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Noether theorem and its inverse for nonlinear dynamical systems with nonstandard Lagrangians

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Abstract The Noether theorem and its inverse theorem for the nonlinear dynamical systems with nonstandard Lagrangians are studied. In this paper, two kinds of nonstandard Lagrangians, namely exponential Lagrangians and power-law Lagrangians, are discussed. For each case, the Hamilton principle based on the action with nonstandard Lagrangians is established, the differential equations of motion for the dynamical systems with nonstandard Lagrangians are obtained, and two basic formulae for the variation in Hamilton action with nonstandard Lagrangians are derived. The definitions and the criteria of the Noether symmetric transformations and the Noether quasi-symmetric transformations are given. The Noether theorem and its inverse theorem are established, which reveal the intrinsic relation between the symmetry and the conserved quantity for the systems with nonstandard Lagrangians. Two examples are given to illustrate the application of the results.

Keywords Nonlinear dynamics · Nonstandard Lagrangians · Noether theorem · Noether inverse theorem · Hamilton principle · Differential equations of motion

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1 Introduction

The Noether symmetry, namely the invariance of Hamilton action under the infinitesimal transformations, is put forward for the first time by Noether [1] in 1918. The presentation of the Noether symmetry is a significant leap in physics. One can find a conserved quantity from a Noether symmetry by using the intrinsic relation between the conserved quantity and the symmetry, which broke through the traditional approaches for finding the conserved quantities by the law of the conservation of energy of the system, the law of the conservation of momentum, and the law of the conservation of angular momentum. Afterward, the theory of symmetry and conserved quantity is extended and applied to various kinds of constrained mechanical systems, such as holonomic nonconservative systems [2,3], nonholonomic systems [3–6], fractional dynamical systems [7–11], and dynamics with time delay [12–15]. But all these results are limited to the systems based on standard Lagrangians, and the systems with nonstandard Lagrangians have not been involved yet.

The nonstandard Lagrangians were entitled "non-natural" by Arnold [16] in 1978, and they play an important role in the nonlinear differential equations, such as the nonlinear second-order Riccati equations [17] and the nonlinear differential equation with Lienard type [18,19], and the dissipative systems [20,21]. In 1984, the nonstandard Lagrangians were applied to the Yang–Mills field theory where they are used to describe large-distance interactions in the region of applicability of classical theory [22]. The nonstandard Lagrangians have various forms, such as exponential form and power-law function, and they completely differ from the standard Lagrangians which are expressed as the difference between kinetic energy and potential energy terms. Recently, many scholars have studied the properties and the applications of nonstandard Lagrangians, such as Musielak [23,24], El-Nabulsi [25–32], Saha [33,34], and Dimitrijevic [35], but the problem of the exploration and the application of nonstandard Lagrangians are still open and require deep analysis.

In this paper, we will present the Noether theorem and its inverse theorem for the systems based on two kinds of action with nonstandard Lagrangians, i.e., with exponential Lagrangians and power-law Lagrangians. The Hamilton principle of the systems is established, the equations of motion of the systems are derived, and the definitions and the criterions of the Noether symmetry and the Noether quasi-symmetry of the systems are given. The intrinsic relation between the Noether symmetry and the conserved quantity is established, and two examples are given to illustrate the application of the results.

2 Noether symmetry and conserved quantity for the systems based on exponential Lagrangians

2.1 Hamilton principle and dynamical equations

Suppose that the configuration of a dynamical system is determined by n generalized coordinates q_k $(k = 1, 2, \ldots, n)$, the standard Lagrangian of the system is $L = L(t, q_k, \dot{q}_k)$, the action with an exponential Lagrangian is defined by [26]

$$S = \int_a^b \exp\left[L\left(t, q_k, \dot{q}_k\right)\right] \mathrm{d}t \tag{1}$$

The isochronal variation principle

$$\delta S = 0 \tag{2}$$

with commutative relation

$$\mathrm{d}\delta q_k = \delta \mathrm{d} q_k \quad (k = 1, 2, \ldots, n), \tag{3}$$

and fixed end-point conditions

$$\delta q_k\,|_{t=a} = \delta q_k|_{t=b} = 0 \quad (k = 1, 2, \ldots, n) \tag{4}$$

can be called the Hamilton principle based on the action with exponential Lagrangian.

From the principle Eqs. (2)–(4), it is easy to obtain

$$\exp(L)\left(\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial \dot{q}_k}\frac{\mathrm{d}L}{\mathrm{d}t}\right) = 0 \quad (k = 1, 2, \ldots, n) \tag{5}$$

Equation (5) is called the Euler–Lagrange equations for the nonlinear dynamical system based on the action with exponential Lagrangian [26].

2.2 Noether symmetry

Let us introduce the infinitesimal transformations of r-parameter finite transformation group with respect to time, generalized coordinates, and generalized velocities, i.e.,

$$\bar{t} = t + \Delta t, \quad \bar{q}_k\left(\bar{t}\right) = q_k(t) + \Delta q_k \quad (k = 1, 2, \ldots, n) \tag{6}$$

and their expansion formulae

$$\bar{t} = t + \varepsilon_\sigma \tau^\sigma\left(t, q_s, \dot{q}_s\right), \quad \bar{q}_k\left(\bar{t}\right) = q_k(t) + \varepsilon_\sigma \xi_k^\sigma\left(t, q_s, \dot{q}_s\right) \quad (k = 1, 2, \ldots, n) \tag{7}$$

where ε_σ $(\sigma = 1, 2, \ldots, r)$ are the infinitesimal parameters, $\tau^\sigma, \xi_k^\sigma$ are the generators of the infinitesimal transformations. Under the infinitesimal transformations, the action (1) will be transformed to

$$S(\bar{\gamma}) = \int_{\bar{a}}^{\bar{b}} \exp\left[L\left(\bar{t}, \bar{q}_k\left(\bar{t}\right), \dot{\bar{q}}_k\left(\bar{t}\right)\right)\right] \mathrm{d}\bar{t} \tag{8}$$

where $\bar{\gamma}$ is a neighbor curve. The variation ΔS in the action S is the principal linear part for ε in the difference $S(\bar{\gamma}) - S(\gamma)$, and we have

$$\Delta S = \delta \int_a^b \exp(L) \mathrm{d}t + \exp(L)\,\Delta t\,\Big|_a^b \tag{9}$$

and

$$\Delta S = \int_a^b \left\{ \Delta \exp(L) + \exp(L) \frac{\mathrm{d}}{\mathrm{d}t} \Delta t \right\} \mathrm{d}t \tag{10}$$

Then, we have

$$\Delta S = \int_a^b \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[\exp(L)\, \Delta t + \exp(L) \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right] + \exp(L) \left(\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial \dot{q}_k} \frac{\mathrm{d}L}{\mathrm{d}t} \right) \delta q_k \right\} \mathrm{d}t \tag{11}$$

and

$$\Delta S = \int_a^b \exp(L) \left\{ \frac{\partial L}{\partial t} \Delta t + \frac{\partial L}{\partial q_k} \Delta q_k + \frac{\partial L}{\partial \dot{q}_k} \Delta \dot{q}_k + \frac{\mathrm{d}}{\mathrm{d}t} \Delta t \right\} \mathrm{d}t \tag{12}$$

Substituting Eq. (7) into Eq. (11), and taking notice that

$$\delta q_k = \Delta q_k - \dot{q}_k \Delta t = \varepsilon_\sigma \left(\xi_k^\sigma - \dot{q}_k \tau^\sigma \right), \tag{13}$$

we obtain

$$\Delta S = \int_a^b \varepsilon_\sigma \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[\exp(L)\, \tau^\sigma + \exp(L) \frac{\partial L}{\partial \dot{q}_k} \left(\xi_k^\sigma - \dot{q}_k \tau^\sigma \right) \right] + \exp(L) \left(\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial \dot{q}_k} \frac{\mathrm{d}L}{\mathrm{d}t} \right) \left(\xi_k^\sigma - \dot{q}_k \tau^\sigma \right) \right\} \mathrm{d}t \tag{14}$$

The formulae (12) and (14) are the basic formulae for the variation in the action (1).

Now, we give the definitions and the criterions of the Noether symmetry and the Noether quasi-symmetry for the nonlinear dynamical system based on the action with exponential Lagrangians.

Definition 1 If the action (1) is an invariant under the infinitesimal transformations (6) of group, i.e., for each of the infinitesimal transformations, the formula

$$\Delta S = 0 \tag{15}$$

always holds, then the transformations (6) are called the Noether symmetric transformations of the dynamical system based on the action with exponential Lagrangians.

By Definition 1 and formula (12), we can get the following criterion.

Criterion 1 For the infinitesimal transformations (6) of group, if the condition

$$\exp(L) \left(\frac{\partial L}{\partial t} \Delta t + \frac{\partial L}{\partial q_k} \Delta q_k + \frac{\partial L}{\partial \dot{q}_k} \Delta \dot{q}_k + \frac{\mathrm{d}}{\mathrm{d}t} \Delta t \right) = 0 \tag{16}$$

is satisfied, then the transformations (6) are the Noether symmetric transformations for the dynamical system based on the action with exponential Lagrangians.

The condition (16) can also be expressed as

$$\exp(L) \left(\frac{\partial L}{\partial t} \tau^\sigma + \frac{\partial L}{\partial q_k} \xi_k^\sigma + \frac{\partial L}{\partial \dot{q}_k} \left(\dot{\xi}_k^\sigma - \dot{q}_k \dot{\tau}^\sigma \right) + \dot{\tau}^\sigma \right) = 0 \quad (\sigma = 1, 2, \ldots, r) \tag{17}$$

when $r = 1$, Eq. (17) is called the Noether identity for the dynamical system based on the action with exponential Lagrangians.

Using Criterion 1 or the Noether identity (17), one can find the Noether symmetry of the system.

Definition 2 If the action (1) is a quasi-invariant under the infinitesimal transformations (6) of group, i.e., for each of the infinitesimal transformations, the formula

$$\Delta S = - \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} (\Delta G)\, \mathrm{d}t \tag{18}$$

always holds, where $G = G(t, q_k, \dot{q}_k)$, then the transformations (6) are called the Noether quasi-symmetric transformations for the dynamical system based on the action with exponential Lagrangians.

By Definition 2 and formula (12), we can get the following criterion.

Criterion 2 For the infinitesimal transformations (6) of group, if the condition

$$\exp(L) \left(\frac{\partial L}{\partial t} \Delta t + \frac{\partial L}{\partial q_k} \Delta q_k + \frac{\partial L}{\partial \dot{q}_k} \Delta \dot{q}_k + \frac{\mathrm{d}}{\mathrm{d}t} \Delta t \right) = - \frac{\mathrm{d}}{\mathrm{d}t} (\Delta G) \tag{19}$$

is satisfied, then the transformations (6) are the Noether quasi-symmetric transformations for the dynamical system based on the action with exponential Lagrangians.

The condition (19) can also be expressed as

$$\frac{\partial L}{\partial t}\tau^{\sigma}+\frac{\partial L}{\partial q_k}\xi_k^{\sigma}+\frac{\partial L}{\partial \dot{q}_k}\left(\dot{\xi}_k^{\sigma}-\dot{q}_k\dot{\tau}^{\sigma}\right)+\dot{\tau}^{\sigma} = -\exp\left(-L\right)\quad \dot{G}^{\sigma}\quad (\sigma=1,2,\ldots,r) \tag{20}$$

where $\Delta G=\varepsilon_{\sigma}G^{\sigma}$. When $r=1$, Eq. (20) is called the generalized Noether identity for the dynamical system based on the action with exponential Lagrangians.

Using Criterion 2 or the generalized Noether identity (20), one can find the Noether quasi-symmetry of the system.

2.3 Noether theorem

Under the Noether symmetric transformations, from Eq. (15) and Eq. (14), we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(L\right)\tau^{\sigma}+\exp\left(L\right)\frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)\right] +\exp\left(L\right)\left(\frac{\partial L}{\partial q_k}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial \dot{q}_k}\frac{\mathrm{d}L}{\mathrm{d}t}\right)\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right) =0\quad (\sigma=1,2,\ldots,r) \tag{21}$$

Substituting Eq. (5) into Eq. (21), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(L\right)\tau^{\sigma}+\exp\left(L\right)\frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)\right]=0 \tag{22}$$

Integrating (22), we obtain the Noether conserved quantity

$$I_N^{\sigma}=\exp\left(L\right)\tau^{\sigma}+\exp\left(L\right)\frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right) =\text{const.}\quad (\sigma=1,2,\ldots,r) \tag{23}$$

Therefore, we have

Theorem 1 *For the dynamical system (5), which is based on the action with exponential Lagrangians, if the infinitesimal transformations (6) of group are the Noether symmetric transformations in the sense of Definition 1, then the system exists with r linearly independent Noether conserved quantities (23).*

Under the Noether quasi-symmetric transformations, from Eq. (18) and Eq. (14), we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(L\right)\tau^{\sigma}+\exp\left(L\right)\frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)+G^{\sigma}\right] +\exp\left(L\right)\left(\frac{\partial L}{\partial q_k}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial \dot{q}_k}\frac{\mathrm{d}L}{\mathrm{d}t}\right)\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right) =0\quad (\sigma=1,2,\ldots,r) \tag{24}$$

Substituting Eq. (5) into Eq. (24), integrating it, we have

$$I_N^{\sigma}=\exp\left(L\right)\tau^{\sigma}+\exp\left(L\right)\frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)+G^{\sigma} =\text{const.}\quad (\sigma=1,2,\ldots,r) \tag{25}$$

Then, we have

Theorem 2 *For the dynamical system (5), which is based on the action with exponential Lagrangians, if the infinitesimal transformations (6) of group are the Noether quasi-symmetric transformations in the sense of Definition 2, then the system exists with r linearly independent Noether conserved quantities (25).*

Theorems 1 and 2 are called the Noether theorem for the dynamical system (5) based on the action with exponential Lagrangians. The Noether theorem shows that if one can find a Noether symmetric transformation or a Noether quasi-symmetric transformation of the system, then one can obtain a conserved quantity of the system.

2.4 Noether inverse theorem

Assume that the dynamical system (5) has r independent first integrals

$$I^{\sigma}\left(t,q_k,\dot{q}_k\right)=c_{\sigma}\quad (\sigma=1,2,\ldots,r) \tag{26}$$

Let us find out the infinitesimal transformations (7) corresponding to the Noether quasi-symmetry of the system.

Multiplying Eq. (5) by $\bar{\xi}_k^{\sigma}=\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}$ and summing up the obtained results over k, we obtain

$$\exp\left(L\right)\left(\frac{\partial L}{\partial q_k}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial \dot{q}_k}\frac{\mathrm{d}L}{\mathrm{d}t}\right)\bar{\xi}_k^{\sigma}=0 \tag{27}$$

Differentiating Eq. (26) with respect to t, we obtain

$$\frac{\partial I^{\sigma}}{\partial t}+\frac{\partial I^{\sigma}}{\partial q_k}\dot{q}_k+\frac{\partial I^{\sigma}}{\partial \dot{q}_k}\ddot{q}_k=0 \tag{28}$$

By adding Eqs. (27) and (28), and let the coefficients of $\ddot{q}_k$ equal to zero, we obtain

$$\frac{\partial I^{\sigma}}{\partial \dot{q}_k}-\exp\left(L\right)\left(\frac{\partial^2 L}{\partial \dot{q}_j\partial \dot{q}_k}+\frac{\partial L}{\partial \dot{q}_j}\frac{\partial L}{\partial \dot{q}_k}\right)\bar{\xi}_j^{\sigma} =0\quad (k=1,2,\ldots,n) \tag{29}$$

In order to make the transformations to be Noether quasi-symmetric transformations, we need to make the integral (26) equal to the Noether conserved quantity (25), i.e.,

$$\exp(L)\,\tau^{\sigma}+\exp(L)\,\frac{\partial L}{\partial\dot{q}_k}\bar{\xi}_k^{\sigma}+G^{\sigma}=I^{\sigma} \tag{30}$$

From Eqs. (29) and (30), one can find the Noether quasi-symmetric transformations. Therefore, we have

Theorem 3 *For the dynamical system (5), which is based on the action with exponential Lagrangian, if r linearly independent first integrals (26) are given, then the infinitesimal transformations determined by formulae (29) and (30) are the Noether quasi-symmetric transformations of the system.*

2.5 Example

Consider a nonlinear dynamic system whose action with an exponential Lagrangian is [26]

$$S=\int_a^b \exp(q\dot{q}t)\mathrm{d}t \tag{31}$$

The Eq. (5) give

$$tq\ddot{q}+q\dot{q}+t\dot{q}^2=-\frac{1}{t} \tag{32}$$

Try to study the Noether symmetry and the conserved quantity of the system.

Firstly, the generalized Noether identity (20) gives

$$q\dot{q}\tau+t\dot{q}\xi+tq\left(\dot{\xi}-\dot{q}\dot{\tau}\right)+\dot{\tau}=-e^{-tq\dot{q}}\dot{G} \tag{33}$$

Equation (33) has the following solutions

$$\tau^1=t,\xi^1=-\frac{1}{q}\ln t,\mathrm{G}^1=0 \tag{34}$$

$$\tau^2=(tq\dot{q}-1)\,e^{-tq\dot{q}},$$
$$\xi^2=\left(tq\dot{q}^2-\dot{q}+\frac{1}{tq}\right)e^{-tq\dot{q}},G^2=\ln t \tag{35}$$

$$\tau^3=1,\quad \xi^3=\dot{q},\quad G^3=-e^{tq\dot{q}} \tag{36}$$

The generator (34) corresponds to the Noether symmetry of the system, and the generators (35) and (36) correspond to the Noether quasi-symmetry of the system. By the Noether theorem we obtained, the system has the following conserved quantities

$$I^1=e^{tq\dot{q}}\left(t-t\ln t-t^2q\dot{q}\right)=c_1 \tag{37}$$

$$I^2=tq\dot{q}+\ln t=c_2 \tag{38}$$

$$I^3=0 \tag{39}$$

The conserved quantity (39) is trivial.

Secondly, by using the Noether inverse theorem, we find out the Noether symmetry from a given integral. Suppose that the system has the integral (37), then Eqs. (29) and (30) give, respectively,

$$e^{tq\dot{q}}tq\left(t-t\ln t-t^2q\dot{q}\right)+e^{tq\dot{q}}\left(-t^2q\right)-e^{tq\dot{q}}\,(tq)^2\,(\xi-\dot{q}\tau)=0 \tag{40}$$

$$e^{tq\dot{q}}\left[\tau+tq\left(\xi-\dot{q}\tau\right)\right]+G=e^{tq\dot{q}}\left(t-t\ln t-t^2q\dot{q}\right) \tag{41}$$

From Eq. (40), we get

$$\xi=\dot{q}\tau-\frac{1}{q}\ln t-t\dot{q} \tag{42}$$

Eqs. (41) and (42) are two algebraic equations with respect to three variables τ, ξ, G, and so, their solutions are not unique. Let

$$G=0, \tag{43}$$

then we have

$$\tau=t,\quad \xi=-\frac{1}{q}\ln t \tag{44}$$

The generator (44) corresponds to the Noether symmetry of the system.

If we take

$$G=-e^{tq\dot{q}}, \tag{45}$$

then we have

$$\tau=t+1,\xi=\dot{q}-\frac{1}{q}\ln t \tag{46}$$

The generator (46) corresponds to the Noether quasi-symmetry of the system.

3 Noether symmetry and conserved quantity for the systems based on power-law Lagrangians

3.1 Hamilton principle and dynamical equations

The action with a power-law Lagrangian is defined by [26]

$$A = \int_a^b L^{1+\alpha}(t, q_k, \dot{q}_k)\,\mathrm{d}t, \quad \alpha \in R \tag{47}$$

The isochronal variation principle

$$\delta A = 0 \tag{48}$$

with commutative relation

$$\mathrm{d}\delta q_k = \delta \mathrm{d} q_k \quad (k = 1, 2, \ldots, n) \tag{49}$$

and boundary conditions

$$\delta q_k\,|_{t=a} = \delta q_k|_{t=b} = 0 \quad (k = 1, 2, \ldots, n) \tag{50}$$

can be called the Hamilton principle based on the action with power-law Lagrangians.

By the principle Eqs. (48)–(50), it is easy to get

$$\int_a^b (1+\alpha)L^\alpha \left(\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\alpha}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}\right) \times \delta q_k \mathrm{d}t = 0 \tag{51}$$

According to the arbitrariness of integral interval and the independence of δq_k $(k = 1, 2, \ldots, n)$, we have

$$(1+\alpha)\,L^\alpha \left(\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\alpha}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \tag{52}$$

If $\alpha \neq -1$, then we have

$$\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\alpha}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} = 0, \quad (k = 1, 2, \ldots, n) \tag{53}$$

Equations (52) and (53) are the Euler–Lagrange equations for the nonlinear dynamical system based on the action with power-law Lagrangians [17]. If $\alpha = 0$, then Eqs. (52) are reduced to the classical Euler–Lagrange equation [26].

3.2 Noether symmetry

Now, let us calculate the variation ΔA in the action (47); we have

$$\Delta A = \delta \int_a^b L^{1+\alpha}\mathrm{d}t + L^{1+\alpha}\Delta t\,\Big|_a^b \tag{54}$$

and

$$\Delta A = \int_a^b \left\{\Delta L^{1+\alpha} + L^{1+\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right\}\mathrm{d}t; \tag{55}$$

then, we have

$$\Delta A = \int_a^b \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[L^{1+\alpha}\Delta t + (1+\alpha)\,L^\alpha \frac{\partial L}{\partial \dot{q}_k}\delta q_k\right] + (1+\alpha)\,L^\alpha \left(\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\alpha}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k\right\}\mathrm{d}t \tag{56}$$

and

$$\Delta A = \int_a^b \left\{(1+\alpha)\,L^\alpha \left(\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\Delta \dot{q}_k\right) + L^{1+\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right\}\mathrm{d}t \tag{57}$$

Substituting Eqs. (7) and (13) into Eq. (56), we obtain

$$\Delta A = \int_a^b \varepsilon_\sigma \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[L^{1+\alpha}\tau^\sigma + (1+\alpha)\,L^\alpha \frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^\sigma - \dot{q}_k\tau^\sigma\right)\right] + (1+\alpha)\,L^\alpha \left(\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\alpha}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}\right)\left(\xi_k^\sigma - \dot{q}_k\tau^\sigma\right)\right\}\mathrm{d}t \tag{58}$$

The formulae (57) and (58) are the basic formulae for the variation in the action (47).

Next, let us give the definitions and the criterions of the Noether symmetry and the Noether quasi-symmetry for the nonlinear dynamical system based on the action with power-law Lagrangians.

Definition 3 If the action (1) is an invariant under the infinitesimal transformations (6) of group, i.e., for each of the infinitesimal transformations, the formula

$$\Delta A = 0 \tag{59}$$

always holds, then the transformations (6) are called the Noether symmetric transformations of the dynamical system based on the action with power-law Lagrangians.

By Definition 3 and formula (57), we can get the following criterion

Criterion 3 For the infinitesimal transformations (6) of group, if the condition

$$(1+\alpha)\,L^{\alpha}\left(\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_k}\Delta q_k+\frac{\partial L}{\partial \dot{q}_k}\Delta \dot{q}_k\right)+L^{1+\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t=0 \tag{60}$$

is satisfied, then the transformations (6) are the Noether symmetric transformations for the dynamical system based on the action with power-law Lagrangians.

The condition (60) can also be expressed as

$$(1+\alpha)\,L^{\alpha}\left[\frac{\partial L}{\partial t}\tau^{\sigma}+\frac{\partial L}{\partial q_k}\xi_k^{\sigma}+\frac{\partial L}{\partial \dot{q}_k}\left(\dot{\xi}_k^{\sigma}-\dot{q}_k\dot{\tau}^{\sigma}\right)\right]+L^{1+\alpha}\dot{\tau}^{\sigma}=0,\ (\sigma=1,2,\ldots,r) \tag{61}$$

when $r=1$, Eq. (61) is called the Noether identity for the dynamical system based on the action with power-law Lagrangians.

Definition 4 If the action (47) is a quasi-invariant under the infinitesimal transformations (6) of group, i.e., for each of the infinitesimal transformations, the formula

$$\Delta A=-\int_a^b\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta G\right)\mathrm{d}t \tag{62}$$

always holds, then the transformations (6) are called the Noether quasi-symmetric transformations for the dynamical system based on the action with power-law Lagrangians.

Criterion 4 For the infinitesimal transformations (6) of group, if the condition

$$\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_k}\Delta q_k+\frac{\partial L}{\partial \dot{q}_k}\Delta \dot{q}_k+\frac{L}{1+\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t=-\frac{1}{(1+\alpha)\,L^{\alpha}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta G\right) \tag{63}$$

is satisfied, then the transformations (6) are the Noether quasi-symmetric transformations for the dynamical system based on the action with power-law Lagrangians.

The condition (63) can also be expressed as

$$\frac{\partial L}{\partial t}\tau^{\sigma}+\frac{\partial L}{\partial q_k}\xi_k^{\sigma}+\frac{\partial L}{\partial \dot{q}_k}\left(\dot{\xi}_k^{\sigma}-\dot{q}_k\dot{\tau}^{\sigma}\right)+\frac{L}{1+\alpha}\dot{\tau}^{\sigma}=-\frac{1}{(1+\alpha)\,L^{\alpha}}\dot{G}^{\sigma}\quad(\sigma=1,2,\ldots,r) \tag{64}$$

when $r=1$, Eq. (64) is called the generalized Noether identity for the dynamical system based on the action with power-law Lagrangians.

Using Criterion 3 or the Noether identity (61), one can find the Noether symmetry of the system. Using Criterion 4 or the generalized Noether identity (64), one can find the Noether quasi-symmetry of the system.

3.3 Noether theorem

Under the Noether symmetric transformations, from Eqs. (59) and (58), we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L^{1+\alpha}\tau^{\sigma}+(1+\alpha)\,L^{\alpha}\frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)\right]+(1+\alpha)\,L^{\alpha}\left(\frac{\partial L}{\partial q_k}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}-\frac{\alpha}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}\right)\times\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)=0\quad(\sigma=1,2,\ldots,r) \tag{65}$$

Substituting Eq. (52) into Eq. (65), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L^{1+\alpha}\tau^{\sigma}+(1+\alpha)\,L^{\alpha}\frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)\right]=0 \tag{66}$$

Integrating (66), we obtain Noether conserved quantity

$$I_N^{\sigma}=L^{1+\alpha}\tau^{\sigma}+(1+\alpha)\,L^{\alpha}\frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)=\text{const.}\quad(\sigma=1,2,\ldots,r) \tag{67}$$

Hence, we have

Theorem 4 *For the dynamical system (52), which is based on the action with power-law Lagrangians, if the infinitesimal transformations (6) of group are the Noether symmetric transformations in the sense of Definition 3, then the system exists r linearly independent Noether conserved quantities (67).*

Under the Noether quasi-symmetric transformations, from Eqs. (62) and (58), we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L^{1+\alpha}\tau^{\sigma}+(1+\alpha)\,L^{\alpha}\frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)+G^{\alpha}\right]+(1+\alpha)\,L^{\alpha}\left(\frac{\partial L}{\partial q_k}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}-\frac{\alpha}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}\right)\times\left(\xi_k^{\sigma}-\dot{q}_k\tau^{\sigma}\right)=0 \tag{68}$$

Substituting Eq. (52) into Eq. (68), and integrating it, we have

$$I_N^\sigma = L^{1+\alpha}\tau^\sigma + (1+\alpha)\, L^\alpha \frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^\sigma - \dot{q}_k\tau^\sigma\right) + G^\sigma$$
$$= \text{const.} \quad (\sigma = 1, 2, \ldots, r) \tag{69}$$

Then, we have

Theorem 5 *For the dynamical system (52), which is based on the action with power-law Lagrangians, if the infinitesimal transformations (6) of group are the Noether quasi-symmetric transformations in the sense of Definition 4, then the system exists r linearly independent Noether conserved quantities (67).*

Theorems 4 and 5 are called the Noether theorem for the dynamical system (52) based on the action with power-law Lagrangians.

3.4 Noether inverse theorem

Assume that the dynamical system (52) has r independent first integrals

$$I^\sigma(t, q_k, \dot{q}_k) = c_\sigma \quad (\sigma = 1, 2, \ldots, r) \tag{70}$$

Let us find out the infinitesimal transformations (7) corresponding to the Noether quasi-symmetry of the system.

Multiplying Eq. (52) by $\bar{\xi}_k^\sigma$ and summing up the obtained results over k, we obtain

$$(1+\alpha)\, L^\alpha \left(\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\alpha}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}\right)\bar{\xi}_k^\sigma = 0 \tag{71}$$

Differentiating Eq. (70) with respect to t, we obtain

$$\frac{\partial I^\sigma}{\partial t} + \frac{\partial I^\sigma}{\partial q_k}\dot{q}_k + \frac{\partial I^\sigma}{\partial \dot{q}_k}\ddot{q}_k = 0 \tag{72}$$

By adding Eqs. (71) and (72), and let the coefficients of $\ddot{q}_k$ equal to zero, we obtain

$$\frac{\partial I^\sigma}{\partial \dot{q}_k} - (1+\alpha)\, L^\alpha \left(\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_k} + \frac{\alpha}{L}\frac{\partial L}{\partial \dot{q}_j}\frac{\partial L}{\partial \dot{q}_k}\right)\bar{\xi}_j^\sigma$$
$$= 0 \quad (k = 1, 2, \ldots, n) \tag{73}$$

In order to make the transformations to be the Noether quasi-symmetric transformations, we need to make the integral (70) equal to the Noether conserved quantity, i.e.,

$$L^{1+\alpha}\tau^\sigma + (1+\alpha)\, L^\alpha \frac{\partial L}{\partial \dot{q}_k}\left(\xi_k^\sigma - \dot{q}_k\tau^\sigma\right) + G^\sigma = I^\sigma \tag{74}$$

According to Eqs. (73) and (74), one can find the Noether quasi-symmetry transformations.

Theorem 6 *For the dynamical system (52), which is based on the action with power-law Lagrangians, if r linearly independent first integrals (70) are given, then the infinitesimal transformations determined by formulae (73) and (74) are the Noether quasi-symmetric transformations of the system.*

3.5 Example

Consider a dynamical system whose action with power-law Lagrangian is [26]

$$A = \int_a^b L^{1+\alpha}(t, q, \dot{q})\,\mathrm{d}t \tag{75}$$

where $L = \dot{q} + e^{-q}$, $\alpha = 1$. The equations (52) give

$$\ddot{q} + e^{-2q} = 0 \tag{76}$$

The generalized Noether identity (64) gives

$$-e^{-q}\xi + \dot{\xi} - \dot{q}\dot{\tau} + \frac{1}{2}\left(\dot{q} + e^{-q}\right)\dot{\tau} = -\frac{1}{2\left(\dot{q} + e^{-q}\right)}\dot{G} \tag{77}$$

Equation (77) has a solution

$$\tau = -1, \xi = 0, G = 0 \tag{78}$$

The generator (78) corresponds to the Noether symmetry of the system. By Theorem 5, we obtain

$$I = \dot{q}^2 - e^{-2q} = \text{const.} \tag{79}$$

The conserved quantity (79) is caused by the Noether symmetry (78).

Next, let us study the inverse problem. Assume the system has the first integral (79). Equations (73) and (74) give

$$\dot{q} - (\xi - \dot{q}\tau) = 0 \tag{80}$$

$$\left(\dot{q} + e^{-q}\right)^2 \tau + 2\left(\dot{q} + e^{-q}\right)(\xi - \dot{q}\tau) + G = \dot{q}^2 - e^{-2q} \tag{81}$$

From Eq. (80), we have

$$\xi = (1 + \tau)\,\dot{q} \tag{82}$$

Substituting Eq. (82) into Eq. (81), we get

$$\left(\dot{q} + e^{-q}\right)^2 \tau + \dot{q}^2 + 2\dot{q}e^{-q} + e^{-2q} + G = 0 \tag{83}$$

If $G = 0$, then we have

$$\tau = -1, \quad \xi = 0 \tag{84}$$

The generator (84) corresponds to the Noether symmetry of the system. If $G = -\left(\dot{q} + e^{-q}\right)^2$, then we have

$$\tau = 0, \quad \xi = \dot{q} \tag{85}$$

The generator (85) corresponds to the Noether quasi-symmetry of the system.

4 Conclusions

Nonstandard Lagrangians can be used to describe the nonlinear problem and the nonlinear differential equations. The study of nonstandard Lagrangians can find some properties that standard Lagrangians do not have. It provides a new modeling solution for the realistic questions. It also presents a new vision for the dynamics research. In this paper, the Hamilton principle for two kinds of nonstandard Lagrangians is presented, the equations of motion of the system are derived, and the Noether theorems with nonstandard Lagrangians are established. The method and results in this paper have universal significance, and they can be applied to other constrained mechanical systems.

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