Nonlinear Dyn (2016) 84:977–990
DOI 10.1007/s11071-015-2543-x

ORIGINAL PAPER

Effect compensation of the presence of a time-dependent interior delay on the stabilization of the rotating disk-beam system

Boumediene Chentouf

Received: 19 April 2015 / Accepted: 26 November 2015 / Published online: 29 December 2015

Abstract In this article, we investigate the compensation problem of the effect of the interior time-dependent delay on the stabilization of a rotating disk-beam system. The physical system consists of a flexible beam free at one end, and attached to the center of the rotating disk whose angular velocity is time-varying. Assuming that a time-dependent interior delay is present in the system, we introduce a dynamic boundary force control at the free end of the beam and a torque control on the disk. Then, we show the destabilizing effect of the delay is compensated. Specifically, it is shown that the presence of such proposed controls assures the exponential stability of the system, provided that some reasonable conditions on the angular velocity of the disk and delay are fulfilled. Numerical examples in the case of constant delay are also provided to highlight the stability result.

Keywords Rotating disk-beam · Interior control · Time-dependent delay · Dissipative boundary force control · Torque control · Stability

List of symbols

ℓ	Length of the beam
ρ	Mass per unit length of the beam
EI	Flexural rigidity of the beam
I_d	Disk's moment of inertia
$y(x,t)$	Beam's displacement at time t with respect to the spatial variable x
$\omega(t)$	Angular velocity of the disk at time t
$\mathcal{U}_I(t)$	Interior control
$\mathcal{U}_F(t)$	Boundary force control
$\mathcal{U}_T(t)$	Torque control
$\alpha \geq 0$	Feedback gain of the interior control
$\beta > 0$	Force control feedback gain
$\gamma > 0$	Torque control feedback gain

B. Chentouf (✉)
Department of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36, 123 Al-Khod, Muscat, Oman
e-mail: chentouf@squ.edu.om

1 Introduction

The system depicted in Fig. 1 consists of a flexible robot beam/arm (B), clamped at one end to the center of a disk (D) and free at the other end. We assume that the center of mass of the disk (D) is fixed in an inertial frame and the disk (D) rotates in that frame with a time-varying angular velocity. Hence, the motion of the whole structure is governed by the following equations (see [1] for more details)

$$\begin{cases} \rho y_{tt} + \mathrm{EI} y_{xxxx} = \rho \omega^2(t) y + \alpha \mathcal{U}_I(t), & x \in (0,\ell), t > 0, \\ y(0,t) = y_x(0,t) = y_{xx}(\ell,t) = 0, & t > 0, \\ \mathrm{EI} y_{xxx}(\ell,t) = \beta \mathcal{U}_F(t), & t > 0, \\ \dot{\omega}(t) = \dfrac{-\gamma \mathcal{U}_T(t) - 2\rho\,\omega(t) \displaystyle\int_0^\ell y y_t \mathrm{d}x}{I_d + \rho \displaystyle\int_0^\ell y^2 \mathrm{d}x}, & t > 0, \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in (0,\ell), \\ \omega(0) = \omega_0 \in \mathbb{R}, & \end{cases} \tag{1}$$

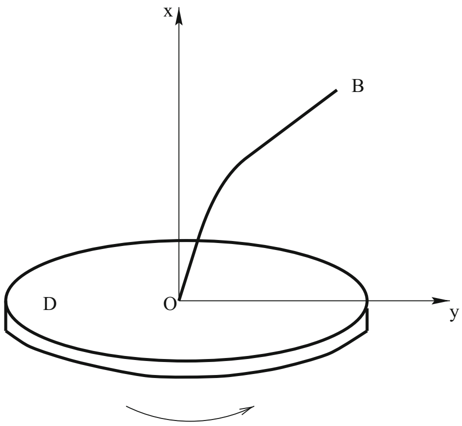

Fig. 1 The disk-beam system

in which $\dot{\omega}(t)$ stands for the time derivative of the angular velocity ω. For sake of clarity, let us recall that y_{tt} is the acceleration, y_t is the velocity, y_x is the rotation, y_{xx} is the bending moment and y_{xxx} is the shear force.

After the pioneer work of Baillieul & Levi [1], a burst of research activity occurred and many papers appeared in the context of stabilization of the system (1) (see, e.g., [2,5,6,11,24–26,33] and the references therein). The most recent works are those related to delay systems, in which the author showed that the effect of the delay occurring in the boundary force control can be compensated through the action of a boundary force control [7]. In fact, motivated by the presence of time delays in applications for several reasons and from different sources, the author recently considered the system (1) without damping ($\alpha = 0$) and established an exponential stabilization result despite the presence of a boundary delay term in the force control $\mathcal{U}_{\mathrm{f}}(t)$ [7]. More precisely, the following delay feedback law has been proposed in [7]

$$\begin{cases} \mathcal{U}_{\mathrm{T}}(t) = \omega(t) - \varpi, \quad \varpi \in \mathbb{R} \\ \mathcal{U}_{\mathrm{f}}(t) = y_t(\ell, t) + \frac{\sigma}{\beta} y_t(\ell, t-\tau), \quad \sigma \in \mathbb{R}, \end{cases}$$

and the closed-loop system, under the presence of the delay $\tau > 0$ but without damping ($\alpha = 0$), is shown to be exponentially stable, provided that $|\sigma| < \beta$ and ϖ is small enough. It turned out that this outcome can be extended even in the more complicated case of time-dependent delay $\tau(t)$ [8].

In the present paper, we suppose that a positive time-dependent delay $\tau(t) > 0$ occurs in the interior control $\mathcal{U}_{\mathrm{I}}(t)$ and the natural question arises: Is it possible to eliminate the effect of such a delay via the action of a boundary control $\mathcal{U}_{\mathrm{f}}(t)$ so that the system (1) remains exponentially stable? As the reader may know, time-dependent time delays arise in a natural way in most controlled systems since the controller needs a certain time to monitor the state of the system, and based on its observations, it makes any necessary adjustments to the system. Obviously, such an adjustment can never be conducted neither instantaneously nor uniformly, and a fortiori a positive time-dependent delay occurs between any observation and the controller action.

The main contribution of this work is to provide an affirmative answer to the above question and show that the feedback law

$$\begin{cases} \mathcal{U}_{\mathrm{I}}(t) = y_t(x, t-\tau(t)), \\ \mathcal{U}_{\mathrm{F}}(t) = y_t(\ell, t) + \dfrac{1}{\beta} e_1^{\top} w(t), \\ \dot{w}(t) = M w(t) + e_2 y_t(\ell, t), \\ \mathcal{U}_{\mathrm{T}}(t) = \omega(t) - \varpi, \quad \varpi \in \mathbb{R}, \end{cases} \tag{2}$$

can exponentially stabilize the system (1), provided that α is small enough. Here, $w \in \mathbb{R}^n$ is the actuator vector state of the dynamic boundary force control $\mathcal{U}_{\mathrm{F}}(t)$, M is an $n \times n$ constant matrix and $e_1,\ e_2 \in \mathbb{R}^n$ are constant vectors. This outcome extends the result obtained in [9], where the delay has been assumed to be constant. The proof of our result is based on the utilization of an appropriate Lyapunov function (see [14–18,31] for variants of the Euler-Bernoulli equation). It is also worth mentioning that the incorporation of dynamical property in the control $\mathcal{U}_{\mathrm{F}}(t)$ (see (2)) provides a wide class of exponentially stabilizing controllers as there are extra degrees of freedom in designing controllers [27].

The rest of the paper is organized as follows: In Sect. 2, we set up the problem as a differential equation in an appropriate Hilbert space. Section 3 is devoted to the proof of stability of solutions of an uncoupled linear system. In Sect. 4, the main result of this work, namely the exponential stability of the system, is stated and proved. This finding is illustrated through numerical examples in Sect. 5. Finally, Sect. 6 concludes the paper with a discussion.

2 Assumptions and problem setup

First of all, one may assume, for sake of simplicity, that $\mathrm{EI} = \rho = \ell = 1$. Indeed, it suffices to change $y(x,t)$ to $y(x\ell, \sqrt{(\ell^4/\mathrm{EI})\rho t})$. Next, we shall assume throughout the remainder of this paper that the following conditions are fulfilled (see [28,29] for heat and wave equations):

There exist constants $d < 1$, $\tau_0 > 0$, and $\bar{\tau} > 0$ such that

$$\tau \in W^{2,\infty}[0, T], \forall T > 0; \tag{3}$$

$$0 < \tau_0 \le \tau(t) \le \bar{\tau}, \ \forall t > 0; \tag{4}$$

$$\dot{\tau}(t) \le d < 1. \tag{5}$$

Furthermore, we suppose, as in [27], that the actuator w satisfies the followings:

H.I: The eigenvalues of the matrix M have negative real parts, and the triplet (M, e_2, e_1) is both observable and controllable.

H.II: The actuator transfer function

$$Y(s) = \beta + e_1^\top (sI - M)^{-1} e_2$$

is strictly positive in the sense that there exists a constant $\hat{\beta} > 0$ such that $\beta > \hat{\beta}$ and $\Re\{Y(ir)\} > \hat{\beta}$, for any $r \in \mathbb{R}$.

Remark 1 As mentioned in [27], It follows from the hypotheses **H.I-H.II** and Kalman-Yakubovich-Popov Lemma [4] that given any $n \times n$ symmetric positive-definite matrix N, there exists an $n \times n$ symmetric positive-definite matrix Q, a constant vector $p \in \mathbb{R}^n$ and $\zeta > 0$ sufficiently small such that:

$$M^\top Q + QM = -pp^\top - \zeta N, \tag{6}$$

$$Qe_2 - \frac{e_1}{2} = \sqrt{\beta - \hat{\beta}}\, p. \tag{7}$$

Now, we invoke the standard change in variables [12]

$$u(x, \eta, t) = y_t(x, t - \eta\tau(t)), \ x, \ \eta \in \Omega = (0, 1),$$

so that the closed-loop system can be brought to the following form

$$\begin{cases} y_{tt} + y_{xxxx} = \omega^2(t) y + \alpha u(x, 1, t), \\ y(0, t) = y_x(0, t) = y_{xx}(1, t) = 0, \\ y_{xxx}(1, t) = \beta y_t(1, t) + e_1^\top w(t), \\ \dot{w}(t) = Mw(t) + e_2 y_t(1, t), \\ \tau(t) u_t(x, \eta, t) + (1 - \dot{\tau}(t)\eta) u_\eta(x, \eta, t) = 0, \\ \dot{\omega}(t) = \dfrac{-\gamma(\omega(t) - \varpi) - 2\,\omega(t) \displaystyle\int_0^1 y y_t \mathrm{d}x}{I_\mathrm{d} + \displaystyle\int_0^1 y^2 \mathrm{d}x}, \\ y(x, 0) = y_0(x), \ \ y_t(x, 0) = y_1(x), \\ u(x, \eta, 0) = f(x, -\eta\tau(0)), \\ w(0) = w_0 \in \mathbb{R}^n, \ \ \omega(0) = \omega_0 \in \mathbb{R}. \end{cases} \tag{8}$$

Let us recall (see [28,29]) that if $t < \tau(t)$, then $y_t(x, t - \tau(t))$ is in the past, and hence, an initial value in the past has to be provided. To do so, we use (5) as well as the mean-value theorem to get $t - \tau(t) > -\tau(0)$. This justifies the initial data $y_t(x, t - \tau(0)) = f(x, t - \tau(0))$ with $(x, t) \in (0, 1) \times (0, \tau(0))$ and so $u(x, \eta, 0) = f(x, -\eta\tau(0))$.

Subsequently, given a real-valued function

$$f : \Omega = (0, 1) \to \mathbb{R},$$

let us recall that

$$L^2(\Omega) = \left\{ f \text{ is measurable and } \int_0^1 |f(x)|^2 \,\mathrm{d}x < \infty \right\}$$

is a Hilbert space endowed with its usual norm

$$\|f\|_{\mathrm{L}^2(\Omega)} = \left(\int_0^1 |f(x)|^2 \,\mathrm{d}x \right)^{1/2},$$

whereas the Sobolev space

$$H^n(\Omega) = \left\{ f : \Omega \to \mathbb{R}; \ f^{(n)} \in L^2(\Omega), \text{for } n \in \mathbb{N} \right\}$$

is equipped with the standard norm

$$\|f\|_{\mathrm{H}^n(\Omega)} = \sum_{i=0}^{i=n} \|f^{(i)}\|_{\mathrm{L}^2(\Omega)}.$$

With regard to our system (8), we adopt the following notations: Let

$$\hat{\Omega} = (0, 1) \times (0, 1),$$

and

$$H_c^m = \left\{ g \in H^m(\Omega); g(0) = g_x(0) = 0 \right\}, \ m = 2, 3, \ldots$$

Moreover, we assume that

$$|\varpi| < 3. \tag{9}$$

Furthermore, consider the state space $\mathcal{X}$ defined by

$$\mathcal{X} = H_c^2 \times L^2(\Omega) \times L^2(\hat{\Omega}) \times \mathbb{R}^n \times \mathbb{R} = \mathcal{Y} \times \mathbb{R}$$

equipped with the following real inner product

$$\begin{aligned} &\langle (y, z, u, w, \omega), (\tilde{y}, \tilde{z}, \tilde{u}, \tilde{w}, \tilde{\omega}) \rangle_{\mathcal{X}} \\ &\quad = \langle (y, z, u, w), (\tilde{y}, \tilde{z}, \tilde{u}, \tilde{w}) \rangle_{\mathcal{Y}} + \omega\tilde{\omega} \\ &\quad = \int_0^1 \left(y_{xx}\tilde{y}_{xx} - \varpi^2 y\tilde{y} + z\tilde{z} \right) \mathrm{d}x + \iint_{\hat{\Omega}} u\tilde{u} \,\mathrm{d}x\mathrm{d}\eta \\ &\qquad + 2\tilde{w}^\top Q w + \omega\tilde{\omega}. \end{aligned} \tag{10}$$

It is worth mentioning that the condition (9) is imposed in order to assure that the state space $\mathcal{X}$ with the inner product (10) is a Hilbert space (see [7] for more details).

Thereafter, the system (8) can be written in $\mathcal{X}$ as follows

$$\begin{cases} \Phi_t(t) = \left[\begin{pmatrix} \mathcal{A}(t) & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{F}\right]\Phi(t), \\ \Phi(0) = \Phi_0 = (y_0, y_1, f(\cdot, -\eta\tau(0)), w_0, \omega_0), \end{cases} \tag{11}$$

where $z = y_t$, $\Phi(t) = (y, z, u, w, \omega)$ and $\mathcal{A}(t)$ is a time-dependent linear operator in $\mathcal{Y}$ defined by

$$\begin{cases} \mathcal{D}(\mathcal{A}(t)) = \Big\{(y, z, u, w) \in H_c^4 \times H_c^2 \times L^2(\hat{\Omega}) \times \mathbb{R}^n; \\ \qquad u_x \in L^2(\hat{\Omega}),\ z = u(\cdot, 0),\ y_{xx}(1) = 0, \\ \qquad y_{xxx}(1) = \beta z(1) + e_1^\top w\Big\} \\ \mathcal{A}(t)(y, z, u, w) = \Big(z, -y_{xxxx} + \varpi^2 y + \alpha u(\cdot, 1), \\ \qquad \dfrac{\eta\dot{\tau}(t) - 1}{\tau(t)} u_\eta, Mw + e_2 u(1, 0)\Big), \end{cases} \tag{12}$$

while $\mathcal{F}$ is a nonlinear operator given by

$$\mathcal{F}(y, z, u, w, \omega) = \Big(0, (\omega^2 - \varpi^2)y, 0, 0, \frac{-\gamma(\omega - \varpi) - 2\omega < y, z >_{L^2(\Omega)}}{I_d + \|y\|^2_{L^2(\Omega)}}\Big), \tag{13}$$

for any $(y, z, u, w, \omega) \in \mathcal{X}$.

We end this section by stating the following remark.

Remark 2 We point out that the feedback gain α of the delayed term is assumed to be nonnegative for sake of simplicity. Notwithstanding, one could pick $\alpha \in \mathbb{R}$ and replace α by $|\alpha|$ from the very beginning of the article (see [7] for a similar situation).

3 Uncoupled linear system

This section is intended to deal with an uncoupled linear system associated with the global nonlinear system (11), namely, let us consider

$$\begin{cases} \varphi_t(t) = \mathcal{A}(t)\varphi(t), \\ \varphi(0) = \varphi_0, \end{cases} \tag{14}$$

where $\varphi = (y, z, u, w)$, $\varphi_0 = (y_0, y_1, f(\cdot, -\eta\tau(0)), w_0)$ and $\mathcal{A}(t)$ is the time-dependent linear operator defined in (12).

As pointed out in [28,29], a general theory of existence and uniqueness of solutions for equations of type (14) is already available in the literature (see, for instance, [10,19–22,30]). One simple way, among others, to prove existence and uniqueness results is to use the following result:

Theorem 1 *Let $A(t)$ be a time-dependent operator on a Hilbert space H such that:*

(i) For all $t \in [0, T]$, the operator $A(t)$ generates a strongly continuous semigroup on H and the family $\{A(t) : t \in [0, T]\}$ is stable with stability constants C and m independent of the time t.

(ii) $\mathcal{D}((A)(t)) = \mathcal{D}(A(0))$, for $t \geq 0$.

(iii) The operator $\dfrac{\mathrm{d}}{\mathrm{d}t}A(t)$ belongs to

$L^\infty([0, T], B(\mathcal{D}(A(0)), H))$,

which is the space of equivalent classes of essentially bounded, strongly measurable functions from $[0, T]$ into the set $B(\mathcal{D}(A(0)), H)$ of bounded operators from $\mathcal{D}(A(0))$ into H.

Then, the Cauchy problem

$$\begin{cases} v_t(t) = A(t)v(t), \\ v(0) = v_0, \end{cases}$$

has a unique mild solution $v \in C([0, T], H)$, for each initial data v_0 in H. Moreover, for all $t \in [0, T]$, there exists a positive constant $c(t)$ such that

$$\|v(t)\|_H \leq c(t)\|U\|_H.$$

In turn, if the initial data v_0 belong to $\mathcal{D}(A(t)) = \mathcal{D}(A(0))$, then the above Cauchy problem has a unique strong solution $v \in C([0, T], \mathcal{D}(A(0)) \cap C^1([0, T], H)$.

Whereupon, our immediate task is to check that the assumptions of Theorem 1 are verified by our operator $\mathcal{A}(t)$. This will be the objective of the following subsection.

3.1 Well-posedness of the problem (14)

Let us begin by claiming that it is obvious that

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)),\ \ t \geq 0. \tag{15}$$

Next, one can readily check that the operator $\mathcal{A}(t)$ is closed and densely defined in $\mathcal{Y}$ (one has merely to argue as in [13,28,29]). Furthermore, we define on

$$\mathcal{Y} = H_c^2 \times L^2(\Omega) \times L^2(\hat{\Omega}) \times \mathbb{R}^n$$

the time-dependent Kato's inner product

$$\begin{aligned}&\langle (y,z,u,w),(\tilde{y},\tilde{z},\tilde{u},\tilde{w})\rangle_t\\ &= \int_0^1 \left(y_{xx}\tilde{y}_{xx} - \varpi^2 y\tilde{y} + z\tilde{z}\right) \mathrm{d}x\\ &+K\tau(t)\iint_{\hat{\Omega}} u\tilde{u}\,\mathrm{d}x\,\mathrm{d}\eta + 2\tilde{w}^\top Q w, \end{aligned}\tag{16}$$

where K is a positive constant. Clearly, thanks to the assumptions (4) and (9), this new inner product is equivalent to that of (10). Thereafter, using exactly the same arguments as in [28,29], it follows from (3)–(4) that for any $\varphi = (y,z,u,w) \in \mathcal{Y}$, we have

$$\frac{\|\varphi\|_t}{\|\varphi\|_s} \le \exp\left(\frac{C}{2\tau_0}|t-s|\right),\tag{17}$$

for any $s,\ t \in [0,T]$ and for some positive constant C. In fact, it is easy to check that (16) yields

$$\begin{aligned}&\|\varphi\|_t - \exp\left(\frac{C}{\tau_0}|t-s|\right)\|\varphi\|_s =\\ &\left(1-\exp\left(\frac{C}{\tau_0}|t-s|\right)\right)\int_0^1\left(y_t^2 + y_{xx}^2 - \varpi^2 y^2\right)\mathrm{d}x\\ &+2\left(1-\exp\left(\frac{C}{\tau_0}|t-s|\right)\right) w^\top Q w\\ &+K\left(\tau(t) - \tau(s)\exp\left(\frac{C}{\tau_0}|t-s|\right)\right)\\ &\times \iint_{\hat{\Omega}} y_t^2(x,t-\eta\tau(t))\,\mathrm{d}x\,\mathrm{d}\eta,\end{aligned}$$

for any $s,\ t \in [0,T]$. Furthermore, we clearly have

$$1-\exp\left(\frac{C}{\tau_0}|t-s|\right) < 0$$

whenever c is positive and $s,\ t \in [0,T]$. On the other hand, it follows from the mean-value theorem that there exists $r \in (s,t)$ such that $\tau(t) - \tau(s) = \tau'(r)(t-s)$. This, together with (3)–(4), implies the desired inequality (17).

Our aim now is to show that for each fixed $t > 0$, the operator $\mathcal{A}(t)$ defined by (12) is dissipative modulo a translation. To this end, let $\varphi = (y,z,u,w) \in \mathcal{D}(\mathcal{A}(t))$. Then, in the light of (10) and (12), a simple integration yields

$$\begin{aligned}\langle \mathcal{A}(t)\varphi,\varphi\rangle_t &= \int_0^1 (y_{xx}z_{xx} - y_{xxxx}z)\,\mathrm{d}x\\ &+\alpha\int_0^1 u(x,1)z\,\mathrm{d}x + K(\dot{\tau}(t)-1)\iint_{\hat{\Omega}} u_\eta u\,\mathrm{d}x\,\mathrm{d}\eta\\ &+\langle (QM+M^\top Q)w + 2Qe_2u(1,0),w\rangle_{\mathbb{R}^n}\\ &=\alpha\int_0^1 u(x,1)z\,\mathrm{d}x + \frac{K}{2}\int_0^1 u^2(x,0)\mathrm{d}x\\ &-y_{xxx}(1)z(1) + \frac{K}{2}(\dot{\tau}(t)-1)\int_0^1 u^2(x,1)\,\mathrm{d}x\\ &-\frac{K}{2}\dot{\tau}(t)\iint_{\hat{\Omega}} u^2\,\mathrm{d}x\,\mathrm{d}\eta\\ &+\langle (QM+M^\top Q)w + 2Qe_2u(1,0),w\rangle_{\mathbb{R}^n}\\ &=-\beta z^2(1) - \beta e_1^\top w z(1) + \alpha\int_0^1 u(x,1)z\,\mathrm{d}x\\ &+\frac{K}{2}\int_0^1 z^2\mathrm{d}x + \frac{K}{2}(\dot{\tau}(t)-1)\int_0^1 u^2(x,1)\,\mathrm{d}x\\ &-\frac{K}{2}\dot{\tau}(t)\iint_{\hat{\Omega}} u^2\,\mathrm{d}x\,\mathrm{d}\eta\\ &+\langle (QM+M^\top Q)w + 2Qe_2u(1,0),w\rangle_{\mathbb{R}^n},\end{aligned}\tag{18}$$

which, together with (6)–(7), implies that

$$\begin{aligned}\langle \mathcal{A}(t)\varphi,\varphi\rangle_t &= -\hat{\beta}z^2(1) - \zeta w^\top N w + \alpha\int_0^1 u(x,1)z\,\mathrm{d}x\\ &+\frac{K}{2}\int_0^1 z^2\mathrm{d}x + \frac{K}{2}(\dot{\tau}(t)-1)\int_0^1 u^2(x,1)\,\mathrm{d}x\\ &-\frac{K}{2}\dot{\tau}(t)\iint_{\hat{\Omega}} u^2\,\mathrm{d}x\,\mathrm{d}\eta - \left[\sqrt{\beta-\hat{\beta}}z(1) - w^\top p\right]^2.\end{aligned}\tag{19}$$

Next, invoking Young's inequality and (5), the identity (19) leads to

$$\begin{aligned}&\langle \mathcal{A}(t)(y,z,u),(y,z,u)\rangle_t \le -\hat{\beta}z^2(1) - \zeta w^\top N w\\ &-\left[\sqrt{\beta-\hat{\beta}}z(1) - w^\top p\right]^2\\ &+\left(\frac{K}{2} + \frac{\alpha}{2\sqrt{1-d}}\right)\int_0^1 z^2\mathrm{d}x\\ &+\frac{\sqrt{1-d}}{2}\left(\alpha - K\sqrt{1-d}\right)\int_0^1 u^2(x,1)\,\mathrm{d}x\\ &+\frac{K}{2}F(t)\iint_{\hat{\Omega}} u^2\,\mathrm{d}x\,\mathrm{d}\eta,\end{aligned}\tag{20}$$

where

$$F(t) = \frac{\sqrt{1+\dot{\tau}(t)^2}}{2\tau(t)} > 0.$$

It follows from (20) that

$$\langle \mathcal{A}(t)\varphi, \varphi\rangle_t \le -\hat{\beta} z^2(1) - \zeta w^\top N w - \left[\sqrt{\beta-\hat{\beta}}z(1) - w^\top p\right]^2 + \frac{\sqrt{1-d}}{2}\left(\alpha - K\sqrt{1-d}\right)\int_0^1 u^2(x,1)\,\mathrm{d}x + \left(\frac{K}{2}(1+F(t)) + \frac{\alpha}{2\sqrt{1-d}}\right)\|\varphi\|_t^2,$$

and choosing

$$K \ge \frac{\alpha}{\sqrt{1-d}}, \tag{21}$$

we can claim that the operator

$$\mathcal{B}(t) = \mathcal{A}(t) - \Upsilon(t) I$$

is dissipative, where

$$\Upsilon(t) = \frac{K}{2}(1+F(t)) + \frac{\alpha}{2\sqrt{1-d}} > 0. \tag{22}$$

The task ahead is to show that the operator $\lambda I - \mathcal{A}(t)$ is onto $\mathcal{Y}$, for some $\lambda > 0$. To do so, we shall use the well-known Lax-Milgram result:

Theorem 2 *(Lax-Milgram) [3] Assume that $a(u, v)$ is a continuous coercive bilinear form on a Hilbert space H. Then, given any $\phi \in H$, there exists a unique element $u \in H$ such that $a(u, v) =< \phi, v >$, $\forall v \in H$.*

First, let $(f, g, h, r) \in \mathcal{Y}$, and let us seek $(y, z, u, w) \in \mathcal{D}(\mathcal{A}(t))$ such that $(\lambda I - \mathcal{A}(t))(y, z, u, w) = (f, g, h, r)$, that is,

$$\begin{cases} y_{xxxx} + (\lambda^2 - \varpi^2)y - \alpha u(x,1) = \lambda f + g, \\ z = \lambda y - f, \\ (1-\dot{\tau}(t)\eta)u_\eta + \lambda\tau(t)u = \tau(t)h, \\ y(0) = y_x(0) = y_{xx}(1) = 0, \\ y_{xxx}(1) = \beta z(1) + e_1^\top w, \\ (\lambda I - M)w = \left[e_2(\lambda y(1) - f(1)) + r\right], \\ z = u(\cdot, 0). \end{cases} \tag{23}$$

Solving the equation of u in the above system, we get

$$u(x,\eta) = (\lambda y(x) - f(x))\mathrm{e}^{-\hat{\tau}\lambda\eta} + \hat{\tau}\int_0^\eta \mathrm{e}^{-\hat{\tau}\lambda(\eta-v)}h(x,v)\,dv, \quad \text{if } \tau(t) = \hat{\tau} \text{ (constant)}, \tag{24}$$

or

$$u(x,\eta) = (\lambda y(x) - f(x))\mathrm{e}^{\lambda q(\eta,t)} + \mathrm{e}^{\lambda q(\eta,t)}\int_0^\eta \frac{h(x,v)}{1-\dot{\tau}(t)v}\mathrm{e}^{-\lambda q(v,t)}\,dv, \quad \text{if } \tau(t) \neq 0, \tag{25}$$

where $q(\eta, t) = \dfrac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)x)$. Using (23), we have

$$w = (\lambda I - M)^{-1}\left[e_2(\lambda y(1) - f(1)) + r\right]$$

and in the light of (24), one has only to seek $y \in H_c^4$ satisfying

$$\begin{cases} y_{xxxx} + \left(\lambda^2 - \varpi^2 - \alpha\lambda\mathrm{e}^{-\hat{\tau}\lambda}\right)y = \alpha Y(x) \\ \quad +\lambda f + g, \quad \text{if } \tau(t) = \hat{\tau} \text{ (constant)}, \\ y(0) = y_x(0) = y_{xx}(1) = 0, \\ y_{xxx}(1) = \lambda\left[\beta + e_1^\top(\lambda I - M)^{-1}e_2\right]y(1) \\ \quad -\left[\beta + e_1^\top(\lambda I - M)^{-1}e_2\right]f(1) \\ \quad + e_1^\top(\lambda I - M)^{-1}r, \end{cases} \tag{26}$$

in which

$$Y(x) = -f(x)\mathrm{e}^{-\hat{\tau}\lambda} + \hat{\tau}\int_0^1 \mathrm{e}^{-\hat{\tau}\lambda(1-v)}h(x,v)\,dv, \quad \text{if } \tau(t) = \hat{\tau} \text{ (constant)}.$$

In turn, using (25)), one should find $y \in H_c^4$ satisfying

$$\begin{cases} y_{xxxx} + \left(\lambda^2 - \varpi^2 - \alpha\lambda\mathrm{e}^{\lambda q(1,t)}\right)y = \alpha Z(x) \\ \quad + \lambda f + g, \quad \text{if } \tau(t) \neq 0, \\ y(0) = y_x(0) = y_{xx}(1) = 0, \\ y_{xxx}(1) = \lambda\left[\beta + e_1^\top(\lambda I - M)^{-1}e_2\right]y(1) \\ \quad -\left[\beta + e_1^\top(\lambda I - M)^{-1}e_2\right]f(1) \\ \quad + e_1^\top(\lambda I - M)^{-1}r, \end{cases} \tag{27}$$

where

$$Z(x) = -f(x)\mathrm{e}^{\lambda\mathrm{e}^{q(1,t)}} + \mathrm{e}^{\lambda q(1,t)}\int_0^1 \frac{h(x,v)}{1-\dot{\tau}(t)v}\mathrm{e}^{-\lambda q(v,t)}\,dv, \quad \text{if } \tau(t) \neq 0.$$

Multiplying the first equation in (26) by $\phi \in H_c^2$, we get: If $\tau(t) = \hat{\tau}$ is constant, then

$$\begin{aligned} \int_0^1 \left(y_{xx}\phi_{xx} + [\lambda^2 - \varpi^2 - \alpha\lambda\mathrm{e}^{-\hat{\tau}\lambda}]y\phi\right)\,\mathrm{d}x &= \lambda\left[\beta + e_1^\top(\lambda I - M)^{-1}e_2\right]y(1)\phi(1) \\ &\quad + \int_0^1 (\alpha Y(x) + \lambda f + g)\,\phi\mathrm{d}x + e_1^\top(\lambda I - M)^{-1}r\phi(1) \\ &\quad + \left[\beta + e_1^\top(\lambda I - M)^{-1}e_2\right]f(1)\phi(1), \end{aligned} \tag{28}$$

which can be written in the variational form

$$\mathcal{L}_1(y, \phi) = \Lambda_1(\phi),$$

where

$$\mathcal{L}_1 : H_c^2 \times H_c^2 \longrightarrow \mathbb{R}$$

is a bilinear form given by

$$\mathcal{L}_1(y,\phi) = \int_0^1 \Big(y_{xx}\phi_{xx} + [\lambda^2 - \varpi^2 - \alpha\lambda e^{-\hat{\tau}\lambda}] y\phi \Big)\, dx$$
$$+\lambda \Big[\beta + e_1^\top(\lambda I - M)^{-1} e_2\Big] y(1)\phi(1),$$

and

$$\Lambda_1 : H_c^2 \longrightarrow \mathbb{R}$$
$$\phi \longmapsto \Lambda_1(\phi) =$$
$$\int_0^1 (\alpha Y(x) + \lambda f + g)\,\phi dx + e_1^\top(\lambda I - M)^{-1} r\phi(1)$$
$$+\Big[\beta + e_1^\top(\lambda I - M)^{-1} e_2\Big] f(1)\phi(1).$$

Using similar arguments for (27), we obtain: If $\tau(t) \neq 0$, then

$$\int_0^1 \Big(y_{xx}\phi_{xx} + \Big(\lambda^2 - \varpi^2 - \alpha\lambda e^{\lambda q(1,t)}\lambda\Big) y\phi \Big)\, dx$$
$$= \lambda \Big[\beta + e_1^\top(\lambda I - M)^{-1} e_2\Big] y(1)\phi(1)$$
$$+ \int_0^1 (\alpha Z(x) + \lambda f + g)\,\phi dx + e_1^\top(\lambda I - M)^{-1} r\phi(1)$$
$$+ \Big[\beta + e_1^\top(\lambda I - M)^{-1} e_2\Big] f(1)\phi(1), \tag{29}$$

which leads to

$$\mathcal{L}_2(y,\phi) = \Lambda_2(\phi),$$

where the bilinear form

$$\mathcal{L}_2 : H_c^2 \times H_c^2 \longrightarrow \mathbb{R}$$

is defined by

$$\mathcal{L}_2(y,\phi) = \int_0^1 \Big(y_{xx}\phi_{xx} + [\lambda^2 - \varpi^2 - \alpha\lambda e^{-\hat{\tau}\lambda}] y\phi \Big)\, dx$$
$$+\lambda \Big[\beta + e_1^\top(\lambda I - M)^{-1} e_2\Big] y(1)\phi(1),$$

and

$$\Lambda_2 : H_c^2 \longrightarrow \mathbb{R}$$
$$\phi \longmapsto \Lambda_2(\phi) = \int_0^1 (\alpha Z(x) + \lambda f + g)\,\phi dx$$
$$+\Big[\beta + e_1^\top(\lambda I - M)^{-1} e_2\Big] f(1)\phi(1)$$
$$+ e_1^\top(\lambda I - M)^{-1} r\phi(1).$$

Invoking the conditions (3)–(7) and (9), one can check that for $\lambda > 0$ large enough, $\mathcal{L}_1$ and $\mathcal{L}_2$ are continuous coercive bilinear forms on $H_c^2 \times H_c^2$, whereas Λ_1 and Λ_2 are continuous linear forms on H_c^2. Applying Theorem 2, we deduce that the operator $\lambda I - \mathcal{A}(t)$ is onto $\mathcal{Y}$ for $\lambda > 0$ large enough and so is the operator

$$\lambda I - \mathcal{B}(t) = (\lambda + \Upsilon(t))I - \mathcal{A}(t)$$

since $\Upsilon(t) > 0$ (see (22)).

Moreover, using (3)–(4), we obtain as in [28,29]

$$\frac{d}{dt}\mathcal{B}(t) \in L_*^\infty([0,T], B(\mathcal{D}(\mathcal{A}(0)); \mathcal{Y})), \tag{30}$$

the space of equivalence classes of essentially bounded and strongly measurable functions from $[0, T]$ into the set $B(\mathcal{A}(0)); \mathcal{Y})$ of bounded operators.
Summarizing the above properties and thanks to Theorem 1 (see also [10,30]), we deduce that for any initial data $\bar{\varphi}_0 \in \mathcal{Y}$, the Cauchy problem

$$\begin{cases} \bar{\varphi}_t = \mathcal{B}(t)\bar{\varphi}, \\ \bar{\varphi}(0) = \bar{\varphi}_0, \end{cases} \tag{31}$$

has a unique solution $\bar{\varphi} \in C([0,\infty); \mathcal{Y})$. In turn, if $\bar{\varphi}_0 \in \mathcal{D}(\mathcal{B}(t))$, then necessarily the solution $\bar{\varphi}$ belongs to $C([0,\infty); \mathcal{D}(\mathcal{A}(t))) \cap C^1([0,\infty); \mathcal{Y})$. This existence and uniqueness result goes for the Cauchy problem involving the operator $\mathcal{A}(t)$ instead of $\mathcal{B}(t)$, thanks to the change in variables $\varphi(t) = e^{\Upsilon(t)}\bar{\varphi}(t)$. We have thus proved the following result

Proposition 1 *For any initial data $\varphi_0 \in \mathcal{Y}$, the Cauchy problem (14) has a unique solution $\varphi(t) \in C([0,\infty); \mathcal{Y})$. In turn, if $\varphi_0 \in \mathcal{D}(\mathcal{A}(t))$, then necessarily the solution $\varphi(t)$ belongs to $C([0,\infty); \mathcal{D}(\mathcal{A}(t))) \cap C^1([0,\infty); \mathcal{Y})$.*

3.2 Exponential stability of the system (14)

In this section, we shall state and prove the exponential stability result of the system (14):

Theorem 3 *Under the assumption (9), namely $|\varpi| < 3$, there exists a positive constant α_0 such that for any $\alpha < \alpha_0$, the semigroup generated by the operator $\mathcal{A}(t)$, defined by (12), is exponentially stable in $\mathcal{Y}$. Whereupon, there exist uniform positive constants M_0 and κ_0 such that the solution of the Cauchy problem (14) obeys to the following estimate*

$$\|\varphi(t)\|_{\mathcal{H}} \le M_0 e^{-\kappa_0 t}\|\varphi_0\|_{\mathcal{H}}, \quad \forall t \ge 0. \tag{32}$$

For sake of clarity, we are going to establish some preliminary results which are needed for the proof of Theorem 3. To proceed, we pick up φ_0 in $\mathcal{D}(\mathcal{A}(t))$ and hence the solution $\varphi = (y, y_t, y_t(\cdot, t - \eta\tau(t)), w)$ of the system (14), namely,

$$\begin{cases} y_{tt} + y_{xxxx} = \varpi^2 y + \alpha y_t(x, t - \tau(t)), \\ y(0, t) = y_x(0, t) = y_{xx}(1, t) = 0, \\ y_{xxx}(1, t) = \beta y_t(1, t) + e_1^\top w(t), \\ \dot{w}(t) = M w(t) + e_2 y_t(1, t), \\ \frac{\partial y_t}{\partial t}(x, t - \eta\tau(t)) = \frac{\dot{\tau}(t)\eta - 1}{\tau(t)} \frac{\partial y_t}{\partial \eta}(x, t - \eta\tau(t)), \end{cases} \tag{33}$$

is regular in the sense that

$$\varphi \in C^1(\mathbb{R}^+; \mathcal{Y}) \cap C(\mathbb{R}^+; \mathcal{D}(\mathcal{A}(t))).$$

On the other hand, in order to make the computations manageable, let us introduce the following functionals:

$$E_1(t) = \frac{1}{2}\int_0^1 \left(y_t^2 + y_{xx}^2 - \varpi^2 y^2\right) \mathrm{d}x + w^\top Q w + \frac{C_1}{2} K\tau(t) \iint_{\hat{\Omega}} y_t^2(x, t - \eta\tau(t)) \,\mathrm{d}x \,\mathrm{d}\eta, \tag{34}$$

$$E_2(t) = 2C_2 \int_0^1 (x y_t y_x) \,\mathrm{d}x, \tag{35}$$

$$\frac{E_3(t)}{K} = C_3\tau(t) \iint_{\hat{\Omega}} \mathrm{e}^{-2\delta\eta\tau(t)} y_t^2(x, t - \eta\tau(t)) \,\mathrm{d}x \,\mathrm{d}\eta, \tag{36}$$

where K is a positive constant satisfying (21), and C_i is a positive constant to be determined, for each $i = 1, 2, 3$. In turn, δ is an arbitrary positive constant.

The first result of this section is

Proposition 2 *Let $\varphi = (y, y_t, y_t(\cdot, t - \eta\tau(t)), w)$ be the regular solution of (14). Then, for any $t \geq 0$ and for any positive constant A_1, we have the following estimates*

$$\min\{1, C_1\}\|\varphi\|_t^2 \leq 2E_1(t) \leq \max\{1, C_1\}\|\varphi\|_t^2, \tag{37}$$

$$\dot{E}_1(t) \leq -\hat{\beta} y_t^2(1, t) - \left[\sqrt{\beta - \hat{\beta}}\, y_t(1, t) - w^\top p\right]^2 - \zeta w^\top N w + \frac{1}{2}(\alpha A_1 + C_1 K) \int_0^1 y_t^2 \,\mathrm{d}x + \frac{1}{2}\left(\frac{\alpha}{A_1} - C_1 K(1 - d)\right) \int_0^1 y_t^2(x, t - \tau(t)) \,\mathrm{d}x. \tag{38}$$

Proof The proof of (37) is obvious. With regard to the second estimate (38), one has merely to argue as for (19). Indeed, differentiating (34) and using (33), we obtain after integration by parts

$$\dot{E}_1(t) \leq -\hat{\beta} y_t^2(1, t) - \left[\sqrt{\beta - \hat{\beta}}\, y_t(1, t) - w^\top p\right]^2 - \zeta w^\top N w + \alpha \int_0^1 y_t y_t(x, t - \tau(t)) \,\mathrm{d}x + \frac{C_1 K}{2} \int_0^1 y_t^2 \mathrm{d}x + \frac{C_1 K}{2}(\dot{\tau}(t) - 1) \int_0^1 y_t^2(x, t - \tau) \,\mathrm{d}x.$$

Now, it suffices to use (5) and the following Young's inequality

$$\int_0^1 y_t y_t(x, t - \tau(t)) \,\mathrm{d}x \leq \frac{A_1}{2} \int_0^1 y_t^2 \,\mathrm{d}x + \frac{1}{2A_1} \int_0^1 y_t^2(x, t - \tau(t)) \,\mathrm{d}x,$$

to get the desired estimate.

The estimates related to $E_2(t)$ are given below.

Proposition 3 *Let $\varphi = (y, y_t, y_t(\cdot, t - \eta\tau(t)), w)$ be the regular solution of (14). Then, for any $t \geq 0$, and for any positive constants A_2 and A_3, the following estimates hold*

$$|E_2(t)| \leq C_2 \int_0^1 \left(y_t^2 + \frac{1}{2} y_{xx}^2\right) \mathrm{d}x, \tag{39}$$

$$\frac{\dot{E}_2(t)}{C_2} \leq (1 + \beta A_2) y_t^2(1, t) + \alpha A_3 \int_0^1 y_t^2(x, t - \tau) \,\mathrm{d}x + \left(\frac{\beta}{A_2} + \frac{\alpha}{2A_3} + \frac{1}{A_4} + \frac{\varpi^2}{3} - 3\right) \int_0^1 y_{xx}^2 \,\mathrm{d}x - \int_0^1 y_t^2 \mathrm{d}x + A_4 \left(e_1^\top w\right)^2. \tag{40}$$

Proof Establishing (39) is a direct consequence of applying Young's inequality

$$2C_2 \int_0^1 (x y_t y_x) \,\mathrm{d}x \leq C_2 \int_0^1 \left(y_t^2 + y_x^2\right) \mathrm{d}x$$

and the Poincaré's inequality

$$\int_0^1 f_x^2 \,\mathrm{d}x \leq \frac{1}{2} \int_0^1 f_{xx}^2 \,\mathrm{d}x, \quad \forall f \in H_c^2. \tag{41}$$

With regard to the proof of (40), we use once again the equations in (33), integrate by parts, to obtain after invoking the boundary conditions in (33)

$$\frac{\dot{E}_2(t)}{C_2} = y_t^2(1, t) - 2\beta y_x(1, t) y_t(1, t) + \varpi^2 y^2(1, t) - \int_0^1 \left\{y_t^2 + 3y_{xx}^2 + \varpi^2 y^2\right\} \mathrm{d}x - 2y_x(1, t) e_1^\top w + 2\alpha \int_0^1 x y_x(x, t) y_t(x, t - \tau(t)) \,\mathrm{d}x. \tag{42}$$

Applying Young's inequality for the cross-product terms $y_x(1,t)y_t(1,t)$ and $xy_x(x,t)y_t(x,t-\tau)$ and by virtue of (41) as well as the well-known estimates

$$f^2(1) \le \frac{1}{3}\int_0^1 f_{xx}^2\,\mathrm{d}x, \quad f_x^2(1) \le \int_0^1 f_{xx}^2\,\mathrm{d}x, \ \forall f \in H_c^2, \tag{43}$$

the identity (42) yields (40).

We also have similar estimates for $E_3(t)$

Proposition 4 *Let $\varphi = (y, y_t, y_t(\cdot, t-\eta\tau(t)), w)$ be the regular solution of (14). Then, for any $t \ge 0$, we have*

$$E_3(t) \le C_3\|\varphi(t)\|_t^2, \tag{44}$$

$$\frac{\dot{E}_3(t)}{KC_3} \le -(1-d)\mathrm{e}^{-2\delta\bar{\tau}}\int_0^1 y_t^2(x,t-\tau(t))\,\mathrm{d}x -2\delta\tau(t)\mathrm{e}^{-2\delta\bar{\tau}}\iint_{\hat{\Omega}} y_t^2(x,t-\eta\tau(t))\,\mathrm{d}x\,\mathrm{d}\eta + \int_0^1 y_t^2\,\mathrm{d}x. \tag{45}$$

Proof Since (44) is trivial, let us focus on (45). Differentiating (36) and using (33), we obtain

$$\frac{\dot{E}_3(t)}{KC_3} = \dot{\tau}(t)\iint_{\hat{\Omega}} \mathrm{e}^{-2\delta\eta\tau(t)} y_t^2(x,t-\eta\tau(t))\,\mathrm{d}x\,\mathrm{d}\eta -2\delta\tau(t)\dot{\tau}(t)\iint_{\hat{\Omega}} \eta\mathrm{e}^{-2\delta\eta\tau(t)} y_t^2(x,t-\eta\tau(t))\,\mathrm{d}x\,d\eta + \iint_{\hat{\Omega}} \mathrm{e}^{-2\delta\eta\tau(t)}(\dot{\tau}(t)\eta-1) \times\frac{\partial}{\partial\eta}\left(y_t^2(x,t-\eta\tau(t))\right)\mathrm{d}x\,\mathrm{d}\eta. \tag{46}$$

Thanks to a simple integration by parts with respect to η, we have

$$\iint_{\hat{\Omega}} \mathrm{e}^{-2\delta\eta\tau(t)}(\dot{\tau}(t)\eta-1)\frac{\partial}{\partial\eta}\left(y_t^2(x,t-\eta\tau(t))\right)\mathrm{d}x\,\mathrm{d}\eta = (\dot{\tau}(t)-1)\mathrm{e}^{-2\delta\tau(t)}\int_0^1 y_t^2(x,t-\tau(t))\,\mathrm{d}x + \int_0^1 y_t^2\,\mathrm{d}x - \iint_{\hat{\Omega}} \mathrm{e}^{-2\delta\eta\tau(t)}\left[\dot{\tau}(t) - 2\delta\tau(t)(\dot{\tau}(t)\eta-1)\right] \times y_t^2(x,t-\eta\tau(t))\,\mathrm{d}x\,\mathrm{d}\eta.$$

Inserting the above identity in (46) and using (4)–(5) leads to the desired estimate.

Now, we are ready to state and prove the following result.

Proposition 5 *Define, along the regular solution of (14), the functional*

$$\mathcal{E}(t) = E_1(t) + E_2(t) + E_3(t), \tag{47}$$

where E_i, $i = 1, 2, 3$ are defined in (34)–(36). If the condition (9) holds, then

(i) there exist positive constants L_1 and L_2, independent of the initial data φ_0 such that for any $t \ge 0$, we have:

$$L_1\|\varphi(t)\|_t^2 \le \mathcal{E}(t) \le L_2\|\varphi(t)\|_t^2; \tag{48}$$

(ii) the functional $\mathcal{E}(t)$ is uniformly exponentially stable for α small. Specifically, there exist positive constants M, κ, α_0, independent of the initial data φ_0 such that for any $t \ge 0$ and for any $\alpha < \alpha_0$, we have:

$$\mathcal{E}(t) \le M\mathrm{e}^{-\kappa t}\mathcal{E}(0). \tag{49}$$

Proof Combining (47) with the estimates (37), (39) and (44), and using the assumption (9), that is, $|\varpi| < 3$, we get

$$\mathcal{E}(t) \le \left(\frac{\max\{1, C_1\}}{2} + C_2 + C_3\right)\|\varphi\|_t^2,$$

$$\mathcal{E}(t) \ge \left(\frac{\min\{1, C_1\}}{2} - C_2\right)\int_0^1 y_t^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 \left(\min\{1, C_1\}\left[1-\frac{\varpi^2}{3}\right] - C_2\right) y_{xx}^2\,\mathrm{d}x + KC_3\tau(t)\iint_{\hat{\Omega}} y_t^2(x,t-\eta\tau(t))\,\mathrm{d}x\,\mathrm{d}\eta + \min\{1, C_1\}w^\top Qw.$$

Thereafter, the estimates (48) hold provided that

$$C_2 < \min\{1, C_1\}\min\left\{1/2, 1-\frac{\varpi^2}{3}\right\}. \tag{50}$$

Let us turn now to the proof of (49). Differentiating (47) and invoking the estimates (38), (40) and (45), we obtain

$$\begin{aligned}\dot{\mathcal{E}}(t) &\le C_2\left[\frac{\varpi^2}{3}-3+\frac{\beta}{A_2}+\frac{\alpha}{2A_3}+\frac{1}{A_4}\right]\int_0^1 y_{xx}^2\,\mathrm{d}x\\
&+[-\hat{\beta}+C_2(1+\beta A_2)]y_t^2(1,t)\\
&+\left[\alpha\left(C_2A_3+\frac{1}{2A_1}\right)-(1-d)K\left(\frac{C_1}{2}+C_3\mathrm{e}^{-2\delta\bar{\tau}}\right)\right]\\
&\times\int_0^1 y_t^2(x,t-\tau)\,\mathrm{d}x\\
&+\left[\frac{\alpha A_1+C_1K}{2}+KC_3-C_2\right]\int_0^1 y_t^2\,\mathrm{d}x\\
&-2KC_3\delta\tau(t)\mathrm{e}^{-2\delta\bar{\tau}}\iint_{\hat{\Omega}} y_t^2(x,t-\eta\tau(t))\,\mathrm{d}x\,\mathrm{d}\eta\\
&+C_2A_4\left(e_1^\top w\right)^2-\zeta w^\top Nw. \qquad (51)\end{aligned}$$

Thanks to the assumption (9) of our theorem, it follows that $\frac{\varpi^2}{3}-3$ is negative, and hence, the coefficient

$$\frac{\varpi^2}{3}-3+\frac{\beta}{A_2}+\frac{\alpha}{2A_3}+\frac{1}{A_4}$$

of $\int_0^1 y_{xx}^2\,\mathrm{d}x$ can be made negative by choosing A_2, A_3 and A_4 large enough. Subsequently, in order to make the coefficient of $\int_0^1 y_t^2(x,t-\tau)\,\mathrm{d}x$ nonpositive, we choose $C_1=3\alpha$ and C_2 small enough such as

$$C_2\le\frac{1}{2A_3}\left(3K(1-d)-\frac{1}{A_1}\right), \qquad (52)$$

where A_1 is chosen large enough so that

$$3K(1-d)-\frac{1}{A_1}>0.$$

In fact, one can consider, for instance,

$$A_1>(3K(1-d))^{-1}.$$

Furthermore, the coefficients

$$-\hat{\beta}+C_2(1+\beta A_2)$$

of $y_t^2(1,t)$ are nonpositive, provided that

$$C_2\le\frac{\hat{\beta}}{1+\beta A_2}. \qquad (53)$$

Now, let us handle the two last terms in (51), namely

$$C_2A_4(e_1^\top w)^2-\zeta w^\top Nw.$$

We have

$$C_2A_4\left(e_1^\top w\right)^2\le C_2A_4\|e_1\|^2\|w\|^2$$

and

$$-\zeta w^\top Nw\le-\zeta\nu_{\min}\|w\|^2,$$

where $\nu_{\min}$ is the smallest positive eigenvalue of N. Whereupon, we conclude that

$$\begin{aligned}&-\zeta w^\top Nw+C_2A_4\left(e_1^\top w\right)^2\\&\quad\le(C_2A_4\|e_1\|^2-\zeta\nu_{\min})\|w\|^2,\end{aligned}$$

which can be made negative by choosing

$$C_2\le\frac{\zeta\nu_{\min}}{A_4\|e_1\|^2}. \qquad (54)$$

Finally, in order to assure the negativity of the coefficient of $\int_0^1 y_t^2\,\mathrm{d}x$, that is,

$$\frac{\alpha A_1+3\alpha K}{2}+KC_3-C_2<0,$$

it suffices to pick up $C_3<C_2/K$ and

$$\alpha<\alpha_0=2\frac{C_2-KC_3}{A_1+3K}. \qquad (55)$$

Thanks to the above choices, there exists a positive constant μ, independent on the initial conditions, such that $\dot{\mathcal{E}}(t)\le-\mu E_1(t)$, which, together with (37) and (48), implies that

$$\dot{\mathcal{E}}(t)\le-\frac{\mu}{2L_1}\mathcal{E}(t).$$

Thereby, the estimate (49) can be easily deduced.

Remark 3 It is worth mentioning that the stability result obtained in (49) and (32) holds without any restriction on the positive constant δ. Whence, one can tune δ such that the stability occurs as fast as possible.

3.2.1 Proof of Theorem 3

It suffices to recall the equivalence of the norms defined in (10) and (16) and use the estimates (48) and (49).

4 Stability of the global system

Now, we are ready to deal with the exponential stability of the original system (8) or equivalently (11). For sake of clarity, we shall provide a very brief presentation to some basic definitions and results which are going to be used for the proof of the main result.

Definition 1 [30] Let $U(t,s)$ be the evolution system corresponding to the operator $A(t)$ defined on a Hilbert space H. A continuous solution u of the integral equation

$$u(t) = U(t,s)v + \int_s^t U(t,r) f(r,u(r))\,dr,$$

will be called a mild solution of the initial-value problem

$$\begin{cases} u_t(t) = \mathcal{A}(t)u(t) + f(t,u(t)), \\ u(s) = v. \end{cases}$$

The following result will be used throughout this section:

Theorem 4 *[30] Let H be a Hilbert space and $f : [0,\infty] \times H \to H$ be continuous in t and locally Lipschitz continuous in u, uniformly in t on bounded intervals. If $U(t,s)$ is the evolution system corresponding to such that the operator $A(t)$, then for every $v \in H$ there exists a $T \le \infty$ such that the above initial-value problem has a unique mild solution u on $[0,T[$.*

The next result provides a sufficient condition for which the mild solution becomes a classical solution of our initial-value problem:

Theorem 5 *[30] Assume that $U(t,s)$ be the evolution system corresponding to the operator $A(t)$. If f is continuously differentiable, then the mild solution of the above initial-value problem with $v \in D(A)$ is a classical solution.*

We also state a result known as Barbalat's lemma:

Lemma 1 *[23] Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous real function on $[0,\infty)$. Suppose that $\int_0^\infty f(s)\,\mathrm{d}s$ exists and is finite. Then, $f(t) \to 0$ as $t \to \infty$.*

Finally, we give another useful inequality in the next lemma:

Lemma 2 *(Gronwall's lemma) Let $a \in L^1(0,T)$ such that $a(t) \ge 0$. If for some $K \ge 0$, the function $g \in L^\infty(0,T)$ satisfies the inequality*

$$g(t) \le K + \int_0^t a(s)g(s)\,\mathrm{d}s,$$

then

$$g(t) \le K \exp\left(\int_0^t a(s)\,\mathrm{d}s\right).$$

Our main result is as follows:

Theorem 6 *Assume that the desired angular velocity ϖ satisfies the smallness condition $|\varpi| < 3$ (see (9)). Then, for any initial data $\Phi_0 \in \mathcal{D}(\mathcal{A}(t)) \times \mathbb{R}$, there exists a positive constant α_0 (see (55)), independent of Φ_0, such that for any $\alpha < \alpha_0$, the solution $\Phi(t)$ of the closed-loop system (11) exponentially tends to the equilibrium point $(0_{\mathcal{Y}}, \varpi)$ in $\mathcal{X}$, as $t \to \infty$.*

Proof First of all, we point out that most of the arguments used in this proof run on much the same lines as in [7,8] (see also [24,33]) with of course a number of changes born out of necessity. Therefore, the exposition will be concise.

Recall that Proposition 1 leads us to claim, thanks to applying the evolution semigroups theory, that for $s \in [0,t]$, there exists a unique evolution system $U(t,s)$ associated with the operator $\mathcal{A}(t)$ [10,30]. This, together with the fact that $\mathcal{F}$ is continuously differentiable [33], yields the following assertion: For any $\Phi_0 = (\phi_0, \omega_0) \in \mathcal{U}$, there exists a unique local mild solution $\Phi(\cdot) = (\phi(\cdot), \omega(\cdot)) \in C([0,T_0];\mathcal{X})$ of (11), for some $T_0 > 0$ (see Definition 1 and Theorem 4). Indeed, Φ is given by the variation in constant formula. Moreover, a regularity result [30] leads us to conclude that each local solution of (11), with initial data in $\mathcal{D}(\mathcal{A}(t)) \times \mathbb{R}$, is a classical one (see Theorem 5). Then, using the same approach as in [32], one can show that that each strong solution exists globally and is bounded. Thus, $\int_0^{+\infty} (\omega(t) - \varpi)^2 \mathrm{d}t$ converges and the solution $(\phi(t), \omega(t))$ is bounded in $\mathcal{X}$. This implies, thanks to Barbalat's result (see Lemma 1), that $\lim_{t\to+\infty} \omega(t) = \varpi$, and hence, for any $\epsilon > 0$, there exists T sufficiently large such that for any $t \ge T$

$$|\omega^2(t) - \varpi^2| < \epsilon. \tag{56}$$

On the other hand, the solution

$$\begin{aligned}\Phi(t) &= (y(\cdot,t), y_t(\cdot,t), u(\cdot,t), w(t), \omega(t)) \\ &= (\Phi_1(t), \omega(t))\end{aligned}$$

of (11) stemmed from $\Phi_s = (\Phi_1(s), \omega(s)) \in \mathcal{D}(\mathcal{A}(t)) \times \mathbb{R}$ ($s \in [0,t]$) can be partially obtained from the variation in constants formula [10,30]

$$\Phi_1(t) = U(t,s)\Phi_1(s) + \int_s^t U(t,r)(\omega^2(r) - \varpi^2) \times P\Phi_1(r)\,dr \tag{57}$$

for any $t \geq s \geq 0$. Here P is the compact operator on $\mathcal{V}$ defined by

$$P(y, z, u, w) = (0, y, 0, 0),$$

for any $(y, z, u, w) \in \mathcal{Y}$. In turn, the second part of the solution $\Phi(t)$, namely $\omega(t)$, is given by the differential equation in (8). Using the boundedness of P and invoking (56) and (32), the identity (57) gives

$$\|\Phi_1(t)\|_{\mathcal{Y}} \leq M_0 e^{-\kappa_0(t-T)} \|\Phi_1(T)\|_{\mathcal{Y}} + \epsilon M_0 \int_T^t e^{-\kappa_0(t-s)} \|\Phi_1(s)\|_{\mathcal{Y}} \, ds, \qquad \forall\, t \geq T. \quad (58)$$

Now, it suffices to applying Gronwall's Lemma 2 to (58) to get

$$\|\Phi_1(t)\|_{\mathcal{Y}} \leq M_0 \|\Phi_1(T)\|_{\mathcal{Y}} e^{-(\kappa_0 - \epsilon M_0)(t-T)}, \quad \forall\, t \geq T. \quad (59)$$

This gives rise to the exponential stability of $\Phi_1(t)$ in $\mathcal{Y}$ as long as $\epsilon < \frac{M_0}{\kappa_0}$, which is possible in view of the arbitrariness of ϵ (see (56)). Finally, going back to (8), one can establish the exponential convergence of $\omega(t) - \varpi$ in $\mathbb{R}$ by arguing as in [24,33] (see also [7,8]).

5 Numerical application

The aim of this section is to illustrate the outcome stated and proved in the previous section via some numerical examples. More precisely, we shall give prominence to the importance of the smallness condition $|\varpi| < 3$ related to the exponential stability result of the closed-loop system (11). To do so, we assume that $\omega(t) = \varpi > 0$, $\tau(t) = \tau > 0$ and the boundary force control $\mathcal{U}_F(t)$ is static. Thereafter, it is a simple task to check that λ is an eigenvalue of the system if and only if there exists a nonzero (y, z, u) such that

$$\begin{cases} -y_{xxxx} + \varpi^2 y + \alpha u(x, 1) = \lambda z, \\ u_\eta(x, \eta) + \lambda \tau u(x, \eta) = 0, \\ z = \lambda y, \\ y(0) = y_x(0) = y_{xx}(1) = 0, \\ y_{xxx}(1) = \lambda \beta y(1), \\ u(x, 0) = z, \end{cases}$$

which implies that one just has to seek a nontrivial solution y of the following system

$$\begin{cases} y_{xxxx} + (\lambda^2 - \varpi^2 - \alpha \lambda e^{-\tau\lambda}) y = 0, \\ y(0) = y_x(0) = y_{xx}(1) = 0, \\ y_{xxx}(1) = \lambda \beta y(1). \end{cases}$$

Let

$$\sigma^4 = \varpi^2 - \lambda^2 + \alpha \lambda e^{-\tau\lambda}. \quad (60)$$

Hereby, one can show that λ is an eigenvalue of the system if and only if λ is a solution of the characteristic equation

$$\sigma^3 (1 + \cosh\sigma \cos\sigma) + \beta\lambda (\cosh\sigma \sin\sigma - \sinh\sigma \cos\sigma) = 0. \quad (61)$$

Next, let $\sigma = \sigma_1 + i\sigma_2$ and $\lambda = \lambda_r + i\lambda_m$, where σ_i as well as λ_r and λ_m are real numbers for $i = 1, 2$. Whereupon, (60) yields

$$\begin{cases} \sigma_1^4 + \sigma_2^4 - \sigma_1^2\sigma_2^2 \\ \quad = \alpha e^{-\tau\lambda_r} [\lambda_r \cos(\tau\lambda_m) + \lambda_m \sin(\tau\lambda_m)] \\ \quad + \varpi^2 - \lambda_r^2 + \lambda_m^2, \\ 4\sigma_1\sigma_2 \left(\sigma_1^2 - \sigma_2^2\right) \\ \quad = \alpha e^{-\tau\lambda_r} [\lambda_m \cos(\tau\lambda_m) - \lambda_r \sin(\tau\lambda_m)] \\ \quad - 2\lambda_r\lambda_m. \end{cases}$$

Inserting the above equations into (61), the latter can be reformulated into the form $\Gamma_1(\lambda_r, \lambda_m) + i\Gamma_2(\lambda_r, \lambda_m) = 0$, in which Γ_1 and Γ_2 are two real functions of the unknowns λ_r and λ_m. Clearly, $\lambda = \lambda_r + i\lambda_m$ is an eigenvalue if and only if both Γ_1 and Γ_2 are zero functions. Lastly, thanks to MAPLE 13, one can get the approximate values of the pairs (λ_r, λ_m) for which the graphs of $\Gamma_1(\lambda_r, \lambda_m) = 0$ and $\Gamma_2(\lambda_r, \lambda_m) = 0$ intersect. Then, it is apparent from Fig. 2 that the spectrum of the system with $\alpha = 1$, $\beta = 3$, $\tau = 0.01$ and $\varpi = 2$

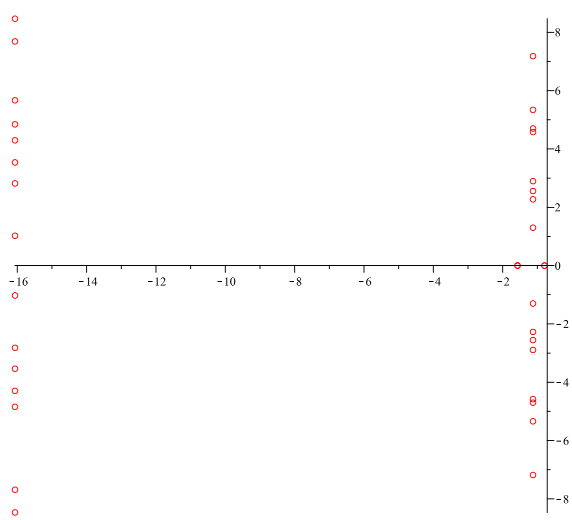

Fig. 2 Spectrum with $\alpha = 1$, $\beta = 3$, $\tau = 0.01$ and $\varpi = 2$

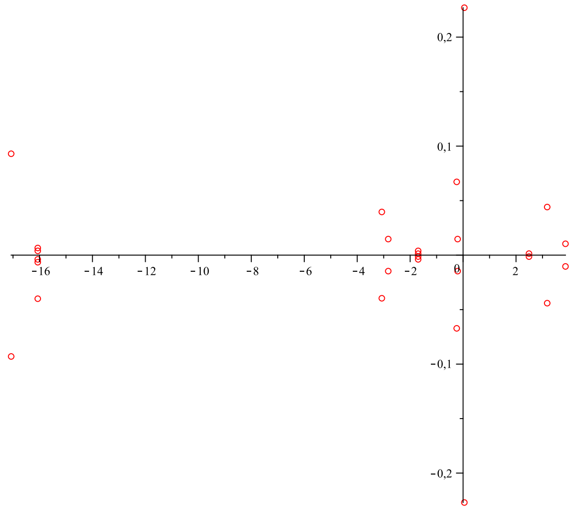

Fig. 3 Spectrum with $\alpha = 1,\ \beta = 3,\ \tau = 0.01$ and $\varpi = 3.2$

has two branches and more importantly has negative real part. Thus, the system is stable. This numerical result validates Theorem 6. Notwithstanding, the case $\varpi = 3.2$ generates some eigenvalues with positive real part as depicted in Fig. 3. This is due to the fact that the condition $|\varpi| < 3$ of Theorem 6 is violated.

6 Concluding discussion

In this article, the well-known rotating disk-beam system has been considered under the presence of a time-dependent interior delayed damping control. Despite the presence of this delay which could be a source of poor performance and instability, it has been proved that the effect of such a delay can be compensated by the action of a boundary force control applied at the free end of the beam, in addition of the torque control exerted on the disk. Indeed, the closed-loop system is shown to have the very desirable property, namely the exponential stability, provided that the delay satisfies appropriate conditions and the angular velocity of the disk does not exceed the value 3. Numerical examples are provided to demonstrate the correctness of the results.

We would like to point out that there is a promising research avenue by considering a time-dependent delay in the boundary force control. Then, it would be very interesting to reduce the impact of such a delay via the presence of an interior damping control.

Acknowledgments The author would like to thank the Associate Editor and the anonymous referees for their constructive comments and valuable suggestions which have led to an improved version of this paper. Also, the author is grateful to one of the referees for having mentioned the recent articles [14–18]. Finally, the author acknowledges the support of Sultan Qaboos University.

References

1. Baillieul, J., Levi, M.: Rotational elastic dynamics. Physica **27**, 43–62 (1987)
2. Bloch, A.M., Titi, E.S.: On the dynamics of rotating elastic beams. In: Conte, Perdon, Wyman (eds.) Proceedings of the Conference on New Trends System Theory, Genoa, Italy. Birkhäuser, Cambridge, MA (1990)
3. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitex, Springer, Berlin (2011)
4. Brogliato, B., Lozano, R., Maschke, B., Egeland, O.: Dissipative Systems Analysis and Control: Theory and Applications. Springer, London (2007)
5. Chentouf, B., Couchouron, J.F.: Nonlinear feedback stabilization of a rotating body-beam without damping. In: ESAIM: COCV, vol. 4, pp. 515–535 (1999)
6. Chentouf, B., Wang, J.M.: Stabilization and optimal decay rate for a non-homogeneous rotating body-beam with dynamic boundary controls. J. Math. Anal. Appl. **318**, 667–691 (2006)
7. Chentouf, B.: Stabilization of the rotating disk-beam system with a delay term in boundary feedback. Nonlinear Dyn. **78**(3), 2249–2259 (2014)
8. Chentouf, B.: Stabilization of a nonlinear rotating flexible structure under the presence of time-dependent delay term in a dynamic boundary control. IMA J. Math. Control Inf. doi:10.1093/imamci/dnu044
9. Chentouf, B.: Compensation of the interior delay effect for a rotating disk-beam system. IMA J. Math. Control Inf. doi:10.1093/imamci/dnv018
10. Chicone, C., Latushkin, Y.: Evolution Semigroups in Dynamical Systems and Differential Equations. Mathematical Surveys and Monographs, vol. 70. American Mathematical Society, Providence, RI (1999)
11. Coron, J.-M., d'Andréa-Novel, B.: Stabilization of a rotating body-beam without damping. IEEE Trans. Autom. Control **43**(5), 608–618 (1998)
12. Dafermos, C.M.: Asymptotic stability in viscoelasticity. Arch. Ration. Mech. Anal. **37**, 297–308 (1970)
13. Fridman, E.E., Nicaise, S., Valein, J.: Stabilization of second order evolution equations with unbounded feedback with time-dependent delay. SIAM J. Control Optim. **48**(8), 5028–5052 (2010)
14. He, W., Ge, S.S., How, B.V.E., Choo, Y.S., Hong, K.S.: Robust adaptive boundary control of a flexible marine riser with vessel dynamics. Automatica **47**, 722–732 (2011)
15. He, W., Ge, S.S.: Vibration control of a nonuniform wind turbine tower via disturbance observer. IEEE/ASME Trans. Mechatron. **20**, 237–244 (2015)
16. He, W., Sun, C., Ge, S.S.: Top tension control of a flexible marine riser by using integral-barrier Lyapunov function. IEEE/ASME Trans. Mechatron. **20**, 497–505 (2015)

17. He, W., Ge, S.S.: Vibration control of a flexible beam with output constraint. IEEE Trans. Ind. Electron. **2**, 5023–5030 (2015)
18. Jin, F.F., Guo, B.Z.: Lyapunov approach to output feedback stabilization for the Euler Bernoulli beam equation with boundary input disturbance. Automatica **52**, 95–102 (2015)
19. Kato, T.: Linear evolution equations of "hyperbolic" type. J. Fac. Sci. Univ. Tokyo Sect I. **17**, 241–258 (1970)
20. Kato, T.: Linear evolution equations of "hyperbolic" type. II. J. Math. Soc. Jpn. **25**, 648–666 (1973)
21. Kato, T.: Linear and quasilinear equations of evolution of hyperbolic type. In: da Prato, G., Geymonat, G. (eds.) Hyperbolicity: Lectures Given at a Summer School of the Centro Internazionale Matematico Estivo (C.I.M.E.), Cortona (Arezzo), Italy, Volume 2, pp. 125–191. Liguori Editore (1977)
22. Kato, T.: Abstract Differential Equations and Nonlinear Mixed Problems, Lezioni Fermiane [Fermi Lectures]. Scuola Normale Superiore, Pisa (1985)
23. Khalil, H.K.: Nonlinear Systems, 3rd edn. Prentice-Hall, Englewood Cliffs (2002)
24. Laousy, H., Xu, C.Z., Sallet, G.: Boundary feedback stabilization of a rotating body-beam system. IEEE Trans. Autom. Control **41**, 241–245 (1996)
25. Morgül, O.: Orientation and stabilization of a flexible beam attached to a rigid body: planar motion. IEEE Trans. Autom. Control **36**, 953–963(1991)
26. Morgül, O.: Constant angular velocity control of a rotating flexible structure. In: Proceedings of the 2nd Conference, ECC'93, pp. 299–302, Groningen, Netherlands (1993)
27. Morgül, O.: An exponential stability result for the wave equation. Automatica **38**, 731–735 (2002)
28. Nicaise, S., Pignotti, C., Valein, J.: Stability of the heat and of the wave equations with boundary time-varying delays. Discret. Contin. Dyn. Syst. Ser. S **4**, 693–722 (2011)
29. Nicaise, S., Valein, J., Fridman, E.: Stability of the heat and of the wave equations with boundary time-varying delays. Discret. Contin. Dyn. Syst. Ser. S **2**, 559–581 (2009)
30. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)
31. Shang, Y.F., Xu, G.Q., Chen, Y.L.: Stability analysis of Euler–Bernoulli beam with input delay in the boundary control. Asian J. Control **14**, 186–196 (2012)
32. Xu, C.Z., Sallet, G.: Boundary stabilization of a rotating flexible system. In: Curtain, R.F., Bensoussan, A., Lions, J.L. (eds.) Lecture Notes in Control and Information Sciences, vol. 185, pp. 347–365. Springer, New York (1992)
33. Xu, C.Z., Baillieul, J.: Stabilizability and stabilization of a rotating body-beam system with torque control. IEEE Trans. Autom. Control **38**, 1754–1765 (1993)