\begin{aligned} \tilde{\mathcal{L}} &= (\tilde{l}_{ij})_{(n-1)\times(n-1)} \\ &= \begin{pmatrix} l_{22} - l_{12} & \cdots & l_{2n} - l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n2} - l_{12} & \cdots & l_{nn} - l_{1n} \end{pmatrix}. \end{aligned} \tag{1}$$

If the graph $\mathcal{G}$ has a spanning tree, then we have, for arbitrary positive definite matrix Q, there exists symmetric positive definite matrices P such that

$$P(-\tilde{\mathcal{L}}) + (-\tilde{\mathcal{L}})^{\mathrm{T}} P = -Q. \tag{2}$$

Proof Let $\mathcal{L}$ be the Laplacian matrix of diagraph $\mathcal{G}$, if there is a spanning tree in $\mathcal{G}$, it follows from [22] that zero is a simple eigenvalue of $\mathcal{L}$ and all the other eigenvalues have positive real parts. Denote the eigenvalues of $\mathcal{L}$ and $\tilde{\mathcal{L}}$ by $\gamma_1, \gamma_2, \ldots, \gamma_n$ and $\mu_1, \mu_2, \ldots, \mu_{n-1}$, respectively, where $0 = \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$, then we have $\mu_1 = \gamma_2,\ \mu_2 = \gamma_3,\ \ldots,\ \mu_{n-1} = \gamma_n$ [23]. It is easy to know that the real parts of eigenvalues of $\tilde{\mathcal{L}}$ are strictly positive, so the set of eigenvalues of matrix $-\tilde{\mathcal{L}}$ is Hurwitz. Therefore, the Lyapunov (2) admits a symmetric positive definite solution P. $\square$

Notations Throughout this paper, the following notations will be used: Let $\mathbb{R}^{m\times n}$ be the set of $m \times n$ real matrices. The superscripts "T" means transpose for real matrices, $\otimes$ denotes the Kronecker product of matrices, $\|\cdot\|$ indicates the Euclidean norm. I_m is used to represent an identity matrix of dimension m. Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ be the minimal and maximal eigenvalue of a symmetric square matrix A, respectively.

Lemma 2 [24] *The Kronecker product $\otimes$ has the following properties. For $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times s}$, $C \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{s\times t}$, we have*

(1) $(\alpha A) \otimes B = A \otimes (\alpha B)$, *where α is a constant;*
(2) $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$;
(3) $(A + B) \otimes C = A \otimes C + B \otimes C$;
(4) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

3 Problem formulation

In this paper, we consider a complex dynamical network consisting of n diffusively coupled identical nodes with time-varying inner coupling. Each of the nodes in the network is a m-dimensional dynamical system, the proposed dynamical network is given by

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j\in\mathcal{N}_i} a_{ij} \Gamma(t)(x_j(t) - x_i(t)), \tag{3}$$
$$i = 1, 2, \ldots, n$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{im}(t)]^{\mathrm{T}} \in \mathbb{R}^m$ are the state variables of node i, $c > 0$ is the coupling strength, $f(x_i(t)) = [f_1(x_i(t)), f_2(x_i(t)), \ldots, f_m(x_i(t))]^{\mathrm{T}} \in \mathbb{R}^m$ is a continuous vector-valued function, $[a_{ij}] \in \mathbb{R}^{n\times n}$ is the coupling configuration weight matrix representing the topological structure of the complex dynamical network, $\Gamma(t) \in \mathbb{R}^{m\times m}$ is the time-varying inner coupling matrix of each node.

Remark 1 Different from the previous common models given in references [25], where the inner coupling matrix was assumed to be time invariant, the time-varying case is widespread in real world. For example, the inner coupling of the multiple autonomous underwater vehicles system maybe time varying due to temperature, density and salinity. The influence of time-varying inner coupling will be studied in this paper.

Definition 1 The synchronization of complex dynamical network is said to be reached exponentially, if there exist positive constants $\kappa > 0$, $\varpi > 0$ and $T > 0$ such that

$$\left\| x_i(t) - x_j(t) \right\| \leq \kappa \mathrm{e}^{-\varpi t}, \quad i, j = 1, 2, \ldots, n$$

for all $t > T$, and ϖ is called the convergence rate.

The purpose of this paper is to obtain the sufficient conditions which guarantee the exponential synchronization of the states of the complex dynamical networks with time-varying inner coupling matrix.

4 Main results

The main results on exponential synchronization of the network (3) are derived in this section. For this purpose, the following assumptions are necessary for the derivation of the main results.

Assumption 1 The time-varying inner coupling matrix $\Gamma(t)$ is assumed to be persistently exciting (PE), i.e., there exist two positive constants T and ε, such that

$$\Gamma(t) \geq \gamma(t) I, \quad \int_t^{t+T} \gamma(\tau) \mathrm{d}\tau \geq \varepsilon.$$

Assumption 2 The nonlinear function $f(\cdot)$ is assumed to satisfy the global Lipschitz condition, that is, there exists a constant $\rho > 0$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in \mathbb{R}^m$$

According to this assumption, the following Lipschitz conditions hold

$$\begin{aligned} &\|f(x_2(t)) - f(x_1(t))\| \leq \rho \|x_2(t) - x_1(t)\|, \\ &\qquad\qquad\qquad \cdots \\ &\|f(x_n(t)) - f(x_1(t))\| \leq \rho \|x_n(t) - x_1(t)\|, \end{aligned}$$

Remark 2 Note that the corresponding dynamical functions of some classical chaotic systems, such as Lorenz system, Chen system, Lü system and Chua's circuit system, are satisfying the above assumption. The Chua's circuit system will be used for simulation in Sect. 5 below.

Now, we can derive the following theorem on exponential synchronization.

Theorem 1 *Suppose that the communication topology $\mathcal{G}$ contains a directed spanning tree and Assumptions 1, 2 hold. Then, we can take the coupling strength*

$$c > \frac{2\rho\lambda_{\max}^2(P)}{\lambda_{\min}(P)} \cdot \frac{T}{\varepsilon}, \tag{4}$$

such that the synchronization of the dynamical network with time-varying inner coupling can be reached exponentially, with the convergence rate

$$\varpi = \frac{c\varepsilon}{2T\lambda_{\max}(P)} - \frac{\rho\lambda_{\max}(P)}{\lambda_{\min}(P)}.$$

Proof Based on Eq. (3), we can define the new variables $\tilde{x}_i = x_{i+1} - x_1$, $i = 1, 2, \ldots, n-1$, and denote $\tilde{x} = [\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}]^{\mathrm{T}}$, then the system model (3) can be written more compactly as

$$\dot{\tilde{x}} = F(\tilde{x}) - c\left(\tilde{\mathcal{L}} \otimes \Gamma(t)\right)\tilde{x},$$

where $F(\tilde{x}) = [F(\tilde{x}_1), F(\tilde{x}_2), \ldots, F(\tilde{x}_{n-1})]^{\mathrm{T}}$, $F(\tilde{x}_i) = f(x_{i+1}) - f(x_1)$, then we have $\|F(\tilde{x}_i)\| \leq \rho\|\tilde{x}_i\|$ by Assumption 2, and $\tilde{\mathcal{L}}$ is defined in Eq. (1).

Since there is a directed spanning tree in the topology, there must be a P satisfying the Eq. (2), we can consider the following Lyapunov function candidate

$$V(t) = \tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})\tilde{x},$$

Take the derivative of $V(t)$, then we get

$$\begin{aligned} \dot{V}(t) &= \dot{\tilde{x}}^{\mathrm{T}}(P \otimes I_{m-1})\tilde{x} + \tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})\dot{\tilde{x}} \\ &= \left[F(\tilde{x})^{\mathrm{T}} - c\tilde{x}^{\mathrm{T}}\left(\tilde{\mathcal{L}} \otimes \Gamma(t)\right)^{\mathrm{T}}\right](P \otimes I_{m-1})\tilde{x} \\ &\quad + \tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})\left[F(\tilde{x}) - c\left(\tilde{\mathcal{L}} \otimes \Gamma(t)\right)\tilde{x}\right] \\ &= F(\tilde{x})^{\mathrm{T}}(P \otimes I_{m-1})\tilde{x} \\ &\quad - c\tilde{x}^{\mathrm{T}}\left(\tilde{\mathcal{L}} \otimes \Gamma(t)\right)^{\mathrm{T}}(P \otimes I_{m-1})\tilde{x} \\ &\quad + \tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})F(\tilde{x}) \\ &\quad - c\tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})\left(\tilde{\mathcal{L}} \otimes \Gamma(t)\right)\tilde{x} \\ &= \Psi_1(\tilde{x}) + \Psi_2(\tilde{x}), \end{aligned}$$

where

$$\begin{aligned} \Psi_1(\tilde{x}) &= F(\tilde{x})^{\mathrm{T}}(P \otimes I_{m-1})\tilde{x} + \tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})F(\tilde{x}), \\ \Psi_2(\tilde{x}) &= -c\tilde{x}^{\mathrm{T}}\left(\tilde{\mathcal{L}} \otimes \Gamma(t)\right)^{\mathrm{T}}(P \otimes I_{m-1})\tilde{x} \\ &\quad - c\tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})\left(\tilde{\mathcal{L}} \otimes \Gamma(t)\right)\tilde{x}. \end{aligned}$$

Since the two terms in $\Psi_1(\tilde{x})$ are scalar numbers, then

$$\Psi_1(\tilde{x}) = 2\tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})F(\tilde{x}),$$

and according to (2), the following inequality can be obtained

$$\begin{aligned} \Psi_2(\tilde{x}) &= -c\tilde{x}^{\mathrm{T}}\Big[(\tilde{\mathcal{L}}^{\mathrm{T}} \otimes \Gamma(t)^{\mathrm{T}})(P \otimes I_{m-1}) \\ &\qquad + (P \otimes I_{m-1})\left(\tilde{\mathcal{L}} \otimes \Gamma(t)\right)\Big]\tilde{x} \\ &= -c\tilde{x}^{\mathrm{T}}\Big[\tilde{\mathcal{L}}^{\mathrm{T}}P \otimes \Gamma(t)^{\mathrm{T}}I_{m-1} \\ &\qquad + P\tilde{\mathcal{L}} \otimes I_{m-1}\Gamma(t)\Big]\tilde{x} \\ &= -c\tilde{x}^{\mathrm{T}}\left[(\tilde{\mathcal{L}}^{\mathrm{T}}P + P\tilde{\mathcal{L}}) \otimes \Gamma(t)\right]\tilde{x} \\ &= -c\tilde{x}^{\mathrm{T}}(Q \otimes \Gamma(t))\tilde{x}. \end{aligned}$$

Then, we yield

$$\dot{V}(t) = 2\tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})F(\tilde{x}) - c\tilde{x}^{\mathrm{T}}(Q \otimes \Gamma(t))\tilde{x}.$$

By Assumption 2, one gets

$$\begin{aligned}
2\tilde{x}^{\mathrm{T}}(P \otimes I_{m-1})F(\tilde{x}) &\le 2\left\|\tilde{x}^{\mathrm{T}}\right\| \|P \otimes I_{m-1}\| \|F(\tilde{x})\| \\
&\le 2\rho\lambda_{\max}(P)\left\|\tilde{x}^{\mathrm{T}}\right\| \|\tilde{x}\| \\
&= 2\rho\lambda_{\max}(P)\tilde{x}^{\mathrm{T}}\tilde{x},
\end{aligned} \tag{5}$$

by Assumption 1 and Eq. (5), it follows that

$$\dot{V}(t) \le 2\rho\lambda_{\max}(P)\tilde{x}^{\mathrm{T}}\tilde{x} - c\gamma(t)\tilde{x}^{\mathrm{T}}(Q \otimes I_m)\tilde{x},$$

we choose the arbitrary positive definite matrix $Q = I_{n-1}$, then

$$\dot{V}(t) \le 2\rho\lambda_{\max}(P)\tilde{x}^{\mathrm{T}}\tilde{x} - c\gamma(t)\tilde{x}^{\mathrm{T}}\tilde{x}.$$

According to the definition of $V(t)$, it is easy obtain

$$\lambda_{\min}(P)\tilde{x}^{\mathrm{T}}\tilde{x} \le V(t) \le \lambda_{\max}(P)\tilde{x}^{\mathrm{T}}\tilde{x}, \tag{6}$$

which implies that

$$\begin{aligned}
\dot{V}(t) \le& \frac{2\rho\lambda_{\max}(P)}{\lambda_{\min}(P)}V(t) - \frac{c\gamma(t)}{\lambda_{\max}(P)}V(t) \\
&= -\left[\frac{c\gamma(t)}{\lambda_{\max}(P)} - \frac{2\rho\lambda_{\max}(P)}{\lambda_{\min}(P)}\right]V(t).
\end{aligned}$$

by using the Gronwall–Bellman inequality, we have

$$V(t) \le V(t_0)\mathrm{e}^{-\int_{t_0}^{t}\left[\frac{c\gamma(\tau)}{\lambda_{\max}(P)} - \frac{2\rho\lambda_{\max}(P)}{\lambda_{\min}(P)}\right]\mathrm{d}\tau},$$

and let $t = t + T$, $t_0 = t$, then one gets

$$\begin{aligned}
V(t+T) \le& V(t)\mathrm{e}^{-\int_{t}^{t+T}\left[\frac{c\gamma(\tau)}{\lambda_{\max}(P)} - \frac{2\rho\lambda_{\max}(P)}{\lambda_{\min}(P)}\right]\mathrm{d}\tau} \\
\le& V(t)\mathrm{e}^{-\left[\frac{c\varepsilon}{\lambda_{\max}(P)} - \frac{2\rho\lambda_{\max}(P)T}{\lambda_{\min}(P)}\right]}.
\end{aligned} \tag{7}$$

From (7), if the condition $\frac{c\varepsilon}{\lambda_{\max}(P)} - \frac{2\rho\lambda_{\max}(P)T}{\lambda_{\min}(P)} > 0$ holds, i.e., $c > \frac{2\rho\lambda_{\max}^2(P)}{\lambda_{\min}(P)} \cdot \frac{T}{\varepsilon}$, and denote $\sigma = \mathrm{e}^{-\left[\frac{c\varepsilon}{\lambda_{\max}(P)} - \frac{2\rho\lambda_{\max}(P)T}{\lambda_{\min}(P)}\right]}$, it is obvious that $\sigma \in (0, 1)$. Thus, we have

$$V(t+T) \le \sigma V(t), \quad \sigma \in (0, 1).$$

Then defining $\eta = -\frac{\ln\sigma}{T} \ge \frac{1-\sigma}{T}$ [26], it is straightforward to show that

$$V(t) \le V(t_0)\mathrm{e}^{-\eta(t-t_0)},$$

By Eq. (6), we have

$$\tilde{x}(t)^{\mathrm{T}}\tilde{x}(t) \le \frac{V(t_0)}{\lambda_{\min}(P)}\mathrm{e}^{-\eta(t-t_0)}.$$

Finally, it is readily seen that

$$\|\tilde{x}(t)\| \le M\mathrm{e}^{-\frac{\eta}{2}(t-t_0)},$$

with $M = \sqrt{\frac{V(t_0)}{\lambda_{\min}(P)}} > 0$, which means that the synchronization of the complex dynamical network can be achieved exponentially with convergence rate $\frac{\eta}{2}$. The proof is thus completed. □

Remark 3 The condition (4) obtained in Theorem 1 is just the sufficient but not necessary condition, i.e., the synchronization of the nonlinear dynamical system can be achieved under some coupling strength c which are not satisfy (4). We will illustrate this property in the following simulations.

5 Numerical simulations

In this section, numerical simulations are presented to verify the effectiveness and feasibility of the exponential synchronization proposed in previous section. Consider the information interactive network with the communication graph $\mathcal{G}$ given in Fig. 1 with 6 nodes. There is a directed spanning tree in the network, and the state variables of ith node are x_{ij}, where $i = 1, 2, \ldots, 6$, $j = 1, 2, 3$, that is, $n = 6$, $m = 3$. The initial conditions of each nodes are randomly generated in the interval $[-20, 20]$.

In simulations, each node on the considered network is assumed to be a Chua's circuit, which is often taken in literature as a paradigm for chaos and for studying synchronization [27,28]. Then, the individual node dynamics are described by

$$\dot{x}_i = \begin{bmatrix} \alpha(-x_{i1} + x_{i2} - \varphi(x_{i1})) \\ x_{i1} - x_{i2} + x_{i3} \\ -\beta x_{i2} \end{bmatrix},$$

with a nonlinear characteristic $\varphi(x_{i1})$ of Chua's diode given by

$$\varphi(x_{i1}) = bx_{i1} + \frac{1}{2}(a-b)\left(|x_{i1}+c| - |x_{i1}-c|\right),$$

and parameters $\alpha = 10, \beta = 18, a = -4/3, b = -3/4$ and $c = 1$. With these parameters, the system produces the double-scroll chaotic attractor. From [29], we know that Assumption 2 holds for $\rho = 4.3871$.

From the interactive network Fig. 1, we can obtain the Laplacian matrix $\mathcal{L}$ and the corresponding $\tilde{\mathcal{L}}$ of the network as follows

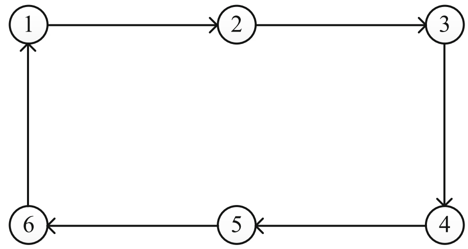

Fig. 1 The information interactive network

$$\mathcal{L}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\tilde{\mathcal{L}}=\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Since matrix Q in (2) is arbitrary, we choose $Q = I_5$, we can obtain the eigenvalues of the symmetric positive definite matrix P are

$$\lambda = \{0.2226, 0.2929, 0.3944, 0.9802, 1.8631\},$$

the smallest and largest eigenvalue are $\lambda_{\min}(P) = 0.2226$, $\lambda_{\max}(P) = 1.8631$, respectively.

The time-varying inner coupling matrix $\Gamma(t)$ was chosen as

$$\Gamma(t)=\begin{bmatrix} \sin^2(t) & 0 & 0 \\ 0 & \sin^2(t) & 0 \\ 0 & 0 & \sin^2(t) \end{bmatrix} \geq \sin^2(t) I_3$$

and

$$\int_t^{t+2\pi} \sin^2(\tau)\mathrm{d}\tau = \pi$$

then, we can choose $\varepsilon = \pi$.

From the above calculations, we can obtain the coupling strength

$$c = \frac{2 \times 4.3871 \times 1.8631^2 \times 2\pi}{0.2226 \times \pi} = 273.6432.$$

Since (4) is just the sufficient condition, we provide three different simulations with the coupling strength: (1) $c = 300$, condition (4) holds and the synchronization can be achieved. (2) $c = 30$, condition (4) is not hold but the synchronization can be achieved. (3) $c = 0.3$, condition (4) is not hold and the synchronization cannot be achieved.

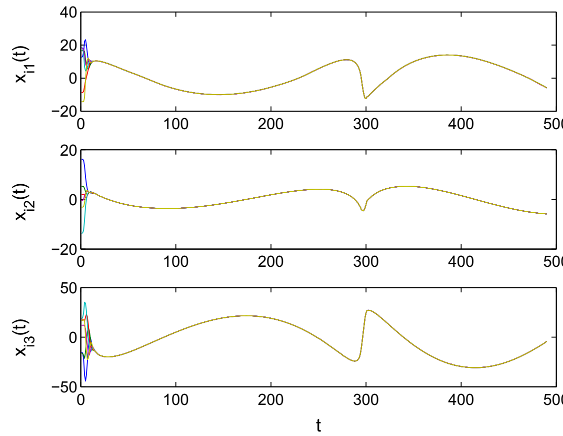

Fig. 2 The evolution of states $x_{ij}(t)$, $i = 1, 2, \ldots, 8$, $j = 1, 2, 3$, $c = 300$

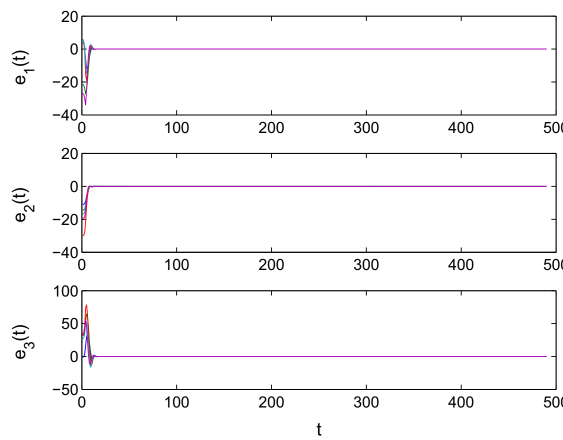

Fig. 3 The evolution of errors $e_j(t)$, $j = 1, 2, 3$, $c = 300$

Figures 2, 5 and 8 show the evolution of all the nodes states $x_{i1}(t)$, $x_{i2}(t)$ and $x_{i3}(t)$, where $i = 1, 2, \ldots, 6$. Figures 3, 6 and 9 are the evolution of errors, $e_j(t) = x_{ij}(t) - x_{1j}(t)$, where $j = 1, 2, 3$. The evolution of errors norm are shown in Figs. 4, 7 and 10, $E_j(t) = \sqrt{\sum_{k \in \mathcal{N}_1} \|x_{kj}(t) - x_{1j}(t)\|}$. It shows that, when $c = 300$ and $c = 30$, all the synchronization errors of the complex dynamical network do globally exponentially converge to zero, i.e., the synchronous solution is exponentially stable for network (3), while for $c = 0.3$, the synchronization cannot be achieved.

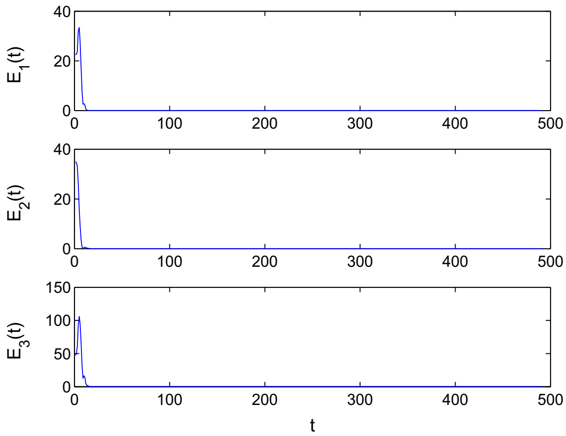

Fig. 4 The evolution of errors norm $E_j(t)$, $j = 1, 2, 3$, $c = 300$

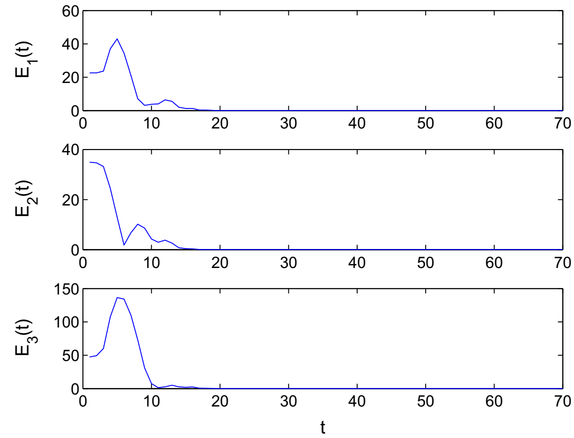

Fig. 7 The evolution of errors norm $E_j(t)$, $j = 1, 2, 3$, $c = 30$

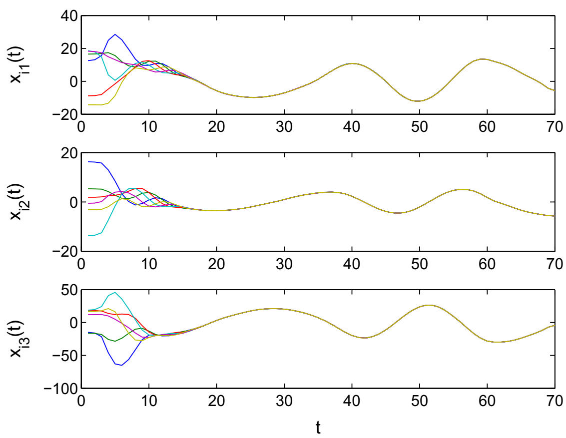

Fig. 5 The evolution of states $x_{ij}(t)$, $i = 1, 2, \ldots, 8$, $j = 1, 2, 3$, $c = 30$

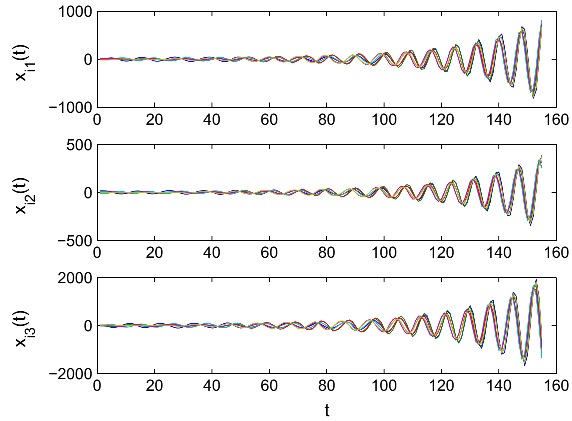

Fig. 8 The evolution of states $x_{ij}(t)$, $i = 1, 2, \ldots, 8$, $j = 1, 2, 3$, $c = 0.3$

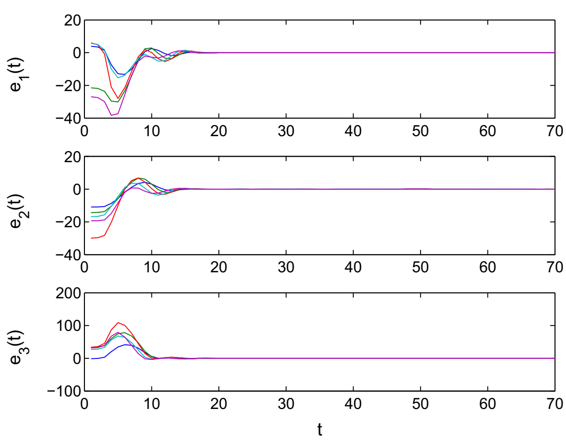

Fig. 6 The evolution of errors $e_j(t)$, $j = 1, 2, 3$, $c = 30$

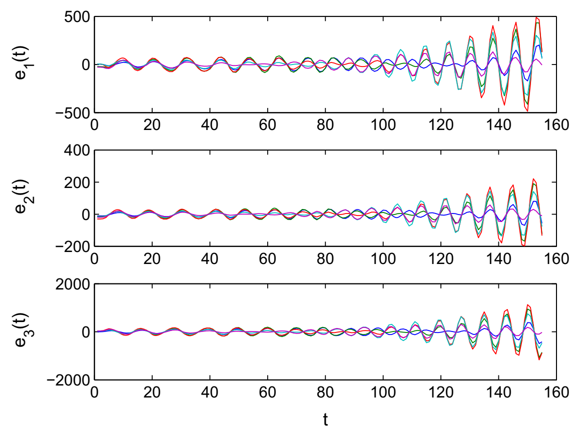

Fig. 9 The evolution of errors $e_j(t)$, $j = 1, 2, 3$, $c = 0.3$

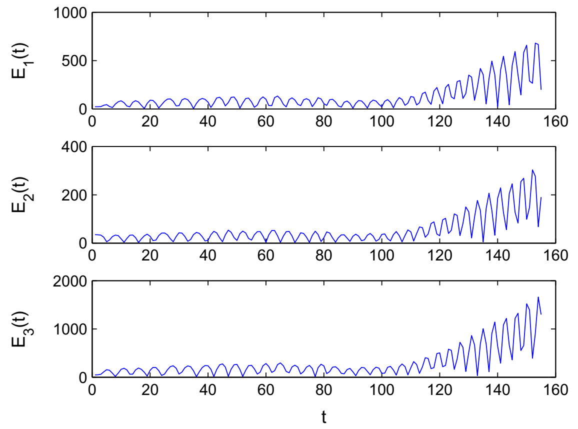

Fig. 10 The evolution of errors norm $E_j(t)$, $j = 1, 2, 3, c = 0.3$

6 Conclusions

The exponential synchronization problem of the complex dynamical networks with time-varying inner coupling matrix is studied in this paper; the sufficient condition is presented based on the graph theory, matrix theory and Lyapunov method. Since (4) is just sufficient condition, we want to give a more compact condition in the future, and to extend the research to the second-order cases.

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