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ORIGINAL ARTICLE

A new (3+1)-dimensional generalized Kadomtsev–Petviashvili equation

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Abstract We introduce a new (3+1)-dimensional generalized Kadomtsev–Petviashvili equation. We use the simplified Hirota's direct method to derive multiple-soliton solutions for the new model with the coefficients of the spatial variables which are left free. We show that the phase shifts depend on all these coefficients. We prove that the new model fails the Painlevé integrability test although it gives multiple-soliton solutions. Moreover, for $x = y = z$, this new model reduces to the potential KdV equation, which we will examine as well.

Keywords Generalized KP equation · simplified Hirota's method · Painlevé test

1 Introduction

It is well known that nonlinear evolution equations play a significant role in describing nonlinear scientific phenomena. The study of these equations involves many domains, which include plasma physics, fluid mechanics, solitary waves theory, hydrodynamics and theory of turbulence, optical fibers, water waves, chaos theory, chemical physics and other applications [1–16]. The work on these nonlinear equations has increased in recent decades to get an insight through qualitative and quantitative features of these equations. The soliton pulse, an important feature of nonlinearity, indicates a perfect balance between nonlinearity and dispersion effects [17–27].

The study of integrable properties for nonlinear evolution equations has become an extremely active area of research. A variety of powerful methods has been used to study the nonlinear evolution equations and the integrability of these equations if it holds, such as the Hirota bilinear method [4], the Bäcklund transformation method, Darboux transformation, Pfaffian technique, the inverse scattering method, the Painlevé analysis, the generalized symmetry method and other methods. The Hirota's bilinear method and the Hereman's simplified form [13,14], are rather heuristic and the most commonly used techniques.

The Kadomtsev–Petviashvili (KP) equation [10] is a nonlinear partial differential equation in two spatial and one temporal coordinate, which describes the evolution of nonlinear, long waves of small amplitude with slow dependence on the transverse coordinate. Kadomtsev and Petviashvili [10] relaxed the restriction that the waves be strictly one dimensional, to derive the completely integrable KP equation in the form

$$(u_t + 6uu_x + u_{xxx})_x + au_{yy} = 0, \tag{1}$$

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which describes the evolution of quasi-one-dimensional shallow-water waves when effects of the surface tension and the viscosity are negligible. A variety of modified and extended KP equations has been examined in the literature [10–16]. For example, the modified KP (mKP) equations [2] read

$$4u_t + u_{xxx} - 6u^2 u_x + 6u_x \partial_x^{-1} u_y + 3u_x^{-1} u_{yy} = 0, \quad (2)$$

which was derived in the study of the propagation of the ion-acoustic waves in a plasma with non-isothermal electrons [1,2]. It can describe the evolution of various solitary waves in multi-temperature electron plasmas, in which there exists a collision less multi-component plasma conceiving cold ions and two temperature electrons having different Maxwellian distributions rendered in the form of two Boltzmann relations.

Recently, a generalized (3+1)-dimensional KP equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0, \quad (3)$$

was presented and examined [1–6]. This equation was investigated in [4] where Wronskian and Grammian formulations were established for this equation. This equation was studied also in [1] where the simplified Hirota's method was used to obtain multiple-soliton solutions, provided specific constraints on the coefficients of the spatial variables.

Under the dependent variable transformation $u = 2(\ln f)_x$, the Eq. (3) is transformed to the Hirota bilinear form [3,4]

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f \times f = 0, \quad (4)$$

where D_t, D_x, D_y and D_z are Hirota's bilinear operators.

A (2+1)-dimensional nonlinear B-type KP (BKP) equation, given by

$$v_t + v_{xxxxx} - 5(v_{xxy} + \partial_x^{-1} v_{yy}) + 15(v_x v_{xx} + v v_{xxx} - v v_y - v_x \partial_x^{-1} v_y) + 45v^2 v_x = 0, \quad (5)$$

was investigated because it possesses many integrable structures such as Lax formulation and the multiple-soliton solutions. The BKP equation was given this name because it is a B-type KP equation.

It is to be noted that other forms of extended KP equations can be found in the literature. Many powerful methods were used to investigate these extended forms.

In this work, we aim to introduce a new form of a (3+1)-dimensional generalized KP equation, given as

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} + u_{tz} - u_{zz} = 0, \quad (6)$$

where one extra term, namely u_{tz}, is added to the generalized form (3). This new form is introduced by making use of our previous work in [1]. It is to be noted that Eq. (6) reduces to the KP equation if $y = x$. However, for $x = y = z$, Eq. (6) reduces to the potential KdV equation

$$3u_t - u_x + 3u_x^2 + u_{xxx} = 0, \quad (7)$$

which will be examined later in this work.

Under the dependent variable transformation $u = 2(\ln f)_x$, Eq. (6) is transformed to the Hirota bilinear form

$$(D_x^3 D_y + D_t D_x + D_t D_y + D_t D_z - D_z^2) f \times f = 0, \quad (8)$$

where D_t, D_x, D_y and D_z are Hirota's bilinear operators, and $f(x, y, z, t)$ is an auxiliary function.

The objectives of this work are twofold. First, we aim to show that this slight change of an additional term will cause a drastic impact on the dispersion relations and on the phase shift. The second goal is to establish multiple-soliton solutions of distinct physical structures for the new model (6), and unlike our work in [1], there are no constraints on the coefficients of the spatial variables. On the contrary, these coefficients will be left as free parameters. Moreover, we will also study the reduced form of this equation if $x = y = z$ as stated earlier. Motivated by the work in [1], the simplified form of Hirota's method developed [13,14] will be used to achieve these goals.

2 The new (3+1)-dimensional generalized KP equation

The new (3+1)-dimensional generalized KP equation reads

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} + u_{tz} - u_{zz} = 0, \quad (9)$$

where $u = u(x, y, z, t)$.

We first substitute

$$u(x, y, z, t) = e^{k_i x + r_i y + s_i z - c_i t}. \tag{10}$$

into the linear terms of (9) to find the dispersion relation as

$$c_i = \frac{(k_i^3 r_i - s_i^2)}{k_i + r_i + s_i}, \quad i = 1, 2, \dots N, \tag{11}$$

and hence the dispersion variable θ_i takes the form

$$\theta_i = e^{k_i x + r_i y + s_i z - \frac{(k_i^3 r_i - s_i^2)}{k_i + r_i + s_i} t}, \quad i = 1, 2, \dots N. \tag{12}$$

Using the dependent variable transformation

$$u(x, y, z, t) = 2(\ln f)_x, \tag{13}$$

where the auxiliary function $f(x, y, z, t)$ is given as

$$f(x, y, z, t) = 1 + f_1(x, y, z, t) = 1 + e^{\theta_1}, \tag{14}$$

the single-soliton solution

$$u(x, y, z, t) = \frac{2k_1 e^{k_1 x + r_1 y + s_1 z - \frac{(k_1^3 r_1 - s_1^2)}{k_1 + r_1 + s_1} t}}{1 + e^{k_1 x + a k_1 y + r_1 z - \frac{(a k_i^4 - r_i^2)}{(1+a) k_i} t}}, \tag{15}$$

follows immediately.

For the two-soliton solutions, we substitute

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{16}$$

into (9), and solving for the phase shift a_{12}, we find for the general case

$$a_{ij} = \frac{N}{M}, \quad 1 \le i < j \le N, \tag{17}$$

where

$$\begin{aligned}
N = {} & 2 k_i k_j^3 r_i r_j + k_j^4 r_i^2 + k_i^4 r_j^2 + k_j^2 s_i^2 \\
& + r_j^2 s_i^2 - 3 k_i k_j^3 r_i^2 - 3 k_i^2 k_j^3 r_i + 2 k_i^3 k_j^2 r_i \\
& - 3 k_i^3 k_j^2 r_j + 2 k_i^2 k_j^3 r_j + 3 k_i^2 k_j^2 r_j^2 + k_i^4 k_j r_j \\
& + 3 k_i^2 k_j^2 r_i^2 + k_i k_j^4 r_i + k_j^4 r_i s_i + 2 k_i r_i s_j^2 \\
& + 2 k_j r_j s_i^2 - k_i^3 r_i s_j^2 - k_j^3 r_j s_i^2 \\
& + k_i^4 r_j s_j + k_j^3 r_i^2 s_j + k_i^3 r_j^2 s_i - 3 k_i^3 k_j r_j^2 \\
& - 2 k_i s_i k_j s_j - 3 k_i^2 k_j^2 r_i s_j - 3 k_i^3 k_j r_j s_j \\
& + k_i k_j^3 r_i s_j + 3 k_i^2 k_j^2 r_j s_j - 3 k_j^2 k_i r_i^2 s_j \\
& + 3 k_i^2 k_j r_i^2 s_j + k_j^3 r_i s_i s_j + k_i^3 r_j s_i s_j \\
& + k_i^3 r_i k_j s_j - k_i^3 r_i r_j s_j + r_i^2 s_j^2 \\
& + k_i^2 s_j^2 - 3 k_j^2 k_i r_i r_j s_i + 3 k_i^2 k_j r_i r_j s_i \\
& + 3 k_j^2 k_i r_i r_j s_j - 3 k_j^2 k_i r_i s_i s_j \\
& - 3 k_i^2 k_j r_i r_j s_j + 3 k_i^2 k_j r_i s_i s_j \\
& - 3 k_i^2 k_j r_j s_i s_j + 3 k_j^2 k_i r_j s_i s_j - 2 r_i s_i r_j s_j \\
& - 2 r_i s_i k_j s_j - 2 k_i s_i r_j s_j \\
& + 3 k_j^2 k_i r_i r_j^2 - 3 k_j^2 k_i r_i^2 r_j + 3 k_i^2 k_j r_i^2 r_j \\
& - 3 k_i^2 k_j r_i r_j^2 - 3 k_i^2 k_j r_j^2 s_i + 3 k_j^2 k_i r_j^2 s_i \\
& - 3 k_i^2 k_j^2 r_j s_i - k_j^3 r_j r_i s_i \\
& - 6 k_i^2 k_j^2 r_i r_j + 3 k_i^2 k_j^2 r_i s_i + k_i^3 k_j r_j s_i \\
& + 2 k_i^3 k_j r_i r_j + k_i k_j^3 r_j s_i - 3 k_i k_j^3 r_i s_i,
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
M = {} & 2 k_i k_j^3 r_i r_j + k_j^4 r_i^2 + k_i^4 r_j^2 + k_j^2 s_i^2 + r_j^2 s_i^2 \\
& + 3 k_i k_j^3 r_i^2 + 3 k_i^2 k_j^3 r_i + 2 k_i^3 k_j^2 r_i + 3 k_i^3 k_j^2 r_j \\
& + 2 k_i^2 k_j^3 r_j + 3 k_i^2 k_j^2 r_j^2 \\
& + k_i^4 k_j r_j + 3 k_i^2 k_j^2 r_i^2 + k_i k_j^4 r_i + k_j^4 r_i s_i \\
& + 2 k_i r_i s_j^2 + 2 k_j r_j s_i^2 \\
& - k_i^3 r_i s_j^2 - k_j^3 r_j s_i^2 + k_i^4 r_j s_j + k_j^3 r_i^2 s_j \\
& + k_i^3 r_j^2 s_i + 3 k_i^3 k_j r_j^2 \\
& - 2 k_i s_i k_j s_j + 3 k_i^2 k_j^2 r_i s_j + 3 k_i^3 k_j r_j s_j \\
& + k_i k_j^3 r_i s_j + 3 k_i^2 k_j^2 r_j s_j + 3 k_j^2 k_i r_i^2 s_j \\
& + 3 k_i^2 k_j r_i^2 s_j + k_j^3 r_i s_i s_j + k_i^3 r_j s_i s_j \\
& + k_i^3 r_i k_j s_j - k_i^3 r_i r_j s_j \\
& + r_i^2 s_j^2 + k_i^2 s_j^2 + 3 k_j^2 k_i r_i r_j s_i \\
& + 3 k_i^2 k_j r_i r_j s_i + 3 k_j^2 k_i r_i r_j s_j \\
& + 3 k_j^2 k_i r_i s_i s_j + 3 k_i^2 k_j r_i r_j s_j + 3 k_i^2 k_j r_i s_i s_j \\
& + 3 k_i^2 k_j r_j s_i s_j + 3 k_j^2 k_i r_j s_i s_j - 2 r_i s_i r_j s_j \\
& - 2 r_i s_i k_j s_j - 2 k_i s_i r_j s_j \\
& + 3 k_j^2 k_i r_i r_j^2 + 3 k_j^2 k_i r_i^2 r_j + 3 k_i^2 k_j r_i^2 r_j \\
& + 3 k_i^2 k_j r_i r_j^2 + 3 k_i^2 k_j r_j^2 s_i \\
& + 3 k_j^2 k_i r_j^2 s_i + 3 k_i^2 k_j^2 r_j s_i - k_j^3 r_j r_i s_i \\
& + 6 k_i^2 k_j^2 r_i r_j + 3 k_i^2 k_j^2 r_i s_i + k_i^3 k_j r_j s_i \\
& + 2 k_i^3 k_j r_i r_j + k_i k_j^3 r_j s_i + 3 k_i k_j^3 r_i s_i.
\end{aligned} \tag{19}$$

It is obvious from this results that, unlike the conclusion in [1] where constraints were imposed on r_i, the coefficients k_m, r_m, s_m of the spatial variables x, y, z are left free parameter. Moreover, the phase shift given above differs completely than the Hirota's type of phase shift only if $k_i \neq r_i \neq s_i$ and depends on all three coefficients. This in turn leads to distinct two-soliton solutions if compared with the results presented in [1]. For example, using $k_i = i, r_i = i + 3, s_i = i + 6, 1 \leq i \leq 3$, the phase shifts are given by

$$a_{12} = \frac{43}{1083},\quad a_{13} = \frac{25}{241},\quad a_{23} = \frac{61}{3301}. \tag{20}$$

To determine the two-soliton solutions explicitly, we substitute the previous results into the formula (13).

It is interesting to point out that for the special case $r_i = s_i = k_i, i = 1, 2, 3$, the phase shift a_{ij} reduces to the Hirota's type in the form

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i < j \leq N. \tag{21}$$

In this case, the generalized KP Eq. (9) does not show any resonant phenomenon [2] because the phase shift term a_{12} in (21) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

In a like manner, we determine the three-soliton solutions by setting

$$\begin{aligned} f(x, y, z, t) &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} \\ &+ a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{22}$$

into (9) and solve for b_{123}, we find that

$$b_{123} = a_{12}a_{13}a_{23}. \tag{23}$$

To determine the three-soliton solutions explicitly, we substitute the last result for f in the formula $u(x, y, z, t) = 2(\ln f(x, y, z, t))_x$. The higher-level soliton solutions for $n \geq 4$ can be obtained in a parallel manner.

2.1 Painlevé analysis

Painlevé analysis is one of the most powerful methods to find the underlying integrable models from a given generalized nonlinear equation [15–17]. Following the WTC–Kruskal approach [15–17], or the Macsyma [14], the first step of the Painlevé test is the leading order analysis to the negative integer α. We second determine the resonances points and finally verify the compatibility conditions, where for every positive resonance there is a compatibility condition which must be identically satisfied. Baldwin et al. presented two packages in Macsyma and Mathematica, respectively [14], which are based on the WTC method and the Kruskal's simplification.

Using the above-mentioned software packages, and considering first the Kruskal simplification and then without it, to test the integrability of the new (3+1)-dimensional generalized KP Eq. (9), four resonant points are found at $j = -1, 2, 4, 6$. In both cases, Eq. (9) does not pass the Painlevé test. The Laurent series has arbitrary coefficients at $r = 1, 4$, and 6, but the compatibility condition is not satisfied at resonance $r = 6$.

It is well known that the presence of multiple-soliton solutions often indicates the integrability of the examined equation. However, this is not sufficient, and the existence of multiple-soliton solutions should be supported by carrying the Painlevé test, or determining Lax pair of the equation, and using other methods as well. In this work, we formally derived multiple-soliton solutions for Eq. (9), but the equation failed the Painlevé test, and this shows that it is a non-integrable equation.

3 The potential KdV equation

For $x = y = z$, the new (3+1)-dimensional generalized KP Eq. (6) reduces, after integrating both sides, to the potential KdV equation given as

$$3u_t - u_x + 3u_x^2 + u_{xxx} = 0, \tag{24}$$

where $u = u(x, t)$.

Following the analysis presented above, we set

$$u(x, t) = e^{k_i x - c_i t}. \tag{25}$$

into the linear terms of (24) to find the dispersion relation as

$$c_i = \frac{1}{3}(k_i^3 - k_i), \quad i = 1, 2, \ldots N, \tag{26}$$

and hence the dispersion relation θ_i takes the form

$$\theta_i = e^{k_i x - \frac{1}{3}(k_i^3 - k_i)t}, \quad i = 1, 2, \ldots N. \tag{27}$$

Using the dependent variable transformation

$$u(x,t) = 2(\ln f(x,t))_x, \tag{28}$$

where the auxiliary function $f(x,t)$ is given as

$$f(x,t) = 1 + f_1(x,t) = 1 + e^{\theta_1}, \tag{29}$$

leads to the single-soliton solution

$$u(x,t) = \frac{2k_1 e^{k_1 x - \frac{1}{3}(k_1^3 - k_1)\,t}}{1 + e^{k_1 x - \frac{1}{3}(k_1^3 - k_1)\,t}}. \tag{30}$$

For the two-soliton solutions, we proceed as before to find that the phase shift a_{ij} reduces to the Hirota's type in the form

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \le i < j \le N. \tag{31}$$

Using

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{32}$$

into (28), we obtain the two-soliton solutions.

In a like manner, we determine the three-soliton solutions by setting

$$\begin{aligned} f(x,t) = 1 &+ e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{theta_1 + \theta_2} \\ &+ a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} + b_{123} e^{\theta_1 + \theta_2 + \theta_3} \end{aligned} \tag{33}$$

into (24), and solve for b_{123}, we find that

$$b_{123} = a_{12} a_{13} a_{23}. \tag{34}$$

To determine the three-soliton solutions explicitly, we substitute the last result for f in the formula $u(x,t) = 2(\ln f(x,t))_x$. The higher-level soliton solutions for $n \ge 4$ can be obtained in a parallel manner.

3.1 Integrability of the potential KdV equation

It is well known in the literature that the potential KdV equation is integrable and passes the Painlevé test. The potential KdV equation is derived from the KdV equation, and both are integrable nonlinear equations.

4 Concluding remarks

A new (3+1)-dimensional generalized KP equation was introduced in this work. The new equation was investigated for multiple-soliton solutions. The simplified form of the Hirota's method is used to formally derive these solutions. The coefficients of the spatial variables are not subjected to any constraint. The Painlevé analysis was used to show that the equation is not integrable, although it provides multiple-soliton solutions.

We also examined the potential KdV equation that can be derived by setting $x = y = z$ in the new (3+1)-dimensional generalized KP equation which was established in this work. Unlike the new (3+1)-dimensional generalized KP equation which is non-integrable, the potential KdV equation is integrable.

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