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ORIGINAL PAPER

On the quasi-periodic response in the delayed forced Duffing oscillator

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Abstract We explore the quasi-periodic (QP) vibrations in a delayed Duffing equation submitted to periodic forcing. The second-step perturbation method is applied on the slow flow of the oscillator to derive the slow–slow flow near the primary resonance. The QP solution corresponding to the nontrivial equilibrium of the slow–slow flow as well as its modulation envelope is predicted analytically. The influence of different system parameters on the QP response is reported and discussed. The analytical results show that for weak nonlinearity and small damping large-amplitude QP vibration induced by destabilization of limit cycle via Neimark–Sacker bifurcation occurs in a broadband of the excitation frequency and in large range of delay parameters.

Keywords Quasi-periodic response · Duffing oscillator · Time delay · Perturbation analysis

1 Introduction

The dynamics and stability analysis of the delayed Duffing oscillator under periodic forcing has been studied by several authors. Hu et al. [1] considered such an oscillator when the delay is present in the position and velocity and investigated analytically the dynamics near the primary and the 1:3 subharmonic resonances using the multiple scales method. In [2,3], the delayed forced Duffing oscillator has been revisited near the primary resonance and the focus was directed toward analyzing analytically the effect of delay parameters on the periodic response in the region where it is stable. Instead, in the region where the periodic solutions turn to unstable via Neimark-Sacker bifurcation giving rise to QP vibrations, only numerical simulation was performed to approximate the location of the QP modulation envelope in the frequency response domain [4]. Usually, the investigation of the QP oscillations present in such a problem is generally ignored or just reported numerically for certain fixed parameters [5,6]. However, in forced nonlinear oscillators under time delay, the interaction between the forcing and the time delay induces not only periodic oscillations, but also QP vibrations covering a broadband of the excitation frequency around the resonance [7]. Likewise, it was shown in certain circumstances that the amplitude of the QP response can be of the same order of magnitude as that of the periodic response and it may even be larger, as reported analytically [8,9] and numerically [4].

Therefore, in order to perform a comprehensive analytical study in the problems of forced oscillators under time delay, as in the delayed forced Duffing oscillator under consideration, the QP response should not be ignored. Instead, it must be systematically consid-

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ered and analyzed, as naturally done for the periodic response near the resonance. The analytical investigation can be performed by applying the second-step perturbation method (on the slow flow) developed first in [10] and applied successfully to different two-period QP systems [7–9,11,12].

It is worth noticing that in the case of three-period QP systems where the slow flow is time-dependant having two different frequencies, it is possible to approximate the QP solutions of the slow flow corresponding to three-period QP oscillations of the original oscillator using a third-step perturbation on the slow–slow flow [13].

In the present paper, we report on the analytical investigation of the QP response of the delayed forced Duffing oscillator using the second-step perturbation method [10]. To this end, we consider the following forced delayed Duffing oscillator

$$\begin{aligned}&\ddot{x}(t)+\delta\dot{x}(t)+\omega_0^2x(t)+\gamma x(t)^3\\&\quad=\alpha x(t-\tau)+f\cos(\lambda t)\end{aligned}\tag{1}$$

keeping, for convenience, the same parameters and the same notations as in [4]. Here, the dot denotes differentiation with respect to the nondimensional time t, δ is a viscous damping coefficient, γ is a small coefficient representing nonlinear stiffness, ω_0 is the natural frequency of the system, α is the delay amplitude, f is the amplitude of the excitation, λ is the excitation frequency and τ is the time delay.

Because we focus our attention on the QP solutions, we shall approximate directly the periodic solutions of the slow flow equations obtained near the primary resonance [4] using the multiple scales method [14]. The second-step perturbation method is then applied on the slow flow system to extract the slow–slow flow for which the nontrivial equilibrium corresponds precisely to the QP response of the original equation (1). Next, the slow–slow flow is determined and the influence of various parameters on the QP vibration and its modulation envelope is examined.

2 The slow–slow flow

The modulation equations of the forced Duffing oscillator with time delay in the position given by Eq. (1) were obtained near the primary resonance [4], i.e., $\lambda=\omega_0+\varepsilon\sigma$, where σ is a detuning parameter. Up to the second order, this slow flow is given by

$$\begin{cases}\dot{a}=S_1a-3S_3a^2\sin(\phi)+S_4a^3\\\qquad+\beta_1\sin(\phi)+\beta_2\cos(\phi)\\a\dot{\phi}=S_2a-S_3a^2\cos(\phi)+S_5a^3+S_6a^5\\\qquad+\beta_1\cos(\phi)-\beta_2\sin(\phi)\end{cases}\tag{2}$$

where a and ϕ are, respectively, the amplitude and the phase of the response, while

$$\begin{aligned}S_1&=-\frac{1}{2}\delta-\frac{1}{2}\alpha\sin(\tau)-\frac{1}{4}\alpha^2\cos(\tau)\sin(\tau)\\S_2&=\lambda-\omega_0+\frac{1}{8}\delta^2+\frac{1}{2}\alpha\cos(\tau)+\frac{1}{8}\alpha^2\cos^2(\tau)\\&\quad-\frac{1}{8}\alpha^2\sin^2(\tau)\\S_3&=\frac{3}{32}\gamma f\\S_4&=\frac{3}{16}\gamma\delta+\frac{3}{8}\alpha\gamma\sin(\tau)\\S_5&=-\frac{3}{8}\gamma-\frac{3}{16}\alpha\gamma\cos(\tau)\\S_6&=\frac{15}{256}\gamma^2\\\beta_1&=\frac{1}{2}f+\frac{1}{8}\alpha f\cos(\tau)\\\beta_2&=\frac{1}{8}\delta f-\frac{1}{8}\alpha f\sin(\tau)\end{aligned}$$

Equilibria of the slow flow (2) corresponding to periodic solutions of the original system (1) were examined in [4]. Here, we study the periodic solutions of the slow flow (2) corresponding to QP responses of (1) and we analyze the influence of different parameters on the QP response. This can be done by transforming the modulation equations from the polar form (2) to the following Cartesian system using the variable change $u=a\cos\phi$ and $v=-a\sin\phi$

$$\begin{cases}\dfrac{du}{dt}=S_2v+\beta_2+\eta\{S_1u+2S_3uv\\\qquad+(S_4u+S_5v)(u^2+v^2)+S_6(u^2+v^2)^2v\}\\\dfrac{dv}{dt}=-S_2u-\beta_1+\eta\{S_1v+S_3v^2+S_3(u^2+v^2)\\\qquad+(S_4u-S_5v)(u^2+v^2)-S_6(u^2+v^2)^2u\}\end{cases}\tag{3}$$

where η is a bookkeeping parameter introduced in damping and nonlinearity so that the unperturbed system of Eq. (3) admits a basic solution. Following [9–12], a periodic solution of the slow flow (3) can be sought in the form

$$u(t) = u_0(T_1, T_2) + \eta u_1(T_1, T_2) + O(\eta^2)$$
$$v(t) = v_0(T_1, T_2) + \eta v_1(T_1, T_2) + O(\eta^2) \tag{4}$$

where $T_1 = t$ and $T_2 = \eta t$. Introducing $D_i = \frac{\partial}{\partial T_i}$ yields $\frac{d}{dt} = D_1 + \eta D_2 + O(\eta^2)$, and substituting (15) into (3) and collecting terms, one obtains up to the first order of η

$$\begin{aligned} D_1^2 u_0 + \nu^2 u_0 &= -\beta_1 S_2 \\ S_2 v_0 &= D_1 u_0 - \beta_2 \end{aligned} \tag{5}$$

$$\begin{aligned} D_1^2 u_1 + \nu^2 u_1 = S_2[&-D_2 v_0 + S_1 v_0 \\ &+ S_3 v_0^2 + S_3(u_0^2 + v_0^2) \\ &+ (S_4 v_0 - S_5 u_0)(u_0^2 + v_0^2) \\ &- S_6(u_0^2 + v_0^2)^2 u_0] - D_1 D_2 u_0 \\ &+ S_1 D_1 u_0 + D_1[2S_3 u_0 v_0 \\ &+ (S_4 u_0 + S_5 v_0)(u_0^2 + v_0^2) \\ &+ S_6(u_0^2 + v_0^2)^2 v_0] \end{aligned} \tag{6}$$

$$\begin{aligned} S_2 v_1 = D_1 u_1 + D_2 u_0 - [&S_1 u_0 + 2S_3 u_0 v_0 \\ &+ (S_4 u_0 + S_5 v_0)(u_0^2 + v_0^2) \\ &+ S_6(u_0^2 + v_0^2)^2 v_0] \end{aligned} \tag{7}$$

where $\nu\ (= S_2)$ is the frequency of slow flow limit cycle corresponding to the frequency of the QP response. It is interesting to point out that the frequency of the modulation ν depends on the delay parameters, the natural frequency, ω_0, the frequency of the excitation, λ, and damping, δ, but it does not depend on the excitation amplitude f (at least up to the leading order).

The solution at the zero-order system (5) can be written as

$$u_0(T_1, T_2) = R(T_2)\cos(S_2 T_1 + \theta(T_2)) - \alpha_1$$
$$v_0(T_1, T_2) = -R(T_2)\sin(S_2 T_1 + \theta(T_2)) - \alpha_2 \tag{8}$$

where $\alpha_1 = \frac{\beta_1}{S_2}, \alpha_2 = \frac{\beta_2}{S_2}$ and R and θ are, respectively, the amplitude and the phase of the slow flow limit cycle. Substituting (8) into (6) and removing secular terms give the following autonomous slow–slow flow system on R and θ

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}t} &= S_4 R^3 + \left(S_1 - \alpha_2 S_3 + 2\alpha_1^2 S_4 + 2\alpha_2^2 S_4\right) R \\ R\frac{\mathrm{d}\theta}{\mathrm{d}t} &= S_6 R^5 + \left(S_5 + 6(\alpha_1^2 + \alpha_2^2) S_6\right) R^3 \\ &\quad + \left((2\alpha_1^2 + 2\alpha_2^2) S_5 + (3\alpha_1^4 + 3\alpha_2^4 \right. \\ &\quad \left. + 6\alpha_1^2\alpha_2^2) S_6\right) R \end{aligned} \tag{9}$$

Equilibria of this slow–slow flow determine the QP solutions of the original equation (1). In addition to the trivial equilibrium, $R = 0$, the nontrivial equilibrium is obtained by setting $\frac{dR}{dt} = 0$ and given by

$$R = \sqrt{\frac{-S_1 + \alpha_2 S_3 - 2\alpha_1^2 S_4 - 2\alpha_2^2 S_4}{S_4}} \tag{10}$$

Thus, the approximate periodic solution of the slow flow (3) is given by

$$u(t) = R\cos(\phi t) - \alpha_1$$
$$v(t) = -R\sin(\phi t) - \alpha_2 \tag{11}$$

and using (11), the approximate amplitude $a(t)$ of the QP oscillations reads

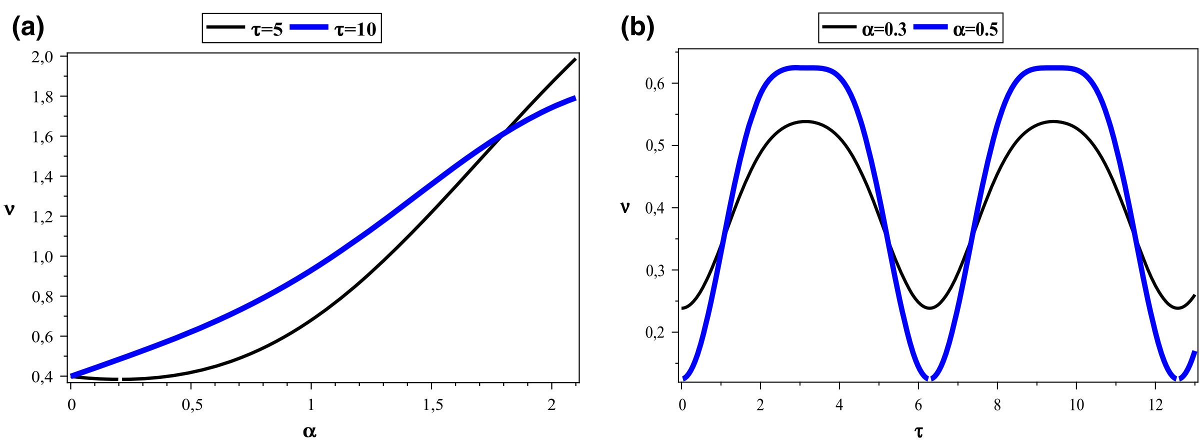

Fig. 1 Variation of the QP modulation frequency versus **a** α, **b** τ for $\lambda = 0.6$. (Color figure online)

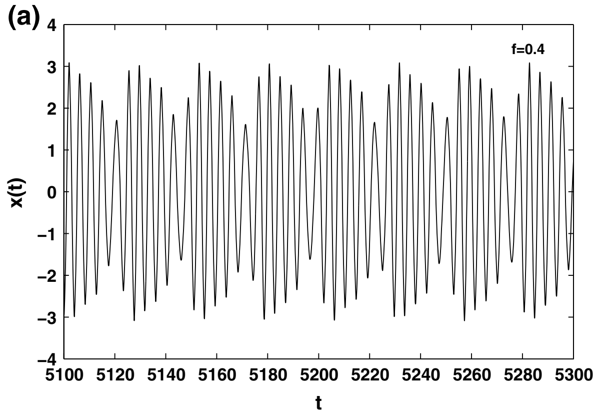

Fig. 2 Time histories for different values of f for $\lambda = 1.6$, $\alpha = 0.5$, $\tau = 4.8$

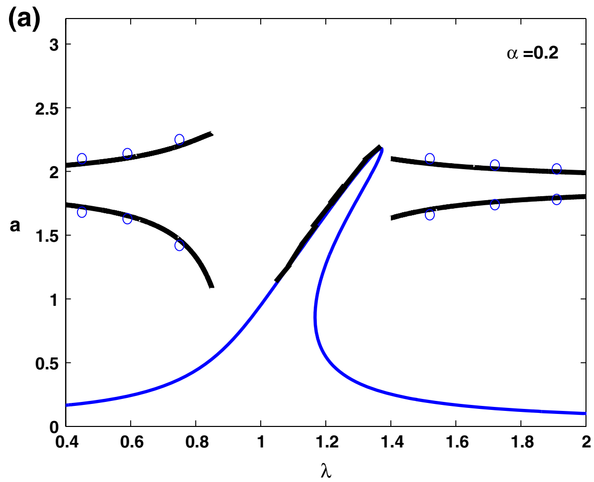

Fig. 3 Periodic and QP frequency responses for $\tau = 4.8$ and different values of α. Analytical prediction (*black lines* for stable and *blue lines* for unstable) and numerical simulation (*circles*). (Color figure online)

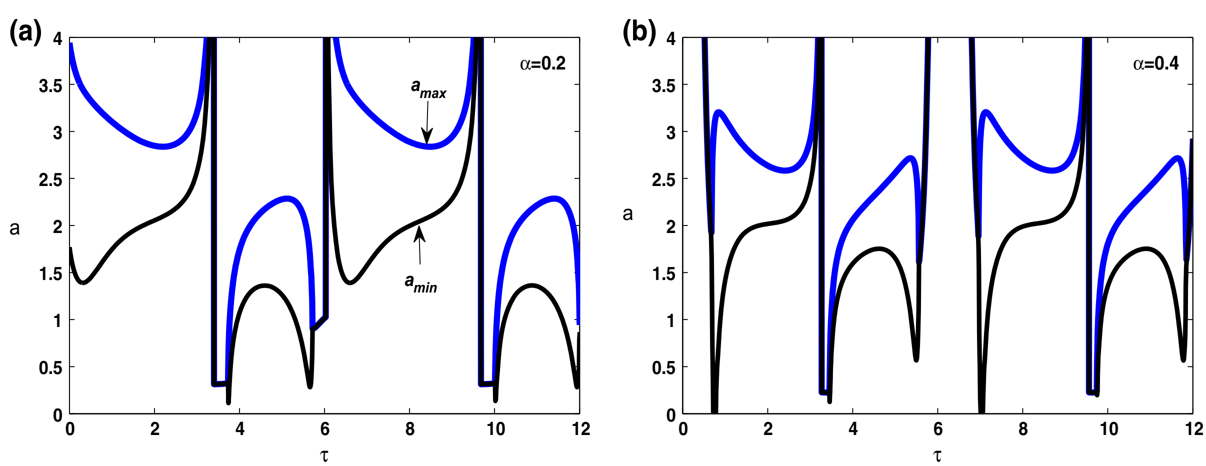

Fig. 4 Variation of the QP envelope versus τ for $\lambda = 0.8$. (Color figure online)

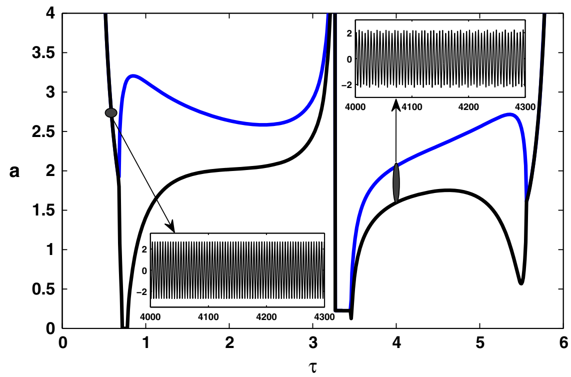

Fig. 5 Variation of the QP envelope versus τ during one period of time delay for $\lambda = 0.8$ and $\alpha = 0.4$. (Color figure online)

$$a(t) = \sqrt{\left[R^2 + \alpha_1^2 + \alpha_2^2\right] - \left[2\alpha_1 R \cos(\phi t) - 2\alpha_2 R \sin(\phi t)\right]} \tag{12}$$

while the modulation envelope is delimited by $a_{\min}$ and $a_{\max}$ given by

$$a_{\min} = \min\left\{\sqrt{\left[R^2 + \alpha_1^2 + \alpha_2^2\right] \pm \left[2\alpha_1 R \pm 2\alpha_2 R\right]}\right\} \tag{13}$$

$$a_{\max} = \max\left\{\sqrt{\left[R^2 + \alpha_1^2 + \alpha_2^2\right] \pm \left[2\alpha_1 R \pm 2\alpha_2 R\right]}\right\} \tag{14}$$

In what follows, we fix the parameters as $\omega_0 = 1$, $\delta = 0.1$, $\gamma = 0.25$ and $f = 0.2$, unless their variation is indicated. Figure 1a shows, for two different values of τ, the variation of the frequency of the QP modulation, ν, as a function of the delay amplitude α. It can be observed that the frequency of the QP modulation increases with increasing α for both values of τ. Figure 1b presents for two values of α the variation of the QP modulation frequency ν with respect to τ. The figure shows that for $\alpha = 0.3$ the frequency of the QP modulation oscillates in a small range around a mean value, while it oscillates in a larger range for increasing α.

It was pointed out above that, up to the leading order, the frequency of the modulation ν $(= S_2)$ does not depend on the excitation frequency f which can be considered as a limitation of the approximation. Indeed, numerical simulations given by time histories shown in Fig. 2a for $f = 0.6$ and in Fig. 2b for $f = 0.8$ indicate clearly the dependence of the modulation frequency ν on the excitation amplitude f. Such a limitation in the approximation of the slow flow (2) deserves to be overcome by pushing the approximation up to the second order.

Figure 3 presents the periodic frequency response [4] and the QP modulation envelope as given by Eqs. (13) and (14). The comparison between the analytical predictions (solid lines) and the numerical simulations of the QP envelope obtained by using dde(23) algorithm [15] (double circles) shows that the analytical results predict well the envelope of the QP modulation. It can be observed that increasing the delay amplitude α, the frequency response turns to linear and decreases drastically, while the amplitude of the QP response increases as shown in Fig. 3b. This result indicates that in practical applications, as in turning and milling process, trying to quench periodic vibrations by tuning the delay

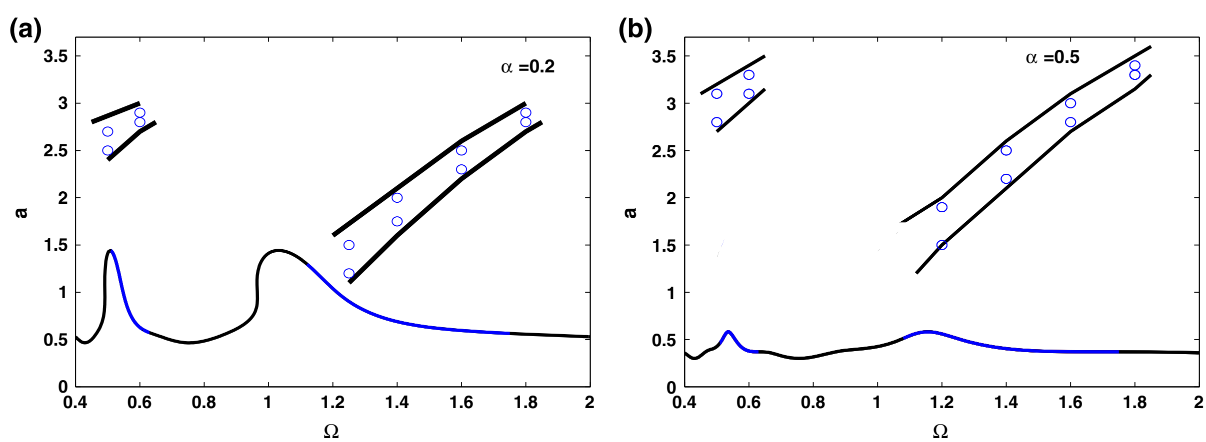

Fig. 6 Periodic and QP responses versus Ω for $\lambda = 0.9$. Analytical prediction (*black lines* for stable and *blue lines* for unstable) and numerical simulation (*circles*). (Color figure online)

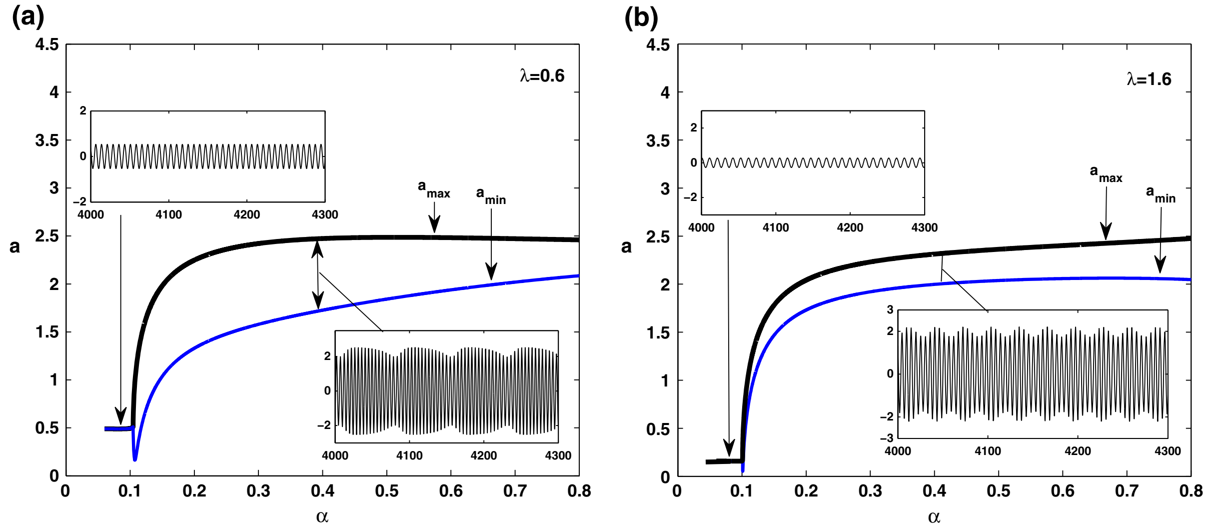

Fig. 7 Variation of the QP envelope versus α for $\tau = 4.8$. (Color figure online)

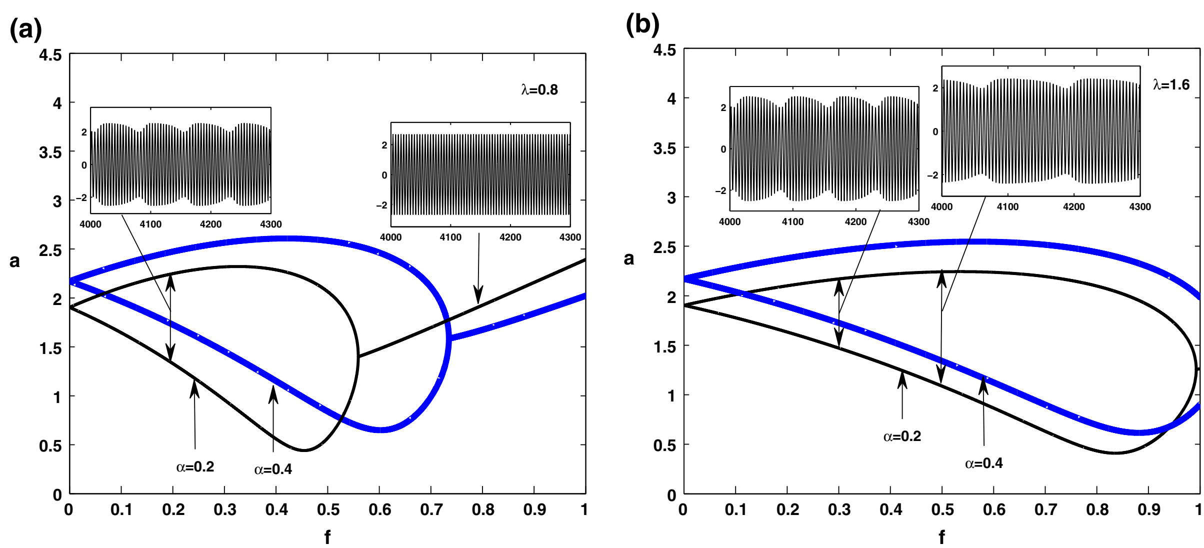

Fig. 8 Variation of the QP envelope versus f for $\tau = 4.8$. (Color figure online)

amplitude, it is very likely that dangerous QP vibrations with large amplitudes can occur in the vicinity of the resonance.

The variation of the response amplitude as a function of the time delay is depicted in Fig. 4 for fixed frequency excitation λ and two different values of the delay amplitude α ($\alpha = 0.2$, Fig. 4a; $\alpha = 0.4$, Fig. 4b). One observes the alternate QP modulation envelopes delimited by a_{max} and a_{min} and connected by small regions of frequency locking. The variation of the amplitude during one period of time delay is depicted in Fig. 5 for $\alpha = 0.4$, and time histories are presented inset in the figure showing the QP response within the modulation region and the frequency-locked response at the lines connecting the QP envelopes. Note that when τ is swept forward or backward, strong jumps between periodic and QP response may occur which can lead to dangerous situations in some engineering applications.

Figure 6 compares the analytical prediction of the QP modulation envelope obtained in the present work

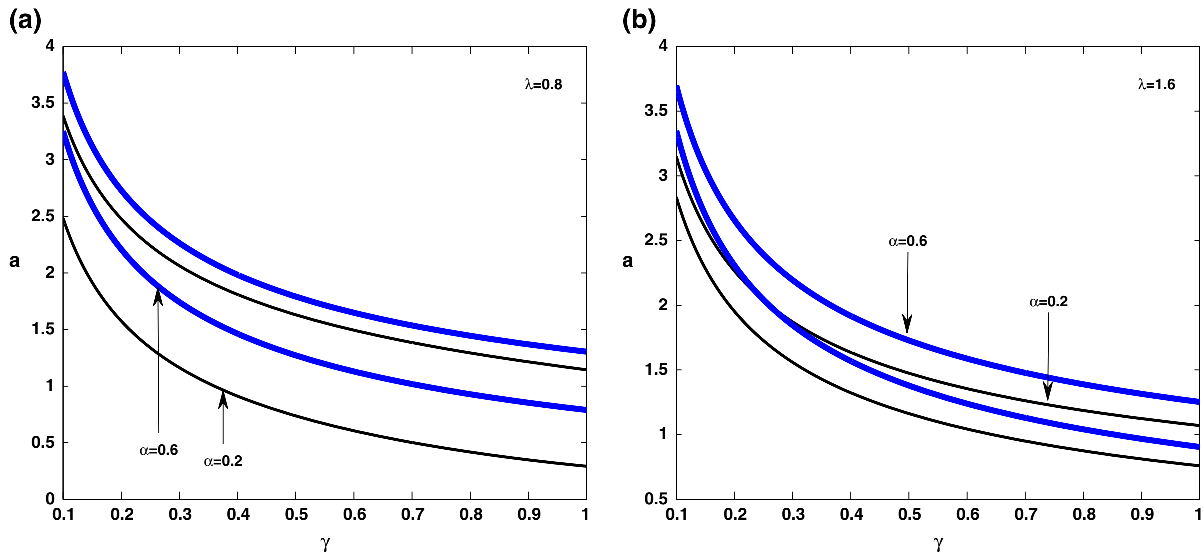

Fig. 9 Variation of the QP envelope versus γ for $\tau = 4.8$. (Color figure online)

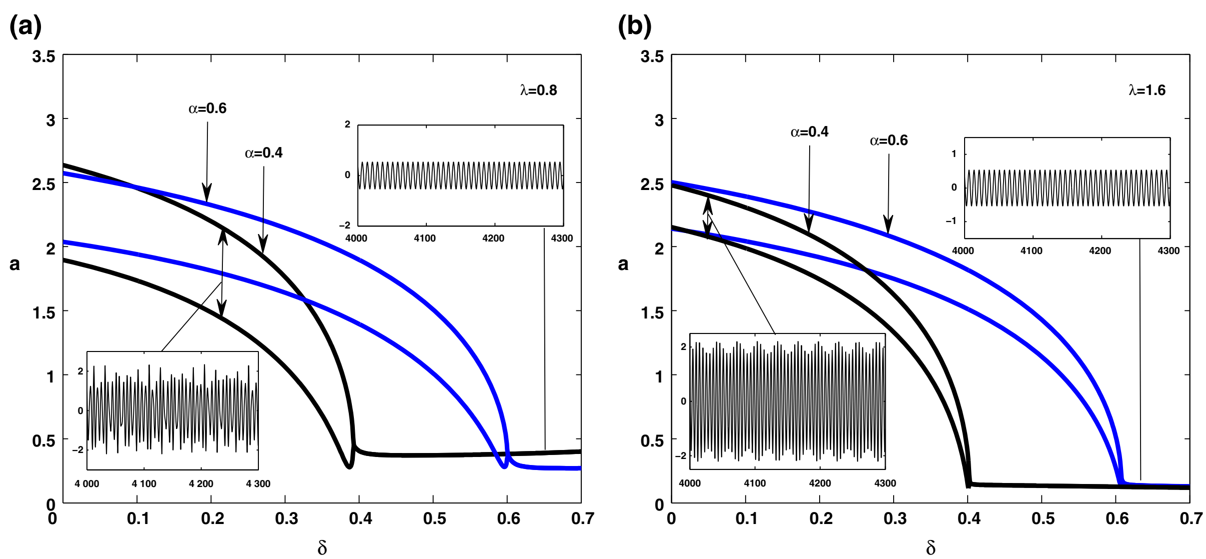

Fig. 10 Variation of the QP envelope versus δ for $\tau = 4.8$. (Color figure online)

(solid lines) with the results obtained by numerical simulations (circles) in the parameter plane ($a, \Omega = \frac{2\pi}{\tau}$) [4] showing a good match. It can be seen from this figure (as in Fig. 5) the alternate characteristic of the QP response appearing in certain ranges of Ω, resulting from destabilization of the periodic response via Neimark-Sacker bifurcation.

In Fig. 7a, b, the variation of the QP envelope versus the delay amplitude α is presented for two different values of the excitation frequency, $\lambda = 0.6$ and $\lambda = 1.6$, respectively. The plots indicate that for small values of α, the system response is periodic with small amplitude of oscillation, as shown by time history inset in the figure. Increasing α, a jump to QP response occurs at certain threshold.

Figure 8 shows the variation of the QP modulation envelope versus the excitation amplitude f for two different values of the frequency, $\lambda = 0.8$ (Fig. 8a) and

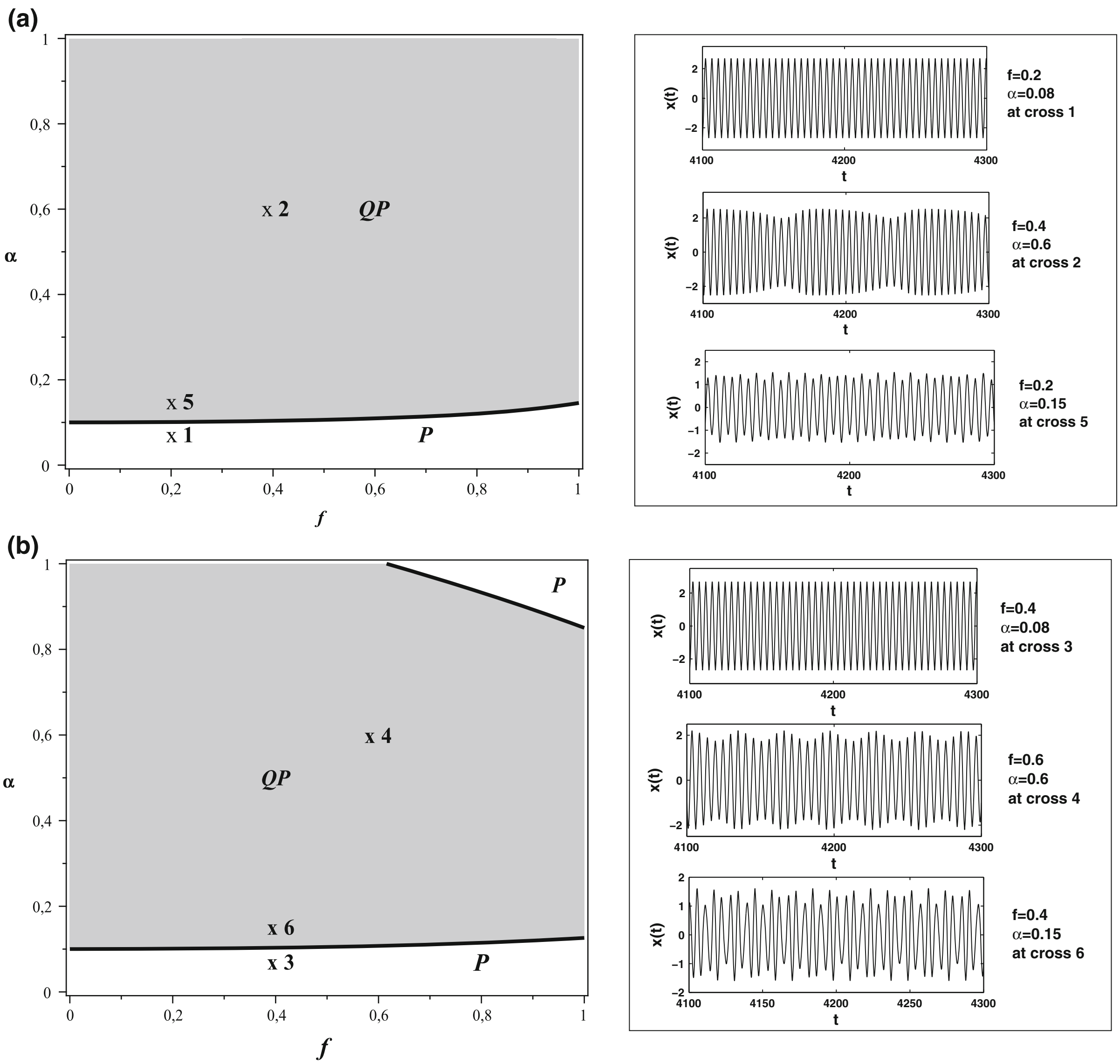

Fig. 11 Region of existence of stable QP solutions in the plan (α, f) for $\tau = 4.8$ and **a** $\lambda = 0.6$, **b** $\lambda = 1.6$

$\lambda = 1.6$ (Fig. 8b). Inset in the figures, time histories corresponding to some values of f are presented. It can be clearly observed that the frequency of modulation depends on the excitation amplitude f (Fig. 8b). This result was not predicted analytically up to the leading order ($\nu = S_2$ which is f-independent).

For completeness of the parametric study, the variation of the QP modulation envelope as a function of nonlinearity and damping is illustrated in Figs. 9 and 10, respectively, for two different values of the excitation frequency (Fig. 9) and α (Fig. 10). Inspection of Fig. 9 shows that the amplitude of the QP vibrations decreases with increasing the nonlinearity. It can also be observed from Fig. 10 that the amplitude of the QP vibrations decreases with damping and can be suppressed at a certain threshold of damping. This suppression onset can be retarded by increasing the delay amplitude α.

Finally, we examine the stability and bifurcation of the QP solutions by calculating the eigenvalues of the Jacobian matrix of system (9). Thus, the region of existence of stable QP oscillations is delimited by the curves given by the conditions

$$-S_1 + \alpha_2 S3 - 2\alpha_1^2 S4 - 2\alpha_2^2 S4 < 0$$
$$S4 < 0 \quad (15)$$

Figure 11a-left shows in the parameter plane (α, f) the regions where stable QP solutions exist, as given by the conditions (15) (green region), while Fig. 11a-right presents time histories corresponding to three values picked at the crosses labeled 1, 2 and 5. It can be observed that the stable QP response exists in a wide region beyond certain critical values of α. Figure 11b illustrates similar results but for a different value of λ. In Fig. 11b-right is shown time histories corresponding to values picked at crosses labeled 3, 4 and 6.

3 Conclusion

The QP response of a delayed forced Duffing oscillator was studied analytically near the primary resonance. The seconde-step perturbation method was applied directly on the slow flow to derive the slow–slow flow. The QP solution, given by the nontrivial equilibrium of the slow–slow flow, as well as its modulation envelope is predicted analytically. Note that the method has not captured the dependence of the modulation frequency of the QP solution on the amplitude of excitation (at least up to the leading order). The influence of the delay and forcing parameters on the QP solution was examined. The results show that in a weakly nonlinear delayed forced Duffing oscillator with small damping, stable large-amplitude QP vibrations exist in certain ranges of the excitation and delay parameters. These QP oscillations may persist even for certain values of delay parameters that decrease the amplitude of the periodic response.

It can be concluded from the present work that to conduct a comprehensive study on the vibration problems in delayed systems under periodic forcing, QP solutions should not be ignored. The existence of large-amplitude QP vibrations in broadband of delay and forcing parameters requires taking into consideration systematically such QP behavior. For instance, in the problem of turning and milling process, which can be modeled by a delayed forced Duffing oscillator, besides the undesirable regenerative effect (self-sustained vibration), large-amplitude QP vibrations may also appear in the system. The intuition that emerges from this study is that such a QP modulation response with a controlled frequency modulation present in the system may be exploited to manufacture circular tools with modulated surfaces.

The present work shows once more the effectiveness of the second-step perturbation method to capture analytically the QP response and its modulation envelope in systems exhibiting such a behavior. It is worthy to notice that to capture the analytical dependence of the modulation frequency of the QP response on the amplitude of the excitation, the second-step perturbation method should be performed to a higher-order approximation.

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