Nonlinear Dyn (2016) 84:1783–1794
DOI 10.1007/s11071-016-2605-8

ORIGINAL PAPER

A partial Lagrangian method for dynamical systems

R. Naz · F. M. Mahomed · A. Chaudhry

Received: 21 September 2015 / Accepted: 4 January 2016 / Published online: 14 January 2016

Abstract We develop a new approach termed as a discount free or partial Lagrangian method for construction of first integrals for dynamical systems of ordinary differential equations (ODEs). It is shown how one can utilize the Legendre transformation in a more general setting to provide the equivalence between a current value Hamiltonian and a partial or discount free Lagrangian when it exists. As a consequence, we develop a discount factor free Lagrangian framework to deduce reductions and closed-form solutions via first integrals for ODEs arising from economics by proving three important propositions. The approach is algorithmic and applies to many state variables of the Lagrangian. In order to show its effectiveness, we apply the method to models, one linear and two nonlinear, with one state variable. We obtain new exact solutions for the last model. The discount free Lagrangian naturally arises in economic growth theory and many other economic models when the control variables can be eliminated at the outset which is not always possible in optimal control theory applications of economics. We explain our method with the help of few widely used economic growth models. We point out the difference between this approach and the more general partial Hamiltonian method proposed earlier for a current value Hamiltonian (Naz et al. in Commun Nonlinear Sci Numer Simul 19:3600–3610, 2014) which is applicable in a general setting involving time, state, costate and control variables.

R. Naz (✉)
Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore 53200, Pakistan
e-mail: rehananaz_qau@yahoo.com

F. M. Mahomed
DST-NRF Centre of Excellence in Mathematical and Statistical Sciences, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa

A. Chaudhry
Department of Economics, Lahore School of Economics, Lahore 53200, Pakistan

Keywords Partial or discount free Lagrangian · Current value Hamiltonian · Partial Noether condition · Economic growth models · First integrals

1 Introduction

At the heart of economic analysis is the concept of optimization under constraints, and the most common application is through dynamic optimization. There are in essence three major approaches, viz. that of dynamic programming, calculus of variations and its extension which is optimal control theory (see, for example, Chiang [1]).

Many of the models utilize the Lagrangian or its generalization, and the current value Hamiltonian whenever the integrand function contains a discount factor. These models range from the simplest to those of neoclassical economic growth [2,3], the optimal firm-level investment [4] and human capital and earnings [5] models.

The Euler–Lagrange equations give necessary conditions for extremal values of the functional which has the Lagrangian as its integrand. This involves the time variable, the state variables and its derivatives up to some finite order. Many of the problems give rise to a Lagrangian which is of the first order. Thus, in this case one obtains a dynamical system of the second order. Further advances have led to the development of optimal control theory (see, for example, [1]) which in addition to the time and state variables also incorporates the control and costate variables. In the latter approach, Pontryagin's maximum principle provides necessary conditions for the resolution of the continuous time optimal control problem which has a current value Hamiltonian, and therefore, a first-order system of dynamical ODEs is formulated for the control, state and costate variables.

There are several dynamic economic models which pervade the existing literature. Different approaches have been advocated to deal with these models. Predominantly qualitative theory and linear approximations around steady states (see [6–10]) have been applied to analyze these model. Numerical methods too have been utilized to solve the nonlinear dynamical systems at hand (see, for example, [11]). Notwithstanding analytical solutions as well as existence and uniqueness, theorems have proved beneficial in the analysis of such models (see, for example, [12–18]).

A separate strand of the literature has looked at the first integrals or conservation laws for differential equations (see, for example, [19–26]). Kara and Mahomed [20] developed the partial Lagrangian approach to constructing conservation laws for partial differential equations (PDEs). All different approaches to constructing conservation laws for PDEs are discussed by Naz et al. [21]. Gan and Qu [22] studied the approximate conservation laws of perturbed PDEs. Kara et al. [23] provided the methodology to establish the first integrals for ODEs. A review of different approaches to deriving first integrals for ODEs is provided by Naz et al. [24]. The approximate first integrals for nonlinear oscillators are discussed in [25,26]. Dorodnitsyn and Kozlov [27] analyzed the invariance and first integrals of continuous and discrete Hamiltonian equations. In a recent study [28,29], the authors proposed a partial Hamiltonian approach for many state, costate and control variables which takes into account the intrinsic and desirable current value Hamiltonian to unearth first integrals and reductions of the economic models under study. This approach provides a general and algorithmic procedure for the search for reductions and solutions of such type of nonlinear economic equations. The characterization of Hamiltonian symmetries and their first integrals is given in [30].

Significant contributions have been made in the analysis and solutions of nonlinear dynamical systems of economic models.

In the present study, we focus on a simple approach which yields first integrals, reductions and closed-form solutions (if there are sufficient operators) for dynamical systems of ODEs arising in economic growth theory. We utilize a partial Lagrangian or what we call a discount free Lagrangian framework for several state variables. This approach is applicable if one is able to construct a partial or discount free Lagrangian for the underlying dynamical system. We point out that this is not always possible as a system of first-order ODEs which is a consequence of optimal control theory may not in general be cast as a second-order system. Therefore, the method we invoke is applicable to an arbitrary system of ODEs provided that a partial or discount free Lagrangian exist. The utility of this method lies in the fact that the control variables, which occurs in the case of the partial Hamiltonian approach, do not arise in the unfolding analysis as the discount free Lagrangian is independent of these control variables from the outset. We apply this approach to a system of two ODEs which involves time, one state variable and its first derivative to show its effectiveness.

The outline of the paper is as follows. In Sect. 2, we provide the overviews of the partial Hamiltonian approach for what follows. The partial or discount free Lagrangian approach for dynamical systems is developed in Sect. 3. Herein we show how it naturally arises from the partial Hamiltonian by means of the partial Legendre transformation. Definitions and propositions are provided on the notion of discount factor free Lagrangians with the third one providing an algorithm on finding first integrals corresponding to this discount free Lagrangian in a simple manner. In Sect. 4, we provide examples using this approach of three fundamental Economic growth models. We compare our results with the partial Hamiltonian method to show its utility. Concluding remarks are finally given in Sect. 5.

2 Preliminaries

Let t be the independent variable which is usually time and $(q, p) = (q^1, \ldots, q^n, p_1, \ldots, p_n)$ the phase space coordinates. In the applications to equations of economics, $q^1, \ldots, q^n$ are the state variables and $p_1, \ldots, p_n$ the costate variables. We provide here the overview of partial Hamiltonian approach. The following definitions and results are adapted from [1,27–29].

Definition 1 The Euler operator $\delta/\delta q^i$ and the variational operator $\delta/\delta p_i$ are defined as

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D\frac{\partial}{\partial \dot{q}^i}, \quad i = 1, 2, \ldots, n, \tag{1}$$

and

$$\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D\frac{\partial}{\partial \dot{p}_i}, \quad i = 1, 2, \ldots, n, \tag{2}$$

where

$$D = \frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + \dot{p}_i\frac{\partial}{\partial p_i} + \cdots \tag{3}$$

is the total derivative operator with respect to the time t. The summation convention applies for repeated indices here and in the sequel.

The variables t, q, p are independent and connected by the differential relations

$$\dot{p}_i = D(p_i), \ \dot{q}^i = D(q^i), \quad i = 1, 2, \ldots, n. \tag{4}$$

In economic analysis, the optimal control problem is stated as (see, for example, [1])

$$\text{Maximize } \mathcal{F} = \int_0^T F(t, q, c)\, \mathrm{e}^{-\rho t}\, \mathrm{d}t$$
$$\text{subject to } \dot{q}^i = f^i(t, q, c), \quad i = 1, \ldots, n, \tag{5}$$

where c is the control vector $c = (c_1, \ldots, c_m)$, $m \leq n$ or $m > n$ as well as appropriate boundary conditions imposed. The integrand contains the discount factor $\mathrm{e}^{-\rho t}$. We can have minimization problems as well.

Definition 2 The present value Hamiltonian is defined as

$$\bar{H} = F(t, q, c)\mathrm{e}^{-\rho t} + \bar{p}_i f^i(t, q, c). \tag{6}$$

The current value Hamiltonian $H(t, q, p, c)$ (see, for example, [1]) is

$$H = F(t, q, c) + p_i f^i(t, q, c), \tag{7}$$

where the following relations hold, viz.

$$p_i = \bar{p}_i\mathrm{e}^{\rho t}, \ \ H = \bar{H}\mathrm{e}^{\rho t}, q^i = \bar{q}^i, c_i = \bar{c}_i. \tag{8}$$

This current value Hamiltonian is independent of the discount factor and deemed more desirable in economic analysis [1].

The maximum principle gives necessary conditions for optimal control for $\bar{H}$

$$\frac{\partial \bar{H}}{\partial \bar{c}_i} = 0, \ \frac{\partial^2 \bar{H}}{\partial \bar{c}_i{}^2} < 0$$
$$\dot{\bar{q}}^i = \frac{\partial \bar{H}}{\partial \bar{p}_i},$$
$$\dot{\bar{p}}^i = -\frac{\partial \bar{H}}{\partial \bar{q}^i}, \quad i = 1, \ldots, n. \tag{9}$$

and these conditions in terms of the current value Hamiltonian H defined in (7) satisfy the system

$$\frac{\partial H}{\partial c_i} = 0, \ \frac{\partial^2 H}{\partial c_i{}^2} < 0$$
$$\dot{q}^i = \frac{\partial H}{\partial p_i},$$
$$\dot{p}^i = -\frac{\partial H}{\partial q^i} + \Gamma_i, \quad i = 1, \ldots, n, \tag{10}$$

where we take Γ_i as a nonzero function of t, p_i, q^i in general. Note that each Γ_i is mostly taken as a linear function of p_i in economic applications in the earlier literature of optimal control theory in which the functional maximized contains the discount factor $\mathrm{e}^{-\rho t}$ in the integrand [1] and this is known as constant time preferences in economic growth theory. For endogenous time preferences, Γ_i is taken more generally. The sufficiency conditions relate the p_i and c_i. In correspondence to mechanics, they deal with non-conservative forces if the Γ_i are nonzero.

We have that the operator X given by

$$X = \xi(t, q, p)\frac{\partial}{\partial t} + \eta^i(t, q, p)\frac{\partial}{\partial q^i} + \zeta_i(t, q, p)\frac{\partial}{\partial p_i} \tag{11}$$

is a generator of point symmetry of the current value Hamiltonian system of ODEs (10) if the following determining system involving the coefficient functions ξ, η^i and ζ_i, viz.

$$\dot{\eta}^i - \dot{q}^i \dot{\xi} - X\left(\frac{\partial H}{\partial p_i}\right) = 0,$$

$$\dot{\zeta}_i - \dot{p}_i \dot{\xi} + X\left(\frac{\partial H}{\partial q^i} - \Gamma_i\right) = 0, \quad i = 1, \ldots, n \quad (12)$$

identically holds on the system (10).

Definition 3 (*see* [28]). An operator X of the form (11) is said to be a partial Hamiltonian operator corresponding to a current value Hamiltonian $H(t, q, p, c)$ which satisfies (7), if there exists a function $B(t, q, p, c)$ such that

$$\zeta_i \frac{\partial H}{\partial p_i} + p_i D(\eta^i) - X(H) - H D(\xi)$$
$$= D(B) + \left(\eta^i - \xi \frac{\partial H}{\partial p_i}\right)(-\Gamma_i) \quad (13)$$

holds.

Note that it is important to point out that if H is a present value Hamiltonian, then Eq. (12) becomes the usual determining equation for symmetries of the Hamiltonian action since $\Gamma_i = 0$ in this case. Also in this case (13) with $\Gamma_i = 0$ (see [27]), the Hamiltonian action with gauge term is invariant with respect to the group generated by operator X of (11).

The following theorem is essential for the construction of first integrals for the system (10).

Theorem 1 (see [28]). *The first integral corresponding to the system (10) associated with a partial Hamiltonian operator X of the current value Hamiltonian $H(t, q, p, c)$ is determined from*

$$I = p_i \eta^i - \xi H - B, \quad (14)$$

where $B(t, p, q, c)$ is a gauge-like function.

If $\Gamma_i = 0$ and $B = B(t, p, q)$, then this formula (14) is valid as well for an invariant Hamiltonian action up to divergence [27].

3 A partial Lagrangian approach for dynamical systems

We provide the connection between a partial Hamiltonian and what we term a partial or discount free Lagrangian when it exists. The dynamic optimization problems can be treated by the calculus of variation techniques and the optimal control theory. In the definition given below, we define the optimal control problem stated in (5) in terms of the calculus of variation problem.

Definition 4 The optimal control problem stated in (5) can be re-cast as the following calculus of variations problem

$$\text{Maximize } \mathcal{F} = \int_0^T F(t, q, \dot{q})\, \mathrm{e}^{-\rho t}\, \mathrm{d}t \quad (15)$$

under the conditions of invertibility given by $p_i = g_i(t, q^i, \dot{q}^i)$ and the relationship between the control vector c and costate vector p which arises from the sufficient conditions of optimal control from the maximum principle yielding $c_i = h_i(p_i)$. Also appropriate boundary conditions are imposed.

Next, we define the partial Legendre transformation in the more general setting with $p_i = \partial L / \partial \dot{q}^i$ which relates the current value Hamiltonian and what we define as a partial or discount free Lagrangian when it exists.

Proposition 1 (Partial Legendre transformation) *If H is the current value Hamiltonian, then under conditions of invertibility given by $p_i = g_i(t, q^i, \dot{q}^i)$ and the relationship between the control vector c and costate vector p which arises from the sufficient conditions of optimal control from the maximum principle yielding $c_i = h_i(p_i)$, there exists the partial or discount free Lagrangian $L(t, q^i, \dot{q}^i)$ satisfying*

$$L\left(t, q^i, \dot{q}^i\right) = p_i \dot{q}^i - H\left(t, q^i, p_i, c_i\right). \quad (16)$$

The relation (16) is called a partial Legendre transformation.

Proof The Legendre transformation [31] connects a standard Lagrangian $\bar{L}$ and canonical Hamiltonian $\bar{H}$ as

$$\bar{L}\left(t, q^i, \dot{q}^i\right) = \bar{p}_i \dot{q}^i - \bar{H}\left(t, q^i, \bar{p}_i, c_i\right). \quad (17)$$

The coordinate transformation $p_i = \bar{p}_i \mathrm{e}^{\rho t}$, $H = \bar{H} \mathrm{e}^{\rho t}$, $q^i = \bar{q}^i$, $c_i = \bar{c}_i$, $L = \bar{L} \mathrm{e}^{\rho t}$, directly transforms (17) to the partial Legendre transformation (16). We term L as a partial or discount free Lagrangian and H is the current value Hamiltonian. □

Proposition 2 (Partial Euler–Lagrange equations) *If H is the current value Hamiltonian, then under conditions of invertibility given by $p_i = g_i(t, q^i, \dot{q}^i)$ and the relationship between the control vector c and costate vector p which arises from the sufficient conditions of optimal control from the maximum principle yielding $c_i = h_i(p_i)$, there exists the discount factor independent Lagrangian $L(t, q^i, \dot{q}^i)$ or discount free Lagrangian via the partial Legendre transformation* (16) *which satisfies*

$$\frac{\delta L}{\delta q^i} = -\Gamma_i. \tag{18}$$

We call Eq. (19) *the partial Euler–Lagrange equations motivated by Kara and Mahomed* [20] *and Kara et al.* [23].

Proof Indeed, the action of the variational operator $\delta/\delta p_i$ on the partial Legendre transformation $p_i\dot{q}^i - H$ (16) after use of (10) yields $\delta L/\delta p_i = 0$. Further, if one acts with the Euler operator $\delta/\delta q^i$ on the partial Legendre transformation (16), then one straightforwardly obtains $-\partial H/\partial q^i - \dot{p}_i$. The latter is equal to $-\Gamma_i$ as a consequence of the second set of (10). We precisely arrive at

$$\frac{\delta L}{\delta q^i} = -\Gamma_i \tag{19}$$

provided one can invert for the costate vector $p_i = g_i(t, q^i, \dot{q}^i)$ and moreover, one requires that the control variables c_i be determined explicitly in terms of t, q^i and $\dot{q}^i$ as H in the partial Legendre transformation (16) has explicit c in it. Note that there is a relation between the variables c and p due to the sufficient conditions for optimal control for the maximum principle. This completes the proof. □

Remark 1 We have an alternative proof of Proposition 2. The standard Lagrangian for the calculus of variations problem as stated in (15) is $\bar{L} = F(t, q^i, \dot{q}^i)\, e^{-\rho t}$. If one acts with the Euler operator $\delta/\delta q^i = 0$ on $\bar{L}$ and using the fact that $\delta\bar{L}/\delta q^i = 0$, one can easily arrive at

$$\frac{\delta F}{\delta q^i} = -\rho \frac{\partial F}{\partial \dot{q}^i}. \tag{20}$$

Setting $L = -F$, $p_i = \frac{\partial L}{\partial \dot{q}^i}$ and $\Gamma_i = \rho p_i$ Eq. (20) provides the partial Euler–Lagrange equations (19). Then, $L = -F$ is the partial or discount free Lagrangian.

The relation (19) to be valid, we must have that L exists as a function of the time, the state variables and their derivatives only. In this case, we say that L is a partial Lagrangian corresponding to the partial Hamiltonian H with the equivalence being provided by the partial Legendre transformation (16).

In the same way that the current value Hamiltonian (which is the partial Hamiltonian) which is free of the discount factor is desirable, the partial or discount free Lagrangian that is defined by means of (19) also displays the same advantages. It is independent of time. We call it a discount free Lagrangian.

We therefore have the following important result for discount factor free Lagrangians that arise in economics.

Proposition 3 *Suppose that the Lagrangian L which is a discount free Lagrangian corresponding to the current value Hamiltonian H satisfies the partial Euler–Lagrange equation* (19) *and X as given by*

$$X = \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i} + \zeta_t^i \frac{\partial}{\partial \dot{q}^i}, \tag{21}$$

where $\zeta_t^i = D\eta^i - \dot{q}^i D\xi$, is a partial Noether operator corresponding to L. Then, X satisfies

$$X(L) + LD(\xi) = D(B) + (\eta^i - \xi\dot{q}^i)(-\Gamma_i), \quad i = 1, \dots, n \tag{22}$$

and the corresponding first integral associated with L is obtained by the explicit formula

$$I = \xi L + \left(\eta^i - \xi\dot{q}^i\right) \frac{\partial L}{\partial \dot{q}^i} - B, \tag{23}$$

for a suitable gauge term $B(t, q^i, \dot{q}^i)$.

Proof The partial Hamiltonian operator determining Eq. (13) with the help of partial Legendre transformation (16) yields

$$\zeta_i\dot{q}^i + p_i D\eta^i - \zeta_i\dot{q}^i - p_i\zeta_t^i + XL - p_i\dot{q}^i D\xi + LD\xi = DB + \left(\eta^i - \xi\dot{q}^i\right)(-\Gamma_i) \tag{24}$$

which easily results in (22) after replacing ζ_t^i with $D\eta^i - \dot{q}^i D\xi$. For the first integral I, Eq. (14) with the aid of partial Legendre transformation (16) yields

$$I = \eta^i \frac{\partial L}{\partial \dot{q}^i} - \xi \left(\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \right) - B, \tag{25}$$

and this is (23). Note that we have $p_i = g_i(t, q, \dot{q})$ in view of Definition 4 and thus Γ_i and B are in terms of the variables $t, q^i, \dot{q}^i$. This completes our proof. □

The approach presented here is termed as a "partial or discount free Lagrangian approach for dynamical systems." This provides a direct methodology to construct first integrals for first-order dynamical or Hamiltonian systems arising from optimal control problems. These types of systems arise naturally in economics. The partial or discount free Lagrangian is always of first order. The partial Lagrangian approach [20,23] can have first-order and higher-order partial Lagrangians. Moreover, it works for even-ordered differential equations or systems of differential equations. One can apply it to odd-ordered differential equations or systems of differential equations by increasing the order of differential equation to an even order.

Remark 2 The partial Hamiltonian approach developed in [28] and the partial Lagrangian approach presented here both are applicable provided the dynamic optimization problem can be stated in optimal control as well as in the calculus of variations form. This depends on invertibility conditions of the control variables. We cannot apply the partial Lagrangian approach if the problem cannot be stated in the calculus of variations form, but we can solve those dynamic optimization problems by utilizing the more generalized partial Hamiltonian approach.

We discuss two typical examples of control problems in economics. One can be expressed as a problem in the calculus of variations in a simple way, and in the second, it is quite cumbersome and the problem is better treated in its original formulation by using a Hamiltonian.

1. Maximize

$$\int_0^\infty \left[\alpha q - \beta q^2 - \alpha u^2 - \gamma u \right] e^{-rt} \mathrm{d}t \tag{26}$$

subject to

$$\dot{q} = u, \tag{27}$$

where α, β, γ are positive constants, r a constant discount factor, $q(t)$ the state variable, and $u(t)$ is the control variable. It is easy to observe that we can state this problem equivalently as an infinite-horizon problem in the calculus of variations as
Maximize

$$\int_0^\infty \left[\alpha q - \beta q^2 - \alpha \dot{q}^2 - \gamma \dot{q} \right] e^{-rt} \mathrm{d}t \tag{28}$$

by simply expressing u in terms of $\dot{q}$ through Eq. (27). This problem can be further simplified by eliminating the discount factor using the present formulation of a discount free Lagrangian. Here both the partial Hamiltonian and partial Lagrangian approaches are applicable.

2. In the model of Lucas–Uzawa [14,15], the representative agents utility function is defined as

$$\mathrm{Max}_{c,u} \quad \int_0^\infty \frac{c^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t}, \ \sigma \neq 1 \tag{29}$$

subject to the constraints of physical and human capital:

$$\begin{aligned} \dot{k}(t) &= \gamma k^\beta (uh)^{1-\beta} - \pi k - c, \ k_0 = k(0), \\ \dot{h}(t) &= \delta(1-u)h, \ h_0 = h(0), \end{aligned} \tag{30}$$

where $1/\sigma$ is the constant elasticity of intertemporal substitution, $\rho > 0$ is the discount factor, β is the elasticity of output with respect to physical capital, $\gamma > 0$ is the technological levels in the goods sector, $\delta > 0$ is the technological levels in the education sector, k is the level of physical capital, h is the level of human capital, c is per capita consumption, and u is the fraction of labor allocated to the production of physical capital. Here we see that it is quite cumbersome to express c and u in terms of the state variables and their derivatives. Thus, we cannot easily cast this problem in the calculus of variations as its description is unduly complicated. Here we can utilize the partial Hamiltonian approach to finding the first integrals and closed-form solution.

We invoke our approach in the next section and demonstrate their usefulness in a simple way.

4 Applications to economics

In this section, we explain our method via three economic growth models: capitalists decision model of endogenous growth, the Ramsey growth model and a one-sector economic growth model with logarithmic utility preferences. First we solve a simple capitalists decision model of endogenous growth to demonstrate how our approach works. Then, we solve the Ramsey growth model and a one-sector economic growth model with logarithmic utility preferences. Moreover, we compare our results for the Ramsey model derived in [28] by the partial Hamiltonian approach.

4.1 Capitalists decision model of endogenous growth

Consider the one-sector capitalists decision model of endogenous growth presented in [32]. The decision problem faced by capitalists is

$$\mathrm{Max}_c \quad \int_0^\infty e^{-\rho t}\frac{c^{1-\theta}-1}{1-\theta}\mathrm{d}t,\ \ \theta>0,\ \ \theta\neq 1 \tag{31}$$

subject to

$$c(t)+i(t)=[1-\tau_k]rk(t)+\tau_k\delta k(t), \tag{32}$$

$$\dot{k}=i(t)-\delta k(t), \tag{33}$$

where ρ is the constant rate of time preference, $1/\theta$ is the intertemporal elasticity of substitution, $c(t)$ is the consumption per worker, $i(t)$ is investment, $k(t)$ is the physical capital per worker, r is rental rate of capital to competitive firms, τ_k is tax rate for gross rental income, and δ is the depreciation rate of capital. Equations (32) and (33) yield

$$\dot{k}=(1-\tau_k)(r-\delta)k-c. \tag{34}$$

The initial and transversality conditions are of the following form:

$$c(0)=c_0,\ \ k(0)=k_0, \tag{35}$$

and

$$\lim_{t\to\infty} e^{-\rho t}c(t)^{-\theta}k(t)=0. \tag{36}$$

The analogue problem in the calculus of variations can be formulated by expressing c in terms of k and $\dot{k}$ from Eq. (34) and substituting this into Eq. (31). The optimal control problem (31) and (34) in the calculus of variations takes the following form:

$$\mathrm{Max}_k \quad \int_0^\infty e^{-\rho t}\frac{\left[-\dot{k}+(1-\tau_k)(r-\delta)k\right]^{1-\theta}-1}{1-\theta}\mathrm{d}t,$$
$$k(0)=k_0. \tag{37}$$

We utilize the partial Lagrangian approach to deriving first integrals for this problem as the partial Lagrangian is time independent, i.e., independent of the discount factor. The desirable partial Lagrangian for our model is

$$L=\frac{\left[-\dot{k}+(1-\tau_k)(r-\delta)k\right]^{1-\theta}}{\theta-1}+\frac{1}{1-\theta}. \tag{38}$$

It is worthy to re-iterate here that the partial Lagrangian is autonomous. The partial Euler–Lagrange equation by Eq. (19) is

$$\frac{\delta L}{\delta k}=-\rho\left[-\dot{k}+(1-\tau_k)(r-\delta)k\right]^{-\theta}. \tag{39}$$

The partial Noether operators determining Eq. (22) in terms variables t and k results in

$$XL+LD_t(\xi)=D_t(B)+(\eta-\xi\dot{k})\frac{\delta L}{\delta k}, \tag{40}$$

where X is the first-order prolonged operator given by

$$X=\xi(t,k)\frac{\partial}{\partial t}+\eta(t,k)\frac{\partial}{\partial k}$$
$$+\left[\eta_t+\dot{k}(\eta_k-\xi_t)-\dot{k}^2\xi_k\right]\frac{\partial}{\partial\dot{k}}. \tag{41}$$

Equation (40) for the partial Lagrangian (38) results in

$$-(1-\tau_k)(r-\delta)\eta+\eta_t+\dot{k}(\eta_k-\xi_t)-\dot{k}^2\xi_k$$
$$+\frac{1}{\theta-1}[-\dot{k}+(1-\tau_k)(r-\delta)k][\xi_t+\dot{k}\xi_k]$$
$$+\frac{1}{1-\theta}(\xi_t+\dot{k}\xi_k)[-\dot{k}+(1-\tau_k)(r-\delta)k]^\theta$$
$$=(B_t+\dot{k}B_k)\left[-\dot{k}+(1-\tau_k)(r-\delta)k\right]^\theta$$
$$+(\eta-\xi\dot{k})(-\rho), \tag{42}$$

where $B(t,k)$ is the gauge term. Separation of Eq. (42) with respect to powers of $\dot{k}$ provides

$$\left[-\dot{k}+(1-\tau_k)(r-\delta)k\right]^\theta:B_t=\frac{1}{1-\theta}\xi_t,\ \ \xi_k=B_k, \tag{43}$$

$$\dot{k}^2:\xi_k=0, \tag{44}$$

$$\dot{k}:\eta_k-\xi_t\frac{\theta}{\theta-1}-\rho\xi=0, \tag{45}$$

$$\dot{k}^0:-(1-\tau_k)(r-\delta)\eta+\eta_t$$
$$+\frac{1}{\theta-1}(1-\tau_k)(r-\delta)k\xi_t+\eta\rho=0. \tag{46}$$

Equations (43)–(45) yield

$$\xi = a_1(t),\ \eta = \left[-\frac{\theta}{1-\theta}\dot{a}_1 + \rho a_1\right]k + a_2(t),$$
$$B = \frac{1}{1-\theta}a_1(t). \quad (47)$$

Equation (46) with ξ, η, B from (47) gives rise to

$$k : \theta\ddot{a}_1 - \left[(1-\tau_k)(r-\delta)(1-\theta) - \rho(1-\theta) - \theta(n-\rho)\right]\dot{a}_1$$
$$-\rho(1-\tau_k)(r-\delta)(\theta-1) + \rho^2(\theta-1)a_1 = 0, \quad (48)$$
$$k^0 : \dot{a}_2 - 1 - \tau_k)(r-\delta)a_2 + \rho a_2 = 0. \quad (49)$$

Equations (47)–(49) lead to the following partial Noether operators and gauge terms:

$$\xi = c_1 e^{-\rho t} + c_2 e^{\frac{\rho(1-\theta)-(1-\tau_k)(r-\delta)(1-\theta)}{\theta}t},$$
$$\eta = \frac{\rho}{1-\theta}c_1 k e^{-\rho t}$$
$$+ (1-\tau_k)(r-\delta)c_2 k e^{\frac{\rho(1-\theta)-(1-\tau_k)(r-\delta)(1-\theta)}{\theta}t}$$
$$+ c_3 e^{((1-\tau_k)(r-\delta)-\rho)t},$$
$$B = \frac{1}{1-\theta}\left[c_1 e^{-\rho t} + c_2 e^{\frac{\rho(1-\theta)-(1-\tau_k)(r-\delta)(1-\theta)}{\theta}t}\right]. \quad (50)$$

The following first integrals corresponding to the operators and gauge terms given in (50) are established from Eq. (23):

$$I_1 = e^{-\rho t}[-\dot{k} + (1-\tau_k)(r-\delta)k]^{-\theta}k$$
$$\times \left[\frac{\rho + (\theta-1)(1-\tau_k)(r-\delta)}{(1-\theta)}\right]$$
$$-\frac{\theta}{1-\theta}\left[-\dot{k} + (1-\tau_k)(r-\delta)k\right]^{1-\theta} e^{-\rho t},$$
$$I_2 = \frac{\theta}{\theta-1}\left[-\dot{k} + (1-\tau_k)(r-\delta)k\right]^{1-\theta} e^{\frac{\rho(1-\theta)-(1-\tau_k)(r-\delta)(1-\theta)}{\theta}t}$$
$$I_3 = [-\dot{k} + (1-\tau_k)(r-\delta)k]^{-\theta} e^{-(\rho-(1-\tau_k)(r-\delta))t}. \quad (51)$$

The partial Lagrangian approach yielded three first integrals. Now we utilize these first integrals to find closed-form solution for $k(t)$. We can utilize either any one of these first integrals or any two first integrals to construct closed-form solution for $k(t)$. Setting $I_3 = B$, we have

$$\left(-\dot{k} + (1-\tau_k)(r-\delta)k\right)^{-\theta} e^{-(\rho-(1-\tau_k)(r-\delta))t} = B, \quad (52)$$

where B is an arbitrary constant. Equation (52) yields first-order ODE in $k(t)$

$$-\dot{k} + (1-\tau_k)(r-\delta)k = B^{-\frac{1}{\theta}} e^{-\frac{(\rho-(1-\tau_k)(r-\delta))t}{\theta}} \quad (53)$$

and this can be solved to obtain closed-form solution for $k(t)$. We can make use of I_1 to directly obtain $k(t)$ without solving ODE (53). We set $I_1 = A$ to arrive at

$$\left[-\dot{k}+(1-\tau_k)(r-\delta)k\right]^{-\theta} k\left[\rho+(\theta-1)(1-\tau_k)(r-\delta)\right]$$
$$-\theta\left[-\dot{k} + (1-\tau_k)(r-\delta)k\right]^{1-\theta} = A(1-\theta)e^{\rho t}, \quad (54)$$

where A is an arbitrary constant. After some simplifications, equation (54) with the help of (53) gives following solution for capital stock $k(t)$

$$k(t) = \frac{\theta B^{-\frac{1}{\theta}} e^{-\frac{\rho-(1-\tau_k)(r-\delta)}{\theta}t} + \frac{A}{B}(1-\theta)e^{(1-\tau_k)(r-\delta)t}}{\rho-(1-\theta)(1-\tau_k)(r-\delta)}. \quad (55)$$

The consumption is $c(t) = -\dot{k} + (1-\tau_k)(r-\delta)k$ and thus

$$c(t) = B^{-\frac{1}{\theta}} e^{-\frac{(\rho-(1-\tau_k)(r-\delta))t}{\theta}}. \quad (56)$$

Equation (32), with the aid of Eqs. (55) and (56), finally gives closed-form solution for investment $i(t)$. Now we make use of initial conditions in order to specify values of arbitrary constant A and B. The initial conditions $c(0) = c_0,\ k(0) = k_0$ give

$$A = \frac{k_0 c_0^{-\theta}}{1-\theta}\left[\rho - (1-\theta)(1-\tau_k)(r-\delta)\right]$$
$$-\frac{\theta}{1-\theta}c_0^{1-\theta},\ B = c_0^{-\theta}. \quad (57)$$

Next we check whether the derived closed-form solutions for $c(t)$ and $k(t)$ satisfy the transversality conditions or not. The transversality condition (36), with the aid of Eqs. (55) and (56), becomes

$$\lim_{t\to\infty}\frac{\theta B^{1-\frac{1}{\theta}} e^{-\frac{\rho-(1-\theta)(1-\tau_k)(r-\delta)}{\theta}t} + A(1-\theta)}{\rho-(1-\theta)(1-\tau_k)(r-\delta)} = 0, \quad (58)$$

and it goes to zero provided $\frac{\rho-(1-\theta)(1-\tau_k)(r-\delta)}{\theta} > 0$ and $A = 0$. Equation (57) for $A = 0$ yields

$$\frac{\rho-(1-\theta)(1-\tau_k)(r-\delta)}{\theta} = \frac{c_0}{k_0}. \quad (59)$$

Now substituting $A = 0,\ B = c_0^{-\theta}$ into (55) and (56), we obtain following final form of solution for $k(t)$ and $c(t)$

$$k(t) = k_0 e^{-\frac{\rho-(1-\tau_k)(r-\delta)}{\theta}t}, \quad c(t) = c_0 e^{-\frac{\rho-(1-\tau_k)(r-\delta)}{\theta}t}, \tag{60}$$

provided parameter restriction (59) holds. The growth rates for capital and consumption are given by

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{(1-\tau_k)(r-\delta)-\rho}{\theta}. \tag{61}$$

4.2 Ramsey neoclassical growth model with CRRA utility function

We consider the following Ramsey neoclassical growth model [6], where the representative consumer's utility maximization problem is defined as

$$\text{Max}_c \quad \int_0^\infty e^{-rt} c^{1-\sigma} \mathrm{d}t, \quad \sigma \neq 0, 1 \tag{62}$$

subject to the capital accumulation equation

$$\dot{k}(t) = k^\beta - \delta k - c, \; k(0) = k_0, \quad 0 < \beta < 1, \tag{63}$$

where $c(t)$ is the consumption per person, $k(t)$ is the capital labor ratio, and β, δ, r are the capital share, depreciation rate, rate of time preferences, respectively. The intertemporal elasticity of substitution is given by $1/\sigma$ and k_0 is the initial capital stock. The optimal control problem (62) and (63) in the calculus of variations takes the following form:

$$\text{Max}_\text{k} \quad \int_0^\infty e^{-rt} \left[k^\beta - \delta k - \dot{k}\right]^{1-\sigma} \mathrm{d}t, \quad k(0) = k_0. \tag{64}$$

The partial Lagrangian for this model is

$$L = -\left[k^\beta - \delta k - \dot{k}\right]^{1-\sigma} \tag{65}$$

which is free of the discount factor and the partial Euler–Lagrange equation is

$$\frac{\delta L}{\delta k} = -r(1-\sigma)\left[k^\beta - \delta k - \dot{k}\right]^{-\sigma}. \tag{66}$$

The partial Noether operators determining Eq. (22) for the partial Lagrangian (65) yields

$$\begin{aligned}&\eta(\sigma-1)(\beta k^{\beta-1}-\delta)+\left[\eta_t+\dot{k}(\eta_k-\xi_t)-\dot{k}^2\xi_k\right](1-\sigma)\\ &\quad -(\xi_t+\dot{k}\xi_k)\left[k^\beta-\delta k-\dot{k}\right]\\ &\quad = (B_t+\dot{k}B_k)[k^\beta-\delta k-\dot{k}]^\sigma+(\eta-\xi\dot{k})r(\sigma-1),\end{aligned} \tag{67}$$

where $B(t,k)$ is the gauge term. By separating equation (67) with respect to powers of $\dot{k}$, we easily have

$$\left[k^\beta - \delta k - \dot{k}\right]^\sigma : B_t = 0, \; B_k = 0, \tag{68}$$

$$\dot{k}^2 : \xi_k = 0, \tag{69}$$

$$\dot{k} : (1-\sigma)\eta_k + \sigma\dot{\xi} - r(1-\sigma)\xi = 0, \tag{70}$$

$$\begin{aligned}\dot{k}^0 &: \eta(\sigma-1)(\beta k^{\beta-1}-\delta)+\eta_t(1-\sigma)\\ &\quad -\dot{\xi}(k^\beta-\delta k)+r(1-\sigma)\eta = 0.\end{aligned} \tag{71}$$

The solution of equations (68)- (71) yield the following partial Noether operator and gauge terms

$$X = e^{\delta\beta(1-\sigma)t}\frac{\partial}{\partial t} - \delta e^{\delta\beta(1-\sigma)t} k \frac{\partial}{\partial k}, \; B = 0, \tag{72}$$

with σ satisfying

$$\sigma = \frac{r+\delta}{\beta\delta}. \tag{73}$$

The first integral corresponding to the partial Noether operator and gauge terms given in (72) is

$$\begin{aligned}I = e^{\delta\beta(1-\sigma)t}&\Big[-\sigma\left[k^\beta-\delta k-\dot{k}\right]^{1-\sigma}+(\sigma-1)\\ &\left[k^\beta - \delta k - \dot{k}\right]^{-\sigma} k^\beta\Big].\end{aligned} \tag{74}$$

Naz et al. [28] derived the following first integral by utilizing the partial Hamiltonian approach

$$I = e^{\delta\beta(1-\sigma)t}[-\sigma c^{1-\sigma} + (\sigma-1)c^{-\sigma}k^\beta]. \tag{75}$$

A closer look at the first integrals given in (74) and (75) shows that both first integrals are identical by replacing

$$c = k^\beta - \delta k - \dot{k} \tag{76}$$

in it. These lead to the solution of the underlying model as shown in [28] and are given by

$$c(t) = \left(1 - \frac{\beta\delta}{r+\delta}\right)k^\beta, \tag{77}$$

$$k(t) = \left[\frac{\beta}{r+\delta} + \left(k_0^{1-\beta} - \frac{\beta}{r+\delta}\right)e^{-(1-\beta)\delta t}\right]^{\frac{1}{1-\beta}}. \tag{78}$$

The steady state for this model is

$$\bar{k} = \left(\frac{r+\delta}{\beta}\right)^{\frac{1}{\beta-1}} \tag{79}$$

and

$$\bar{c} = \left(1 - \frac{\beta\delta}{r+\delta}\right)\bar{k}^{\beta}. \tag{80}$$

In order to determine the dynamics of the system, we now differentiate (77) with respect to t

$$\frac{\dot{c}}{c} = \beta\frac{\dot{k}}{k}. \tag{81}$$

From (81), we have the growth rate of capital and consumption is the same as for $\beta > 0$. The growth rate of capital is given by

$$\frac{\dot{k}}{k} = \frac{-\delta\left(k_0^{1-\beta} - \frac{\beta}{r+\delta}\right)e^{-(1-\beta)\delta t}}{\frac{\beta}{r+\delta} + \left(k_0^{1-\beta} - \frac{\beta}{r+\delta}\right)e^{-(1-\beta)\delta t}} \tag{82}$$

or

$$\frac{\dot{k}}{k} = \delta\left[\left(\frac{k}{\bar{k}}\right)^{\beta-1} - 1\right], \tag{83}$$

where $\bar{k} = \left(\frac{r+\delta}{\beta}\right)^{\frac{1}{\beta-1}}$. From (83), clearly the growth rate of capital decreases over time and goes to zero as the steady state is reached for $t \mapsto \infty$. The growth rate of consumption also decreases over time as the steady state is approached. For the case when the capital's share is equal to the reciprocal of the intertemporal elasticity of substitution studied by [33], the growth rate of capital decreases over time, but the growth rate of consumption grows linearly with capital stock.

Secondly, the gross saving rate $s = 1 - c/k^{\beta}$ for this model is constant and is given by

$$s = \frac{\beta\delta}{r+\delta} = \sigma. \tag{84}$$

4.3 Economic growth model with logarithmic utility function

We consider the following neoclassical growth model [6], where the representative consumer's utility maximization problem is defined as

$$\text{Max}_c \quad \int_0^{\infty} e^{-rt}\ln(c)\mathrm{d}t, \tag{85}$$

subject to the capital accumulation equation

$$\dot{k}(t) = k^{\beta} - (\delta - A)k - c, \; k(0) = k_0, \; 0 < \beta < 1, \tag{86}$$

where $c(t)$ is the consumption per person, $k(t)$ the capital labor ratio, $A > 0$ the marginal product of capital, β the capital share, δ the depreciation rate, and r is the rate of time preferences. In the capital accumulation equation (86), we consider a Cobb–Douglas version of the A-K technology: $y = k^{\beta} + Ak$. The current value Hamiltonian for this model is

$$H = \ln(c) + \lambda(k^{\beta} - (\delta - A)k - c) \tag{87}$$

where λ is the costate variable. The necessary first-order conditions for optimal control are

$$\lambda = \frac{1}{c}, \tag{88}$$

$$\dot{k}(t) = k^{\beta} - (\delta - A)k - c, \tag{89}$$

$$\dot{\lambda} = -(\beta k^{\beta-1} - \delta + A - r)\lambda, \tag{90}$$

and the growth rate of consumption is

$$\dot{c} = \left(\beta k^{\beta-1} - \delta + A - r\right)c. \tag{91}$$

We do not apply the partial Hamiltonian approach to solving this model. We solve this model by the partial Lagrangian approach, and first we transform the problem to an equivalent calculus of variations problem.

The optimal control problems (85) and (86) in the calculus of variations can be expressed as

$$\text{Max}_k \quad \int_0^{\infty} e^{-rt}\ln\left(k^{\beta} - \delta k + Ak - \dot{k}\right)\mathrm{d}t, \; k(0) = k_0. \tag{92}$$

The partial Lagrangian for this model is

$$L = -\ln\left(k^{\beta} - \delta k + Ak - \dot{k}\right) = \ln\left(\frac{1}{k^{\beta} - \delta k + Ak - \dot{k}}\right) \tag{93}$$

and the partial Euler–Lagrange equation is

$$\frac{\delta L}{\delta k} = -r\left(k^{\beta} - \delta k + Ak - \dot{k}\right)^{-1}. \tag{94}$$

For the partial Lagrangian (93), the partial Noether operators determining Eq. (22) results in

$$\begin{aligned}&-\eta(\beta k^{\beta-1}-\delta+A)+\eta_t+\dot{k}(\eta_k-\xi_t)-\dot{k}^2\xi_k\\&\quad+(\xi_t+\dot{k}\xi_k)(k^\beta-\delta k+Ak-\dot{k})\ln\\&\quad\times\left(\frac{1}{k^\beta-\delta k+Ak-\dot{k}}\right)\\&=(B_t+\dot{k}B_k)(k^\beta-\delta k+Ak-\dot{k})-r(\eta-\xi\dot{k}),\end{aligned}\tag{95}$$

where $B(t,k)$ is the gauge term. The term $(\xi_t+\dot{k}\xi_k)(k^\beta-\delta k+Ak-\dot{k})\ln(k^\beta-\delta k+Ak-\dot{k})^{-1}$ in Eq. (95) after power series expansion in $\dot{k}$ contains higher powers of $\dot{k}$ and thus yields $\xi_k=0,\ \xi_t=0$. Now separating the rest of the terms in Eq. (95) with respect to powers of $\dot{k}$, we have

$$\dot{k}^2: B_k=0,\tag{96}$$

$$\dot{k}: \eta_k+B_t-r\xi=0,\tag{97}$$

$$\begin{aligned}&\dot{k}^0: -\eta(\beta k^{\beta-1}-\delta+A)+\eta_t-B_t\left(k^\beta-\delta k+Ak\right)\\&\quad+r\eta=0.\end{aligned}\tag{98}$$

The solution of Eqs. (96)–(98) yields the following partial Noether operator and gauge terms

$$\xi=c_1,\quad \eta=\frac{rc_1k}{1-\beta},\quad B=\frac{c_1\beta tr}{\beta-1}+c_2,\tag{99}$$

with β satisfying

$$\beta=\frac{r+\delta-A}{\delta-A}.\tag{100}$$

The first integral corresponding to the partial Noether operator and gauge terms given in (96) under the parameter restriction (100) is

$$\begin{aligned}I&=\frac{\beta rt}{\beta-1}+\ln(k^\beta-\delta k+Ak-\dot{k})\\&\quad+k^\beta(k^\beta-\delta k+Ak-\dot{k})^{-1}-1.\end{aligned}\tag{101}$$

We set this integral to be a constant as

$$\begin{aligned}I&=\frac{\beta rt}{\beta-1}+\ln\left(k^\beta-\delta k+Ak-\dot{k}\right)\\&\quad+k^\beta\left(k^\beta-\delta k+Ak-\dot{k}\right)^{-1}-1=e_1\end{aligned}\tag{102}$$

in which e_1 is an arbitrary constant. From this equation (102), we deduce k which is written in terms of the control variable c as

$$k=[(e_1+1-\ln c+\delta\beta t-A\beta t)c]^{\frac{1}{\beta}}.\tag{103}$$

Now Eq. (91) after substitution of the value of k in (103) becomes

$$\begin{aligned}&\dot{c}+(\delta-A)\beta c\\&\quad=\beta c\left[(e_1+1-\ln c+\delta\beta t-A\beta t)\,c\right]^{\frac{\beta-1}{\beta}}.\end{aligned}\tag{104}$$

We set $S=c\exp(\delta-A)\beta t$. Then, (104) becomes

$$\dot{S}=\beta S(e_1+1-\ln S)^{\frac{\beta-1}{\beta}}S^{\frac{\beta-1}{\beta}}\exp(-rt)\tag{105}$$

which is integrable by quadratures

$$\int\frac{\mathrm{d}S}{\beta S^{\frac{2\beta-1}{\beta}}(e_1+1-\ln S)^{\frac{\beta-1}{\beta}}}=-\frac{1}{r}\exp(-rt)+e_2,\tag{106}$$

where e_2 is another arbitrary constant. This is a new solution.

5 Concluding remarks

We have developed a new approach termed as a partial or discount free Lagrangian approach for dynamical systems. We have shown how a partial Lagrangian approach provides solutions for a range of equations of economics. This naturally arises from the allied notion of a current value Hamiltonian via the partial Legendre transformation in a generalized setting when the costate and control variables can be expressed in terms of the state variables and their derivatives. This approach, when it works, is simpler than the more general partial Hamiltonian approach as the control variables do not enter into the calculations which are therefore much more simplified. Notwithstanding, the partial or discount free Lagrangian that arises is independent of the discount factor which makes it desirable in the same way as the current value Hamiltonian is desirable for not having the discount factor.

We worked out three model examples, one linear and two nonlinear, using this approach. We have demonstrated the simplicity of this method as compared to the worked out example of [28] which involves more variables in the form of the control variables. In the third example, we derived a first integral and the new exact solutions for the agents maximization problem with logarithmic time preference.

The utility of the method advocated here for economic models is thus advantageous when a discount free Lagrangian exists. This is indeed the case for a number of economic models which need investigation in future research. It is worthy to mention here that for the case of more control variables it is sometimes convenient to apply the more generalized approach of partial Hamiltonians.

Acknowledgments FMM is thankful to Dr. Shahid Amjad Chaudhry, Rector, Lahore School of Economics, Pakistan, for his warm hospitality during which time this work was commenced. He also is grateful to the NRF of South Africa for enabling research support through grants. RN is thankful to Lahore School of Economics for providing funds to complete this work.

References

1. Chiang, A.C.: Elements of Dynamic Optimization. McGraw Hill, New York (1992)
2. Ramsey, F.: A mathematical theory of saving. Econ. J. **38**, 543–559 (1928)
3. Lucas, R.: On the mechanics of economic development. J. Monet. Econ. **22**, 3–42 (1988)
4. Eisner, R., Strotz, R.: The Determinants of Business Investment. Impacts of Monetary Policy. Prentice-Hall, Englewood cliffs (1963)
5. Ben-Porath, Y.: The production of human capital and the life cycle of earnings. J. Polit. Econ. **75**, 352–365 (1967)
6. Barro, R.J., Sala-i-Martin, X.: Economic Growth. The MIT Press, Cambridge (2004)
7. Rodriguez, A.: On the local stability of the solution to optimal control problems. J. Econ. Dyn. Control **28**, 2475–2484 (2004)
8. Brock, W.A., Scheinkman, J.: Global asymptotic stability of optimal control systems with applications to the theory of economic growth. J. Econ. Theory **12**, 164–190 (1976)
9. Rodriguez, A.: On the local stability of the stationary solution to variational problems. J. Econ. Dyn. Control **20**, 415–431 (1996)
10. Ragni, S., Diele, F., Marangi, C.: Steady-state invariance in high-order Runge–Kutta discretization of optimal growth models. J. Econ. Dyn. Control **34**, 1248–1259 (2010)
11. Mulligan, C.B., Sala-i-Martin, X.: Transitional dynamics in two-sector models of endogenous growth. Q. J. Econ. **108**, 739–773 (1993)
12. Ruiz-Tamarit, J.R.: The closed-form solution for a family of four-dimension nonlinear MHDS. J. Econ. Dyn. Control **32**, 1000–1014 (2008)
13. Chilarescu, C.: An analytical solutions for a model of endogenous growth. Econ. Model. **25**, 1175–1182 (2008)
14. Chilarescu, C.: A closed-form solution to the transitional dynamics of the Lucas–Uzawa model. Econ. Model. **26**, 135–138 (2009)
15. Hiraguchi, R.: A note on the closed-form solution to the Lucas–Uzawa model with externality. J. Econ. Dyn. Control **33**, 1757–1760 (2009)
16. Guerrini, L.: A closed-form solution to the Ramsey model with logistic population growth. Econ. Model. **27**, 1178–1182 (2010)
17. Diele, F., Marangi, C., Ragni, S.: Exponential Lawson integration for nearly Hamiltonian systems arising in optimal control. Math. Comput. Simul. **81**, 1057–1067 (2011)
18. Ruiz-Tamarit, J.R., Ventura-Marco, M.: Solution to nonlinear MHDS arising from optimal growth problems. Math. Soc. Sci. **61**, 86–96 (2011)
19. Yasar, E.: Variational principles and conservation laws to the Burridge–Knopoff equation. Nonlinear Dyn. **54**, 307–312 (2008)
20. Kara, A.H., Mahomed, F.M.: Noether-type symmetries and conservation laws via partial Lagrangians. Nonlinear Dyn. **45**, 367–383 (2006)
21. Naz, R., Mahomed, F.M., Mason, D.P.: Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics. Appl. Math. Comput. **205**, 212–230 (2008)
22. Gan, Y., Qu, C.: Approximate conservation laws of perturbed partial differential equations. Nonlinear Dyn. **61**, 217–228 (2010)
23. Kara, A.H., Mahomed, F.M., Naeem, I., Wafo Soh, C.: Partial Noether operators and first integrals via partial Lagrangians. Math. Methods Appl. Sci. **30**, 2079–2089 (2007)
24. Naz, R., Freire, I.L., Naeem, I.: Comparison of different approaches to construct first integrals for ordinary differential equations. Abstr. Appl. Anal. (2014). doi:10.1155/2014/978636
25. Naeem, I., Mahomed, F.M.: Approximate partial Noether operators and first integrals for coupled nonlinear oscillators. Nonlinear Dyn. **57**, 303–311 (2009)
26. Unal, G.: Approximate first integrals of weakly nonlinear, damped-driven oscillators with one degree of freedom. Nonlinear Dyn. **26**, 309–329 (2001)
27. Dorodnitsyn, V., Kozlov, R.: Invariance and first integrals of continuous and discrete Hamiltonian equations. J. Eng. Math. **66**, 253–270 (2010)
28. Naz, R., Mahomed, F.M., Chaudhry, A.: A partial Hamiltonian approach for current value Hamiltonian systems. Commun. Nonlinear Sci. Numer. Simul. **19**, 3600–3610 (2014)
29. Naz, R., Chaudhry, A., Mahomed, F.M.: Closed-form solutions for the Lucas–Uzawa model of economic growth via the partial Hamiltonian approach. Commun. Nonlinear Sci. Numer. Simul. **30**(1), 299–306 (2016)
30. Mahomed, F.M., Roberts, J.A.G.: Characterization of Hamiltonian symmetries and their first integrals. Int. J. Non-Linear Mech. **74**, 84–91 (2015)
31. Legendre, A.M.: Réflexions sur differéntes manières de démostrer la théorie des paralléles ou le théorème sur la somme des trois angles du triangle. Mémoires de l'Academie des Sciences de Paris **13**, 213–220 (1833)
32. Lansing, K.J.: Optimal redistributive capital taxation in a neoclassical growth model. J. Public Econ. **73**(3), 423–453 (1999)
33. Smith, W.T.: A closed form solution to the Ramsey model. Contrib. Macroecon. **6**, 1–28 (2006)